Asymptotic and numerical stability of systems of neutral differential equations with many delays

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Abstract

We are concerned with the asymptotic stability of a system of linear neutral differential equations with many delays in the form

\[ y'(t) = Ly(t) + \sum_{i=1}^{d} M_i y(t - \tau_i) + \sum_{i=1}^{d} N_i y'(t - \tau_i), \]

where \( L, M_i, N_i \in \mathbb{C}^{N \times N} \) (\( i = 1, 2, \ldots, d \)) are constant complex matrices, \( \tau_i > 0 \) (\( i = 1, 2, \ldots, d \)) are constant delays and \( y(t) = (y_1(t), y_2(t) \ldots y_N(t))^T \) is an unknown vector-valued function for \( t > 0 \). We first establish a new result for the distribution of the roots of its characteristic function, next we obtain a sufficient condition for its asymptotic stability and then we investigate the corresponding numerical stability of linear multistep methods applied to such systems. One numerical example is given to testify our numerical analysis.

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1. Introduction

Consider the following system of linear neutral differential equations (NDEs) with many delays

\[ y'(t) = Ly(t) + \sum_{i=1}^{d} M_i y(t - \tau_i) + \sum_{i=1}^{d} N_i y'(t - \tau_i), \quad t > 0, \tag{1.1} \]

\[ y(t) = \phi(t), \quad t \leq 0, \tag{1.2} \]

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where \( L, M_i, N_i \) \((i = 1, 2, \ldots, d)\) are complex \( N \times N \) matrices, \( \phi(t) \) is a given continuous vector-valued function, \( y(t) \) is an unknown vector-valued function for \( t > 0 \) and \( 0 < \tau_1 < \tau_2 < \cdots < \tau_d \) are constant delays.

Neutral delay differential equations have appeared frequently in mechanics, physics, biology, medicine, economics, and engineering systems, such as distributed networks containing lossless transmission lines, population dynamic models and control of epidemics \([2,6]\). For example, a population dynamic model of two species interacting in a closed environment can be described by a neutral differential system with delays \([9]\). The stability analysis of neutral differential systems has received considerable attention due to its theoretical interests and has been one of the most interesting topics in control theory since it is a powerful tool for practical system analysis and design \([12,14]\).

There are many research works on the asymptotic and numerical stability of System \((1.1)\) of NDEs with a single delay (i.e. \( d = 1 \)). In 1967, Brayton and Willoughby \([1]\) first proved that such a system is asymptotically stable if \( L, M \) and \( N \) are real symmetric matrices and \( I \pm N, L \pm M \) are positive definite and discussed the numerical stability of the linear \( \theta \)-method. Later in 1984, Jackiewicz \([3]\) studied the asymptotic stability of \((1.1)\) and the numerical stability of the linear \( \theta \)-method in the case where \( L, M \) and \( N \) are complex numbers. In 1994, Kuang, Xiang and Tian \([8]\) investigated the asymptotic stability of \((1.1)\) and the numerical stability of the linear \( \theta \)-method in the case where \( L, M \) and \( N \) are complex matrices. In 1995, Hu and Mistui \([4]\) gave a sufficient condition such that \((1.1)\) is asymptotically stable and dealt with the numerical stability of explicit Runge–Kutta methods. Further, in 1999, Qiu, Yang, and Kuang \([15]\) reconsidered the asymptotic stability of \((1.1)\) and focused on the numerical stability of implicit Runge–Kutta methods.

Meanwhile, several authors studied the analytic and numerical stability of System \((1.1)\) of NDEs with multiple delays (i.e. \( d \geq 2 \)). In 1998, Zhang and Zhou \([21]\) presented a sufficient condition similar to \([4]\) for the asymptotic stability of \((1.1)\) and discussed the numerical stability of multistep Runge–Kutta methods. In 2002, Zhang \([20]\) further considered the numerical stability of the \((A, B, C)\) method. Recently, Sun \([16]\) studied the asymptotic stability of \((1.1)\) using the property of logarithmic norms but did not deal with its numerical treatment.

The organization of this paper is as follows. In Section 2 we derive a new sufficient condition for the matrices \( L, M_i, N_i \) \((i = 1, 2, \ldots, d)\) such that \((1.1)\) is asymptotically stable. This sufficient condition differs from the results given in the literature (for example, \([4,21]\)) and may be adopted more conveniently for analysis of the asymptotic stability of the numerical methods. In Section 3 we apply a linear multistep method, combined with Lagrange interpolation, to \((1.1)\) and investigate the asymptotic stability of this process. A numerical example is given to confirm our main result.

2. Asymptotic stability of system \((1.1)\)

Depending on whether or not the stability criterion itself contains the size of the delays, stability criteria of NDEs can be classified into delay-independent criteria or delay-dependent criteria. Generally speaking, the former ones which establish stability for all possible delay values are comprehensive but conservative, while the latter ones which can give the maximum allowable bound of the time delay are more selective and flexible.

We first introduce some definitions for the asymptotic stability of System \((1.1)\).

**Definition 2.1** (Delay-Dependent Stability). System \((1.1)\) is said to be delay-dependently asymptotically stable if, for arbitrarily fixed \( 0 < \tau_1 < \tau_2 < \cdots < \tau_d \) and any initial function \( \phi(t) \), its analytic solution \( y(t) \) satisfies

\[
\lim_{t \to \infty} y(t) = 0.
\]

**Definition 2.2** (Delay-Independent Stability). System \((1.1)\) is said to be delay-independently asymptotically stable if, for any initial function \( \phi(t) \), its analytic solution \( y(t) \) satisfies

\[
\lim_{t \to \infty} y(t) = 0
\]

for any \( 0 < \tau_1 < \tau_2 < \cdots < \tau_d \).

It is well known \([2]\) that System \((1.1)\) is asymptotically stable if and only if all zeros of its characteristic function

\[
p(\xi, \tau_1, \tau_2, \ldots, \tau_d) = \det \left\{ \xi I - L - \sum_{j=1}^{d} e^{-\xi \tau_j} M_j - \xi \sum_{i=1}^{d} e^{-\xi \tau_i} N_i \right\}
\]

(2.1)
lie in the left half complex plane and are uniformly bounded away from the imaginary axis. It is a difficult task to study the location of all zeros of the quasi-polynomial (2.1) because the well-known Routh–Hurwitz theorem for the location of zeros of a polynomial can not be applied directly. It follows from [13] that

- if System (1.1) is delay-dependently asymptotically stable, then for arbitrarily fixed $0 < \tau_1 < \tau_2 < \cdots < \tau_d$
  
  $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) < 0$; \hspace{1cm} (2.2)

- if for arbitrarily fixed $0 < \tau_1 < \tau_2 < \cdots < \tau_d$, there exists some positive real number $\gamma$ such that
  
  $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) \leq -\gamma$, \hspace{1cm} (2.3)

then System (1.1) is delay-dependently asymptotically stable;

- if System (1.1) is delay-independently asymptotically stable, then
  
  $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) < 0$ \hspace{1cm} (2.4)

for any $0 < \tau_1 < \tau_2 < \cdots < \tau_d$;

- if there exists some positive real number $\gamma$ such that
  
  $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) \leq -\gamma$ \hspace{1cm} (2.5)

for any $0 < \tau_1 < \tau_2 < \cdots < \tau_d$, then System (1.1) is delay-independently asymptotically stable.

In order to prove that System (1.1) is asymptotically stable, we only need to show that all zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d)$ lie in the left half complex plane and leave the imaginary axis uniformly, that is, there is some positive real number $\tilde{\gamma} > 0$ such that all the zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d)$ satisfy $|\Re(\zeta)| \geq \tilde{\gamma}$.

The following lemma describes the distribution of the zeros of the characteristic function.

**Lemma 2.1.** 1. Suppose that there exists some integer $k \in \{1, 2, \ldots, d\}$ such that

\[
\sup_{\Re(\zeta)=0} \rho \left( N_k + \sum_{i=1, i\neq k}^d e^{\zeta (\tau_i - \tau_k)} N_i \right) < 1, \hspace{1cm} (2.6)
\]

where $\rho(X)$ stands for the spectral radius of the matrix $X$, then the zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d)$ leave the imaginary axis uniformly except some finite possible exception points.

2. Suppose that the coefficient matrix $N_d$ is invertible, then the real part of the zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d)$ is bounded, that is, there is some positive real number $C_0 > 0$ such that all the zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \tau_2, \ldots, \tau_d)$ satisfy $|\Re(\zeta)| \leq C_0$.

**Proof.** For the integer $k$ in the above statement, define $\eta_i = \tau_k - \tau_i, i = 1, 2, \ldots, d$. From (2.1), we have

\[
p(\zeta, \tau_1, \ldots, \tau_d) = \det \left\{ \zeta I - L - e^{-\zeta \tau_k} \left( \sum_{i=1}^d e^{\eta_i \zeta} M_i + \zeta \sum_{i=1}^d e^{\eta_i \zeta} N_i \right) \right\}.
\]

When the modulus of $\zeta$ is sufficiently large, $\zeta I - L$ is invertible and thus

\[
p(\zeta, \tau_1, \ldots, \tau_d) = 0 \iff e^{\zeta \tau_k} \in \sigma \left[ (\zeta I - L)^{-1} \left( M_k + \sum_{i=1, i\neq k}^d e^{\eta_i \zeta} M_i + \zeta N_k + \zeta \sum_{i=1, i\neq k}^d e^{\eta_i \zeta} N_i \right) \right], \hspace{1cm} (2.7)
\]

here $\sigma[X]$ stands for the spectrum of matrix $X$. In this case, we denote

\[
A_s(\zeta) = (\zeta I - L)^{-1} \left( M_s + \sum_{i=1, i\neq s}^d e^{\eta_i \zeta} M_i + \zeta N_s + \zeta \sum_{i=1, i\neq s}^d e^{\eta_i \zeta} N_i \right), \hspace{0.5cm} s = 1, 2, \ldots, d,
\]

then (2.7) becomes

\[
p(\zeta, \tau_1, \ldots, \tau_d) = 0 \iff e^{\zeta \tau_k} \in \sigma[A_k(\zeta)].
\]
For sufficiently large $|\zeta|$ we obtain
\[ A_k(\zeta) = \left( I + \frac{1}{\zeta} L + \frac{1}{\zeta^2} L^2 + \cdots \right) \left( N_k + \sum_{i=1, i \neq k}^{d} e^{\zeta_0^i} N_i + \frac{1}{\zeta} M_k + \frac{1}{\zeta^2} \sum_{i=1, i \neq k}^{d} e^{\zeta_0^i} M_i \right). \] (2.8)

Since $p(\zeta, \tau_1, \ldots, \tau_d)$ is a non-polynomial entire function, it follows from [18] that the characteristic function has countably infinite zeros $\{\zeta_n\}_{n=1}^{\infty}$ satisfying $\zeta_n \to \infty$ as $n \to \infty$. Suppose that these zeros do not leave the imaginary axis uniformly, then there exist a sub-sequence $\{\zeta_{n_j}\}_{j=1}^{\infty}$ such that $\Re \zeta_{n_j} \to 0$ and $|\Im \zeta_{n_j}| \to \infty$ as $j \to \infty$.

From (2.8), we have
\[ A_k(\zeta_{n_j}) = N_k + \sum_{i=1, i \neq k}^{d} e^{\zeta_0^i} N_i + O\left( \frac{1}{\zeta_{n_j}} \right), \quad j \to \infty, \] (2.9)
where the symbol $O\left( \frac{1}{\zeta_{n_j}} \right)$ means the term of a matrix satisfying
\[ \left\| O\left( \frac{1}{\zeta_{n_j}} \right) \right\| \leq \frac{1}{|\zeta_{n_j}|} R_0 \quad \text{for some positive number } R_0. \]

Since $\Re \zeta_{n_j} \to 0$ as $j \to \infty$, then
\[ \lim_{j \to \infty} \left| e^{\zeta_{n_j}} r_k \right| = 1. \] (2.10)

It follows from (2.7) that there is an eigenvalue $\lambda_{A_k(\zeta_{n_j})}$ such that
\[ e^{\zeta_{n_j}} \rho = \lambda_{A_k(\zeta_{n_j})}, \quad j = 1, 2, \ldots. \]

So we have
\[ \lim_{j \to \infty} \left| \lambda_{A_k(\zeta_{n_j})} \right| = \lim_{j \to \infty} \left| \lambda \left( N_k + \sum_{i=1, i \neq k}^{d} e^{\zeta_0^i} N_i + O\left( \frac{1}{\zeta_{n_j}} \right) \right) \right| \]
\[ \leq \sup_{\Re (\zeta) = 0} \rho \left[ N_k + \sum_{i=1, i \neq k}^{d} e^{\zeta_0^i} N_i \right] \]
\[ < 1, \]
which contradicts (2.10). We conclude that the zeros $\zeta$ of the characteristic function $p(\zeta, \tau_1, \ldots, \tau_d)$ leave the imaginary axis uniformly except some finite possible exception points.

It remains to prove the second statement. If the real part of the zeros of $p(\zeta, \tau_1, \ldots, \tau_d)$ is not bounded, then there exists a subsequence of zeros $\{\zeta_{n_k}\}_{k=1}^{\infty}$ such that $\Re (\zeta_{n_k}) \to -\infty$ or $\Re (\zeta_{n_k}) \to +\infty$ as $k \to \infty$. Using the same reasoning as above we obtain
\[ p(\zeta_{n_k}, \tau_1, \ldots, \tau_d) = 0 \Leftrightarrow e^{\zeta_{n_k}} t_d \in \sigma \left[ A_d(\zeta_{n_k}) \right], \quad k = 1, 2, \ldots \]
and
\[ A_d(\zeta_{n_k}) = N_d + \sum_{i=1}^{d-1} e^{\zeta_{n_k}(\tau_d-\tau_i)} N_i + O\left( \frac{1}{\zeta_{n_k}} \right), \quad \Re (\zeta_{n_k}) \to -\infty. \]

which are similar to (2.7) and (2.9). This implies that there is an eigenvalue $\lambda_{A_d(\zeta_{n_k})}$ such that
\[ e^{\zeta_{n_k}} t_d = \lambda_{A_d(\zeta_{n_k})}. \]
It is easy to see that
\[
\lim_{\Re(\zeta_{nk}) \to -\infty} |e^{\zeta_{nk} t_1}| = 0,
\]
while
\[
\lim_{\Re(\zeta_{nk}) \to -\infty} |\lambda_{Ad}(\zeta_{nk})| = |\lambda_{Nd}| \neq 0,
\]
is a contradiction. On the other hand, we have
\[
e^{\zeta_{nk} t_1} \in \sigma\left[A_1(\zeta_{nk})\right]
\]
and
\[
A_1(\zeta_{nk}) = N_1 + \sum_{i=2}^{d} e^{\zeta_{nk}(t_1-t_i)} N_i + O\left(\frac{1}{\zeta_{nk}}\right), \quad \text{as } \Re(\zeta_{nk}) \to +\infty.
\]
It follows that
\[
\lim_{\Re(\zeta_{nk}) \to +\infty} |e^{\zeta_{nk} t_1}| = \infty,
\]
while
\[
\lim_{\Re(\zeta_{nk}) \to +\infty} |\lambda_{A_1(\zeta_{nk})}| = |\lambda_{N_1}|,
\]
where \(\lambda_{N_1}\) is an eigenvalue of \(N_1\). This is a contradiction again. This completes the proof. \(\square\)

For any matrix \(A = (a_{ij}) \in \mathbb{C}^{N \times N}\), define \(|A| = (|a_{ij}|)\) and \(\|\cdot\|\) to represent any (consistent) matrix norm. The following corollary generalizes some results obtained in [20, 21].

**Corollary 2.1.** If one of the following conditions

1. \(\rho\left[\sum_{i=1}^{d} |\xi_i| N_i\right] < 1, \forall \xi_i \in \mathbb{C}, |\xi_i| \leq 1, i = 1, 2, \ldots, d,\)
2. \(\rho\left(\sum_{i=1}^{d} |N_i|\right) < 1,\)
3. \(\sum_{i=1}^{d} \|N_i\| < 1\)

is satisfied, then the first statement in Lemma 2.1 holds.

**Proof.** For any \(\xi = (\xi_1, \xi_2, \ldots, \xi_d)^T \in \mathbb{C}^d,\) define its norm as \(\|\xi\|_\infty = \max_{i=1,2,\ldots,d} |\xi_i|\). The unit circle \(D = \{\xi \in \mathbb{C}^d, \|\xi\|_\infty \leq 1\}\) is compact in \(\mathbb{C}^d\). Since the spectral radius \(\rho[\sum_{i=1}^{d} \xi_i N_i]\) is a continuous function of \(\xi \in D,\) there exists some \(\xi^0 = (\xi^0_1, \xi^0_2, \ldots, \xi^0_d)^T \in D\) such that \(\rho[\sum_{i=1}^{d} \xi^0_i N_i] = \max_{\xi \in D} \rho[\sum_{i=1}^{d} \xi_i N_i]\). It follows from Condition (A) that \(\rho[\sum_{i=1}^{d} \xi^0_i N_i] < 1.\) This gives
\[
\sup_{\Re(\zeta) = 0} \rho\left(N_k + \sum_{i=1, i \neq k}^{d} e^{\zeta(t_k-t_i)} N_i\right) \leq \rho\left[\sum_{i=1}^{d} \xi^0_i N_i\right] < 1.
\]
This implies that (2.6) holds and thus the first statement in Lemma 2.1 holds. For any matrices \(M, N \in \mathbb{C}^{N \times N}\) it follows from [11] that \(\rho[M] \leq \rho[N]\) if \(|M| \leq N\), then Condition (B) implies that (2.6) holds and thus the first statement in Lemma 2.1 holds. Finally, Condition (C) implies that (2.6) holds and thus the first statement in Lemma 2.1 hold. This completes the proof. \(\square\)

We now give a sufficient condition such that the system (1.1) is delay-dependently asymptotically stable.

**Theorem 2.1.** Suppose that the coefficient matrices \(L, M_i, N_i (i = 1, 2, \ldots, d)\) of System (1.1) satisfy the following conditions

1. \(\sum_{i=1}^{d} \|N_i\| < 1,\)
2. \(\lambda \in \sigma[L] \Rightarrow \Re(\lambda) < 0,\)

Then System (1.1) is delay-dependently asymptotically stable.
(iii) \[ \rho \left[ (\zeta I - L)^{-1} \left( \sum_{i=1}^{d} e^{\zeta \omega_i} M_i + \zeta \sum_{i=1}^{d} e^{\zeta \omega_i} N_i \right) \right] < 1, \forall \zeta \in \mathbb{C}, \Re(\zeta) = 0, \zeta \neq 0, \]

(iv) \[-1 \notin \sigma \left[ L^{-1} \sum_{i=1}^{d} M_i \right],\]

where \( \omega_i = \tau_1 - \tau_i < 0, i = 2, \ldots, d, \) then System (1.1) is delay-dependently asymptotically stable.

**Proof.** To prove that System (1.1) is delay-dependently asymptotically stable, we show that there is a real number \( \gamma > 0 \) such that

\[ p(\zeta, \tau_1, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) \leq -\gamma < 0. \] (2.11)

We first verify that

\[ p(\zeta, \tau_1, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) < 0. \] (2.12)

Denote

\[ C_0 = \{ \zeta \in \mathbb{C} : \Re(\zeta) = 0 \}, \]
\[ C^- = \{ \zeta \in \mathbb{C} : \Re(\zeta) < 0 \}, \]
\[ C^+ = \{ \zeta \in \mathbb{C} : \Re(\zeta) > 0 \}. \]

From Condition (ii), we have that \( \zeta \notin \sigma[L] \) and \( \zeta I - L \) is invertible whenever \( \Re(\zeta) \geq 0 \), meanwhile, Condition (iii) implies

\[ \sup_{\Re(\zeta)=0} \rho \left[ (\zeta I - L)^{-1} \left( \sum_{i=1}^{d} e^{\zeta \omega_i} M_i + \zeta \sum_{i=1}^{d} e^{\zeta \omega_i} N_i \right) \right] \leq 1. \] (2.13)

For any given \( \zeta_0 \) satisfying \( \Re(\zeta_0) > 0 \), there exists a disk \( C_r \), centered at the origin with sufficiently large radius \( r > 0 \), such that it contains \( \zeta_0 \). Denote \( D_r = C_r \cap C^+ \), \( \Gamma_r = \partial D_r \) and \( \tilde{\Gamma}_r = \Gamma_r \cap C^+ \). It follows from Condition (i) that there is a positive number \( \rho_0 \) such that

\[ \sup_{\Re(\zeta) \geq 0} \rho \left[ N_1 + \sum_{i=2}^{d} e^{\zeta \omega_i} N_i \right] \leq \sum_{i=1}^{d} \| N_i \| = \rho_0 < 1. \] (2.14)

For sufficiently large \( |\zeta| \)

\[ A_1(\zeta) = N_1 + \sum_{i=2}^{d} e^{\zeta \omega_i} N_i + O \left( \frac{1}{\zeta} \right). \] (2.15)

Note that \( \rho[A_1(\zeta)] \) is a continuous function of \( \zeta \) with \( \Re(\zeta) > 0 \). For sufficiently large \( r > 0 \) and \( \zeta \in \tilde{\Gamma}_r \), there is an \( \varepsilon_r > 0 \) such that

\[ \rho[A_1(\zeta)] \leq \rho \left[ N_1 + \sum_{i=2}^{d} e^{\zeta \omega_i} N_i \right] + \varepsilon_r \]

\[ \leq \sup_{\Re(\zeta) \geq 0} \rho \left[ N_1 + \sum_{i=2}^{d} e^{\zeta \omega_i} N_i \right] + \varepsilon_r \]

\[ = \rho_0 + \varepsilon_r. \]

Since \( \varepsilon_r \to 0 \) as \( r \to \infty \), there exist a sufficiently large \( r_0 \) and a small enough \( \varepsilon \) such that

\[ \rho[A_1(\zeta)] \leq \rho_0 + \varepsilon < 1, \quad \forall \zeta \in \tilde{\Gamma}_r, \] and thus

\[ \sup_{\zeta \in \tilde{\Gamma}_r} \rho[A_1(\zeta)] \leq \rho_0 + \varepsilon < 1. \]
The above result and the inequality (2.13) imply that
\[
\sup_{\zeta \in \Gamma_r} \rho[A_1(\zeta)] \leq 1. \tag{2.16}
\]

For \( \zeta \in \mathbb{C}, \Re(\zeta) > 0 \), it follows from [11] that every eigenvalue of \( A_1(\zeta) \) is an analytic function of \( \zeta \) or \( \sqrt[\zeta]{} \) \((p \geq 2)\).

An application of the maximum modulus principle on \( D_r \) gives
\[
\rho[A_1(\zeta_0)] < \sup_{\zeta \in \Gamma_r} \rho[A_1(\zeta)] \leq 1, \quad r \geq r_0
\]

Since \( \zeta_0 \) is arbitrarily chosen in \( \mathbb{C}^+ \), we have
\[
\rho[A_1(\zeta)] < 1, \quad \forall \zeta \in \mathbb{C}^+.
\]

Combining the above results with (2.13) we get
\[
\rho[A_1(\zeta)] = \rho \left[ (\zeta I - L)^{-1} \left( M_1 + \sum_{i=2}^{d} e^{\zeta \omega_i} M_i + \zeta N_1 + \zeta \sum_{i=2}^{d} e^{\zeta \omega_i} N_i \right) \right] \leq 1, \quad \forall \zeta \in \mathbb{C}^+ \cup \mathbb{C}_0. \tag{2.17}
\]

Suppose that \( \rho(\zeta, \tau_1, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) \geq 0 \). Noting that \( \rho(\zeta, \tau_1, \ldots, \tau_d) = 0 \Leftrightarrow e^{\zeta \tau_1} \in \sigma[A_1(\zeta)] \), then if \( \Re(\zeta) > 0, |e^{\zeta \tau_1}| > 1 \), a contradiction with (2.17). If \( \Re(\zeta) = 0 \), then \( |e^{\zeta \tau_1}| = 1 \), also a contradiction with (iii) and (iv). Thus we have proved \( p(\zeta, \tau_1, \ldots, \tau_d) = 0 \Rightarrow \Re(\zeta) < 0 \). An application of Lemma 2.1 yields the desired result. □

The following two corollaries on delay-dependent or delay-independent asymptotic stability can be easily obtained by Theorem 2.1.

**Corollary 2.2.** Suppose that the coefficient matrices \( L, M_i, N_i (i = 1, 2, \ldots, d) \) satisfy

(I) \( \sum_{i=1}^{d} ||N_i|| < 1 \),

(II) \( \lambda \in \sigma[L] \Rightarrow \Re(\lambda) < 0 \),

(III) \( \sup_{\zeta \in \mathbb{C}_0} \rho[(\zeta I - L)^{-1}(\sum_{i=1}^{d} e^{\zeta (\tau_1 - \tau_i)} M_i + \zeta \sum_{i=1}^{d} e^{\zeta (\tau_1 - \tau_i)} N_i)] < 1 \).

Then System (1.1) is delay-dependently asymptotically stable.

**Corollary 2.3.** If the coefficient matrices \( L, M_i, N_i (i = 1, 2, \ldots, d) \) satisfy Conditions (I) and (II) of Corollary 2.2 and

(III') \( \sup_{\zeta \in \mathbb{C}_0} \rho[(\zeta I - L)^{-1}(\sum_{i=1}^{d} \xi_i M_i + \zeta \sum_{i=1}^{d} \xi_i N_i)] < 1, \quad \forall \xi_i \in \mathbb{C}, |\xi_i| \leq 1, \)

then System (1.1) is delay-independently asymptotically stable.

3. **NGP\(_m\)-stability of linear multistep methods**

Consider the following system of ordinary differential equations (ODEs)
\[
x'(t) = f(t, x(t)), \quad t > 0, \tag{3.1}
\]
\[
x(0) = x_0, \tag{3.2}
\]

where \( x(t) \) and \( f(t, x(t)) \) are vector-valued functions and \( x(t) \) is unknown for \( t > 0 \). A linear multistep method solving (3.1) and (3.2) is defined as
\[
\sum_{j=0}^{k} \alpha_j x_{n+j} = h \sum_{j=0}^{k} \beta_j f(t_{n+j}, x_{n+j}), \tag{3.3}
\]

where \( \alpha_j, \beta_j \) (\( j = 0, 1, \ldots, k \)) are the coefficients with \( \alpha_k \neq 0, \alpha_0^2 + \beta_0^2 \neq 0, h > 0 \) is a fixed step-length, and \( x_n \) is the approximation of \( x(t_n) \) when \( t_n = nh, n = 1, 2, \ldots \). The first and second characteristic polynomials are defined as
\[
\psi_1(z) = \sum_{j=0}^{k} \alpha_j z^j, \quad \psi_2(z) = \sum_{j=0}^{k} \beta_j z^j.
\]
The linear multistep method is called \( A \)-stable if, for any step length \( h > 0 \), its numerical solution, when applied to the following linear equation
\[
x'(t) = \lambda x(t), \quad t > 0 \quad (\Re(\lambda) < 0),
\]
satisfies
\[
\lim_{n \to \infty} x_n = 0.
\]

**Lemma 3.1** ([7]). The following statements are equivalent
(a) a linear multistep method is \( A \)-stable,
(b) \( \psi_1(z) - \overline{h}\psi_2(z) \) is a Schur polynomial whenever \( \overline{h} \equiv \lambda h \in \mathbb{C}^- \),
(c) \( \Re \left( \frac{\psi_1(z)}{\psi_2(z)} \right) \geq 0 \) whenever \( z \in \mathbb{C}, |z| \geq 1 \),
(d) \( \psi_1(z)I - \psi_2(z)X \) is invertible whenever \( z \in \mathbb{C}, |z| \geq 1, X \in \mathbb{C}^{N \times N}, \sigma(X) \subseteq \mathbb{C}^- \).

When numerically solving a neutral differential equation with many delays, \( t_n - \tau_i \quad (i = 1, 2, \ldots, d) \) may not be mesh points simultaneously. For any given \( h > 0 \) there exist positive integers \( m_i \) such that
\[
\tau_i = (m_i - \delta_i)h, \quad \delta_i \in [0, 1), \quad i = 1, 2, \ldots, d.
\]

In [19] the authors introduced the following discretization scheme combining a linear multistep method with a Lagrange interpolation for (1.1) as
\[
\sum_{j=0}^{k} \alpha_j \left( y_{n+j} - \sum_{i=1}^{d} N_i y_{n+j-m_i+\delta_i} \right) = h \sum_{j=0}^{k} \beta_j \left( Ly_{n+j} + \sum_{i=1}^{d} M_i y_{n+j-m_i+\delta_i} \right), \tag{3.4}
\]
where
\[
y_{n+j-m_i+\delta_i} = \sum_{p=-q}^{s} L_p(\delta_i) y_{n+j-m_i+p}, \tag{3.5}
\]
\[
L_p(\delta_i) = \prod_{j=-q, j \neq p}^{s} \frac{\delta_i - j}{p - j}.
\]
When \( m_i \geq s + 1, \alpha_q I - \beta_k L \) is invertible, and \( y_n \quad (n = -m_i, \ldots, k - 1) \) are given, the discretization scheme can be solved for \( y_n, n = k, k + 1, \ldots \).

Let
\[
\gamma_i(z, \delta_i) = \sum_{p=-q}^{s} L_p(\delta_i) z^{p+q},
\]
\[
\widetilde{\gamma}_i(z, \delta_i) = \sum_{p=-q}^{s} L_p(\delta_i) z^{-m_i+p} = \gamma_i(z, \delta_i) z^{-m_i-q}.
\]

The characteristic polynomial of (3.4) and (3.5) is
\[
G(z) \equiv \det \left\{ \psi_1(z)I - \psi_2(z)\overline{L} - \psi_1(z) \sum_{i=1}^{d} N_i \widetilde{\gamma}_i(z, \delta_i) - \psi_2(z) \sum_{i=1}^{d} M_i \widetilde{\gamma}_i(z, \delta_i) \right\}, \tag{3.6}
\]
where \( \overline{L} = hL \) and \( \overline{M}_i = hM_i \quad (i = 1, 2, \ldots, d) \).

Denote
\[
m_0 = \max\{m_1, m_2, \ldots, m_d\},
\]
\[
Q(z) = \psi_1(z)I - \psi_2(z)\overline{L},
\]
\[
P(z) = \psi_2(z) \sum_{i=1}^{d} \overline{M}_i \widetilde{\gamma}_i(z, \delta_i) z^{-m_i+m_0} + \psi_1(z) \sum_{i=1}^{d} N_i \widetilde{\gamma}_i(z, \delta_i) z^{-m_i+m_0},
\]
\[
\]
then (3.6) can be rewritten as
\[ G(z) = \det(z^{m_0+q} I)^{-1} \times \det(z^{m_0+q} Q(z) - P(z)). \] (3.7)

According to Corollary 2.3, we introduce the set
\[ H = \{(\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d) : (\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d) \text{ satisfies (I), (II) and (III)} \} \).

The following definitions of numerical stability for NDEs with multiple delays have been adopted in [15,20,21].

**Definition 3.1.** The difference process (3.4) and (3.5) is called \((\delta_1, \ldots, \delta_d)\)-stable at \((\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d)\) if and only if (3.4) and (3.5) is solvable and the solution sequence \(\{y_n\}_{n=1}^{\infty} \) satisfies \(\lim_{n \to \infty} y_n = 0\) whenever \(m_i \geq s + 1, i = 1, 2, \ldots, d\). Denote \[ S_{\delta_1, \ldots, \delta_d} = \{(\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d) : (3.4) \text{ and (3.5) is } (\delta_1, \ldots, \delta_d) \text{-stable at } (\overline{L}, M_i, N_i, i = 1, \ldots, d)\} \] and \[ S = \prod_{\delta_i \in (0, 1), i=1, \ldots, d} S_{\delta_1, \ldots, \delta_d}. \]

**Definition 3.2.** The linear multistep method (3.3) for System (1.1) is called \(N P_m\)-stable if and only if \(H \subseteq S_{0, \ldots, 0}\).

**Definition 3.3.** The linear multistep method (3.3) for System (1.1) is called \(N G P_m\)-stable if and only if \(H \subseteq S\).

**Remark.** \(N G P_m\)-stability implies \(N P_m\)-stability.

It is well-known from [4,8,15,19] that the difference process (3.4) and (3.5) is \((\delta_1, \ldots, \delta_d)\)-stable at \((\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d)\) if and only if the characteristic polynomial \(G(z)\) in (3.7) satisfies
\[ \det(z^{m_0+q} Q(z) - P(z)) = 0 \implies |z| < 1, \quad \forall m_0 \geq s + 1, \]  (3.8)
that is, \(G(z)\) is a Schur polynomial.

**Lemma 3.2 ([15]).** The characteristic polynomial \(G(z)\) is a Schur polynomial if
\[ Q(z) \text{ is invertible whenever } z \in \mathbb{C}, |z| \geq 1, \] (3.9)
\[ \sup_{|z|=1} \rho(Q(z)^{-1} P(z)) \leq 1, \] (3.10)
\[ \det(z^{m_0+q} Q(z) - P(z)) \neq 0, \quad \forall m_0 \geq s + 1, |z| = 1, \rho(Q(z)^{-1} P(z)) = 1. \] (3.11)

Furthermore, if \(G(z)\) is a Schur polynomial, then \(Q(z)\) is invertible whenever \(z \in \mathbb{C}, |z| \geq 1\).

The following lemma gives conditions such that
\[ |\gamma(t, \delta_i)| \leq 1. \] (3.12)

**Lemma 3.3 ([17]).** For any given \(|z| = 1, \delta_i \in [0, 1)(i = 1, \ldots, d), |\gamma(t, \delta_i)| \leq 1 \text{ if and only if } q \leq s \leq q + 2. \) Furthermore, if \(s + q > 0, q \leq s \leq q + 2, |z| = 1 \text{ and } \delta_i \in (0, 1)(i = 1, 2, \ldots, d), \) then \(|\gamma(t, \delta_i)| = 1 \text{ if and only if } z = 1\).

We first prove the following result.

**Lemma 3.4.** For \((\overline{L}, \overline{M}_i, N_i, i = 1, \ldots, d) \in H, \)
\[ \sup_{\zeta \in \mathbb{C}^+} \rho \left( (\zeta I - \overline{L})^{-1} \left( \sum_{i=1}^{d} \xi_i \overline{M}_i + \zeta \sum_{i=1}^{d} \xi_i N_i \right) \right) < 1, \quad \forall \xi_i \in \mathbb{C}, |\xi_i| \leq 1. \]
Proof. Denote

\[ A(\zeta, \xi_1, \ldots, \xi_d) = (\zeta I - L)^{-1} \left( \sum_{i=1}^{d} \xi_i M_i + \zeta \sum_{i=1}^{d} \xi_i N_i \right) \].

For sufficiently large \(|\zeta|\),

\[ A(\zeta, \xi_1, \ldots, \xi_d) = \left( I + \frac{1}{\zeta} L + \frac{1}{\zeta^2} L^2 + \cdots \right) \left( \sum_{i=1}^{d} \xi_i N_i + \frac{1}{\zeta} \sum_{i=1}^{d} \xi_i M_i \right), \quad \forall |\xi_i| \leq 1. \]

Letting \(\zeta \to \infty\) gives

\[ \lambda_A(\zeta, \xi_1, \ldots, \xi_d) = \frac{\lambda_d}{\sum_{i=1}^{d} \xi_i N_i}, \]

\[ \rho[A(\zeta, \xi_1, \ldots, \xi_d)] = \rho \left( \sum_{i=1}^{d} \xi_i N_i \right) \leq \sum_{i=1}^{d} \|N_i\| < 1. \]

Since \(\lambda_A(\zeta, \xi_1, \ldots, \xi_d)\) is an algebraic function of \(\zeta\) for \(|\xi_i| \leq 1\), application of the maximum modulus principle on the unbounded region \(\mathbb{C}^+ \cup \mathbb{C}_0\) yields

\[ \rho[A(\zeta, \xi_1, \xi_2, \ldots, \xi_d)] < \max \left\{ \sum_{i=1}^{d} \|N_i\|, \sup_{\zeta \in \mathbb{C}_0} \rho[A(\zeta, \xi_1, \xi_2, \ldots, \xi_d)] \right\} < 1, \quad \forall \zeta \in \mathbb{C}^+. \]

Then

\[ \sup_{\zeta \in \mathbb{C}^+} \rho[A(\zeta, \xi_1, \ldots, \xi_d)] \leq \max \left\{ \sum_{i=1}^{d} \|N_i\|, \sup_{\zeta \in \mathbb{C}_0} \rho[A(\zeta, \xi_1, \xi_2, \ldots, \xi_d)] \right\} < 1 \]

whenever \(|\xi_i| \leq 1, i = 1, \ldots, d\). This completes the proof. \(\square\)

The following theorem fully characterizes the \(NGP_m\)-stability of linear multistep methods when applied to the neutral differential equations with many delays.

**Theorem 3.1.** Suppose that the Lagrange interpolation (3.5) satisfies \(q \leq s \leq q+2\) and \(m_i \geq s+1\) \((i = 1, 2, \ldots, d)\), then the linear multistep method (3.3) for System (1.1) is \(NGP_m\)-stable if and only if the linear multistep method is \(A\)-stable for ODEs.

**Proof.** Suppose that the linear multistep method (3.3) is \(A\)-stable. Let \((\bar{L}, \bar{M}_i, N_i, i = 1, \ldots, d) \in H\), so we only prove \((\bar{L}, \bar{M}_i, N_i, i = 1, \ldots, d) \in S_{\delta_1, \ldots, \delta_d}\). From Lemma 3.1(c), \(|\zeta| \geq 1 \Rightarrow \Re[\psi_2(z)] \geq 0\). Since \(\frac{\psi_1(z)}{\psi_2(z)} \rightarrow \frac{a_k}{b_k} > 0\) as \(z \rightarrow \infty\), then \(a_k I - \beta_k \bar{L}\) is invertible and the difference equations (3.4) and (3.5) is solvable. Also from Lemma 3.1(d), \(Q(z)\) is invertible whenever \(|z| \geq 1\).

If \(\psi_2(z) \neq 0\) for some \(z\) with \(|z| = 1\), then \(\psi_1(z) \neq 0\) and \(\rho[Q(z)^{-1} P(z)] \leq \sum_{i=1}^{d} \|N_i\| < 1\).

If \(\psi_2(z) \neq 0\) for some \(z\) with \(|z| = 1\), then it follows from Lemma 3.1(c) that \(\Re[\psi_2(z)] \geq 0\). In the case when \(\Re[\psi_2(z)] = 0\), from the definition of \(H\), we have

\[ \rho[Q(z)^{-1} P(z)] = \rho \left[ \left( \frac{\psi_1(z)}{\psi_2(z)} I - \bar{L} \right)^{-1} \left( \frac{\psi_1(z)}{\psi_2(z)} \sum_{i=1}^{d} \xi_i N_i + \sum_{i=1}^{d} \xi_i M_i \right) \right] < 1, \]

and thus

\[ \sup_{|z|=1} \rho \left[ Q(z)^{-1} P(z) \right] < 1, \]

where \(\xi_i = \gamma_i(z, \delta_i)z^{-m_i+m_0}, |\xi_i| \leq 1, i = 1, \ldots, d\). In the case when \(\Re[\psi_2(z)] > 0\), from Lemma 3.4 we get

\[ \sup_{|z|=1} \rho \left[ Q(z)^{-1} P(z) \right] = \sup_{|z|=1} \rho \left[ \left( \frac{\psi_1(z)}{\psi_2(z)} I - \bar{L} \right)^{-1} \left( \frac{\psi_1(z)}{\psi_2(z)} \sum_{i=1}^{d} \xi_i N_i + \sum_{i=1}^{d} \xi_i M_i \right) \right] < 1. \]
Since \( \sup|z|=1 \rho[Q(z)^{-1}P(z)] < 1 \) implies (3.10) and (3.11), it follows from Lemma 3.2 that \((\bar{L}, \bar{M}_i, N_i, i = 1, \ldots, d) \in S_{\delta_1, \delta_2, \ldots, \delta_d} \). This indicates that \( H \subseteq S \).

Suppose that \( H \subseteq S \). Since \((\bar{L}, \bar{M}_i, N_i, i = 1, \ldots, d) \in H \) implies that \((\bar{L}, M_i, N_i, i = 1, \ldots, d) \in S \), then \( Q(z)^{-1} \) exists whenever \(|z| \geq 1 \). It follows from Lemma 3.1(d) that the linear multistep method is \( A \)-stable for ODEs. This completes the proof. \( \square \)

4. Numerical experiment

The numerical experiment described in this section consists of applying two linear two-step methods to the same system of linear neutral differential equations. The purpose is to demonstrate the effect of linear stability theory in the preceding section.

Consider the following two-dimensional neutral differential system with two delays

\[
y'(t) = Ly(t) + M_1y(t - \tau_1) + M_2y(t - \tau_2) + N_1y'(t - \tau_1) + N_2y'(t - \tau_2),
\]

where \( \tau_1 = 1, \tau_2 = 2 \) and the coefficient matrices are given by

\[
L = \begin{bmatrix}
-5/4 & -3/4 \\
1/4 & -9/4
\end{bmatrix}, \quad
M_1 = \begin{bmatrix}
9/20 & -9/20 \\
-9/20 & 9/20
\end{bmatrix}, \quad
M_2 = \begin{bmatrix}
-17/40 & -17/40 \\
-17/40 & -17/40
\end{bmatrix}, \quad
N_1 = \begin{bmatrix}
1/5 \\
-1/5
\end{bmatrix}, \quad
N_2 = \begin{bmatrix}
1/4 \\
1/4
\end{bmatrix}.
\]

It can be proved that Conditions (I), (II) and (III)' are satisfied and therefore this neutral differential system is asymptotically stable. The initial function, for simplicity, is taken as \( \phi(t) = \begin{bmatrix} 10 \\ 50 \end{bmatrix} \) for \( t \in [1-\tau_2, 0] \).

We solve System (4.1) using the two-step BDF

\[
x_{n+2} - 4/3x_{n+1} + 1/3x_n = 2h/3 f(t_{n+2}, x_{n+2})
\]

with

\[
\psi_1^{BDF}(z) = z^2 - 4/3z + 1/3 \quad \text{and} \quad \psi_2^{BDF}(z) = 2/3 z^2
\]

and the two-step Adams–Bashforth method (ABM)

\[
x_{n+2} - x_{n+1} = \frac{h}{2} \left( 3f(t_{n+1}, x_{n+1}) - f(t_n, x_n) \right)
\]

with

\[
\psi_1^{ABM}(z) = z^2 - z \quad \text{and} \quad \psi_2^{ABM}(z) = 3 z^2 - \frac{1}{2}.
\]
respectively. It follows from [10] that BDF (4.2) is $A$-stable, while ABM (4.3) is not $A$-stable. We use the linear interpolation ($q = 0, s = 1$) since both methods are of second order. Numerical results at a range of times $t$ with step length $h = 1/4$ are given in Table 1.

The numerical solution in the second column produced by BDF (4.2) tends to zero, while the numerical solution in the third column generated by ABM (4.3) does not tend to zero. This phenomenon happens even if we solve the problem choosing other step sizes. At this stage, we conclude that the numerical results in the table agree with Theorem 3.1.

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