Evaluation and Characters of Dual Binary Pseudoframes with Filter Functions

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Abstract

The frame theory has been one of powerful tools for researching into wavelets. In the article, the notion of affine bivariate pseudoframes is introduced. The concept of a bivariate generalized multiresolution analysis is developed. A novel approach for constructing one GMRA of Paley-Wiener subspaces of \( L^2(\mathbb{R}^2) \) is presented. The sufficient condition for the existence of a class of bivariate pseudoframes with filter banks is obtained by virtue of a generalized multiresolution analysis. The pyramid decomposition scheme is provided based on such a generalized multiresolution analysis.

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1. Introduction

As a redundant wavelet system, a wavelet frame may have many desirable properties and is of interest in application such as signal processing and numerical analysis. The rise of frame theory in applied mathematics is due to the flexibility and redundancy of frames. The structured frames are much easier to construc than structured orthonormal bases. The concept of frames was introduced by Duffin and Schaeffer¹ and popularized greatly by the work of Daubechies and her coauthors²³. After this ground breaking work, the theory of frames began to be more widely studied both in theory and in applications⁴⁻⁷, such as signal processing, image processing, data compression and sampling theory. Frames have become the focus of active research, both in theory and in applications. Every frame (or Bessel sequence) determines an analysis operator, the range of which is important for a lumber of applications.

The notion of Frame Multiresolution Analysis (FMRA) as described by [5] generalizes the notion of
Multiresolution Analysis by allowing non-exact affine frames. However, subspaces at different resolutions in a FMRA are still generated by a frame formed by translates and dilates of a single function. This article is motivated from the observation that standard methods in sampling theory provide examples of multiresolution structure which are not FMRAs. Inspired by [5] and [7], we introduce the notion of a Generalized Multiresolution Structure (GMRA) of \( L^2(\mathbb{R}^2) \) generated by several functions of integer translates in \( L^2(\mathbb{R}^2) \). We have that the GMRA has a pyramid decomposition scheme and obtain a frame-like decomposition based on such a GMRA. It also lead to new constructions of affine frames of \( L^2(\mathbb{R}^2) \).

By \( H \), we denote a separable Hilbert space. We recall that a sequence \( \{ f_v : v \in \mathbb{Z}^2 \} \subseteq H \) is a frame for \( H \), if there exist positive real numbers \( C_1, C_2 \) such that

\[
\forall g \in H, C_1 \|g\|^2 \leq \sum_{v \in \mathbb{Z}^2} |\langle g, f_v \rangle|^2 \leq C_2 \|g\|^2.
\]  

(1)

A sequence \( \{ f_v \} \subseteq H \) is a Bessel sequence if (only) the upper inequality of (1) follows. If only for all \( g(t) \in \Gamma \subseteq H \), the upper inequality of (1) holds, the sequence \( f_v \subseteq H \) is a Bessel sequence with respect to (w.r.t.) the subspace \( \Gamma \). If \( \{ f_v \} \) is a frame of a Hilbert space \( H \), there exist a dual frame \( \{ f_v^* \} \) such that

\[
\forall g \in H, g = \sum_{v \in \mathbb{Z}^2} \langle g, f_v \rangle f_v^* = \sum_{v \in \mathbb{Z}^2} \langle g, f_v^* \rangle f_v.
\]

(2)

2. A Generalized multiresolution analysis

We consider the case of generators, which yield pseudoframes of integer grid translates for subspaces of \( L^2(\mathbb{R}^2) \). Let \( \{ \tau_n \phi \} \) and \( \{ \tau_n \tilde{\phi} \} \) \( (n \in \mathbb{Z}^2) \) be two sequences in \( \subseteq L^2(\mathbb{R}^2) \). Let \( \Omega \) be a closed subspace of \( L^2(\mathbb{R}^2) \). We say that \( \{ \tau_n \phi \} \) forms a pseudoframe for the subspace \( \Omega \) with respect to (w.r.t.) \( \{ \tau_n \tilde{\phi} \} \) \( (n \in \mathbb{Z}^2) \) if

\[
\forall h(t) \in \Omega, h(t) = \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \phi \rangle \tau_n \tilde{\phi}(t).
\]  

(3)

where we define a translate operator, \( (\tau_n f)(t) = f(t - n), \ n \in \mathbb{Z}^2, \) for an any function \( f(t) \in L^2(\mathbb{R}^2) \). It is important to note that \( \{ \tau_n \phi(v) \} \) and \( \{ \tau_n \tilde{\phi}(v) \} \) \( (n \in \mathbb{Z}^2) \) need not be contained in \( \Omega \). Consequently, the positions of \( \tau_n \phi(t) \) and \( \tau_n \tilde{\phi}(t) \) are not generally “commutable” [7], i.e., there exists \( h(t) \in \Omega \) such that

\[
\sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \tilde{\phi} \rangle \tau_n \phi(t) \neq \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \phi \rangle \tau_n \tilde{\phi}(t) = h(t).
\]

However, in the context of the affine structure, the commutativity in the above sense is easily achievable.

**Definition 1.** A Generalized multiresolution analysis (GMRA) \( \{ V_n, \phi(t), \tilde{\phi}(t) \} \) is a sequence of closed linear subspaces \( \{ V_n \}_{n \in \mathbb{Z}} \) of \( L^2(\mathbb{R}^2) \) and 2 elements \( \phi(t), \tilde{\phi}(t) \in L^2(\mathbb{R}^2), v \in \Lambda \) such that

(a) \( V_n \subseteq V_{n+1}, \forall n \in \mathbb{Z} ; \) (b) \( \bigcap_{n \in \mathbb{Z}} V_n = \{0\} ; \) \( \bigcup_{n \in \mathbb{Z}} V_n \) is dense in \( L^2(\mathbb{R}^2) \);
(c) \( h(t) \in V_n \) if and only if \( Dh(t) \in V_{n+1}, \forall n \in \mathbb{Z} \), where \( Df(t) = 2f(2t) \), for \( \forall f(t) \in L^2(\mathbb{R}^2) \);

(d) \( h(t) \in V_0 \) implies \( \tau_n h(t) \in V_0 \), for all \( n \in \mathbb{Z}^2 \); (e) \( \{\tau_n \phi(t), n \in \mathbb{Z}^2\} \) constitutes a pseudoframe for \( V_0 \) with respect to \( \{\tau_n \phi(t), n \in \mathbb{Z}^2\} \).

3. A Generalized multiresolution analysis of paley-wiener subspaces

A necessary and sufficient condition for the construction of pseudoframe of translation for Paley-Wiener subspaces is presented as follow.

**Theorem 1.** Let \( \phi(t) \in L^2(\mathbb{R}^2) \) be such that \( |\hat{\phi}(\omega)| > 0 \) a.e. on a connected neighborhood of \( 0 \) in \( [-\frac{1}{2}, \frac{1}{2}]^2 \) and \( |\hat{\phi}(\omega)| = 0 \) a.e. otherwise. Define \( \Delta = \bigcap \{\omega \in \mathbb{R}^2, |\hat{\phi}(\omega)| \geq c > 0\} \), and \( V_0 = PW_\Delta = \{h(t) \in L^2(\mathbb{R}^2) : \text{supp}(\hat{h}) \subseteq \Delta\} \). Then, for \( \hat{\phi} \in L^2(\mathbb{R}^2) \), \( \{\tau_n \phi, n \in \mathbb{Z}^2\} \) is a pseudoframe of translates for \( V_0 \) with respect to \( \{\tau_n \hat{\phi}, n \in \mathbb{Z}^2\} \) if and only if

\[
\mathcal{F}(\phi(t) \hat{\phi}(\omega) \cdot \chi_\Delta(\omega) = \chi_\Delta(\omega) \quad \text{a. e.,}
\]

where \( \chi_\Delta \) is the characteristic function on \( \Delta \), and the Fourier transform of an integrable function \( g(t) \in L^1(\mathbb{R}^2) \cap L^2(\mathbb{R}^2) \) defined by

\[
\mathcal{F}g(\omega) = \hat{g}(\omega) = \int_{\mathbb{R}^2} g(t)e^{-2\pi i t \omega} dt, \omega \in R^2
\]

which, as usual, can be naturally extended to functions in \( L^1(\mathbb{R}^2) \). For a sequence \( c = \{c(n)\} \in l^2(\mathbb{Z}^2) \), we define the discrete-time Fourier transform as the function in \( L^1(0,1)^2 \) given by

\[
\mathcal{F}c(\omega) = C(\omega) = \sum_{k \in \mathbb{Z}^2} c(n)e^{-2\pi i k \omega},
\]

Note that the discrete-time Fourier transform is 1-periodic.

Moreover, if \( \hat{\phi} \) are also such that \( |\hat{\phi}| > 0 \) a.e. on a connected neighborhood of \( 0 \) in \( [-\frac{1}{2}, \frac{1}{2}]^2 \), and \( |\hat{\phi}| = 0 \) a.e. otherwise, that (6) holds, then \( \{\tau_n \phi(t), n \in \mathbb{Z}^2\} \) and \( \{\tau_n \hat{\phi}(t), n \in \mathbb{Z}^2\} \) are a pair of commutative pseudoframes for \( \Omega \), i.e.,

\[
\forall h(t) \in \Omega, h(t) = \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \hat{\phi} \rangle \tau_n \phi(t)
\]

\[
= \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \hat{\phi} \rangle \tau_n \hat{\phi}(t)
\]
Proof of Theorem 1. For all \( h(t) \in PW_\Lambda \), consider
\[
\mathcal{F}\left( \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \hat{\phi} \rangle \tau_n \phi \right) = \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \hat{\phi} \rangle \mathcal{F}(\tau_n \phi) \\
= \sum_{n \in \mathbb{Z}^2} \int_{\mathbb{R}} \hat{h}(\mu) \hat{\phi}(\mu) e^{2\pi i \mu \cdot \frac{1}{2}} d\mu \hat{\phi}(\omega) e^{-2\pi i \omega \cdot \frac{1}{2}} \\
= \sum_{n \in \mathbb{Z}^2} \int_{\mathbb{R}} \hat{h}(\mu + n) \hat{\phi}(\mu + n) e^{2\pi i \mu \cdot \frac{1}{2}} d\mu \hat{\phi}(\omega) e^{-2\pi i \omega \cdot \frac{1}{2}} \\
= \phi(\omega) \sum_{n \in \mathbb{Z}^2} \hat{h}(\omega + n) \hat{\phi}(\omega + n) = \hat{h}(\omega) \hat{\phi}(\omega) \hat{\phi}(\omega)
\]
where we have used the fact that \( \hat{\phi} \neq 0 \) only on \( \left[ -\frac{1}{2}, \frac{1}{2} \right] \) and that \( \sum_{n \in \mathbb{Z}^2} \hat{h}(\omega + n) \hat{\phi}(\omega + n) \) is \( Z^2 \)-periodic. Therefore \( \hat{\phi} \cdot \chi_\Lambda = \chi_\Lambda \) a.e., is a necessary and sufficient condition for \( \{\tau_n \phi, n \in Z^2\} \) to be a pseudo frame for \( V_0 \) with respect to \( \{\tau_n \hat{\phi}, n \in Z^2\} \). Direct calculation also shows that (7) is satisfied if \( \hat{\phi} \) and \( \hat{h} \) satisfy supported conditions specified in Theorem 1. Thereof, \( \{\tau_n \phi, n \in Z^2\} \) and \( \{\tau_n \hat{\phi}, n \in Z^2\} \) are a pair of multiple pseudo frames for \( \Omega \).

Proposition 1[8]. Let \( \{\tau_n \phi(t)\} \) be a pseudo frame for \( V_0 \) with respect to \( \{\tau_n \hat{\phi}(t)\} \). Define \( V_l \) by
\[
V_l = \{ f(t) \in L^2(\mathbb{R}^2) : f(t / 2^l) \in V_0 \}, l \in Z^2,
\]
Then, \( \{\phi_{l,n}\} \) is a pseudo frame of translates for \( V_l \) with respect to \( \{\phi_{l,n}\} \), where \( f(t) \equiv 2^{l/2} f(2^l t - n), \) for \( \forall f(t) \in L^2(\mathbb{R}^2) \).

Theorem 2. Let \( \phi(t), \hat{\phi}(t) \in L^2(\mathbb{R}^2) \) have the properties specified in Theorem 1 such that the condition (6) is satisfied. Assume that \( V_n \) is defined by (8). Then \( \{V_n, \phi(t), \hat{\phi}(t)\} \) forms a GMRA.

Proof. We need to prove four axioms in Definition 1. The inclusion \( V_n \subseteq V_{n+1} \) follows from the fact that \( V_n \) defined by (8) is equivalent to \( PW_{2^\Lambda} \), and \( PW_{2^\Lambda} \subseteq PW_{2^\Lambda} \). Condition (b) is satisfied because the set of all band-limited signals is dense in \( L^2(\mathbb{R}^2) \). On the other hand, the intersection of all band-limited signals is the trivial function. Condition (c) is an immediate consequence of (8). For condition (d), if \( h(t) \in V_0 \), then \( h(t) = \sum_{n \in \mathbb{Z}^2} \langle h, \tau_n \hat{\phi}(t) \rangle \tau_n \phi(t) \). By taking variable substitution, for \( \forall n \in Z^2 \),
\[
h(t-n) = \sum_{k \in \mathbb{Z}^2} \langle h(\cdot), \tau_n \hat{\phi}(\cdot-k) \rangle \phi(t-k-n) \\
= \sum_{k \in \mathbb{Z}^2} \langle h(\cdot-k), \tau_n \hat{\phi}(\cdot-k) \rangle \phi(t-k) \\
That is, \( \tau_n h = \sum_{k \in \mathbb{Z}^2} \langle \tau_n h, \tau_k \hat{\phi} \rangle \tau_k \phi \).
\]
Or, it is a fact \( \mathcal{F}(\tau_k \phi(t)) \) has support in \( \Omega \) for arbitrary \( k \in Z^2 \). Therefore, \( \tau_n h \in V_0 \).
Example Let $\varphi(t) \in L^2(R^2)$ be such that

$$\hat{\varphi}(\omega) = \begin{cases} 1, & \text{a.e., } \|\omega\| \leq \frac{1}{4}, \\ (3 - 8\|\omega\|), & \text{a.e., } \frac{1}{4} \leq \|\omega\| < \frac{3}{8}, \\ 0, & \text{otherwise.} \end{cases}$$

where $\|\omega\|$ means the norm of $\omega$, and $j \in \Lambda$. Choose

$$\Delta \equiv \{\omega \in R^2 : |\hat{\varphi}(\omega)| \geq 1\} = \left[\frac{1}{4}, \frac{1}{4}\right],$$

and define $V_0 = PW_\Delta$. Now, select $\hat{\varphi}_j \in L^2(R^2)$ such that

$$\hat{\varphi}_j(\omega) = \begin{cases} 1, & \text{a.e., } \|\omega\| \leq \frac{1}{4}, \\ 5 - 16\|\omega\|, & \text{a.e., } \frac{1}{4} \leq \|\omega\| < \frac{5}{16}, \\ 0, & \text{otherwise.} \end{cases}$$

Then, since $\sum_{v \in \Lambda} \hat{\varphi}_v \cdot \hat{\varphi}_v = 1$ a.e., on $\Delta$, by Theorem 1, $\{\tau_n \varphi_v\}$ and $\{\tau_n \tilde{\varphi}_v\}$ form a pair of pseudoframes for $V_0 = PW_\Delta$. Further, define $V_n$ as in (9), $\{V_n, \varphi_v, \tilde{\varphi}_v\}$ forms a generalized multi-resolution analysis.

4. The scaling relationships with filter banks

The familiar scaling relationships are associated with GMRA between dilates of the function $\varphi(t)$ as well as that of $\hat{\varphi}(t)$ still hold in GMRA. Define filter functions $P(\omega)$ and $\tilde{P}(\omega)$ by

$$P(\omega) = \sum_{n \in Z^2} p(n) e^{-2\pi i n \omega} \quad \text{and} \quad \tilde{P}(\omega) = \sum_{n \in Z^2} \tilde{p}(n) e^{-2\pi i n \omega}$$

of the sequences $p = \{p(n)\}$ and $\tilde{p} = \{\tilde{p}(n)\}$, respectively, wherever the sum is defined.

Proposition 2 Let $\{p(n)\}_{n \in Z^2}$ be such that $P(0) = 2$ and $P(\omega) \neq 0$ in a neighborhood of 0. Assume also that $|P(\omega)| \leq 2$. Then there exist $\varphi \in L^2(R)$ such that

$$\varphi(t) = 2 \sum_{n \in Z^2} p(n) \varphi(2t - n). \quad (9)$$

Similarly, there exist one scaling relationship for $\hat{\varphi}(t)$ under the same conditions as that of $p$ for a seq. $\tilde{p}$, i.e.,

$$\tilde{\varphi}(t) = 2 \sum_{n \in Z^2} \tilde{p}(n) \tilde{\varphi}(2t - n). \quad (10)$$
In terms of filters $p(\omega)$ and $\check{p}(\omega)$, Theorem 1 becomes the following:

**Corollary 1**[8] Suppose that $p(\omega)$ and $\check{p}(\omega)$ generate $\phi(t)$ and $\check{\phi}(t)$ as in equations (9) and (10), respectively. Assume $\phi(t)$, $\check{\phi}(t)$ have the properties specified in Theorem 1. Then

\[ \{\tau_n\phi(t), n \in \mathbb{Z}^2\} \]

forms a pseudoframe of translates for $V_0$ with respect to $\{\tau_n\check{\phi}(t), n \in \mathbb{Z}\}$ if and only if

\[ \overline{P}(\omega)\check{P}(\omega) \chi_{\mathbb{N}/2}(\omega) = 4 \chi_{\mathbb{N}/2}(\omega) \text{ a. e.}, \tag{11} \]

5. Affine frames of space $L^2(\mathbb{R}^2)$

**Definition 2.** Let $\{V_n, \phi(t), \check{\phi}(t)\}$ be a given GMRA, and let $\psi_1(t)$ and $\check{\psi}_1(t)$ be 2 band-pass functions in $L^2(\mathbb{R}^2)$. We say $\{\tau_n\phi(t), \tau_n\check{\phi}(t), n \in 1, 2, 3\}$ forms a pseudoframe of translates for $V_1$ w. r. t. $\{\tau_n\phi(t), \tau_n\check{\phi}(t), n \in 1, 2, 3\}$ if for

\[ f(t) = \sum_{n \in \mathbb{Z}^2} \langle f, \tau_n\phi(t) \rangle \tau_n\phi(t) + \sum_{l \in \Lambda} \sum_{n \in \mathbb{Z}^2} \langle f, \tau_n\check{\phi}(t) \rangle \tau_n\check{\phi}(t), \tag{12} \]

where $\Lambda = \{1, 2, 3\}$. Accordingly, $\{\tau_n\phi(t), \tau_n\psi_1(t), l \in \Lambda\}$ is called a dual pseudo-frame to $\{\tau_n\phi(t), \tau_n\check{\phi}(t), n \in \Lambda\}$ in the sense of the above formula (12).

**Proposition 3**[8]. Let $\{V_n, \phi(t), \check{\phi}(t)\}$ be a given GMRA, and let $\{\tau_n\phi(t), \tau_n\check{\psi}_1(t), l \in \Lambda\}$ be a pseudoframe of translates for $V_1$ w. r. t. $\{\tau_n\phi(t), \tau_n\check{\psi}_1(t), l \in \Lambda\}$. Then, for each $n \in \mathbb{Z}^2$, the family of functions $\{\phi_{n,k}, \psi_{ln,k}, l \in \Lambda\}$ forms a pseudoframe of translates for $V_{n+1}$ with respect to $\{\tilde{\phi}_{l,k}, \psi_{ln,k}, l \in \Lambda\}$, i.e.,

\[ \forall f(t) \in V_{n+1}, \]

\[ f(t) = \sum_{l \in \Lambda} \sum_{k \in \mathbb{Z}^2} \langle f, \tilde{\phi}_{l,k} \rangle \tilde{\phi}_{l,k} + \sum_{l \in \Lambda} \sum_{k \in \mathbb{Z}^2} \langle f, \psi_{ln,k} \rangle \psi_{ln,k}. \tag{13} \]

To characterize the condition for which $\{\tau_n\phi(t), \tau_n\check{\psi}_1(t), l \in \Lambda\}$ forms a pseudoframe of translates for $V_1$ with respect to $\{\tau_n\phi(t), \tau_n\check{\psi}_1(t), l \in \Lambda\}$, we begin with developing the “wavelet equations” associated with “band-pass” functions $\psi_1(t)(l \in \Lambda)$ and $\check{\psi}_1(t)(l \in \Lambda)$ based on a GMRA, namely,

\[ \psi_1(t) = 2 \sum_{k \in \mathbb{Z}^2} q_1(k)\phi(2t-k) \text{ in } L^2(\mathbb{R}^2), l \in \Lambda \tag{13} \]

and

\[ \check{\psi}_1(t) = 2 \sum_{k \in \mathbb{Z}^2} q_1(k)\check{\phi}(2t-k) \text{ in } L^2(\mathbb{R}^2), l \in \Lambda \tag{14} \]

Similar to Proposition 3, we have the following fact.

**Proposition 4**[7]. Let $\{q_l(k), l \in \Lambda\}$ be such that $Q_l(0) = 0$ and $Q_l(\omega) \in L^\infty(T)$, where $T = [0, 1]^2$. Let $\phi(t) \in L^2(\mathbb{R}^2)$ and be defined by (10). Assume that $\{p(k)\}$ satisfies the conditions in Proposition 3. Then there exist $\psi_1(t) \in L^2(\mathbb{R}^2), l \in \Lambda$ generated from (14).
Let $\chi_\Delta(\omega)$ be the characteristic function of the interval $\Delta$ defined in Proposition 1. We shall use the below 1-periodic function
\[
\Gamma_\Delta(\omega) = \sum_k \chi_\Delta(\omega + k)
\] (15)

**Theorem 3**[8]. Let $\Delta$ be the bandwidth of the subspace $V_0$ defined in Theorem 1. \{$\tau_\omega \phi(t), \tau_\omega \psi(t), l \in \Lambda$\} forms a pseudo-frame of translates for $V_1$ with respect to \{$\tau_\omega \phi(t), \tau_\omega \psi(t), l \in \Lambda$\} if and only if there exist functions $B(\omega)$ and $D_l(\omega), l \in \Lambda$ in $L^2([0,1])$ such that
\[
\frac{B(\omega)}{2} \hat{P}(\omega) + \sum_{l \in \Lambda} D_l(\omega) \hat{Q}_l(\omega) \Gamma_\Delta(\omega) = 4 \Gamma_\Delta(\omega); \\
\frac{B(\omega)}{2} \hat{P}(\omega + \frac{1}{2}) + \sum_{l \in \Lambda} D_l(\omega) \hat{Q}_l(\omega + \frac{1}{2}) \Gamma_\Delta(\omega) = 0
\] (16.1) (16.2)

**Theorem 4**. Let $\phi(t), \tilde{\phi}(t), \psi(t), l \in \Lambda$ be functions in $L^2(R^2)$ defined by (14), (15), (18) and (19), respectively. Assume that conditions in Theorem 3 are satisfied. Then, for any function $h \in L^2(R^2)$, and any integer $n$,
\[
\sum_{k \in \mathbb{Z}} \langle h, \tilde{\phi}_{n,k} \rangle \phi_{n,k}(t) = \sum_{l=1}^{g-1} \sum_{k \in \mathbb{Z}} \sum_{v=-\infty}^{\infty} \langle h, \tilde{\psi}_{l,v,k} \rangle \psi_{l,v,k}(t).
\] (18)
Furthermore,
\[
\forall h \in L^2(R^2), h(t) = \sum_{l=1}^{\infty} \sum_{v=-\infty}^{\infty} \sum_{k \in \mathbb{Z}} \langle h, \tilde{\psi}_{l,v,k} \rangle \psi_{l,v,k}(t).
\] (19)

Consequently, if $\{\psi_{l,v,k}\}$ and $\{\tilde{\psi}_{l,v,k}\}$, $(l \in \Lambda, v \in \mathbb{Z}, k \in \mathbb{Z}^2)$ are also Bessel sequences, they are a pair of affine frames for $L^2(R^2)$.

**Proof.** (i) Consider, for $\sigma \geq 0, \sigma \in \mathbb{Z}$, the operator $E_\sigma : L^2(R^2) \rightarrow L^2(R^2)$ such that
\[
E_\sigma h(t) \equiv h_\sigma(t) \equiv \sum_{k \in \mathbb{Z}^2} \langle h, \phi_{(-\sigma),k} \rangle \phi_{(-\sigma),k}(t)
\]
Then the operator $E_\sigma$ are well defined and uniformly bounded in the norm on $L^2(R^2)$. To show that $E_\sigma \rightarrow 0$ as $\sigma \rightarrow \infty$, it is sufficient to show that, for all $g(t)$ in any dense subspace of band-limited functions in $L^2(R^2),$
\[
\sum_{k \in \mathbb{Z}^2} \langle g, \phi_{(-\sigma),k} \rangle \phi_{(-\sigma),k} \rightarrow 0 \text{ as } \sigma \rightarrow \infty.
\]
In particular, we may choose the dense set of functions $g(t)$, whose Fourier transform have compact support, is continuous, and vanishes in a neighborhood of 0.
\[
\sup_{h \in H^1} \left\| \sum_{k} \langle g, \phi_{(-\sigma),k} \rangle \phi_{(-\sigma),k} \right\| = \sup_{h \in H^1} \left\| \sum_{k} \langle g, \phi_{(-\sigma),k} \rangle \phi_{(-\sigma),k}, h \right\|
\leq \sup_{h \in H^1} \left( \sum_{k} \left\| g, \phi_{(-\sigma),k} \right\|^2 \right)^{1/2} \left( \sum_{k} \left\| h, \phi_{(-\sigma),k} \right\|^2 \right)^{1/2}
\leq B^{1/2} \left( \sum_{k} \left\| g, \phi_{(-\sigma),k} \right\|^2 \right)^{1/2}
\]
(20)
where $B$ is the Bessel bound of $\{\phi_{(-\sigma),k}\}$. Implementing standard calculation of the right-hand side of (20), we have
\[
\sum_{k} \left| \langle g, \hat{\phi}_{\sigma - \omega, k} \rangle \right|^2 \\
= \sum_{k} \int \hat{g}(\omega) \hat{\phi}(4^\sigma \omega) \cdot 4^\sigma e^{2\pi i \omega \cdot n} \int \hat{\phi}(4^\sigma \gamma) \cdot 2^\sigma e^{-2\pi i \gamma \cdot v} d\gamma d\omega \\
= \int \left( \sum_{n} \hat{g}(\omega + 2^\sigma u) \hat{\phi}(2^\sigma \omega + u) \hat{\phi}(4^\sigma \omega) \hat{g}(\omega) \right) d\omega \\
\leq \left( \sum_{n} \left| \hat{g}(\omega + 2^\sigma u) \right|^2 \right)^{1/2} \left( \sum_{n} \left| \hat{\phi}(4^\sigma \omega + u) \right|^2 \right)^{1/2} \left| \hat{\phi}(4^\sigma \omega) \hat{g}(\omega) \right| d\omega \\
\leq (B^*)^{1/2} \left( 4^\sigma \sum_{n} \left| \hat{g}(\omega + 2^\sigma u) \right|^2 \right)^{1/2} \left| \hat{\phi}(4^\sigma \omega) \hat{g}(\omega) \right| d\omega 
\]

where \( B^* \) is the Bessel bound of \( \{ \hat{\phi}_{(-\sigma), k} \} \). Following the lead of [8] and since \( \hat{f}(\omega) \) is continuous with compact support, the term \( \left( 4^\sigma \sum_{n} \left| \hat{g}(\omega + 2^\sigma u) \right|^2 \right)^{1/2} \leq C^2 < +\infty \), being a Riemann sum to the finite integral \( \int \left| \hat{g}(\omega + t) \right| dt \). Moreover, since \( \hat{g}(\omega) \) vanishes in a neighborhood of 0 for all \( \|\omega\| < \delta_g \), we get that

\[
\sum_{k} \left| \langle g, \hat{\phi}_{(-\sigma), k} \rangle \right|^2 \leq (B^*)^{1/2} C \left( \left| \hat{\phi}(4^\sigma \omega) \hat{g}(\omega) \right| d\omega \right)^{1/2} \\
\leq (B^*)^{1/2} C \left\| g \right\| \left( \int_{|\cdot| \leq \delta_g} \left| \hat{\phi}(4^\sigma \omega) \hat{g}(\omega) \right|^2 d\omega \right)^{1/2} 
\]

Note that the last integral at the right-hand side tends to 0 as \( \sigma \to \infty \). This proves the first part of the theorem since, by using (18) recursively, we have

\[
h_{\sigma}(t) = \sum_{v=1}^{3} \sum_{k} \left\langle h, \hat{\phi}_{v, n,k} \right\rangle \phi_{v, n,k}(t) \\
- \sum_{l=1}^{3} \sum_{n=-\sigma}^{\sigma} \sum_{k \in \mathbb{Z}} \left\langle h, \hat{\psi}_{l,n,k} \right\rangle \psi_{l,n,k}(t) 
\]

(ii) Since \( \bigcup V_l = L^2(\mathbb{R}^2) \), for any \( h \in L^2(\mathbb{R}^2) \) and any \( \varepsilon > 0 \) there exists \( n_0 = n_0(\varepsilon) > 0 \), and for any \( n > n_0 \) there exists \( g \in V_{n_0} \subset V_n \) such that

\[
g(t) = \sum_{k} \left\langle g, \hat{\phi}_{v, k} \right\rangle \phi_{v, k}(t). 
\]

Moreover, for \( C = \sqrt{BB^*} \), \( \left\| h - g \right\|_2 < (1 + C)^{-1} \varepsilon \). Now, by (18), for all \( n > n_0 \), we have

\[
\left\| h - \sum_{v=1}^{3} \sum_{n=-\sigma}^{\sigma} \sum_{k \in \mathbb{Z}} \left\langle h, \hat{\psi}_{l,n,k} \right\rangle \psi_{l,n,k} \right\|_2 \\
\leq \left\| h - \sum_{k \in \mathbb{Z}} \left\langle h, \hat{\phi}_{v, k} \right\rangle \phi_{v, k} \right\|_2 \\
\leq \left\| h - g \right\|_2 + C \left\| h - g \right\|_2 = \left\| h - g \right\|_2 (1 + C) < \varepsilon 
\]
If \( \{ \phi_{r,k} \} \) and \( \{ \tilde{\phi}_{r,k} \} \) are Bessel sequences, then equation (19) implies that both \( \{ \phi_{r,k} \} \) and \( \{ \tilde{\phi}_{r,k} \} \) will be affine frames. In fact, the lower frame bound of \( \{ \phi_{r,k} \} \) or \( \{ \tilde{\phi}_{r,k} \} \) is implied by the upper Bessel bound of the other. Therefore, this completes the proof of the second part of Theorem 4.

6. Conclusion

The special affine frames for the subspaces of \( L^2(\mathbb{R}^2) \) are described, and a method for constructing a GMRA of Paley-Wiener subspaces of \( L^2(\mathbb{R}^2) \) is presented. As a major new contribution the construction of affine frames for space \( L^2(\mathbb{R}^2) \) based on a GMRA is proposed. The pyramid decomposition scheme is derived based on such a GMRA.

References


