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An Inequality for Complete Elliptic Integrals

G. D. ANDERSON

Department of Mathematics, Michigan State University, East Lansing, Michigan 48824

PETER DUREN

Department of Mathematics, University of Michigan, Ann Arbor, Michigan 48109

AND

M. K. VAMANAMURTHY

Department of Mathematics, University of Auckland, Auckland, New Zealand

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In a recent study of Robin capacity [5], a Schwarz–Christoffel mapping was constructed from the exterior of the unit disk onto the exterior of a rectangle, preserving infinity. The mapping was indexed by a parameter k $(0 < k < 1)$, and the side-lengths of the rectangle were found to be

$$2J = \frac{1}{k}\,[E - (1-k)\,K]; \qquad L = \frac{1}{2k}\,(E' - kK'),$$

where

$$K = K(k) = \int_0^{\pi/2} \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}$$

and

$$E = E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \sin^2 t}\, dt$$

are the complete elliptic integrals of first and second kinds, respectively (see [4]). In standard notation, $K' = K'(k) = K(k')$ and $E' = E'(k) = E(k')$,

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where $k' = \sqrt{1-k^2}$ is the complementary modulus. It is easily shown (details below) that $2kJ$ increases from 0 to 1 and $2kL$ decreases from 1 to 0 as k increases from 0 to 1. Thus the ratio $m = 2J/L$, called the *exterior modulus* of the rectangle, increases from 0 to ∞. The exterior modulus is a conformal invariant of a quadrilateral with respect to mappings of the exterior that preserve infinity.

The *Robin modulus* μ of the rectangle was defined in [5] as the Robin capacity of the two sides of length $2J$ divided by the Robin capacity of the two sides of length L. Again it is a conformal invariant of a quadrilateral. For the given rectangle it was found to be $\mu = 2\sqrt{k}/(1-k)$. It was shown in [5] that

$$c_1 m(Q) \leqslant \mu(Q)^2 \leqslant c_2 m(Q) \tag{1}$$

for arbitrary quadrilaterals Q, where c_1 and c_2 are some positive absolute constants. In the present note the best values of the constants are found.

The differentiation formulas are

$$kk'^2 \frac{dK}{dk} = E - k'^2 K; \qquad k\frac{dE}{dk} = E - K.$$

Easy calculations give

$$(1+k)\frac{d}{dk}(2kJ) = E > 0$$

and

$$(1+k)\frac{d}{dk}(2kL) = E' - K' < 0,$$

showing that the exterior modulus $m = 2J/L$ increases with k. However, the Robin modulus $\mu = 2\sqrt{k}/(1-k)$ also increases, so the behavior of the ratio

$$R(k) = \frac{\mu^2}{m} = \frac{2k}{(1-k)^2}\,\frac{E' - kK'}{E - (1-k)K}$$

is unclear.

THEOREM. *The ratio $R(k)$ decreases from $4/\pi$ to $\pi/4$ as k increases from 0 to 1. Hence the best constants in (1) are $c_1 = \pi/4$ and $c_2 = 4/\pi$.*

For the proof it is advantageous to make the change of parameter

$k \mapsto (1-k)/(1+k)$. Known transformation formulas (see [2, p. 319] or [3, pp. 12, 16]) lead to the elegant expression

$$R^*(k) = R\left(\frac{1-k}{1+k}\right) = \frac{k'^2}{k^2}\frac{E - k'^2 K}{E' - k^2 K'}.$$

The relation $R^*(k') = 1/R^*(k)$ reflects the geometrically obvious symmetries of the moduli m and μ under interchange of the sides of the rectangle. In particular, $R^*(1/\sqrt{2}) = 1$ or $R(3-2\sqrt{2}) = 1$, a consequence of the fact that the rectangle is a square when the original parameter k is $3-2\sqrt{2}$. In order to show that $R^*(k)$ increases from $\pi/4$ to $4/\pi$, it will suffice to prove the following lemma.

LEMMA. *The quantity* $G(k) = k^{-2}(E - k'^2 K)$ *increases from* $\pi/4$ *to* 1 *as* k *increases from* 0 *to* 1.

This result appears in [1], but we include the simple proof here. It follows directly from the definitions of K and E that

$$G(k) = \int_0^{\pi/2} \frac{\cos^2 t}{\sqrt{1 - k^2 \sin^2 t}}\, dt,$$

which is clearly increasing from $G(0) = \pi/4$ to $G(1) = 1$.

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