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Note

Cycles of Length 0 Modulo k in Directed Graphs

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We show that every digraph D with minimum outdegree δ and maximum indegree Δ contains a directed cycle of length $0(\text{mod}\, k)$, provided $e[\Delta d+1](1-1/k)^{\delta}<1$. In particular, if $\Delta<(2^{\delta}-e)/e\delta$ then D contains an even cycle. © 1989 Academic Press, Inc.

1. INTRODUCTION

The problem of deciding whether a given directed graph contains an even directed cycle shows up in various contexts (see [BT, Th1, Th2]). There are several results that supply sufficient conditions for a digraph to contain such a cycle. In particular, Thomassen [Th1] showed that any simple digraph with n vertices and minimum outdegree $\lfloor \log_2 n \rfloor + 1$ contains an even cycle, and that there are digraphs on n vertices with minimum outdegree $\lfloor \frac{1}{2} \log_2 n \rfloor$ which contain no even cycles. A recent result of Friedland [Fr] asserts that for every $d \geqslant 7$, every d-regular simple digraph (i.e., a digraph with no loops and no multiple edges in which every indegree and every outdegree is d) contains an even cycle. The proof given in [Fr] relies on Van der Waerden conjecture (first proved in [Fa]) and on the Hadamard inequality and the fact that the graph is regular

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seems to be essential for the proof. It also does not seem that the proof can be extended to give cycles of length $0(\mathrm{mod}\, k)$ for $k>2$. In this note we present a completely different, simple method that enables us to extend the above result in several directions. In particular we show that every d-regular digraph contains a cycle of length $0(\mathrm{mod}\, k)$ for every $k \leqslant d/(1+\ln(d^2+1))$. We also show that every digraph with minimum outdegree δ and maximum indegree Δ contains an even cycle, provided $\Delta < (2^\delta - e)/e\delta$. These are two special cases of our main result, presented and proved in the next section. Several extensions of this result are given in Section 3.

2. The Main Result

All digraphs considered are simple, i.e., contain no loops and no parallel directed edges. For a digraph $D=(V,E)$, let $\delta_{\mathrm{in}}=\delta_{\mathrm{in}}(D)$ and $\Delta_{\mathrm{in}}=\Delta_{\mathrm{in}}(D)$ denote the minimum and the maximum indegrees of a vertex of D, respectively. Similarly, let $\delta_{\mathrm{out}}=\delta_{\mathrm{out}}(D)$ and $\Delta_{\mathrm{out}}=\Delta_{\mathrm{out}}(D)$ denote the minimum and the maximum outdegrees, respectively. D is d-regular if $\delta_{\mathrm{in}}=\Delta_{\mathrm{in}}=\delta_{\mathrm{out}}=\Delta_{\mathrm{out}}=d$. For $v\in V$, put $N_{\mathrm{in}}(v)=\{u\in V\colon (u,v)\in E\}$, $N_{\mathrm{out}}(v)=\{u\in V\colon (v,u)\in E\}$. A vertex coloring of D by k colors is simply a function $f\colon V\to\{0,1,...,k-1\}$ (without any additional requirement). The following simple observation is useful in producing cycles of length $0(\mathrm{mod}\, k)$ in a digraph.

Proposition 2.1. *Let $D=(V,E)$ be a digraph and let $k\geqslant 2$ be an integer. If there is a vertex coloring $f\colon V\to\{0,1,...,k-1\}$ of D such that for every $v\in V$ there is a $u\in N_{\mathrm{out}}(v)$ with $f(u)=(f(v)+1)(\mathrm{mod}\, k)$ then D contains a directed cycle of length $0(\mathrm{mod}\, k)$. Similarly, if there is a vertex coloring $h\colon V\to\{0,1,...,k-1\}$ such that for every $v\in V$ there is a $u\in N_{\mathrm{in}}(v)$ with $h(u)=(h(v)+1)(\mathrm{mod}\, k)$ then D contains a directed cycle of length $0(\mathrm{mod}\, k)$.*

Proof. Given the vertex coloring f, choose, for every vertex $v\in V$, some vertex $p(v)\in N_{\mathrm{out}}(v)$ that satisfies $f(p(v))=(f(v)+1)(\mathrm{mod}\, k)$. Suppose $v\in V$ and consider the sequence

$$v_0=v,\qquad v_1=p(v),\qquad v_2=p(v_1),\qquad v_3=p(v_2),\ \ldots.$$

Let j be the minimum index such that there is an $i<j$ with $v_j=v_i$. The cycle $v_i, v_{i+1}, v_{i+2}, ..., v_{j-1}, v_j=v_i$ is clearly a directed cycle of length $0(\mathrm{mod}\, k)$. The proof when the vertex coloring h is given is analogous. ∎

Proposition 2.1 enables us to obtain cycles of the desired type in a digraph $D=(V,E)$ by proving the existence of the required vertex colorings. The existence of such colorings will be proved by applying the Lovász local lemma, first proved in [EL] (see also [GRS]), which is the following:

LEMMA 2.2 (Lovász Local Lemma: Symmetric Case [EL]). *Let $A_1, A_2, \ldots, A_n$ be events in a probability space. Suppose that for all i, $\Pr(A_i) \leqslant p$ and that each A_i is mutually independent of all but at most b of the other events A_j. If $ep(b+1) < 1$ then $\Pr(\bigcap_{i=1}^{n} \bar{A}_i) > 0$.*

We can now prove our main result.

THEOREM 2.3. *Let $D = (V, E)$ be a digraph and let $k \geqslant 2$ be an integer. Put $\delta_{\text{in}} = \delta_{\text{in}}(D)$, $\Delta_{\text{in}} = \Delta_{\text{in}}(D)$, $\delta_{\text{out}} = \delta_{\text{out}}(D)$, $\Delta_{\text{out}} = \Delta_{\text{out}}(D)$. If*

$$e(\Delta_{\text{in}} \delta_{\text{out}} + 1)(1 - 1/k)^{\delta_{\text{out}}} < 1 \tag{2.1}$$

or if

$$e(\Delta_{\text{out}} \delta_{\text{in}} + 1)(1 - 1/k)^{\delta_{\text{in}}} < 1 \tag{2.2}$$

then D contains a directed cycle of length $0 (\bmod k)$.

Proof. We assume that (2.1) holds. The proof when case (2.2) holds is analogous. We first omit, if necessary, edges from D until every outdegree is precisely δ_{out}. Now let $f \colon V \to \{0, 1, \ldots, k-1\}$ be a random vertex coloring of V by k colors, where for each $v \in V$ independently, $f(v) \in \{0, 1, \ldots, k-1\}$ is chosen according to a uniform distribution. For every vertex $v \in V$, let A_v be the event that no vertex in $N_{\text{out}}(v)$ has the color $(f(v) + 1) (\bmod k)$. Clearly $\Pr(A_v) = (1 - 1/k)^{\delta_{\text{out}}}$. Also, each event A_v is mutually independent of all the events A_u that satisfy

$$(u \cup N_{\text{out}}(u)) \cap N_{\text{out}}(v) = \varnothing. \tag{2.3}$$

As the maximum indegree in D (before, and hence certainly after deleting some of its edges to make every outdegree precisely δ_{out}) is at most Δ_{in}, one can easily check that for each fixed v, at most $b = \Delta_{\text{in}} \delta_{\text{out}}$ vertices u violate (2.3). Therefore, Lemma 2.2 and Inequality (2.1) imply that with positive probability no event A_v holds. The desired result now follows from Proposition 2.1. ∎

Theorem 2.3 implies that for every $d \geqslant 8$, every d-regular diagraph contains an even directed cycle. We note that Friedland's result [Fr] shows that this holds even for 7-regular digraphs, and as mentioned in [Fr] it probably holds even for every $d \geqslant 3$. Although this does not follow from Theorem 2.3, notice that this theorem proves the existence of even cycles in digraphs that are far from being regular. For example, any digraph with minimum outdegree 8 and maximum indegree 10 (without any restriction on the maximum outdegree or on the minimum indegree) contains an even cycle. Similarly, any digraph with minimum outdegree 60 and maximum indegree at most 3900 contains a directed cycle of length $0(\bmod k)$ for any $k \leqslant 5$.

Two immediate corollaries of Theorem 2.3 are the following.

COROLLARY 2.4. *For any digraph D and any integer $k \geqslant 2$ if either*

$$\Delta_{\text{in}} < \frac{(k/(k-1))^{\delta_{\text{out}}} - e}{e\delta_{\text{out}}}$$

or

$$\Delta_{\text{out}} < \frac{(k/(k-1))^{\delta_{\text{in}}} - e}{e\delta_{\text{in}}}$$

then D contains a directed cycle of length $0(\bmod k)$.

COROLLARY 2.5. *Every d-regular digraph contains a directed cycle of length* $0(\bmod k)$ *for every* $k \leqslant d/(1 + \ln(d^2 + 1))$.

3. CONCLUDING REMARKS

1. A weighted digraph is a digraph with an integral weight associated with each of its edges. The weight of a cycle is the sum of the weights of its edges. Theorem 2.3 can be easily estended to the weighted case. To do so, one first proves the following simple extension of Proposition 2.1.

PROPOSITION 3.1. *Let $D = (V, E)$ be a weighted digraph with a weight $w(e)$ associated with each edge $e \in E$. If $k \geqslant 2$ and there is a vertex coloring $f\colon V \to \{0, 1, 2, ..., k-1\}$ of D such that for every $v \in V$ there is a $u \in N_{\text{out}}(v)$ with $f(u) = (f(v) + w((v, u)))(\bmod k)$ then D contains a directed cycle of weight* $0(\bmod k)$.

The proof of Proposition 3.1 is analogous to that of Proposition 2.1 and is thus omitted. This proposition, with a simple modification in the proof of Theorem 2.3 clearly gives the following.

THEOREM 3.2. *Let $D = (V, E)$ be a weighted digraph, and let $k \geqslant 2$, $\delta_{\text{in}}, \Delta_{\text{in}}, \delta_{\text{out}}, \Delta_{\text{out}}$ be as in Theorem 2.3. If either (2.1) or (2.2) hold then D contains a directed cycle of weight* $0(\bmod k)$.

2. In [Th1] Thomassen denotes by $g(n)$ the smallest integer g such that any digraph with n vertices and minimum outdegree g contains an

even cycle. Similarly, he denotes by $h(n)$ the smallest integer h such that for any digraph $D=(V, E)$ with n vertices and minimum outdegree h and for any set of edges $E' \subseteq E$, D has a cycle containing an even number of the edges of E'. Thomassen proves that for every $n \geqslant 2$, $\lfloor \frac{1}{2} \log n \rfloor \leqslant g(n) \leqslant h(n) = \lfloor \log_2 n \rfloor + 1$.

The following observation improves the upper estimate for $g(n)$ and shows that it is strictly smaller then $h(n)$ (for all sufficiently large n).

PROPOSITION 3.3. *There exists a constant c such that for every $n \geqslant 2$*

$$g(n) \leqslant \log_2 n - \tfrac{1}{3} \log_2 \log_2 n - c.$$

Proof. As proved by Beck [Be], there is a constant $d>0$ such that for every $m \geqslant 2$, every hypergraph with at most $d \cdot m^{1/3} \cdot 2^m$ edges, each containing at least m vertices, is two colorable. Suppose now that

$$n = \lfloor dm^{1/3} \cdot 2^m \rfloor \tag{3.1}$$

and let $D=(V, E)$ be a digraph with n vertices and minimum outdegree at least $m-1$. Let H be the hypergraph on the set of vertices V, whose n edges are the sets $\{v \cup N_{\text{out}}(v) \colon v \in V\}$. As each edge of H is of size at least m, Beck's result implies it is two colorable. But this means that there is a vertex coloring $f\colon V \to \{0, 1\}$ of D such that for every $v \in V$ there is a $u \in N_{\text{out}}(v)$ with $f(u) = (f(v)+1) (\text{mod } 2)$. Therefore by Proposition 2.1, D contains an even cycle. Solving for m from (3.1) we conclude that

$$g(n) \leqslant m-1 \leqslant \log_2 n - \tfrac{1}{3} \log_2 \log_2 n - c$$

for a properly chosen constant c.

3. Let $D=(V, E)$ be a digraph, suppose $k \geqslant 2$, and let $\bar{\varepsilon} = (\varepsilon_0, \varepsilon_1, ..., \varepsilon_{k-1}) \in \{0, 1\}^k$ be a binary vector of length k. A cycle $C = v_0, v_1, v_2, ..., v_r = v_0$ in D is called a cycle of type $\bar{\varepsilon}$ if k divides r and if for every i, $1 \leqslant i \leqslant r$, the edge in C between v_{i-1} and v_i is directed from v_{i-1} to v_i if and only if $\varepsilon_{i (\text{mod } k)} = 0$ (otherwise it is directed from v_i to v_{i-1}). It is not too difficult to extend the results of Section 2 and obtain sufficient conditions for a digraph to contain cycles of type k. For $k=2$ the only cycle of type k which is not a directed cycle is a cycle whose edges alternate in direction, and an easy argument shows that every digraph $D=(V, E)$ on n vertices and more than $4(n-1)$ edges contains such a cycle. Indeed, a simple averaging argument shows that there is a partition of V, $V = V_1 \cup V_2$, such that the number of edges that start at V_1 and end at V_2 is at least $|E|/4 \geqslant n$. Hence, these edges must contain a cycle, which has the desired type. For $k>2$ one can modify the proof of Proposition 2.1 and prove the following.

PROPOSITION 3.4. *Let $D=(V,E)$ be a digraph. Suppose $k>2$ and let $\bar{\varepsilon}=(\varepsilon_0, \varepsilon_1, ..., \varepsilon_{k-1}) \in \{0,1\}^k$ be a type vector. Suppose there is a vertex coloring*

$$f: V \to \{0, 1, ..., k-1\}$$

with the following property: For every $v \in V$, if $\varepsilon_{f(v)}=0$ then there is a $u \in N_{\text{out}}(v)$ such that $f(u)=(f(v)+1) \bmod k$; and if $\varepsilon_{f(v)}=1$ then there is a $w \in N_{\text{in}}(v)$ such that $f(w)=(f(v)+1)(\bmod k)$. Then D contains a cycle of type $\bar{\varepsilon}$.

It is also possible to modify Theorem 2.3 and obtain a version of it that guarantees cycles of a given type. In paticular one can prove the following special case, whose detailed proof, which is very similar to that of Theorem 2.3, is omitted.

PROPOSITION 3.5. *Every d-regular digraph contains a cycle of type $\bar{\varepsilon}=(\varepsilon_0, \varepsilon_1, ..., \varepsilon_{k-1})$ for any binary vector $\bar{\varepsilon}$ of length k, provided*

$$2 \leqslant k \leqslant \frac{d}{1+\ln(4d^2+1)}.$$

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