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A Note on I-Automorphisms

J. C. Beidleman

Department of Mathematics, University of Kentucky, Lexington, Kentucky 40506-0027

and

H. Heineken[1]

Mathematisches Institut, Universität Würzburg, Am Hubland, 97074 Würzburg, Germany

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1. INTRODUCTORY REMARKS

Let G be a group and denote by $\mathrm{PAut}(G)$ the group of power automorphisms of G (see [4]). An automorphism of G is called an I-automorphism of G if it maps every infinite subgroup of G onto itself. The set $\mathrm{IAut}(G)$ of all I-automorphisms of G is a normal subgroup of $\mathrm{Aut}(G)$ containing $\mathrm{PAut}(G)$. $\mathrm{IAut}(G)$ has been investigated by Curzio et al. [5], where they often consider the case when $\mathrm{IAut}(G) = \mathrm{PAut}(G)$ or at least $\mathrm{IAut}(G)$ is abelian.

Here we consider the structural restriction on G when $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. One of the major outcomes of this work is that for many groups H in fact $\mathrm{IAut}(H) = \mathrm{PAut}(H)$ holds. We also characterize infinite Chernikov groups G for which $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$.

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For completeness and convenience we begin with two statements connected mainly with the corresponding inner automorphisms: We characterize those groups all of whose infinite subgroups are normal and the intersection of all normalizers of infinite subgroups of a group. In analogy to the intersection of all normalizers being called the norm of a group and denoted by $\mathrm{N}(G)$, we use for the intersection of the normalizers of infinite subgroups the notation $\mathrm{IN}(G)$.

2. INFINITE NORM

Chernikov [3] has characterized the class of groups H satisfying the following two conditions: (a) H possesses an infinite abelian subgroup, (b) all infinite subgroups of H are normal. The characterization of our class of groups in Theorem 1 may be known, however, we provide a proof for sake of completeness. We make use of the notion of a minimal infinite group to mean a group of infinite order with only finite proper subgroups.

THEOREM 1. *Let G be a group of infinite order with the property that all subgroups of infinite order of G are normal subgroups of G. Then one of the following three statements is true*:

(1) *G is a Dedekind group.*

(2) *G is an extension of a Prüfer p-group by a finite Dedekind group.*

(3) *G is an extension of a non-abelian minimal infinite group M by a Dedekind group and all subgroups of infinite order of G contain M.*

Proof. If U is any subgroup of infinite order of G, all subgroups V satisfying $U \subseteq V \subseteq G$ are normal in G, so G/U is a Dedekind group and is either abelian or nilpotent of class 2, periodic, with commutator subgroup of order 2 and 2-subgroup a direct product of an elementary abelian and possibly one quaternion group (see[7, 5.3.1]). If G is not periodic, there is an infinite cyclic group $\langle x\rangle \subseteq G$, and, for every natural number k, the quotient group $G/\langle x^{8k}\rangle$ is a Dedekind group possessing elements of order 8 and therefore is abelian. We deduce for this case that G is abelian. In the arguments now following we restrict ourselves to periodic groups.

We consider the intersection M of all subgroups of infinite order of G. As a first step we assume that M is finite; we will show that G satisfies statement (1) in this case. If again U is a subgroup of infinite order, the same is true for $U \cap C(M)$, and we deduce that M is abelian. Since G/U is nilpotent of class 2 for every infinite subgroup U of G, G/M is nilpotent and $U \cap C(M)$ is a nilpotent subgroup of infinite order. Hence it contains an abelian normal subgroup L of infinite order. If $M \neq 1$, we have that L is periodic, there are only finitely many primes p such that L possesses elements of order p, and L is of finite rank; furthermore, the divisible part of L is a locally cyclic p-group P containing M. It is now easy to see that P must be of finite index in G and $M = P$, contradicting the finiteness of M. We have deduced $M = 1$, and so $G_3 = (G_2)^2 = 1$. It remains to show that G is either abelian or the Sylow 2-subgroup T of G is a direct product of a quaternion group of order 8 and an elementary abelian group. Choose an element $y \in G_2$ different from the identity. Since $M = 1$, there is a subgroup U of infinite order such that $y \notin U$. The quotient group G/U is a nonabelian Dedekind group; it possesses a (normal) subgroup K/U such that G/K is isomorphic to the quaternion group of order 8. If $T^2 \cap K \neq 1$ there is an element $z \in T$ such that $z^2 \neq 1$ and $z^2 \in K$. On the other hand, there is a subgroup V of infinite order such that $z^2 \notin V$, so G/V possesses elements of order 4. Again G/V is a Dedekind group, and there is a subgroup J/V such that G/V is either finite and cyclic of order $2^n > 2$ or isomorphic to the quaternion group of order 8. Now K and J are of finite index, so $K \cap J$ is a subgroup of infinite order, but $G/(K \cap J)$ is not a Dedekind group, a contradiction. So for finite M we obtain $M = 1$ and G satisfies statement (1).

The intersection M of all subgroups of infinite order surely has only proper subgroups of finite order. If M is abelian, we obtain statement (2), since $C(M)$ is of finite index in the periodic group G and a nilpotent quotient group $C(M)/M$ would have an abelian normal subgroup of infinite order, leading to an abelian subgroup $S \subseteq C(M)$ of infinite order with $S \cap M = 1$, a contradiction. If M is nonabelian, it is perfect and is an extension of a finite center by a simple minimal infinite group.

EXAMPLES I. (A) Choose a Prüfer p-group Q with element x of order p and a finite p-group P such that $P' = Z(P)$ is cyclic of order p, generated by y. If R is any finite Dedekind group, $Q \times P \times R/\langle xy \rangle$ is a group mentioned in statement (2) of Theorem 1. (B) For $p = 2$, Q may be substituted by the infinite generalized quaternion group in (A). (C) For statement (3) in Theorem 1, choose a minimal infinite group M with $Z(M)$ of order p and substitute M for Q in (A). The work of Adjan [1] hints at the existence of such groups M.

We turn our attention now to the "infinite norm" IN(G). The following statement is well known and gives a reason for our interest.

LEMMA 1. *Let Ξ be a characteristic class of subgroups of a group G and assume that $\alpha \in$ Aut(G) fixes all subgroups belonging to Ξ. If $x \in G$, then $x^{-1}x^{\alpha} \in \bigcap_{U \in \Xi} N(U)$.*

Proof. Let $U \in \Xi$, then also $xUx^{-1} \in \Xi$ and $(xUx^{-1})^{\alpha^{-1}} = xUx^{-1}$. Furthermore, $(x^{-1}(xUx^{-1})^{\alpha^{-1}}x)^{\alpha} = U$. But also $(x^{-1}(xUx^{-1})^{\alpha^{-1}}x)^{\alpha} = (x^{-1})^{\alpha}(xUx^{-1})x^{\alpha} = (x^{-1}x^{\alpha})^{-1}U(x^{-1}x^{\alpha})$.

COROLLARY 1. *Let x be an element of an infinite group G and let $\alpha \in \operatorname{Aut}(G)$.*

(a) *If* $\alpha \in \operatorname{PAut}(G)$, *then* $x^{-1}x^{\alpha} \in \operatorname{N}(G)$.

(b) *If* $\operatorname{N}(G) = 1$, *then* $\operatorname{PAut}(G) = 1$.

(c) *If* $\alpha \in \operatorname{IAut}(G)$, *then* $x^{-1}x^{\alpha} \in \operatorname{IN}(G)$.

(d) *If* $\operatorname{IN}(G) = 1$, *then* $\operatorname{IAut}(G) = 1$.

Remark. Cooper [4] has shown $x^{-1}x^{\alpha} \in Z(G)$ in case (a).

THEOREM 2. *Let G be a group of infinite order. Then one of the following statements is true*:

(1) $\operatorname{IN}(G)$ *is finite and nilpotent of class* 2.

(2) $\operatorname{IN}(G)$ *is an infinite Dedekind group*.

(3) $\operatorname{IN}(G)$ *is an extension of a Prüfer p-group by a finite Dedekind group*.

(4) $\operatorname{IN}(G)$ *is an extension of a nonabelian minimal infinite group M by a Dedekind group*, *and every infinite subgroup of G contains M*.

Proof. Again we consider the intersection M of all subgroups of infinite order of G. If M has infinite order, statement (3) or (4) is true by statements (2) and (3) of Theorem 1. If M has finite order and $\operatorname{IN}(G)$ has infinite order, we obtain as in Theorem 1 that M is abelian and that $C(M) \cap \operatorname{IN}(G)$ is nilpotent and of infinite order. Again $M = 1$, and statement (2) is true by statement (1) of Theorem 1. The case that $\operatorname{IN}(G)$ is finite remains to be considered.

In this case, consider any subgroup U of infinite order in G. Since $\operatorname{IN}(G)$ is a finite group, $V = U \cap C(\operatorname{IN}(G))$ is also of infinite order. Now $\operatorname{IN}(G)V/V \cong \operatorname{IN}(G)/(\operatorname{IN}(G) \cap V)$ is a Dedekind group and

$$\operatorname{IN}(G) \cap V = \operatorname{IN}(G) \cap C(\operatorname{IN}(G)) \cap U = Z(\operatorname{IN}(G)) \cap U.$$

So $\operatorname{IN}(G)/Z(\operatorname{IN}(G))$ is a Dedekind group. Assume that this quotient group is nonabelian. Let $Z = Z(\operatorname{IN}(G))$ and let $\{a_i Z | a_i \in \operatorname{IN}(G)\}$ be a set of generators for this quotient group. Furthermore, choose an element bZ of order 4. Then $\{c_i Z | c_i \in \operatorname{IN}(G)\}$ generates the quotient group if $c_i Z = a_i Z$ for all $a_i Z$ of order divisible by 4 and $c_i Z = a_i bZ$ otherwise. Let wZ be the nontrivial commutator. Now wZ is a power of every generator $c_i Z$ so that $[w, c_i] = 1$ and $w \in Z$. This is in contradiction to the construction, the central quotient group is abelian, and statement (1) is true.

EXAMPLES II. For statements (2), (3), and (4), examples can be taken that are the same as for Theorem 1. Examples for $\operatorname{IN}(G)$ to be finite non-Dedekind are now established.

(A) Choose two nonabelian minimal infinite groups M, N such that $|Z(M)| = |Z(N)|$ is an odd prime p and $M/Z(M) \ncong N/Z(N)$. Also choose a finite p-group R such that $R' = Z(R)$ and $|Z(R)| = p$. Let the three centers be generated by elements a, b, and c, respectively. Put $K = (M \times N \times R)/\langle ab, ac\rangle$. Then $\mathrm{IN}(K) \cong R$.

(B) For $p = 2$, choose M and N as before and let $R \cong S \cong Q_8$, with generators of the centers a, b, c, and d. Put $K = (M \times R/\langle ac\rangle) \times (N \times S/\langle bd\rangle)$. Then $\mathrm{IN}(G) \cong Q_8 \times Q_8$.

(C) The direct product of two quaternion groups also appears for $\mathrm{IN}(L)$ if $L = Q \times N \times S/\langle bd\rangle$, where Q is the infinite generalized quaternion group and N, S, b, d are as in (B).

Examples with commutator subgroups of higher order depend on the possibilities of $Z(M)$ for the nonabelian minimal infinite groups M. In (A), the nonisomorphy condition on $M/Z(M)$ and $N/Z(N)$ can be replaced by the following condition: if $M \cong N$, $\mathrm{Out}(G)$ does not operate transitively on $Z(M)$.

PROPOSITION 1. (a) *If G is a nonperiodic group, then* $\mathrm{IN}(G)$ *is abelian.*

(b) *If* $\mathrm{IN}(G)$ *is nonperiodic, then* $\mathrm{IN}(G) = Z(G)$.

(c) *In both cases* $\mathrm{IAut}(G)$ *is abelian.*

Proof. If $\mathrm{IN}(G)$ is nonperiodic, it satisfies the conditions of Theorem 1, so $\mathrm{IN}(G)$ is abelian. If G is nonperiodic and $\mathrm{IN}(G)$ is periodic, we take an element $x \in G$ of infinite order and an element $a \in \mathrm{IN}(G)$. Now also xa is of infinite order modulo the normal subgroup $\mathrm{IN}(G)$, and $\mathrm{IN}(G)$ normalizes $\langle x\rangle$ and $\langle xa\rangle$, also $\mathrm{IN}(G) \cap \langle x\rangle = \mathrm{IN}(G) \cap \langle xa\rangle = 1$. So x and xa centralize $\mathrm{IN}(G)$, and so does a; $\mathrm{IN}(G)$ is abelian. This shows (a).

For (b), choose an element $y \in \mathrm{IN}(G)$ of infinite order and another element $z \in G$. The subgroup $\langle y, z\rangle$ is infinite and, being finitely generated metabelian, residually finite. So by Lemma 2.1 of [5], the I-automorphism induced in this subgroup by the conjugation with y is a power automorphism, that is, $y^{-1}zy \in \langle z\rangle$. But then there is a natural number t such that $\langle z, y^t\rangle$ is abelian and nonperiodic. Again the I-automorphism induced by conjugation with y is a power automorphism, it fixes the element y^t of infinite order, and it is the identity. So $y^{-1}zy = z$ and all elements y of infinite order of $\mathrm{IN}(G)$ are in $Z(G)$. Now $\mathrm{IN}(G) = Z(G)$ since $\mathrm{IN}(G)$ is generated by its elements of infinite order.

For the commutativity of $\mathrm{IAut}(G)$, assume first the existence of an element x of infinite order such that $\langle x\rangle \cap \mathrm{IN}(G) = 1$. Arguing as in (a) we obtain commutativity for $\langle x, \mathrm{IN}(G)\rangle = B$. Every $\alpha \in \mathrm{IAut}(G)$ induces an I-automorphism in B which then is a power automorphism in B. Since $x^{-1}x^{\alpha} = 1$, we have that α fixes every element of $\mathrm{IN}(G)$ and stabilizes the series $1 \subseteq \mathrm{IN}(G) \subseteq G$. So $\mathrm{IAut}(G)$ is abelian in this case. If, on the other hand, $\mathrm{IN}(G)$ is nonperiodic, we have that $\langle u, \mathrm{IN}(G)\rangle$ is abelian and nonperiodic for every $u \in G$, and I-automorphisms of G induce here power automorphisms. So $u^\alpha \in \langle u\rangle$ and $\mathrm{IAut}(G) = \mathrm{PAut}(G)$, which is abelian. We have also shown (c).

Note that Theorem B of [5] is a consequence of Proposition 1.

3. IN(G) AND IAut(G)

We consider the structure of $\mathrm{IN}(G)$ more closely for the case where $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$.

THEOREM 3. *Let G be an infinite group and assume* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. *Then every infinite abelian normal subgroup A of G has the following structure: A is periodic, the finite residual R of A is a p-group of finite rank, and A is Chernikov.*

Proof. Let $h \in G$ and $\alpha \in \mathrm{IAut}(G)$ such that $h^\alpha \notin \langle h\rangle$. We may assume that h is of p-power order for some prime p. If t is an element of infinite order in A, then $\langle t\rangle^{\langle h\rangle}$ is a finitely generated h-invariant abelian group, and so $\langle t, h\rangle$ is a residually finite group. By Lemma 2.1 of [5], $\mathrm{IAut}(\langle h, t\rangle) = \mathrm{PAut}(\langle h, t\rangle)$, so that $h^\alpha \in \langle h\rangle$, contrary to construction. So A is periodic.

Let $A[p] = \langle a \in A | a^p = 1\rangle$ for $p \in \Pi$, where Π is the set of primes q such that A has an element of order q, and let C be the product of all $A[p]$ with $p \in \Pi$. If C is infinite, then $\langle C, h\rangle = C\langle h\rangle$ is a residually finite group and, again, $h^\alpha \in \langle h\rangle$, contrary to construction. So C is finite and A is the direct product of its finite residual R and a finite group. If R is not a p-group, let E and F be maximal p-, q-subgroups of R. Then

$$\langle h\rangle = \langle h, E\rangle \cap \langle h, F\rangle = \langle h, E\rangle^\alpha \cap \langle h, F\rangle^\alpha = \langle h\rangle^\alpha,$$

contrary to the definition of h. So R is a p-group of finite rank.

EXAMPLES III. (A) Let P be a Prüfer p-group, let C be a group of order p, and let $G \cong P \times C$. Then $\mathrm{PAut}(G) \cong \mathrm{Aut}(P)$, an uncountable abelian group, while $\mathrm{IAut}(G) \cong \mathrm{Aut}(P) \times \mathrm{Hol}(C)$, where $\mathrm{Hol}(C)$ is the holomorph of C.

(B) Let $G = \langle a, b_i | a^2 = (ab_i)^2 = b_1^p = b_{i+1}^p b_i^{-1} = 1\rangle$ with $\langle b_i\rangle = P$. If $p \neq 2$, $\mathrm{PAut}(G) = \mathrm{N}(G) = 1$ and $\mathrm{IAut}(G) \cong \mathrm{Hol}(P)$. If $p = 2$, $\mathrm{PAut}(G) = 1$, $\mathrm{N}(G) = Z(G)$, and $\mathrm{IAut}(G) \cong \mathrm{Hol}(P)$. Also note that in both cases $\mathrm{Inn}(G) \subseteq \mathrm{IAut}(G)$.

(C) Begin with an elementary abelian noncyclic p-group A and with an automorphism σ of order a prime $q \neq p$ of A such that A does not possess proper σ-invariant subgroups. The automorphism σ can be extended to a divisible p-group P where σ operates as in A on the socle of P. Let $G = \langle x, P | x^q = 1,\ x^{-1}ux = u^\sigma,\ \forall\, u \in P\rangle$. Then $\mathrm{PAut}(G) = \mathrm{N}(G) = 1$, while $\mathrm{IAut}(G)$ is the split extension of P by $\mathrm{PAut}(P)$.

(D) If, instead, $p = q$ in (C) and A is of rank $p - 1$, and if, furthermore, σ fixes only p elements of A, the same construction as in (C) can be done. In this case $\mathrm{PAut}(G) = 1$, $\mathrm{N}(G) = Z(G)$ is of order p, and $\mathrm{IAut}(G)$ is isomorphic to the split extension of P by $\mathrm{PAut}(P)$ since $\mathrm{Inn}(G) \cong G/Z(G) \cong G$.

COROLLARY 2. *Let G be a group such that* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$ *and let A be an infinite abelian normal subgroup of G. Let h be an element of G and let $h \in$* $\mathrm{IAut}(G)$ *such that $h^\alpha \notin \langle h\rangle$. If $\langle h\rangle$ is subnormal in $A\langle h\rangle$, then $\langle h, A\rangle$ is the direct product of a Prüfer p-group and a finite group.*

Proof. By Theorem 3, A is a product of a divisible group D and a finite group, and D is a p-group for some prime p. We may assume that $\langle h\rangle$ is a q-group for some prime q. Since $\langle h\rangle$ is subnormal in $D\langle h\rangle$, we have that $D\langle h\rangle$ is abelian and $p = q$. Assume now that D is not locally cyclic; then $D \cap \langle h\rangle$ is contained in some locally cyclic subgroup E of D. If the intersection is trivial, take any locally cyclic subgroup E of D. Then $D = E \times F$ for some divisible subgroup $F \subseteq D$, and

$$\langle h\rangle = \langle h, E\rangle \cap \langle h, F\rangle = \langle h, E\rangle^\alpha \cap \langle h, F\rangle^\alpha = \langle h\rangle^\alpha,$$

a contradiction. Hence D is locally cyclic.

From Corollary 2 we obtain

COROLLARY 3. *Let G be an infinite group such that* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. *Then the following are true*:

(1) *If G is nilpotent, then it is an extension of a Prüfer p-group by a finite group.*

(2) *The center of G is finite or an extension of a Prüfer p-group by a finite group.*

COROLLARY 4. *Let G be a group which contains an infinite abelian subgroup A. Then* $\mathrm{IAut}(G)$ *is metabelian. In particular, if G is a locally finite infinite group then* $\mathrm{IAut}(G)$ *is metabelian.*

Proof. By Proposition 1 we can assume that A is a torsion group. We can also assume $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. Now A contains an infinite subgroup B which is either Prüfer or non-Chernikov. By Theorem 3, $I = \mathrm{IAut}(G)$ operates as group of power automorphisms on B. Therefore, I' stabilizes the series $1 \subseteq B^G \subseteq G$, and I is metabelian. If G is infinite and locally finite, then G contains an infinite abelian subgroup.

Another consequence of Theorem 3 is the following:

Remark. Let G be a locally finite infinite group with the property that $\mathrm{IAut}(N(A)) \neq \mathrm{PAut}(N(A))$ for all infinite abelian subgroups A of G. Then G is a Chernikov group whose divisible part has one primary component.

Proof. Let A be an infinite abelian subgroup of G. By Theorem 3, A is Chernikov, and hence G is Chernikov by Theorem 5.8 of [6].

COROLLARY 5. *Let G be an infinite group and let* $\mathrm{IN}(G)$ *be soluble. Then* $\mathrm{IAut}(G)$ *is metabelian.*

Proof. Again we may assume that $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. First, suppose that $\mathrm{IN}(G)$ is infinite. Since $\mathrm{IN}(G)$ is soluble, it follows by parts (2), (3), and (4) of Theorem 2 that $\mathrm{IN}(G)$ contains a nontrivial divisible normal subgroup. Hence $\mathrm{IAut}(G)$ is metabelian by Corollary 4.

Next assume that $\mathrm{IN}(G)$ is finite and let $\alpha \in \mathrm{IAut}(G)$. Consider the subgroup F_α of all elements of G that are fixed by α. We obtain $x^{-1}x^\alpha = y^{-1}y^\alpha$ if and only if $F_\alpha x = F_\alpha y$. Since $x^{-1}x^\alpha \in \mathrm{IN}(G)$ we have that F_α is of finite index in G. Similarly, the intersection T_α of all of the conjugates of F_α is a normal subgroup of finite index in G. We form $L = \bigcap_{\alpha \in \mathrm{IAut}(G)} T_\alpha$. Either L is infinite and $\alpha \in \mathrm{IAut}(G)$ induces a power automorphism on G/L or G/L is infinite and residually finite, and again α induces a power automorphism. It follows that $(\mathrm{IAut}(G))'$ stabilizes the series $1 \subseteq L \subseteq G$, so that $\mathrm{IAut}(G)$ is metabelian.

LEMMA 2. *Let G be an infinite group and assume that* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. *If G has more than one nontrivial divisible abelian normal subgroup, then the following statements are true*:

(1) *The product D of all divisible abelian normal subgroups of G is an abelian p-group.*

(2) *No two divisible abelian normal subgroups are operator-isomorphic. In particular, the divisible part of $D \cap Z(G)$ is a Prüfer group or is trivial.*

(3) *If T_i is the* (*only*) *normal subgroup of order p of G which is contained in the minimal divisible normal subgroup M_i of G, the rank of the product of all T_i is at most* 2.

(4) *Every set of divisible normal subgroups of G having pairwise trivial intersection consists of at most p members.*

(5) $\mathrm{IN}(G)/Z(G)$ *is finite.*

Proof. D is abelian, so (1) follows from Theorem 3.

For the following, the reader is reminded that minimal divisible normal subgroups of G need not intersect trivially; they do so if their intersections with the center of G do. Also $L \cap Z(G)$ is cyclic for every minimal divisible noncentral normal subgroup L of G. Choose an element $h \in G$ and $\alpha \in \mathrm{IAut}(G)$ such that $h^\alpha \notin \langle h \rangle$. Then h has finite order, and we may take it to be of order a power of a prime q. We show (2) first for $\langle h \rangle D$ instead of G. Assume that there are two minimal divisible h-invariant subgroups E, F of D which are operator isomorphic in $\langle D, h \rangle$; operator isomorphism allows the choice of E, F with trivial intersection. We consider $\langle h \rangle EF$. Either $\langle h \rangle \cap EF = 1$, or there is a minimal divisible h-invariant subgroup T of D such that $D \cap \langle h \rangle = T \cap \langle h \rangle$ and, without loss of generality, $T \cap F = 1$. Substituting T for E in both cases we have $\langle h \rangle = \langle h, T \rangle \cap \langle h, F \rangle$ and $\langle h \rangle = (\langle h \rangle)^\alpha$ as intersection of two infinite subgroups, a contradiction. This shows (2) for $\langle D, h \rangle$ and for G.

In the course of the argument we have also deduced $\langle h \rangle \cap D \neq 1$, and so $p = q$. A minimal nonlocally cyclic h-invariant divisible subgroup of D will always split into minimal h^p-invariant divisible subgroups which are operator-isomorphic in $\langle h^p \rangle D$. We therefore obtain $G = \langle h \rangle C(D)$.

If the rank mentioned in (3) is more than 2, we have three divisible normal subgroups A, B, C in D such that $ABC = A \times B \times C$; in this case $\langle h \rangle = \langle h, A \rangle \cap \langle h, B \rangle \cap \langle h, C \rangle$ is fixed by α, a contradiction. This shows (3).

If (4) is not satisfied, every cyclic subgroup in the group mentioned in (3) occurs as some T_i. We know that $\langle h \rangle$ intersects nontrivially with this subgroup. Let, for instance, $T_i \subseteq \langle h \rangle$, $T_i \subseteq M_i$, and let M_j be some other minimal divisible normal subgroup in D such that $M_i \cap M_j = 1$. Now $\langle h \rangle = \langle h, M_i \rangle \cap \langle h, M_j \rangle$, and $\langle h \rangle$ is fixed by α, contrary to construction. This contradiction shows that (4) is true.

Assume now that the rank of the subgroup mentioned in (3) is 2. Then there are two h-invariant divisible subgroups U, V of D such that $U \cap V = 1$. We find $\mathrm{IN}(G)U/U \subseteq \mathrm{N}(G/U) \subseteq Z_2(G/U)$ and $\mathrm{IN}(G)V/V \subseteq \mathrm{N}(G/V) \subseteq Z_2(G/V)$. Since $U \cap V = 1$, we obtain $\mathrm{IN}(G) \subseteq Z_2(G)$. If all T_i coincide, two minimal h-invariant divisible subgroups U, V of D have finite intersection, and as before we obtain $\mathrm{IN}(G)(U \cap V)/(U \cap V) \subseteq Z_2(G/(U \cap V))$. By finiteness we have $U \cap V \subseteq Z_s(G)$ for some finite s, and $\mathrm{IN}(G) \subseteq Z_{s+2}(G)$, since $G = C(D)\langle h \rangle$. Assume now that $\mathrm{IN}(G)$ is infinite; then the infinite subgroup $D \cap \mathrm{IN}(G)$ contains an abelian divisible subgroup, and this subgroup can only be contained in $Z(G)$. Now the quotient group $\mathrm{IN}(G)/Z(G)$ has to be finite. Thus (5) is established and the lemma is proved.

EXAMPLES IV. (A) Let P, Q be Prüfer p-groups and let $A = \langle a \rangle$, $B = \langle b \rangle$ be cyclic groups of order p^2. We form the standard wreath products $Pwr\,A = P^A A$ and $Qwr\,B = Q^B B$. Let $X = P^A \cap C(a^p)$ and $Y = [b, Q^B]$ and choose two elements x, y of order p such that $x \in X \cap Z(XA)$; $y \in Y \cap Z(YB)$. Now

$$G = \langle X, Y, z | z^{p^3} = 1;\ z^{-1}uz = a^{-1}ua\ \forall\, u \in X;\ z^{-1}vz = b^{-1}vb\ \forall v \in Y \rangle$$

is defined. Let $L = \langle z^{p^2} xy \rangle$. Then the automorphism α fixing all elements of XYL/L and mapping zL onto zxL belongs to $\mathrm{IAut}(G/L)$ but not to $\mathrm{PAut}(G/L)$.

(B) With the hypotheses of (A) (except that instead of L we consider $M = \langle z^{p^2}, xy \rangle$), we have that $XM \cap YM \cap D = M \cap D = \langle x \rangle M/M$ is the intersection of all minimal divisible zM-invariant subgroups of DM/M, and the automorphism β fixing all elements of DM/M and mapping zM onto zxM is contained in $\mathrm{IAut}(G/M)$ but not in $\mathrm{PAut}(G/M)$.

THEOREM 4. *Let G be an infinite group such that* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$, $\mathrm{IN}(G)$ *is a Dedekind group, and* $\mathrm{IN}(G)/Z(G)$ *is infinite. Then the following statements are true*:

(1) *The maximal abelian divisible subgroup D of* $\mathrm{IN}(G)$ *is a nontrivial p-group, and it is the only nontrivial abelian divisible normal subgroup of G.*

(2) *If $x \in G \backslash C(D)$, all infinite x-invariant subgroups of G contain D.*

(3) *$G/C(D)$ is cyclic.*

(4) *If $G/C(D)$ is a p-group, it is of order p and D has rank $p-1$.*

(5) *If $G/C(D)$ is not a p-group and D has rank m, the order of $G/C(D)$ is divisible by primes dividing $p^m - 1$ but not dividing any $p^n - 1$ with $1 \leq n < m$.*

Proof. The maximal abelian divisible normal subgroup of G is a p-subgroup, so the same is true for $D \subseteq \mathrm{IN}(G)$. If G possesses more than one divisible abelian normal subgroup, by Lemma 2, $\mathrm{IN}(G)/Z(G)$ is finite. So (1) is true, and D is the only nontrivial abelian divisible normal subgroup of G. We assume now the existence of an element x of order a power of p such that $|\langle x \rangle C(D) : C(D)| = p^k > p$. Then D possesses nontrivial x^p-invariant divisible subgroups, and conjugation by elements of D are no longer elements of $\mathrm{IN}(G)$, by Lemma 2. So $x^p \in C(D)$, and the rank of D must be $p-1$ since there is only one operator isomorphism class for D. This shows (4); (3) and (2) are also true in this case.

The alternative is that $G/C(D)$ is not a p-group; by the argument above this means that there is no element of order p in $G/C(D)$. Now D splits into a direct product of divisible x-invariant subgroups for every $x \in G \backslash C(D)$, which in turn yields that there is no such proper splitting for any $x \in G \backslash C(D)$, proving (2) for this case. This means in particular that every $x \in G \backslash C(D)$ operates by conjugation without any fixed points on the socle S of D. Accordingly, abelian subgroups of $G/C(D)$ have to be cyclic. Since nonabelian supersoluble subgroups of $G/C(D)$ would contain elements inducing power automorphisms in S, we obtain that S itself is cyclic (and $G/C(D)$ is cyclic) or $G/C(D)$ is cyclic, since these nonabelian supersoluble subgroups of $G/C(D)$ do not exist. So in this case we see that (2), (3), and (5) are true (the number theoretic condition being a consequence of (2)). Theorem 4 is proved.

4. RADICAL GROUPS

In this section we show that the condition $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$ for an infinite group G yields that the structures of the Hirsch–Plotkin radical and the Baer radical are very restricted. We begin with

THEOREM 5. *Let G be an infinite group satisfying* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$ *and let L be a locally radical normal subgroup of G. Then L is a soluble Chernikov group whose finite residual has one primary component.*

Proof. We can assume that L is infinite. Let h be an element of G and let $\alpha \in \mathrm{IAut}(G)$ such that $h^{\alpha} \notin \langle h \rangle$. Then h has finite order. Put $B = L\langle h \rangle$ and note that B is locally radical and $\mathrm{IAut}(B) \neq \mathrm{PAut}(B)$. Hence B satisfies the minimum condition by Theorem A of [5]. Thus L is periodic and locally radical, so L is locally soluble. By I.E.6 of [6] , L is a soluble Chernikov group. The rest follows from Theorem 3.

COROLLARY 6. *Let G be an infinite group satisfying* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. *Then*

(1) *The Hirsch–Plotkin radical R of G is a soluble Chernikov group which is hypercentral. Moreover, if S is the last term of the upper Hirsch–Plotkin series, then S/R is finite and S is a soluble Chernikov group.*

(2) *The Baer radical B of G is a nilpotent Chernikov group. If A is the subsoluble radical of G, then A is a soluble Chernikov group and A/B is finite.*

Proof. By Theorem 5 we know that S is a soluble Chernikov group whose finite residual is contained in R. Thus S/R is finite, and, by I.H.4 of [6], R is hypercentral. So (1) is true.

By Lemma 2 of [2], B is a nilpotent Chernikov group. By (1) A is a soluble Chernikov group. It is clear that the finite residual of A is contained in B, so we have shown (2).

5. CHERNIKOV GROUPS

We now characterize Chernikov groups G in which $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$.

THEOREM 6. *Let G be an infinite Chernikov group and let D be the maximal divisible normal subgroup of G. Then* $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$ *if and only if one of the following three statements holds true*:

(1) $\mathrm{IN}(G)$ *is infinite and $Z(G)$ is finite. If D is not locally cyclic, G is described in Theorem* 4. *If D is locally cyclic, then $G/C(D) \neq 1$.*

(2) $\mathrm{IN}(G)/Z(G)$ *is finite, $D \not\subseteq Z(G)$, and G is described in Lemma* 2.

(3) $Z(G)$ *is infinite, $D \subseteq Z(G)$, and D is a locally cyclic p-group; furthermore,*

(a) *G possesses a normal subgroup of index p,*

or

(b) *there is an element $x \in G'$ of order a power of p such that $o(x) >$* $\mathrm{Max}(2, |D \cap G'|)$.

Proof. It is clear that $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$ whenever (1) or (2) holds. Now assume that $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. By Lemma 2, $Z(G) \cap D$ is finite or finite by locally cyclic. Note that (1) and (2) apply according to whether $D \not\subseteq Z(G)$ is the only abelian divisible normal subgroup or not.

It remains to prove that (3) holds if and only if $\mathrm{IAut}(G) \neq \mathrm{PAut}(G)$. We begin by showing the sufficiency. In case (a), there is a normal subgroup N such that $|G : N| = p$, and there is an element x of p-power order such that $G = \langle x, N\rangle$. Since $\langle x, D\rangle$ is abelian and D is divisible, $\langle x, D\rangle = \langle y\rangle \times D$ for some element y, and $G = \langle y, N\rangle$. Now define $\alpha \in \mathrm{Aut}(G)$, fixing all of the elements of N and mapping y onto yt, where t is an element of order p from D. Then $\alpha \in \mathrm{IAut}(G) \setminus \mathrm{PAut}(G)$. Assume now that (a) is not satisfied. Then the only minimal supplement L of D in G is generated by elements of order prime to p; also $G' = (DL)' = L'$. Let K be the Hall p'-subgroup of $Z(G)$ and note that since D is a p-subgroup contained in $Z(G)$ we also have that KD/D is the Hall p'-subgroup of $Z(G/D)$. Every $\alpha \in \mathrm{IAut}(G)$ induces a power automorphism in G/D, and $G/D = LD/D \cong L/(D \cap L)$ is generated by elements of order prime to p. By Cooper [4], the automorphism induced by α in G/D is central. So if y is an element of order prime to p, we have $(yD)^\alpha = ywD$, where $w \in K$, and in fact $y^\alpha = yw$. We see therefore that α induces the identity on L/K; in particular, elements of order a power of p in L are fixed by α. This means that α will induce a power automorphism in D which fixes all elements of $D \cap L$, for example, $t^\alpha = t^{1+m}$ for $m = \mathrm{Max}(2, |D \cap L|)$. If $z \in L$ is an element of order a power of p such that $|\langle z\rangle| > \mathrm{Max}(2, |D \cap L|)$, the following automorphism β belongs to $\mathrm{IAut}(G)\backslash\mathrm{PAut}(G)$:

$$u^{\beta} = u, \quad \forall\, u \in L; \qquad u^{\beta} = u^{1+m}, \quad \forall\, u \in D.$$

(Here we note that the restriction of β to $\langle z, D\rangle$ is not a power automorphism.) To prove necessity we have to show that $\mathrm{IAut}(G) = \mathrm{PAut}(G)$ if neither (a) nor (b) is true. In this case we use the preceding argument to find the minimal supplement L of D to be generated by all elements of order prime to p, and if $\alpha \in \mathrm{IAut}(G)$, all of the elements in L of order a power of p are fixed. If $p = 2$ and the only elements in L of order 2^r have order 2, then $\mathrm{IAut}(G) = \mathrm{PAut}(G)$. If this case does not arise and there is no element in L of order a power of p and bigger than $|D \cap L|$, and if, furthermore, $\alpha \in \mathrm{IAut}(G)$, then $\langle t^{\alpha}\rangle = \langle t\rangle$ and $t^{-1}t^{\alpha} \in \langle t^m\rangle$, $\forall t \in D$, where $m = |D \cap L|$. Moreover, α restricted to $\langle y, D\rangle$ is a power automorphism for all elements of order p^s since $p^s \leq m$. This proves statement (3) and completes the proof of Theorem 6.

An example for (1) is Example III(C); for (2) we have Example IV(A). For statement (3) case (a) we have Example III(A). For case (b) and $p = 2$ choose a Prüfer 2-group A with element a of order 2 and $B \cong SL(2, q)$ with $q \neq 2$, with element $b \in Z(B)$ of order 2. Now $(A \times B)/\langle ab\rangle$ is an example. For $p \neq 2$ choose A a Prüfer p-group with element a of order p, and $B \cong SL(p, q)$ with p^2 dividing $q - 1$ and central element $b \neq 1$. Again $(A \times B)/\langle ab\rangle$, but $A \times (B/Z(B)$ are also examples.

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