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Stable manifolds for nonautonomous equations without exponential dichotomy

Luis Barreira*, Claudia Valls

Departamento de Matemática, Instituto Superior Técnico, Av. Rovisco Pais 1049-001 Lisboa, Portugal

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Abstract

We consider *nonautonomous* ordinary differential equations $v' = A(t)v$ in Banach spaces and, under fairly general assumptions, we show that for any sufficiently small perturbation f there exists a stable invariant manifold for the perturbed equation $v' = A(t)v + f(t, v)$, which corresponds to the set of negative Lyapunov exponents of the original linear equation. The main assumption is the existence of a *nonuniform exponential dichotomy* with a small nonuniformity, i.e., a small deviation from the classical notion of (uniform) exponential dichotomy. In fact, we showed that essentially *any* linear equation $v' = A(t)v$ admits a nonuniform exponential dichotomy and thus, the above assumption only concerns the smallness of the nonuniformity of the dichotomy. This smallness is a rather common phenomenon at least from the point of view of ergodic theory: almost all linear variational equations obtained from a measure-preserving flow admit a nonuniform exponential dichotomy with *arbitrarily small* nonuniformity. We emphasize that we do not need to assume the existence of a uniform exponential dichotomy and that we never require the nonuniformity to be arbitrarily small, only sufficiently small. Our approach is related to the notion of Lyapunov regularity, which goes back to Lyapunov himself although it is apparently somewhat forgotten today in the theory of differential equations.

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* Corresponding author.
E-mail addresses: barreira@math.ist.utl.pt (L. Barreira), cvalls@math.ist.utl.pt (C. Valls)
URL: http://www.math.ist.utl.pt/∼barreira/ (L. Barreira).

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1. Introduction

We are interested in the construction of stable and unstable invariant manifolds without assuming the existence of an exponential dichotomy, here sometimes called a *uniform exponential dichotomy*, for the associated linear variational equation. We consider both finite-dimensional and infinite-dimensional spaces. Our main objective is to find the weakest possible setting in which one can construct the above invariant manifolds. We still require some amount of hyperbolicity or, equivalently, some amount of exponential dichotomy to establish the existence of the invariant manifolds. Namely, we show that under fairly general assumptions on the linear variational equation, the generalized notion of *nonuniform exponential dichotomy* is in fact the weakest possible context which allows us to establish the existence of stable and unstable invariant manifolds.

In the theory of differential equations, both in finite-dimensional and infinite-dimensional spaces, the notion of *exponential dichotomy*, together with some of its variants, extensions, and modifications, plays a central role in the study of hyperbolic invariant manifolds. In particular, the existence of an exponential dichotomy for a linear variational equation

$$v' = A(t)v \tag{1}$$

precludes the existence of stable and unstable invariant manifolds for the solution of the nonlinear differential equation originating (1), up to mild additional assumptions on the nonlinear part of the vector field. The theory of exponential dichotomies and its applications are widely developed. We refer to the books [5–8,20] for details and further references.

On the other hand, the notion of exponential dichotomy demands considerably from the dynamics and it is of considerable interest to look for more general types of hyperbolic behavior. Nevertheless, we emphasize that there exist large classes of linear differential equations possessing exponential dichotomies. In this respect, we can mention, for example, the classical series of papers of Sacker and Sell [15–19] that in particular discuss sufficient conditions for the existence of exponential dichotomies, also in the infinite-dimensional setting. For a detailed discussion, further references, and historical comments, we recommend the book [5]. In a different direction, geodesic flows on compact smooth Riemannian manifolds with strictly negative sectional curvature have the whole unit tangent bundle as a hyperbolic set, i.e., they define Anosov flows. Furthermore, time changes and small C^1 perturbations of flows with a hyperbolic set also possess a hyperbolic set. We refer to the book [9] for details.

In order to illustrate our approach we will first formulate a rigorous statement on the existence of stable manifolds.

Consider a Banach space X and a continuous function $t \mapsto A(t)$ such that $A(t)$ is a bounded linear operator on X for each $t \geqslant 0$. We assume that all solutions of (1) are global in the future, i.e., are defined for every $t \geqslant 0$. Let $T(t, s)$ be the evolution operator associated with Eq. (1). This is the operator satisfying $T(t, s)v(s) = v(t)$ for every solution $v(t)$ of (1) and every $t \geqslant s$. For simplicity, we assume that the evolution operator $T(t, s)$ has a block decomposition

$$T(t, s) = (U(t, s), V(t, s))$$

into evolution operators with respect to some invariant decomposition $X = E \times F$ (which is time independent). We say that Eq. (1) admits a *weak nonuniform exponential dichotomy* if there exist constants $\lambda < 0 \leqslant \mu$ and a, b, $K > 0$, such that for every $t \geqslant s \geqslant 0$,

$$\|U(t, s)\| \leqslant K e^{\lambda(t-s)+as} \quad \text{and} \quad \|V(t, s)^{-1}\| \leqslant K e^{-\mu(t-s)+bt}. \tag{2}$$

We note that a nonuniform exponential dichotomy is a particular case of this notion (see Section 2.2 for details). The weaker version considered here is sufficient for the existence of stable manifolds (see Theorem 1).

The constants λ and μ play the role of Lyapunov exponents, while a and b measure the nonuniformity of the dichotomy. The assumption $\lambda < 0$ means that there exists at least one negative Lyapunov exponent.

We now consider the perturbed equation

$$v' = A(t)v + f(t, v), \tag{3}$$

where $f(t, v)$ is a continuous function defined for $t \geqslant 0$ and $v \in X$, such that $f(t, 0) = 0$ for every $t \geqslant 0$ (and thus the origin is also a solution of (3)). We will use the notation $\mathbb{R}_0^+$ for $[0, +\infty)$.

We now formulate our main result on the existence of stable manifolds.

Theorem 1. *Assume that Eq.* (1) *admits a weak nonuniform exponential dichotomy, and that there exist* $c > 0$ *and* $q > 0$ *such that*

$$\|f(t, u) - f(t, v)\| \leqslant c\|u - v\|(\|u\|^q + \|v\|^q)$$

for every $t \geqslant 0$ *and* u, $v \in X$. *If the conditions*

$$q\lambda + a < 0 \quad \textit{and} \quad \lambda + b < \mu \tag{4}$$

are satisfied, then there exists a Lipschitz manifold $\mathcal{W} \subset \mathbb{R} \times X$ *which is the graph of a Lipschitz function* $\varphi: U \to F$, *where* $U \subset \mathbb{R}_0^+ \times E$ *is an open neighborhood of the line* $\mathbb{R}_0^+ \times \{0\}$, *and the following properties hold*:

(1) $(t, 0) \in \mathcal{W}$ *for every* $t \geqslant 0$;
(2) $\mathcal{W}$ *is forward invariant under the semiflow* Ψ_τ *on* $\mathbb{R}_0^+ \times X$ *generated by the system*

$$t' = 1, \quad v' = A(t)v + f(t, v);$$

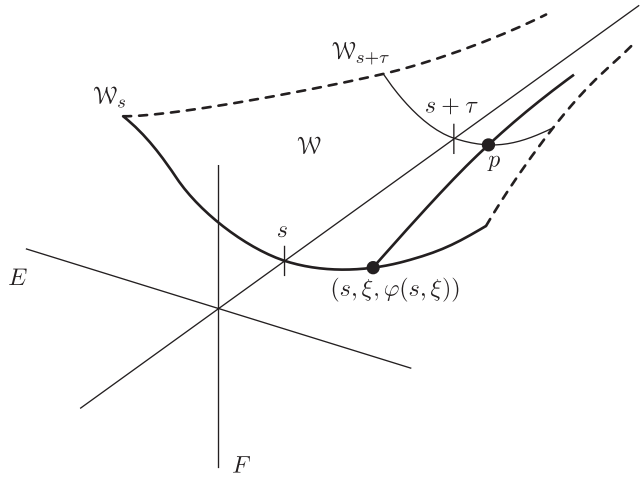

Fig. 1. A local stable manifold $\mathcal{W}$ of the origin. In order that $\mathcal{W}$ is invariant under the semiflow Ψ_τ we require that $p = \Psi_\tau(s, \xi, \varphi(s, \xi))$.

(3) *there exists* $D > 0$ *such that for every* (s, u), $(s, v) \in \mathcal{W}$ *and* $\tau \geqslant 0$, *we have*

$$\|\Psi_\tau(s, u) - \Psi_\tau(s, v)\| \leqslant D e^{\lambda\tau + as} \|u - v\|.$$

Theorem 1 is an immediate consequence of Theorem 3. See Fig. 1 for an illustration. We refer to Section 3 for a detailed formulation, and in particular for an explicit description of the class of Lipschitz functions φ under consideration.

Note that the first inequality in (4) is satisfied for a given a provided that q, the order of the perturbation, is sufficiently large (note that $\lambda < 0$). Furthermore, both inequalities in (4) are automatically satisfied when a and b are sufficiently small. The "small" exponentials e^{as} and e^{bt} in (2), which are not present in the case of a uniform exponential dichotomy, are in fact the main cause of difficulties in our construction of invariant manifolds. Their occurrence means that the exponential behavior may be spoiled along the orbits, although with small exponential speed provided that a and b are sufficiently small.

The former exposition leads us to describe a gradation of types of dichotomies, as follows:

1. uniform exponential dichotomy, the classical notion of dichotomy;
2. nonuniform exponential dichotomy with "controlled" nonuniformity, such as for example as given by the conditions in (4);
3. nonuniform exponential dichotomy, without any a priori control on the nonuniformity.

It is shown in [4] that *any* equation $v' = A(t)v$ with $A(t)$ as above admits a nonuniform exponential dichotomy. Thus, in specific applications, such as in Theorem 1, we never need to assume the existence of a nonuniform exponential dichotomy (since this is always the case), but instead we look for "smallness" conditions on a and b which ensure the desired results. Although this could be seen as suggesting to leave aside the notion of nonuniform exponential dichotomy, since as described above *any* linear equation with global solutions admits one, it may be very difficult to obtain sharp estimates for a and b other than when $A(t)$ is lower triangular or upper triangular for every $t \geqslant 0$ with respect to some fixed basis (without knowing explicitly the solutions of (1), which in general are virtually impossible to obtain). Thus, it is very convenient to proceed somewhat axiomatically and deal with the minimal necessary structure. This is the main reason why we consider the notion of nonuniform exponential dichotomy.

In view of the unavoidable assumption of sufficiently small constants a and b, it is important to estimate in quantitative terms how a nonuniform exponential dichotomy deviates from a uniform one—more precisely, from the one for which the uniform exponential contraction and the uniform exponential expansion are the same as the corresponding contraction and expansion in the uniform exponential dichotomy, but with the extra small exponentials replaced by *constants*. Fortunately, there exists a device, introduced by Lyapunov, that allows one to measure this deviation. It is the so-called regularity coefficient in the context of the Lyapunov–Perron regularity theory (see Section 6.1). In particular, we showed in [4] how to obtain sharp bounds for the regularity coefficient in terms of the entries of the matrix $A(t)$ and its solutions.

On the other hand, it turns out that the smallness of the nonuniformity is a rather common phenomenon at least from the point of view of ergodic theory: almost all linear variational equations obtained from a measure-preserving flow on a smooth Riemannian manifold admit a nonuniform exponential dichotomy with arbitrarily small nonuniformity (see Proposition 2).

Our work can also be considered a contribution to the theory of nonuniformly hyperbolic dynamics. We refer to [1] and to the supplement in [9] for detailed expositions of parts of the theory and to the survey [2] for a detailed description of its contemporary status. The theory goes back to the landmark works of Oseledets [11] and Pesin [12,13]. Since then it became a major part of the general theory of dynamical systems and a principal tool in the study of stochastic behavior. The nonuniform hyperbolicity conditions can be expressed in terms of the Lyapunov exponents. For example, almost all trajectories of a dynamical system preserving an invariant measure with nonzero Lyapunov exponents are nonuniformly hyperbolic. Our definition of weak nonuniform exponential dichotomy in (2) is inspired in the notion of uniform exponential dichotomy but also in the notion of nonuniformly hyperbolic trajectory (see for example [1]). There are however specific independent reasons to consider the notion in (2). This is the case of Proposition 6 in Section 6 below (the proof of which is inspired in related work in [1]), showing that *any* linear equation $v' = A(t)v$ with global solutions admits a (weak) nonuniform exponential dichotomy.

Invariant manifolds were first obtained for nonuniformly hyperbolic trajectories by Pesin [12]. The first related results in Hilbert spaces were established by Ruelle [14]. The case of transformations in Banach spaces under some compactness assumptions was considered by Mañé [10] (including the case of differentiable maps with compact derivative at each point). These results were extended in [21] for a class of transformations satisfying a certain asymptotic compactness.

On the other hand, there are several differences between our approach and the usual approach in the theory of nonuniformly hyperbolic dynamics. In particular, we start from a linear equation $v' = A(t)v$ instead of a linear variational equation $v' = A_x(t)v$ with $A_x(t) = d_{\varphi_t x}F$, obtained from a particular solution $\varphi_t x$ of a given autonomous equation $x' = F(x)$. Here φ_t is the flow generated by the autonomous equation. Thus, we depart from the linear to the nonlinear behavior, seen as a perturbation, as in (3), instead of departing from a given nonuniformly hyperbolic trajectory of the original map or flow, with the hyperbolicity conditions expressed in terms of the differentials along the orbit. Another difference is the method of proof. On one hand, our approach to the proof of Theorem 1 could be considered classical, and consists in using the differential equation (3) to express the forward invariance of the manifold $\mathcal{W}$ under the dynamics to conclude that φ must satisfy a fixed point problem. However, the extra small exponentials in a nonuniform exponential dichotomy substantially complicate this approach and the implementation requires several new ideas (see Section 4 for details). We also obtain in a very direct manner explicit quantitative information on the size of the neighborhoods on which we must choose an initial condition so that the solution satisfies a given bound. In fact, this information is put from the beginning in the space on which we look for the fixed point.

Another advantage of our approach is that we deal directly with flows instead of considering time-one maps as it is sometimes customary in the theory of hyperbolic dynamics (uniform or nonuniform). This allows us to give a clean and direct proof, dealing *simultaneously* with all the times along each orbit. This observation is also relevant in the study of the regularity of the manifolds. We intend to show in [3] that the invariant manifolds constructed in Theorem 1 are as smooth as the perturbation f (the material in [3] is omitted in this paper since the proofs are considerably long and use methods that are substantially different). Furthermore, since we are dealing with semiflows instead of flows it is in general impossible to introduce adapted Lyapunov norms.

The structure of the paper is the following. In Section 2 we describe our setup and we discuss the notion of nonuniform exponential dichotomy and its relation to ergodic theory. Our main results concerning the existence of stable manifolds are formulated in Section 3 and are proved in Section 4. The case of unstable manifolds is considered in Section 5, as a consequence of the former existence results by reversing time. In Section 6 we show how our results can be applied to nonautonomous linear differential equations with negative Lyapunov exponents to obtain corresponding stable and unstable manifolds. In particular, we use the classical Lyapunov–Perron regularity theory to estimate the nonuniformity of a nonuniform exponential dichotomy.

2. Nonuniform exponential dichotomies

2.1. Setup

Let $A: \mathbb{R}_0^+ \to B(X)$ be a continuous function, where $B(X)$ is the set of bounded linear operators on X. Consider the initial value problem

$$v' = A(t)v, \quad v(s) = v_s, \tag{5}$$

with $s \geqslant 0$ and $v_s \in X$. We assume that

$$\text{each solution of (5) is defined for every } t \geqslant s. \tag{6}$$

We want to study nonlinear perturbations of Eq. (5). That is, consider a continuous function $f: \mathbb{R}_0^+ \times X \to X$ such that

$$f(t, 0) = 0 \text{ for every } t \geqslant 0. \tag{7}$$

We assume that there exist $c > 0$ and $q > 0$ such that

$$\|f(t,u) - f(t,v)\| \leqslant c\|u - v\|(\|u\|^q + \|v\|^q) \tag{8}$$

for every $t \geqslant 0$ and $u, v \in X$. In applications, condition (8) may be obtained by choosing an appropriate cut-off of the perturbation in a neighborhood of $0 \in X$ and using a Taylor expansion of f, provided that the perturbation is sufficiently regular. Consider now the initial value problem

$$v' = A(t)v + f(t, v), \quad v(s) = v_s, \tag{9}$$

with $s \geqslant 0$ and $v_s \in X$. Note that $v(t) \equiv 0$ is a solution of (9).

Our main objective is to study stable and unstable invariant manifolds for Eq. (9). We assume in the paper that there exists a decomposition $X = E \times F$ (independent of t), with respect to which $A(t)$ has the block form

$$A(t) = \begin{pmatrix} B(t) & 0 \\ 0 & C(t) \end{pmatrix}. \tag{10}$$

The blocks $B(t)$ and $C(t)$ will correspond, respectively, to the stable and center-unstable components of $A(t)$ (see Section 2.2 for the standing assumptions). The existence of the decomposition in (10) is certainly a restriction to $A(t)$. The general case may be treated at the expense of reducing $A(t)$ to the block form in (10) and then proceeding virtually without additional technical complications. Unfortunately, this reduction is rarely explicit (and often time-dependent). We refer to Section 2.2 for a related discussion.

Due to the block form in (10), the unique solution of (5) can be written in the form

$$v(t) = (U(t,s)\xi, V(t,s)\eta) \text{ for } t \geqslant s, \tag{11}$$

with $v_s = (\xi, \eta) \in E \times F$, where $U(t,s)$ and $V(t,s)$ are the evolution operators associated, respectively, with the blocks $B(t)$ and $C(t)$.

We also write $f = (g, h) \in E \times F$. Clearly,

$$g(t,0) = 0 \text{ and } h(t,0) = 0 \text{ for every } t \geqslant 0, \tag{12}$$

and

$$\begin{aligned} \|g(t,u) - g(t,v)\| &\leqslant c\|u - v\|(\|u\|^q + \|v\|^q), \\ \|h(t,u) - h(t,v)\| &\leqslant c\|u - v\|(\|u\|^q + \|v\|^q) \end{aligned} \tag{13}$$

for every $t \geqslant 0$ and u, $v \in X$.

We now write $v = (x, y) \in E \times F$. Due to the block form in (10), given $s \geqslant 0$ and $v_s = (\xi, \eta) \in E \times F$, the problem (9) is equivalent to the system

$$x' = B(t)x + g(t,x,y), \quad y' = C(t)y + h(t,x,y),$$

with $(x(s), y(s)) = (\xi, \eta)$. We denote by $(x(\cdot, s, \xi, \eta), y(\cdot, s, \xi, \eta))$ the unique solution of this problem or, equivalently, of the problem

$$\begin{aligned} x(\rho) &= U(\rho,s)\xi + \int_s^\rho U(\rho,r)g(r,x(r),y(r))\,dr, \\ y(\rho) &= V(\rho,s)\eta + \int_s^\rho V(\rho,r)h(r,x(r),y(r))\,dr \end{aligned} \tag{14}$$

for $\rho \geqslant s$. For each $\tau \geqslant 0$, we write

$$\Psi_\tau(s,\xi,\eta) = (s+\tau, x(s+\tau,s,\xi,\eta), y(s+\tau,s,\xi,\eta)).$$

This is the semiflow generated by Eq. (9).

2.2. *Nonuniform exponential dichotomies*

We now recall the concept of nonuniform exponential dichotomy (as introduced in [4], although only considered in that paper for finite-dimensional spaces).

We write the unique solution of the initial value problem in (5) in the form $v(t) = T(t,s)v(s)$, where $T(t,s)$ is the associated evolution operator. Consider constants

$$\underline{a} \leqslant \overline{a} < 0 \leqslant \underline{b} \leqslant \overline{b} \quad \text{and} \quad a, b \geqslant 0. \tag{15}$$

We say that the linear equation $v' = A(t)v$ admits a *nonuniform exponential dichotomy* if there exists a function $P\colon \mathbb{R}_0^+ \to B(X)$ such that $P(t)$ is a projection for each $t \geqslant 0$ with

$$P(t)T(t,s) = T(t,s)P(s)$$

for every $t \geqslant s \geqslant 0$, and there exist constants $\underline{a}$, $\overline{a}$, a, $\underline{b}$, $\overline{b}$, b as in (15) and D_1, $D_2 \geqslant 1$ such that for every $t \geqslant s \geqslant 0$,

$$\begin{aligned} &\|T(t,s)P(s)\| \leqslant D_1 \mathrm{e}^{\overline{a}(t-s)+as}, \quad &&\|T(t,s)^{-1}P(t)\| \leqslant D_1 \mathrm{e}^{-\underline{a}(t-s)+at}, \\ &\|T(t,s)Q(s)\| \leqslant D_2 \mathrm{e}^{\overline{b}(t-s)+bs}, \quad &&\|T(t,s)^{-1}Q(t)\| \leqslant D_2 \mathrm{e}^{-\underline{b}(t-s)+bt}, \end{aligned} \tag{16}$$

where $Q(t) = \mathrm{Id} - P(t)$ is the complementary projection for each $t \geqslant 0$. The first four constants in (15) play the role of Lyapunov exponents for the solutions of the linear system in (5), while a and b measure the nonuniformity of the dichotomy. We note that we can indeed obtain estimates for the actual values of the six numbers in (15) in terms of the Lyapunov exponents (see Proposition 6). The inequality $\overline{a} < 0$ in (15) means that there exists at least one negative Lyapunov exponent; we can of course formulate an entirely analogous version when instead $\overline{a} \leqslant 0 < \underline{b}$.

We also say that the linear equation $v' = A(t)v$ admits a *weak nonuniform exponential dichotomy* if there exist a function $P\colon \mathbb{R}_0^+ \to B(X)$ as above and constants $\overline{a} < 0 \leqslant \underline{b}$ and $D_1, D_2 \geqslant 1$ such that the first and the last inequalities in (16) hold, i.e., for every $t \geqslant s \geqslant 0$,

$$\|T(t,s)P(s)\| \leqslant D_1 \mathrm{e}^{\overline{a}(t-s)+as}, \quad \|T(t,s)^{-1}Q(t)\| \leqslant D_2 \mathrm{e}^{-\underline{b}(t-s)+bt}. \tag{17}$$

Clearly, any nonuniform exponential dichotomy is also a weak nonuniform exponential dichotomy. On the other hand, we can define in an equivalent manner a weak nonuniform exponential dichotomy as a nonuniform exponential dichotomy with $\underline{a} = -\infty$ and $\overline{b} = +\infty$. This generalized notion is particularly relevant in the infinite-dimensional setting in view of the possibility of an infinite number of Lyapunov exponents. In particular, one should expect the more restrictive notion of nonuniform exponential dichotomy to be necessary in the study of the regularity of invariant manifolds, with the appearance of the usual spectral gaps (see [3] for a related study in the context of nonuniform exponential dichotomies). We emphasize that for the construction of *Lipschitz* invariant manifolds the generalized notion of *weak* nonuniform exponential dichotomy is sufficient.

When $A(t)$ has the block form in (10), we can rephrase the notion of nonuniform exponential dichotomy in the following (equivalent) manner: the evolution operators $U(t,s)$ and $V(t,s)$ define a nonuniform exponential dichotomy if there exist constants as in (15) and $D_1, D_2 \geqslant 1$ such that for every $t \geqslant s \geqslant 0$,

$$\begin{aligned} \|U(t,s)\| &\leqslant D_1 e^{\overline{a}(t-s)+as}, \quad \|U(t,s)^{-1}\| \leqslant D_1 e^{-\underline{a}(t-s)+at}, \\ \|V(t,s)\| &\leqslant D_2 e^{\overline{b}(t-s)+bs}, \quad \|V(t,s)^{-1}\| \leqslant D_2 e^{-\underline{b}(t-s)+bt}. \end{aligned} \tag{18}$$

A similar observation applies to the notion of weak nonuniform exponential dichotomy, which corresponds to assume that the first and the last inequalities in (18) hold. We showed in [4] that for *any* equation $v' = A(t)v$ in a finite-dimensional space, having $A(t)$ the block form in (10) for every $t \geqslant 0$, if there exists at least one negative Lyapunov exponent, then the evolution operators $U(t,s)$ and $V(t,s)$ define a nonuniform exponential dichotomy.

On the other hand, in our construction of invariant manifolds, we need the numbers a and b in (16) or (18) to be *sufficiently small* when compared to the "Lyapunov exponents", i.e., to the constants $\underline{a}$, $\overline{a}$, $\underline{b}$, and $\overline{b}$. This assumption means that the nonuniformity of the dichotomy is sufficiently small when compared to these constants. It turns out that at least from the point of view of ergodic theory, the constants a and b can be made arbitrarily small almost always. To formulate a rigorous statement, we recall that a flow $\Psi_t\colon M \to M$ is said to preserve a measure μ on M if $\mu(\Psi_t A) = \mu(A)$ for every measurable set $A \subset M$ and every $t \in \mathbb{R}$.

Proposition 2 (*see [4]*). *Assume that F is a vector field on a smooth Riemannian manifold M, and that it defines a flow Ψ_t preserving a finite measure μ on M such that*

$$\int_M \sup_{-1 \leqslant t \leqslant 1} \log^+ \|d_x \Psi_t\| \, d\mu(x) < \infty.$$

For μ-almost every $x \in M$, if the linear variational equation

$$v' = A_x(t)v, \ \ \text{with} \ A_x(t) = d_{\Psi_t x} F \tag{19}$$

has at least one negative Lyapunov exponent, then the evolution operator defined by (19) *admits a nonuniform exponential dichotomy with arbitrarily small a and b.*

3. Existence of stable manifolds

We now present a rigorous formulation of the existence of a stable manifold $\mathcal{W}$ for the origin in Eq. (9). We make the following assumptions:

H1. The function $A\colon \mathbb{R}_0^+ \to B(X)$ is continuous and satisfies (6) and (10) for every $t \geqslant 0$;

H2. The function $f\colon \mathbb{R}_0^+ \times X \to X$ is continuous and satisfies (7) and (8) for some $c > 0$ and $q > 0$.

We first describe the class of Lipschitz functions that will be considered. We denote by $Q(\delta) \subset E$ the closed ball of radius $\delta > 0$ centered at zero. Fix now $\delta > 0$, $\kappa > 0$, and let

$$\beta = a(1 + 1/q) + b/q, \tag{20}$$

with a and b as in (15). The number β specifies the size of the neighborhood $Q\left(\delta e^{-\beta s}\right)$ in which we take the initial condition. We consider the set

$$Z_\beta = \left\{(s, \xi) : s \geqslant 0 \text{ and } \xi \in Q\left(\delta e^{-\beta s}\right)\right\} \subset \mathbb{R}_0^+ \times E,$$

and we denote by $\mathcal{X}_\beta$ the space of continuous functions $\varphi \colon Z_\beta \to F$ such that for each $s \geqslant 0$, we have $\varphi(s, 0) = 0$ and

$$\|\varphi(s, x) - \varphi(s, y)\| \leqslant \kappa \|x - y\| \text{ for every } x, \ y \in Q\left(\delta e^{-\beta s}\right).$$

Given a function $\varphi \in \mathcal{X}_\beta$ we consider the graph of φ,

$$\mathcal{W} = \{(s, \xi, \varphi(s, \xi)) : (s, \xi) \in Z_\beta\} \subset \mathbb{R}_0^+ \times X. \tag{21}$$

We simply refer to $\mathcal{W}$ as a Lipschitz manifold. Note that $(s, 0) \in \mathcal{W}$ (since $\varphi(s, 0) = 0$) for every $s \geqslant 0$. In particular, the set $\mathcal{W}$ contains the line $\mathbb{R}_0^+ \times \{0\}$ and the Lipschitz graph $\mathcal{W}_s = \left\{(s, \xi, \varphi(s, \xi)) : \xi \in Q\left(\delta e^{-\beta s}\right)\right\}$ for each fixed $s \geqslant 0$. See Fig. 1 for an illustration.

We also consider the constant

$$\gamma = a(2 + 1/q) + b/q, \tag{22}$$

and the corresponding sets Z_γ and $\mathcal{X}_\gamma$ defined as before, simply replacing β by γ everywhere. We will show that there exists a function $\varphi \in \mathcal{X}_\beta$ such that for every initial condition $(s, \xi) \in Z_\gamma \subset Z_\beta$ the corresponding solution of (9) is entirely contained in $\mathcal{W}$. This means that for this particular φ the set $\mathcal{W}$ is forward invariant under the semiflow Ψ_τ.

We will assume the conditions

$$q\overline{a} + a < 0 \quad \text{and} \quad \overline{a} + b < \underline{b}. \tag{23}$$

Note that both inequalities in (23) are automatically satisfied when a and b are sufficiently small, and that the first is satisfied for a given a provided that q is sufficiently large (i.e., provided that the order of the perturbation is sufficiently large).

We can now present our main result on the existence of stable manifolds.

Theorem 3. *Assume that* H1–H2 *hold. If the equation $v' = A(t)v$ in the Banach space X admits a weak nonuniform exponential dichotomy and the conditions in* (23) *hold, then there exist $\delta > 0$ and a unique function $\varphi \in \mathcal{X}_\beta$ such that the set $\mathcal{W}$ in* (21) *is forward invariant under the semiflow Ψ_τ, i.e.,*

$$\text{if } (s, \xi) \in Z_\gamma \text{ then } \Psi_\tau(s, \xi, \varphi(s, \xi)) \in \mathcal{W} \text{ for every } \tau \geqslant 0. \tag{24}$$

Furthermore:

1. *for every $(s, \xi) \in Z_\gamma$, we have*

$$\varphi(s, \xi) = -\int_s^{+\infty} V(\tau, s)^{-1} h(\Psi_{\tau-s}(s, \xi, \varphi(s, \xi)))\, d\tau;$$

2. *there exists $D > 0$ such that for every $s \geqslant 0$, $\xi, \bar{\xi} \in Q\left(\delta e^{-\gamma s}\right)$, and $\tau \geqslant 0$,*

$$\|\Psi_\tau(s, \xi, \varphi(s, \xi)) - \Psi_\tau(s, \bar{\xi}, \varphi(s, \bar{\xi}))\| \leqslant D e^{\bar{a}\tau + as} \|\xi - \bar{\xi}\|. \tag{25}$$

We call $\mathcal{W}$ a *local stable manifold* or simply a *stable manifold* of the origin. In particular, setting $\bar{\xi} = 0$ in (25) we see that any solution of the initial value problem in (9) starting in $\mathcal{W}$, i.e., with $v(s) = (\xi, \varphi(s, \xi))$ for some $\xi \in Q\left(\delta e^{-\gamma s}\right)$, approaches the zero solution with exponential speed $\bar{a}$ (which is independent of ξ).

The fact that the initial condition ξ must be taken in a neighborhood of exponentially decreasing size $Q\left(\delta e^{-\gamma s}\right)$, with respect to the initial time s, is a manifestation of the exponential terms e^{as} and e^{bt} in the norm bounds in (17) for the operators $U(t, s)$ and $V(t, s)$. Roughly speaking, this means that the size of the neighborhood of initial conditions for which there exists a bounded solution of the differential equation $v' = A(t)v + f(t, v)$ decreases essentially with the same exponential speed which increases our loss of control of the norm of the operators $U(t, s)$ and $V(t, s)$ for the linear equation $v' = A(t)v$. It should be noted that although these sizes may vary with exponential speed, if the constants a and b are sufficiently small, the speed will be small when compared to the Lyapunov exponents.

We now explain how Theorem 3 can be used to establish the existence of stable manifolds for nonuniformly hyperbolic solutions of a differential equation. Consider a continuous function $F: \mathbb{R}_0^+ \times \mathbb{R}^n \to \mathbb{R}^n$ and the equation

$$v' = F(t, v). \tag{26}$$

Let now $v_0(t)$ be a solution of (26). We say that $v_0(t)$ is *nonuniformly hyperbolic* if the matrix function

$$A(t) = \frac{\partial F}{\partial v}(t, v_0(t))$$

admits a nonuniform exponential dichotomy. We continue to assume that $A(t)$ has the block form in (10) with respect to the invariant decomposition $\mathbb{R}^n = E \times F$.

Theorem 4. *Assume that F is of class C^1 and let $v_0(t)$ be a nonuniformly hyperbolic solution of* (26) *such that there exist $c > 0$ and $q > 0$ such that for every $t \geqslant 0$ and $y \in \mathbb{R}^n$,*

$$\left\| \frac{\partial F}{\partial v}(t, y + v_0(t)) - A(t) \right\| \leqslant C \|y\|^q.$$

If conditions (23) *hold, then there exist $\delta > 0$ and a unique function $\varphi \in \mathcal{X}_\gamma$ such that the set*

$$\mathcal{W} = \{(s, \xi, \varphi(s, \xi)) + (0, v_0(s)) : (s, \xi) \in Z_\gamma\}$$

satisfies the following properties:

1. *$\mathcal{W}$ is forward invariant under solutions of* (26), *i.e., if $(s, v_s) \in \mathcal{W}$ then $(t, v(t)) \in \mathcal{W}$ for every $t \geqslant s$, where $v(t) = v(t, v_s)$ is the unique solution of* (26) *for $t \geqslant s$ with $v(s) = v_s$;*
2. *there exists $D > 0$ such that for every $s \geqslant 0$, (s, v_s), $(s, \overline{v}_s) \in \mathcal{W}$, and $t \geqslant s$ we have*

$$\|v(t, v_s) - v(t, \overline{v}_s)\| \leqslant D\, \mathrm{e}^{\overline{a}(t-s)+as} \|v_s - \overline{v}_s\|.$$

Proof. We shall reduce the study of Eq. (26) to that of (9). For this we consider the change of variables $(t, y) = (t, v - v_0(t))$. Letting $y(t) = v(t) - v_0(t)$, where $v(t)$ is a solution of (26), we obtain

$$y'(t) = A(t)y(t) + G(t, y(t)),$$

where

$$G(t, y) = F(t, y + v_0(t)) - F(t, v_0(t)) - A(t)y. \tag{27}$$

Hypothesis $A(t)$ satisfies assumption H1. Furthermore, it follows from (27) that G is continuous and clearly $G(t, 0) = 0$ for every $t \geqslant 0$. It remains to establish property (8). For this we note that

$$\|G(t, y) - G(t, z)\| \leqslant \sup_{r \in [0,1], i=1,\ldots,n} \left\| \frac{\partial G_i}{\partial y}(t, y + r(z - y)) \right\| \cdot \|y - z\|,$$

where $G = (G_1, \ldots, G_n)$. Since

$$\frac{\partial G}{\partial y}(t, y) = \frac{\partial F}{\partial v}(t, y + v_0(t)) - A(t)$$

for every $t \geqslant 0$ and $y \in \mathbb{R}^n$, we obtain

$$\begin{aligned}\|G(t, y) - G(t, z)\| &\leqslant c \sup_{r\in[0,1]} \|y + r(z - y)\|^q \|y - z\| \\ &\leqslant c \max\{\|y\|^q, \|z\|^q\}\|y - z\| \leqslant c(\|y\|^q + \|z\|^q)\|y - z\|.\end{aligned}$$

Thus, function G satisfies assumption H2. We can now apply Theorem 3 to obtain the desired statement. □

4. Proof of Theorem 3

4.1. Preliminaries

Conceptually, our method of proof of Theorem 3 is classical, and consists in looking for $\mathcal{W}$ as the graph of a Lipschitz function φ, while using the differential equation to express the forward invariance of the graph under the dynamics to conclude that φ must satisfy a fixed point problem. Nevertheless, to implement this approach in our context presents additional difficulties, of course due to the nonuniformity of the exponential behavior of the evolution operators. We emphasize that we may not have a uniform exponential dichotomy. In particular, this causes that the customary application of Gronwall's lemma need not always provide a control of the stable component of the solution.

In view of the desired forward invariance of $\mathcal{W}$ under solutions of the equation in (9) (see (24)), any solution with initial condition in $\mathcal{W}$ at time $s \geqslant 0$ must remain in $\mathcal{W}$ for every $t \geqslant s$ and thus must be of the form $(x(t), \varphi(t, x(t)))$ for every $t \geqslant s$. In particular, the equations in (14) can be replaced by

$$x(t) = U(t, s)x(s) + \int_s^t U(t, \tau)g(\tau, x(\tau), \varphi(\tau, x(\tau)))\, d\tau, \tag{28}$$

$$\varphi(t, x(t)) = V(t, s)\varphi(s, x(s)) + \int_s^t V(t, \tau)h(\tau, x(\tau), \varphi(\tau, x(\tau)))\, d\tau. \tag{29}$$

Without loss of generality we will always consider the norm $\|(x, y)\| = \|x\| + \|y\|$ for $(x, y) \in E \times F$. We equip the space $\mathcal{X}_\beta$ (see Section 3 for the definition) with the norm

$$\|\varphi\| = \sup\left\{\|\varphi(t, x)\|/\|x\| : t \geqslant 0 \text{ and } x \in Q\left(\delta e^{-\beta t}\right) \setminus \{0\}\right\}. \tag{30}$$

Note that $\|\varphi\| \leqslant \kappa$ for every $\varphi \in \mathcal{X}_\beta$. Furthermore, given $t \geqslant 0$ and $x \neq 0$, we have

$$\|\varphi(t,x)\| \leqslant \delta e^{-\beta t} \|\varphi(t,x)\|/\|x\| \leqslant \delta \|\varphi\| \leqslant \delta\kappa$$

for every $\varphi \in \mathcal{X}_\beta$. This readily implies that $\mathcal{X}_\beta$ is a complete metric space with the distance induced by $\|\cdot\|$. For technical reasons, we also consider the space $\mathcal{X}_\beta^*$ of continuous functions $\varphi\colon \mathbb{R}_0^+ \times E \to F$ such that $\varphi|Z_\beta \in \mathcal{X}_\beta$ and

$$\varphi(s,\xi) = \varphi(s, \delta e^{-\beta s}\xi/\|\xi\|) \text{ whenever } \xi \notin Q\left(\delta e^{-\beta s}\right).$$

Clearly, there is a one-to-one correspondence between functions in $\mathcal{X}_\beta$ and functions in $\mathcal{X}_\beta^*$. In particular $\mathcal{X}_\beta^*$ is also a Banach space with the norm $\mathcal{X}_\beta^* \ni \varphi \mapsto \|\varphi|Z_\beta\|$. Furthermore, one can easily verify that given $\varphi \in \mathcal{X}_\beta^*$ and $s \geqslant 0$ we have

$$\|\varphi(s,x) - \varphi(s,y)\| \leqslant \kappa\|x-y\| \text{ for every } x, y \in E.$$

4.2. Solution on the stable direction

The proof of Theorem 3 is obtained in several steps. We first establish the existence of a unique function $x(t) = x_\varphi(t)$ satisfying (28) for each given $\varphi \in \mathcal{X}_\beta^*$. By (23) we have

$$T_1 = q\bar{a} + a < 0. \tag{31}$$

Lemma 1. *There exists $R > 0$ such that for every $\delta > 0$ sufficiently small*:

1. *for each $\varphi \in \mathcal{X}_\beta^*$, given $(s,\xi) \in Z_\beta$ there exists a unique continuous function $x = x_\varphi\colon [s,+\infty) \to E$ with $x_\varphi(s) = \xi$ satisfying* (28) *for every $t \geqslant s$*;
2. *the function x_φ satisfies*

$$\|x_\varphi(t)\| \leqslant R e^{\bar{a}(t-s)+as}\|\xi\| \text{ for every } t \geqslant s. \tag{32}$$

Proof. Given $\delta > 0$ and $s \geqslant 0$, we consider the space

$$\mathcal{B} = \{x\colon [s,+\infty) \to E : \ x \text{ is continuous and } \|x\|' \leqslant \delta\, e^{-\beta s}\},$$

with the norm

$$\|x\|' = \frac{1}{2D_1} \sup\{\|x(t)\| e^{-\bar{a}(t-s)-as} : t \geqslant s\},$$

where $D_1 \geqslant 1$ is the constant in (17). One can easily verify that with the distance induced by this norm $\mathcal{B}$ is a complete metric space. Given $\varphi \in \mathcal{X}_\beta^*$ and $s \geqslant 0$, we can define the operator

$$(Jx)(t) = \int_s^t U(t,\tau) g(\tau, x(\tau), \varphi(\tau, x(\tau)))\, d\tau$$

for each $x \in \mathcal{B}$. Given x, $y \in \mathcal{B}$ and $\tau \geqslant s$, we obtain

$$\|(x(\tau), \varphi(\tau, x(\tau)))\| = \|(x(\tau), \varphi(\tau, x(\tau)) - \varphi(\tau, 0))\| \leqslant (1+\kappa)\|x(\tau)\|, \tag{33}$$

and

$$\|(x(\tau), \varphi(\tau, x(\tau))) - (y(\tau), \varphi(\tau, y(\tau)))\| \leqslant (1+\kappa)\|x(\tau) - y(\tau)\|. \tag{34}$$

Therefore, by (13),

$$\begin{aligned}
&\|g(\tau, x(\tau), \varphi(\tau, x(\tau))) - g(\tau, y(\tau), \varphi(\tau, y(\tau)))\| \\
&\quad \leqslant c(1+\kappa)^{q+1}\|x(\tau) - y(\tau)\|(\|x(\tau)\|^q + \|y(\tau)\|^q) \\
&\quad \leqslant 2^{2+q} D_1^{1+q} c\delta^q (1+\kappa)^{q+1} \mathrm{e}^{\overline{a}(q+1)(\tau-s)-bs}\|x-y\|'.
\end{aligned} \tag{35}$$

By the first inequality in (18) we obtain

$$\begin{aligned}
&\|(Jx)(t) - (Jy)(t)\| \\
&\quad \leqslant \int_s^t \|U(t,\tau)\| \cdot \|g(\tau, x(\tau), \varphi(\tau, x(\tau))) - g(\tau, y(\tau), \varphi(\tau, y(\tau)))\|\, d\tau \\
&\quad \leqslant 2^{2+q} D_1^{1+q} c\delta^q (1+\kappa)^{q+1}\|x-y\|' \int_s^t D_1 \mathrm{e}^{\overline{a}(t-\tau)+a\tau} \mathrm{e}^{\overline{a}(q+1)(\tau-s)-bs}\, d\tau \\
&\quad \leqslant 2^{2+q} D_1^{2+q} c\delta^q (1+\kappa)^{q+1}\|x-y\|' \mathrm{e}^{\overline{a}(t-s)+as-bs} \int_s^\infty \mathrm{e}^{(q\overline{a}+a)(\tau-s)}\, d\tau \\
&\quad \leqslant 2^{2+q} D_1^{2+q} c\delta^q (1+\kappa)^{q+1}\|x-y\|' \mathrm{e}^{\overline{a}(t-s)+as} \int_s^\infty \mathrm{e}^{T_1(\tau-s)}\, d\tau,
\end{aligned}$$

with $T_1 < 0$ as in (31). Therefore

$$\|Jx - Jy\|' \leqslant \theta \|x-y\|', \tag{36}$$

where

$$\theta = 2^{1+q} c D_1^{1+q} \delta^q (1+\kappa)^{q+1}/|T_1|.$$

We now choose $\delta > 0$ sufficiently small, independently of s, so that $\theta < 1/2$. Given $\xi \in Q\left(\delta e^{-\beta s}\right)$ we consider the operator $\bar{J}$ on the space $\mathcal{B}$ defined by

$$(\bar{J}x)(t) = z(t) + (Jx)(t),$$

where $z(t) = U(t,s)\xi$. For $y = 0$ we obtain $Jy = 0$ (note that since $\varphi \in \mathcal{X}_\beta^*$ we have $\varphi(t,0) = 0$ for every $t \geqslant 0$), and thus, by (36), $\|Jx\|' \leqslant \theta \|x\|'$. By the first inequality in (18) we obtain $\|z\|' \leqslant \|\xi\|/2$ and hence,

$$\|\bar{J}x\|' \leqslant \|z\|' + \|Jx\|' \leqslant \tfrac{1}{2}\|\xi\| + \theta\|x\|' \leqslant \tfrac{1}{2}\delta e^{-\beta s} + \tfrac{1}{2}\delta e^{-\beta s} = \delta e^{-\beta s}.$$

Therefore, $\bar{J}: \mathcal{B} \to \mathcal{B}$ is a well-defined operator. In view of (36),

$$\|\bar{J}x - \bar{J}y\|' = \|Jx - Jy\|' \leqslant \theta\|x - y\|',$$

and $\bar{J}$ is a contraction. Therefore, there exists a unique function $x = x_\varphi \in \mathcal{B}$ such that $\bar{J}x = x$. Furthermore, x can be obtained by

$$x(t) = \lim_{n\to\infty} (\bar{J}^n 0)(t) = \sum_{n=0}^{\infty} (J^n z)(t).$$

It follows from (36) that

$$\|x\|' \leqslant \sum_{n=0}^{\infty} \|J^n z\|' \leqslant \sum_{n=0}^{\infty} \theta^n \|z\|' = \frac{\|z\|'}{1-\theta} \leqslant \frac{\|\xi\|}{2(1-\theta)}.$$

Therefore, for every $t \geqslant s$,

$$\|x(t)\| \leqslant 2D_1 e^{\bar{a}(t-s)+as} \frac{\|\xi\|}{2(1-\theta)}.$$

We obtain the desired result with $R = D_1/(1-\theta)$. $\square$

The fact that the initial condition ξ must be taken in a neighborhood of exponentially decreasing size $Q\left(\delta e^{-\beta s}\right)$ is a manifestation of the exponential term e^{as} in the norm bound in (18) for the operator $U(t,s)$.

4.3. Auxiliary results

We now establish some auxiliary results that describe the asymptotic behavior of the function x_φ given by Lemma 1, as time approaches infinity, when we change the initial condition ξ or the function φ. Given $\delta > 0$ sufficiently small, $\varphi \in \mathcal{X}^*_\beta$, $s \geqslant 0$, and initial conditions $\xi, \bar{\xi} \in Q\left(\delta e^{-\beta s}\right)$, we denote by x_φ and $\bar{x}_\varphi$, respectively, the unique continuous functions given by Lemma 1 such that $x_\varphi(s) = \xi$ and $\bar{x}_\varphi(s) = \bar{\xi}$.

Lemma 2. *There exists $K_1 > 0$ such that for every $\delta > 0$ sufficiently small, $\varphi \in \mathcal{X}^*_\beta$, $s \geqslant 0$, and $\xi, \bar{\xi} \in Q\left(\delta e^{-\beta s}\right)$ we have*

$$\|x_\varphi(t) - \bar{x}_\varphi(t)\| \leqslant K_1 e^{\overline{a}(t-s)+as} \|\xi - \bar{\xi}\| \text{ for every } t \geqslant s. \tag{37}$$

Proof. Proceeding in a similar manner to that in (33)–(35), for every $\tau \geqslant s$ we obtain

$$\begin{aligned} &\|g(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - g(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\| \\ &\quad \leqslant c(1+\kappa)^{q+1} \|x_\varphi(\tau) - \bar{x}_\varphi(\tau)\| (\|x_\varphi(\tau)\|^q + \|\bar{x}_\varphi(\tau)\|^q). \end{aligned} \tag{38}$$

By Lemma 1 (see (32)), we thus have

$$\begin{aligned} &\|g(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - g(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\| \\ &\quad \leqslant \eta e^{q\overline{a}(\tau-s)-as-bs} \|x_\varphi(\tau) - \bar{x}_\varphi(\tau)\|, \end{aligned}$$

where

$$\eta = 2c(1+\kappa)^{q+1} R^q \delta^q \leqslant 2c(1+\kappa)^{q+1} R^q, \tag{39}$$

provided that $\delta \leqslant 1$. Note that the last constant is independent of δ. Setting $\rho(t) = \|x_\varphi(t) - \bar{x}_\varphi(t)\|$ and using the first inequality in (18), it follows from (28) that

$$\begin{aligned} \rho(t) &\leqslant \|U(t,s)\| \cdot \|\xi - \bar{\xi}\| + \int_s^t \|U(t,\tau)\| \eta e^{q\overline{a}(\tau-s)-as-bs} \rho(\tau)\, d\tau \\ &\leqslant D_1 e^{\overline{a}(t-s)+as} \|\xi - \bar{\xi}\| + D_1 \eta \int_s^t e^{\overline{a}(t-\tau)+T_1(\tau-s)} \rho(\tau)\, d\tau, \end{aligned} \tag{40}$$

with $T_1 < 0$ as in (31). We can now use Gronwall's lemma for the function $e^{-\overline{a}(t-s)}\rho(t)$ and (39) to obtain (assuming that $\delta \leqslant 1$)

$$\rho(t) \leqslant D_1 e^{as + D_1 \eta e^{T_1 s}/|T_1|} e^{\overline{a}(t-s)} \|\xi - \bar{\xi}\| \leqslant K_1 e^{\overline{a}(t-s)+as} \|\xi - \bar{\xi}\|,$$

with $K_1 = D_1 \exp[2c(1+\kappa)^{q+1} D_1 R^q / |T_1|]$. This completes the proof. $\square$

Given $\delta > 0$ sufficiently small, $\varphi, \psi \in \mathcal{X}^*_\beta$, and $(s, \xi) \in Z_\beta$, we denote by x_φ and x_ψ the continuous functions given by Lemma 1 such that $x_\varphi(s) = x_\psi(s) = \xi$.

Lemma 3. *There exists $K_2 > 0$ such that for every $\delta > 0$ sufficiently small, $\varphi, \psi \in \mathcal{X}^*_\beta$, and $(s, \xi) \in Z_\beta$ we have*

$$\|x_\varphi(t) - x_\psi(t)\| \leqslant K_2 e^{\overline{a}(t-s)+(a-b)s} \|\xi\| \cdot \|\varphi - \psi\| \text{ for every } t \geqslant s. \tag{41}$$

Proof. Proceeding in a similar manner to that in (35), we obtain

$$\begin{aligned}
&\|g(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - g(\tau, x_\psi(\tau), \psi(\tau, x_\psi(\tau)))\| \\
&\quad \leqslant c(1+\kappa)^q \|(x_\varphi(\tau) - x_\psi(\tau), \varphi(\tau, x_\varphi(\tau)) - \psi(\tau, x_\psi(\tau)))\| \\
&\qquad \times (\|x_\varphi(\tau)\|^q + \|x_\psi(\tau)\|^q).
\end{aligned} \tag{42}$$

Furthermore,

$$\begin{aligned}
&\|\varphi(\tau, x_\varphi(\tau)) - \psi(\tau, x_\psi(\tau))\| \\
&\quad \leqslant \|\varphi(\tau, x_\varphi(\tau)) - \psi(\tau, x_\varphi(\tau))\| + \|\psi(\tau, x_\varphi(\tau)) - \psi(\tau, x_\psi(\tau))\| \\
&\quad \leqslant \|x_\varphi(\tau)\| \cdot \|\varphi - \psi\| + \kappa \|x_\varphi(\tau) - x_\psi(\tau)\|.
\end{aligned} \tag{43}$$

By (32) in Lemma 1 we conclude that

$$\begin{aligned}
&\|g(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - g(\tau, x_\psi(\tau), \psi(\tau, x_\psi(\tau)))\| \\
&\quad \leqslant 2c(1+\kappa)^q R^q \delta^q e^{q\overline{a}(\tau-s)-as-bs} \\
&\qquad \times (\|x_\varphi(\tau)\| \cdot \|\varphi - \psi\| + (1+\kappa)\|x_\varphi(\tau) - x_\psi(\tau)\|).
\end{aligned} \tag{44}$$

We now proceed in a similar manner to that in (40) in order to apply Gronwall's lemma. Set

$$\bar{\rho}(t) = \|x_\varphi(t) - x_\psi(t)\| \quad \text{and} \quad \bar{\eta} = 2c(1+\kappa)^q R^q \delta^q. \tag{45}$$

Note that $\bar{\eta} \leqslant 2c(1+\kappa)^q R^q$ provided that $\delta \leqslant 1$. Note also that the last constant is independent of δ. Using the first inequality in (18), (32) in Lemma 1, and (44), it follows from (28) that

$$\begin{aligned}
\bar{\rho}(t) \leqslant\ & \bar{\eta} \int_s^t \|U(t,\tau)\| e^{q\overline{a}(\tau-s)-as-bs} \|x_\varphi(\tau)\| \cdot \|\varphi - \psi\| \, d\tau \\
& + \bar{\eta} \int_s^t \|U(t,\tau)\| e^{q\overline{a}(\tau-s)-as-bs} (1+\kappa) \|x_\varphi(\tau) - x_\psi(\tau)\| \, d\tau \\
\leqslant\ & \bar{\eta} D_1 R \|\xi\| \cdot \|\varphi - \psi\| \int_s^t e^{\overline{a}(t-s)+a\tau+q\overline{a}(\tau-s)-bs} \, d\tau \\
& + \bar{\eta} D_1 (1+\kappa) \int_s^t e^{\overline{a}(t-\tau)+a\tau} e^{q\overline{a}(\tau-s)-as} \bar{\rho}(\tau) \, d\tau.
\end{aligned}$$

We conclude that

$$
\begin{aligned}
\mathrm{e}^{-\overline{a}(t-s)}\bar{\rho}(t) \leqslant\ & \bar{\eta} D_1 R \mathrm{e}^{(a-b)s} \int_s^{\infty} \mathrm{e}^{T_1(\tau-s)}\, d\tau \|\xi\| \cdot \|\varphi-\psi\| \\
& + \bar{\eta} D_1 (1+\kappa) \int_s^t \mathrm{e}^{T_1(\tau-s)} \mathrm{e}^{-\overline{a}(\tau-s)} \bar{\rho}(\tau)\, d\tau,
\end{aligned}
$$

with $T_1 < 0$ as in (31). We can now use Gronwall's lemma for the function $\mathrm{e}^{-\overline{a}(t-s)}\bar{\rho}(t)$ to obtain (assuming that $\delta \leqslant 1$)

$$
\begin{aligned}
\bar{\rho}(t) &\leqslant \frac{\bar{\eta} D_1 R}{|T_1|} \mathrm{e}^{\bar{\eta} D_1 (1+\kappa)/|T_1|} \mathrm{e}^{\overline{a}(t-s)+(a-b)s} \|\xi\| \cdot \|\varphi-\psi\| \\
&\leqslant K_2 \mathrm{e}^{\overline{a}(t-s)+(a-b)s} \|\xi\| \cdot \|\varphi-\psi\|,
\end{aligned}
$$

where

$$
K_2 = \frac{2c(1+\kappa)^q D_1 R^{q+1}}{|T_1|} \exp\left(\frac{2c(1+\kappa)^{q+1} D_1 R^q}{|T_1|} \right) = \frac{2c(1+\kappa)^q R^{q+1}}{|T_1|} K_1.
$$

This completes the proof of the lemma. $\square$

An immediate consequence of Lemmas 2 and 3 is a description of how the functions given by Lemma 1 may change asymptotically when we vary *simultaneously* the initial condition ξ and the function φ. Namely, using the same notations as above, it follows from Lemmas 2 and 3 that

$$
\begin{aligned}
\|x_\psi(t) - \bar{x}_\varphi(t)\| &\leqslant \|x_\varphi(t) - \bar{x}_\varphi(t)\| + \|x_\varphi(t) - x_\psi(t)\| \\
&\leqslant K_1 \mathrm{e}^{\overline{a}(t-s)+as} \|\xi - \bar{\xi}\| + K_2 \delta \mathrm{e}^{\overline{a}(t-s)-as/q-b(1+1/q)s} \|\varphi - \psi\|.
\end{aligned}
$$

4.4. Equivalent problem

In order to show the existence of a function $\varphi \in \mathcal{X}_\beta^*$ satisfying (29) when $x = x_\varphi$, we first transform this problem in an equivalent problem. We recall that x_φ is the function given by Lemma 1 with $x_\varphi(s) = \xi$. Let

$$
T_2 = \overline{a} - \underline{b} + b < 0. \tag{46}
$$

Lemma 4. *Given $\delta > 0$ sufficiently small and $\varphi \in \mathcal{X}_\beta^*$, the following properties hold:*

1. *if*

$$
\begin{aligned}
\varphi(t, x_\varphi(t)) =\ & V(t,s)\varphi(s,\xi) \\
& + \int_s^t V(t,\tau) h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\, d\tau
\end{aligned} \tag{47}
$$

for every $(s, \xi) \in Z_\beta$ *and* $t \geqslant s$, *then*

$$\varphi(s, \xi) = -\int_s^\infty V(\tau, s)^{-1} h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\, d\tau \tag{48}$$

for every $(s, \xi) \in Z_\beta$ (*including the requirement that the integral is well-defined*);
2. *if* (48) *holds for every* $(s, \xi) \in Z_\beta$, *then* (47) *holds for every* $(s, \xi) \in Z_\gamma$ *and* $t \geqslant s$.

Proof. To establish the first property, we start by showing that the integral in (48) is well-defined for each $(s, \xi) \in Z_\beta$. By (32) in Lemma 1 and (13), we have

$$\begin{aligned}
&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\| \\
&\quad \leqslant c\|(x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\|^{q+1} \leqslant c(1+\kappa)^{q+1}\|x_\varphi(\tau)\|^{q+1} \\
&\quad \leqslant c(1+\kappa)^{q+1} R^{q+1} \mathrm{e}^{(q+1)\overline{a}(\tau-s)+a(q+1)s}\|\xi\|^{q+1}.
\end{aligned}$$

It follows from the last inequality in (18) that

$$\begin{aligned}
&\int_s^\infty \|V(\tau, s)^{-1} h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\|\, d\tau \\
&\quad \leqslant D_2 c(1+\kappa)^{q+1} R^{q+1} \delta^{q+1} \mathrm{e}^{-a(1+1/q)s-b(1+1/q)s} \\
&\qquad \times \int_s^\infty \mathrm{e}^{-\underline{b}(\tau-s)+b\tau+(q+1)\overline{a}(\tau-s)}\, d\tau \\
&\quad = D_2 c(1+\kappa)^{q+1} R^{q+1} \delta^{q+1} \mathrm{e}^{bs-a(1+1/q)s-b(1+1/q)s} \int_s^\infty \mathrm{e}^{(T_2+q\overline{a})(\tau-s)}\, d\tau.
\end{aligned}$$

Since $\overline{a} < 0$, we have $T_2 + q\overline{a} < 0$, and thus the last integral is finite. Therefore, the integral in (48) is well-defined.

We now assume that (47) holds for every $(s, \xi) \in Z_\beta$ and $t \geqslant s$, and we rewrite the identity in the equivalent form

$$\varphi(s, \xi) = V(t, s)^{-1}\varphi(t, x_\varphi(t)) - \int_s^t V(\tau, s)^{-1} h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\, d\tau. \tag{49}$$

By (32) in Lemma 1, we have

$$\begin{aligned}
\|V(t, s)^{-1}\varphi(t, x_\varphi(t))\| &\leqslant D_2 \mathrm{e}^{-\underline{b}(t-s)+bt}\kappa\|x_\varphi(t)\| \\
&\leqslant D_2 \mathrm{e}^{-\underline{b}(t-s)+bt}\kappa R \mathrm{e}^{\overline{a}(t-s)}\delta\, \mathrm{e}^{-as/q-bs/q} \\
&= D_2 \kappa R \delta \mathrm{e}^{T_2(t-s)+[b(1-1/q)-a/q]s}.
\end{aligned}$$

Thus, letting $t \to \infty$ in (49), we obtain (48) for every $(s,\xi) \in Z_\beta$ and $t \geqslant s$. This establishes the first property.

We now assume that (48) holds for every $(s,\xi) \in Z_\beta$ and $t \geqslant s$. Since $V(t,s)V(\tau,s)^{-1} = V(t,\tau)$ we readily obtain

$$V(t,s)\varphi(s,\xi) + \int_s^t V(t,\tau)h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\,d\tau$$
$$= -\int_t^\infty V(\tau,t)^{-1}h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\,d\tau. \tag{50}$$

We now show that the right-hand side of (50) is equal to $\varphi(t, x_\varphi(t))$. We first define a semiflow F_τ for each $\tau \geqslant 0$ and $(s,\xi) \in Z_\beta$ by

$$F_\tau(s,\xi) = (s+\tau, x_\varphi(s+\tau, s, \xi)).$$

This semiflow is obtained from the autonomous equation

$$\begin{cases} t' = 1, \\ x' = B(t)x + g(t, x, \varphi(t,x)) \end{cases}$$

considering a new time variable. Note that in view of (48),

$$\varphi(s,\xi) = -\int_s^\infty V(\tau,s)^{-1}h(F_{\tau-s}(s,\xi), \varphi(\tau, F_{\tau-s}(s,\xi)))\,d\tau. \tag{51}$$

Given $\tau \geqslant t \geqslant s$, we have

$$F_{\tau-t}(t, x_\varphi(t)) = F_{\tau-t}(F_{t-s}(s,\xi)) = F_{\tau-s}(s,\xi) = (\tau, x_\varphi(\tau)).$$

Furthermore, when $(s,\xi) \in Z_\gamma$ it follows from (32) in Lemma 1 that

$$\|x_\varphi(t)\| \leqslant R\mathrm{e}^{\overline{a}(t-s)+as}\|\xi\| \leqslant R\delta\mathrm{e}^{-\beta s}, \tag{52}$$

and thus $(t, x_\varphi(t)) \in Z_\beta$ for every $t \geqslant s$. This shows that eventually making δ smaller if necessary, we can replace (s,ξ) by $(t, x_\varphi(t))$ in (51). This yields

$$\varphi(t, x_\varphi(t)) = -\int_t^\infty V(\tau,t)^{-1}h(F_{\tau-t}(t, x_\varphi(t)), \varphi(\tau, F_{\tau-t}(t, x_\varphi(t))))\,d\tau$$
$$= -\int_t^\infty V(\tau,t)^{-1}h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\,d\tau. \tag{53}$$

Combining (50) and (53), we conclude that (47) holds for every $(s,\xi) \in Z_\gamma$ and $t \geqslant s$. This completes the proof of the lemma. $\square$

4.5. Final step

We now put together all the information given by the former lemmas to establish the existence of a function $\varphi \in \mathcal{X}_\beta^*$ satisfying (29) when $x = x_\varphi$ (with the function x_φ given by Lemma 1).

Lemma 5. *Given $\delta > 0$ sufficiently small, there exists a unique function $\varphi \in \mathcal{X}_\beta^*$ such that* (48) *holds for every $(s, \xi) \in Z_\beta$.*

Proof. We look for a fixed point of the operator Φ defined for each $\varphi \in \mathcal{X}_\beta^*$ by

$$(\Phi\varphi)(s, \xi) = -\int_s^\infty V(\tau, s)^{-1} h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau)))\, d\tau \tag{54}$$

when $(s, \xi) \in Z_\beta$, where $x_\varphi\colon [s, +\infty) \to E$ is the unique continuous function given by Lemma 1 such that $x_\varphi(s) = \xi$, and by

$$(\Phi\varphi)(s, \xi) = (\Phi\varphi)(s, \delta \mathrm{e}^{-\beta s} \xi / \|\xi\|) \tag{55}$$

otherwise. We recall that $\mathcal{X}_\beta^*$ is a complete metric space for the distance induced by the norm $\|\cdot\|$ in (30) (or more precisely by the norm $\varphi \mapsto \|\varphi | Z_\beta\|$).

Note that when $\xi = 0$ we have $x_\varphi(t) = 0$ for every $\varphi \in \mathcal{X}_\beta^*$ and $t \geqslant s$. Thus, in view of (12), $(\Phi\varphi)(t, 0) = 0$. Given

$$\varphi \in \mathcal{X}_\beta^*,\ s \geqslant 0, \quad \text{and} \quad \xi,\ \bar{\xi} \in Q\left(\delta \mathrm{e}^{-\beta s}\right),$$

let now x_φ and $\bar{x}_\varphi$ be the unique continuous functions given by Lemma 1 such that, respectively, $x_\varphi(s) = \xi$ and $\bar{x}_\varphi(s) = \bar{\xi}$. Proceeding in a similar manner to that in (38), replacing g by h, we obtain

$$\begin{aligned}
&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\| \\
&\quad \leqslant c(1+\kappa)^{q+1} \|x_\varphi(\tau) - \bar{x}_\varphi(\tau)\| (\|x_\varphi(\tau)\|^q + \|\bar{x}_\varphi(\tau)\|^q).
\end{aligned}$$

Using (32) in Lemma 1, and (37) in Lemma 2 this implies that

$$\begin{aligned}
&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\| \\
&\quad \leqslant c(1+\kappa)^{q+1} K_1 R^q \mathrm{e}^{(q+1)\bar{a}(\tau - s) + a(q+1)s} \|\xi - \bar{\xi}\| (\|\xi\|^q + \|\bar{\xi}\|^q).
\end{aligned}$$

Since $\xi,\ \bar{\xi} \in Q\left(\delta e^{-\beta s}\right)$ we obtain

$$\begin{aligned}&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\|\\&\quad\leqslant \eta K_1 \delta^q e^{(q+1)\overline{a}(\tau-s)-bs}\|\xi - \bar{\xi}\|,\end{aligned}$$

with η as in (39). Using the last inequality in (18) we conclude that

$$\begin{aligned}&\|(\Phi\varphi)(s,\xi) - (\Phi\varphi)(s,\bar{\xi})\|\\&\quad\leqslant \int_s^\infty \|V(\tau,s)^{-1}\| \cdot \|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, \bar{x}_\varphi(\tau), \varphi(\tau, \bar{x}_\varphi(\tau)))\|\, d\tau\\&\quad\leqslant D_2 \eta K_1 \delta^q \|\xi - \bar{\xi}\| \int_s^\infty e^{-\underline{b}(\tau-s)+b\tau} e^{(q+1)\overline{a}(\tau-s)-bs}\, d\tau\\&\quad= D_2 \eta K_1 \delta^q \|\xi - \bar{\xi}\| \int_s^\infty e^{(T_2+q\overline{a})(\tau-s)}\, d\tau,\end{aligned}$$

with $T_2 < 0$ as in (46). Choosing $\delta > 0$ sufficiently small we have

$$\sigma = \frac{D_2 \eta K_1 \delta^q}{|T_2 + q\overline{a}|} = \frac{D_2 2c(1+\kappa)^{q+1} R^q K_1 \delta^q}{|T_2 + q\overline{a}|} < \kappa.$$

In particular,

$$\|(\Phi\varphi)(t,\xi) - (\Phi\varphi)(t,\bar{\xi})\| \leqslant \kappa \|\xi - \bar{\xi}\|$$

for every $\xi, \bar{\xi} \in Q\left(\delta e^{-\beta s}\right)$, and thus, in view of (55), for every $\xi, \bar{\xi} \in E$. This shows that $\Phi(\mathcal{X}_\beta^*) \subset \mathcal{X}_\beta^*$, and hence, $\Phi\colon \mathcal{X}_\beta^* \to \mathcal{X}_\beta^*$ is well-defined.

We now show that $\Phi\colon \mathcal{X}_\beta^* \to \mathcal{X}_\beta^*$ is a contraction. Given $\varphi,\ \psi \in \mathcal{X}_\beta^*$, and $(s,\xi) \in Z_\beta$, let x_φ and x_ψ be the unique continuous functions given by Lemma 1 such that $x_\varphi(s) = x_\psi(s) = \xi$. Proceeding in a similar manner to that in (42) and (43), with g replaced by h, we obtain

$$\begin{aligned}&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, x_\psi(\tau), \psi(\tau, x_\psi(\tau)))\|\\&\quad\leqslant c(1+\kappa)^q \|(x_\varphi(\tau) - x_\psi(\tau), \varphi(\tau, x_\varphi(\tau)) - \psi(\tau, x_\psi(\tau)))\|\\&\qquad\times(\|x_\varphi(\tau)\|^q + \|x_\psi(\tau)\|^q)\\&\quad\leqslant c(1+\kappa)^q (\|x_\varphi(\tau)\| \cdot \|\varphi - \psi\| + (1+\kappa)\|x_\varphi(\tau) - x_\psi(\tau)\|)\\&\qquad\times(\|x_\varphi(\tau)\|^q + \|x_\psi(\tau)\|^q).\end{aligned}$$

It follows from (41) in Lemma 3 that

$$\begin{aligned}
&\|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, x_\psi(\tau), \psi(\tau, x_\psi(\tau)))\| \\
&\quad \leqslant \bar{\eta} \mathrm{e}^{q\overline{a}(\tau-s)-as-bs}(\|x_\varphi(\tau)\| \cdot \|\varphi-\psi\| + (1+\kappa)\|x_\varphi(\tau)-x_\psi(\tau)\|) \\
&\quad \leqslant \bar{\eta} \mathrm{e}^{(q+1)\overline{a}(\tau-s)-bs}(R + (1+\kappa)K_2\mathrm{e}^{-bs})\|\xi\| \cdot \|\varphi-\psi\|,
\end{aligned}$$

with $\bar{\eta}$ as in (45). Setting $G = R + (1+\kappa)K_2$, we conclude that

$$\begin{aligned}
&\|(\Phi\varphi)(s,\xi) - (\Phi\psi)(s,\xi)\| \\
&\quad \leqslant \int_s^\infty \|V(\tau,s)^{-1}\| \cdot \|h(\tau, x_\varphi(\tau), \varphi(\tau, x_\varphi(\tau))) - h(\tau, x_\psi(\tau), \psi(\tau, x_\psi(\tau)))\|\, d\tau \\
&\quad \leqslant D_2\bar{\eta}G\|\xi\| \cdot \|\varphi-\psi\| \int_s^\infty \mathrm{e}^{-\underline{b}(\tau-s)+b\tau}\mathrm{e}^{(q+1)\overline{a}(\tau-s)-bs}\, d\tau \\
&\quad = D_2\bar{\eta}G\|\xi\| \cdot \|\varphi-\psi\| \int_s^\infty \mathrm{e}^{(T_2+q\overline{a})(\tau-s)}\, d\tau \\
&\quad \leqslant \frac{D_2\bar{\eta}G}{|T_2+q\overline{a}|}\|\xi\| \cdot \|\varphi-\psi\|.
\end{aligned}$$

Provided that $\delta > 0$ is sufficiently small, we have

$$\bar{\theta} = \frac{D_2\bar{\eta}G}{|T_2+q\overline{a}|} = \frac{D_2 2c(1+\kappa)^q R^q \delta^q (R\delta + (1+\kappa)K_2)}{|T_2+q\overline{a}|} < 1.$$

Therefore

$$\|\Phi\varphi_1 - \Phi\varphi_2\| \leqslant \bar{\theta}\|\varphi_1 - \varphi_2\|,$$

and $\Phi\colon \mathcal{X}_\beta^* \to \mathcal{X}_\beta^*$ is a contraction in the complete metric space $\mathcal{X}_\beta^*$. Hence, there exists a unique function $\varphi \in \mathcal{X}_\beta^*$ satisfying $\Phi\varphi = \varphi$. In particular, in view of (54), the identity (48) holds for every $(s,\xi) \in Z_\beta$. This completes the proof of the lemma. $\square$

We can now establish Theorem 3.

Proof of Theorem 3. As explained in the beginning of Section 4.1, in view of the required forward invariance property in (24), to show the existence of a (Lipschitz) stable manifold $\mathcal{W}$ is equivalent to find a function φ satisfying (28) and (29) in some appropriate domain. If follows from Lemma 1 that for each fixed $\varphi \in \mathcal{X}_\beta^*$ there exists a unique function $x = x_\varphi$ satisfying (28) and thus it remains to solve (29) setting $x = x_\varphi$ or, equivalently, to solve (47) in Lemma 4. This lemma indicates that this is essentially equivalent to solve the equation in (48), i.e., to find $\varphi \in \mathcal{X}_\beta^*$ such that (48) holds for every $(s, \xi) \in Z_\beta$. More precisely, it follows from the second property in Lemma 4 that if (48) holds for every $(s, \xi) \in Z_\beta$, then (47) holds for every $(s, \xi) \in Z_\gamma$ and $t \geqslant s$. Finally, Lemma 5 shows that there exists a unique function $\varphi \in \mathcal{X}_\beta^*$ such that (48) holds for every $(s, \xi) \in Z_\beta$. From now on we consider the continuous function $x = x_\varphi$ given by Lemma 1 for this particular φ (note that $Z_\gamma \subset Z_\beta$). Furthermore, by the second property in Lemma 4, the identity in (47) holds for every $(s, \xi) \in Z_\gamma$ and $t \geqslant s$. Thus, the same happens with the identity in (29), with $x = x_\varphi$, for every $t \geqslant s$. Furthermore, by (52), provided that δ is sufficiently small and $(s, \xi) \in Z_\gamma$ we have $(t, x_\varphi(t)) \in Z_\beta$ for every $t \geqslant s$. This ensures that we can replace the function φ in (28)–(29) by the restriction $\varphi | Z_\beta$. In other words, there exists a unique function $\varphi \in \mathcal{X}_\beta^*$ such that the corresponding set $\mathcal{W}$ in (21) obtained from the function $\varphi | Z_\beta$ is forward invariant under the semiflow Ψ_τ for initial conditions with $(s, \xi) \in Z_\gamma$, and we obtain the property in (24).

It remains to establish the two remaining properties in the theorem. The first is an immediate consequence of the above discussion (or of the first property in Lemma 4). To prove the second property, we denote by x_φ and $\bar{x}_\varphi$ the unique continuous functions given by Lemma 1 such that, respectively, $x_\varphi(s) = \xi$ and $\bar{x}_\varphi(s) = \bar{\xi}$. It follows from Lemma 2 that

$$\begin{aligned}
&\|\Psi_\tau(s, \xi, \varphi(s, \xi)) - \Psi_\tau(s, \bar{\xi}, \varphi(s, \bar{\xi}))\| \\
&\quad = \|(t, x_\varphi(t), \varphi(t, x_\varphi(t))) - (t, \bar{x}_\varphi(t), \varphi(t, \bar{x}_\varphi(t)))\| \\
&\quad \leqslant (1+\kappa)\|x_\varphi(t) - \bar{x}_\varphi(t)\| \leqslant (1+\kappa) K_1 e^{\bar{a}\tau + as} \|\xi - \bar{\xi}\| \qquad (56)
\end{aligned}$$

for every $\tau = t - s \geqslant 0$. Again, since $\xi, \bar{\xi} \in Q\left(\delta e^{-\gamma s}\right)$, in view of (52) we can replace the function φ in (56) by its restriction to Z_β. This completes the proof of the theorem. $\square$

5. Existence of unstable manifolds

We now consider the case of unstable manifolds. The theory is entirely analogous and the proofs can be obtained by reversing time in the former notions and arguments. As such, we formulate the corresponding results of existence of unstable manifolds without proof.

We first briefly describe the corresponding setup. Consider a continuous function $A: \mathbb{R}_0^- \to B(X)$, with $\mathbb{R}_0^- = (-\infty, 0]$, satisfying

$$\limsup_{t \to -\infty} \frac{1}{|t|} \log^+ \|A(t)\| = 0.$$

As in the case of positive time, we assume that there exists a decomposition $X = E \times F$ (independent of t), with respect to which $A(t)$ has the block form in (10) for every $t \leqslant 0$.

Then the solution of the initial value problem in (5) with $s \leqslant 0$ and $v_s = (\xi, \eta) \in E \times F$ is global in the past and can be written in the form

$$v(t) = (U(t,s)\xi, V(t,s)\eta) \text{ for } t \leqslant s,$$

where $U(t,s)$ and $V(t,s)$ are the evolution operators associated, respectively, with the blocks $B(t)$ and $C(t)$ (see (10)). In an analogous manner to that for positive time, we say that the evolution operators $U(t,s)$ and $V(t,s)$ define a *weak nonuniform exponential dichotomy* if there exist constants $\overline{b} < 0 \leqslant \underline{a}$, a, $b \geqslant 0$, and D_1, $D_2 \geqslant 1$ such that for every $t \leqslant s \leqslant 0$,

$$\|U(t,s)^{-1}\| \leqslant D_1 \mathrm{e}^{-\underline{a}|t-s|+a|t|} \quad \text{and} \quad \|V(t,s)\| \leqslant D_2 \mathrm{e}^{\overline{b}|t-s|+b|s|}.$$

We also consider a continuous function $f: \mathbb{R}_0^- \times X \to X$ such that $f(t,0) = 0$ for every $t \leqslant 0$, and there exist $c > 0$ and $q > 0$ such that (8) holds for every $t \leqslant 0$ and u, $v \in X$. We continue to write $f = (g,h) \in E \times F$. We consider the semiflow Ψ_τ (now with $\tau \leqslant 0$) generated by Eq. (9) or equivalently by the system in (14) for $\rho \leqslant s$ in the corresponding maximal interval of definition.

Again we look for an unstable manifold as the graph of a Lipschitz function. For this we define the new constants

$$\beta = b(1 + 1/q) + a/q \quad \text{and} \quad \gamma = b(2 + 1/q) + a/q,$$

and given $\delta > 0$ and $\kappa > 0$, we consider a space $\mathcal{X}_\beta^u$ of continuous functions obtained as in Section 3, replacing positive time by negative time: consider the set

$$Z_\beta^u = \left\{(s,\xi) : s \leqslant 0 \text{ and } \xi \in Q\left(\delta \mathrm{e}^{-\beta|s|}\right)\right\} \subset \mathbb{R}_0^- \times F, \tag{57}$$

and let $\mathcal{X}_\beta^u$ be the space of continuous functions $\psi: Z_\beta^u \to E$ such that for each $s \leqslant 0$, we have $\psi(s,0) = 0$ and

$$\|\psi(s,x) - \psi(s,y)\| \leqslant \kappa \|x - y\| \text{ for every } x,\ y \in Q\left(\delta \mathrm{e}^{-\beta|s|}\right).$$

We also consider the set Z_γ^u obtained as in (57) with β replaced by γ.

We can now formulate our result on the existence of unstable manifolds.

Theorem 5. *If there exists a weak nonuniform exponential dichotomy for the equation $v' = A(t)v$ in the Banach space X, and the conditions $q\overline{b} + b < 0$ and $\overline{b} + a < \underline{a}$ hold, then there exist $\delta > 0$ and a unique function $\psi \in \mathcal{X}_\beta^u$ such that the set*

$$\mathcal{W}^u = \left\{(s, \psi(s,\xi), \xi) : (s,\xi) \in Z_\beta^u\right\} \subset \mathbb{R}_0^- \times X$$

is invariant under the semiflow Ψ_τ, *i.e.*,

$$\text{if } (s,\xi) \in Z^u_\gamma \text{ then } \Psi_\tau(s,\psi(s,\xi),\xi) \in \mathcal{W}^u \text{ for every } \tau \leqslant 0.$$

Furthermore:

1. *for every* $(s,\xi) \in Z^u_\gamma$, *we have*

$$\psi(s,\xi) = \int_{-\infty}^{s} U(\tau,s)^{-1} g(\Psi_{\tau-s}(s,\psi(s,\xi),\xi))\,d\tau;$$

2. *there exists* $D > 0$ *such that for every* $s \leqslant 0$, ξ, $\bar{\xi} \in Q\left(\delta e^{-\gamma|s|}\right)$, *and* $\tau \leqslant 0$,

$$\|\Psi_\tau(s,\psi(s,\xi),\xi) - \Psi_\tau(s,\psi(s,\bar{\xi}),\bar{\xi})\| \leqslant D e^{\bar{b}|\tau|+b|s|}\|\xi - \bar{\xi}\|.$$

6. Equations with negative Lyapunov exponents

The purpose of this section is to describe how the former existence results of stable and unstable manifolds can be applied to nonautonomous linear differential equations with negative Lyapunov exponents. We emphasize that we only consider finite-dimensional spaces in this section. Our study is based on the fact that essentially all such equations admit a nonuniform exponential dichotomy (see Proposition 6). There are several difficulties presented by a corresponding generalization to the infinite-dimensional setting. In the case of finite-dimensional systems, we can apply the classical Lyapunov–Perron regularity theory to measure in an effective manner the deviation of a nonuniform exponential dichotomy from the classical notion of uniform exponential dichotomy. In particular, this allows us to obtain sharp estimates for the numbers λ, μ, a, and b in (2) solely expressed in terms of Lyapunov exponents and regularity coefficients (see Proposition 6). On the other hand, the full extent generalization of our approach to infinite-dimensional systems would require an appropriate development of the regularity theory in this setting. It does not exist and presents some additional technical difficulties, essentially due to the fact that a Lyapunov exponent may then take infinitely many values, not to mention that the ambient space may not have a countable basis. We intend to consider this problem elsewhere.

We assume that the functions $A\colon \mathbb{R}^+_0 \to M_n(\mathbb{R})$, where $M_n(\mathbb{R})$ is the set of $n \times n$ matrices with real entries, and $f\colon \mathbb{R}^+_0 \times \mathbb{R}^n \to \mathbb{R}^n$ are continuous and satisfy the conditions H1–H2 in Section 3 with $X = \mathbb{R}^n$.

6.1. Lyapunov exponents and Lyapunov regularity

We define the Lyapunov exponent $\chi\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ for Eq. (5) by

$$\chi(v_0) = \limsup_{t\to+\infty} \frac{1}{t} \log\|v(t)\|,$$

where $v(t)$ is the solution of (5) with $s = 0$. It follows from the abstract theory of Lyapunov exponents (see [1] for a detailed description) that the function χ takes at most $r \leqslant n$ distinct values on $\mathbb{R}^n \setminus \{0\}$, say

$$-\infty \leqslant \chi_1 < \cdots < \chi_k < 0 \leqslant \chi_{k+1} < \cdots < \chi_r, \tag{58}$$

for some $r \leqslant n$ and some $0 \leqslant k \leqslant n$. Note that $k \geqslant 1$ if and only if there is at least one negative Lyapunov exponent. We emphasize that χ_{k+1} may be zero. Moreover, for each $i = 1, \ldots, r$ the set

$$E_i = \{v_0 \in \mathbb{R}^n : \chi(v_0) \leqslant \chi_i\} \tag{59}$$

is a linear space. We assume from now on that there is at least one negative Lyapunov exponent.

For simplicity of the exposition we always assume that there exists a subspace $F \subset \mathbb{R}^n$ such that $E = E_k$ (see (58)) and F give a decomposition $\mathbb{R}^n = E \times F$, with respect to which $A(t)$ has the block form in (10). Note that (for any norm in $\mathbb{R}^n$)

$$\chi(v_0) \leqslant \chi_k < 0 \text{ for every } v_0 \in E,$$

and $\chi(v_0) \geqslant \chi_{k+1} \geqslant 0$ for every $v_0 \in \mathbb{R}^n \setminus E$ (and thus for every $v_0 \in F \setminus \{0\}$).

We now introduce the classical notion of Lyapunov regularity. For this we need to consider the initial value problem

$$w' = -A(t)^* w, \quad w(0) = w_0, \tag{60}$$

with $w_0 \in \mathbb{R}^n$, where $A(t)^*$ denotes the transpose of $A(t)$. We also consider the associated Lyapunov exponent $\widetilde{\chi}\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\widetilde{\chi}(w_0) = \limsup_{t \to +\infty} \frac{1}{t} \log \|w(t)\|,$$

where $w(t)$ is the solution of (60). We define the *regularity coefficient* of χ and $\widetilde{\chi}$ by

$$\gamma(\chi, \widetilde{\chi}) = \min \max\{\chi(v_i) + \widetilde{\chi}(w_i) : 1 \leqslant i \leqslant n\},$$

where the minimum is taken over all bases $v_1, \ldots, v_n$ and $w_1, \ldots, w_n$ of $\mathbb{R}^n$ such that $\langle v_i, w_j \rangle = \delta_{ij}$ for each i and j (here δ_{ij} is the Kronecker symbol). We say that Eq. (5) is *Lyapunov regular* or simply *regular* if $\gamma(\chi, \widetilde{\chi}) = 0$.

We also consider the Lyapunov exponents associated to the blocks $B(t)$ and $C(t)$ in (10), i.e., to the pair $x' = B(t)x$ and $x' = -B(t)^* x$, as well as to the pair $y' = C(t)y$ and $y' = -C(t)^* y$. The corresponding regularity coefficients are, respectively,

$$\gamma_U = \gamma(\chi|E, \widetilde{\chi}|E) \quad \text{and} \quad \gamma_V = \gamma(\chi|F, \widetilde{\chi}|F).$$

The following statement shows that the evolution operators $U(t,s)$ and $V(t,s)$ in (11) always define a nonuniform exponential dichotomy. Furthermore, the six numbers in (15) can be related to the Lyapunov exponents and to the regularity coefficients.

Proposition 6 (*Barreira and Valls [4]*). *Assume that the matrix $A(t)$ has the block form in* (10) *for every $t \geqslant 0$, and that the equation $v' = A(t)v$ has at least one negative Lyapunov exponent. Then for each $\varepsilon > 0$, the evolution operators $U(t,s)$ and $V(t,s)$ define a nonuniform exponential dichotomy with*

$$\underline{a} = \chi_1 + \varepsilon, \quad \overline{a} = \chi_k + \varepsilon, \quad a = \gamma_U + 2\varepsilon, \tag{61}$$

$$\underline{b} = \chi_{k+1} + \varepsilon, \quad \overline{b} = \chi_r + \varepsilon, \quad b = \gamma_V + 2\varepsilon. \tag{62}$$

As in Section 2.1, the assumption in (10) is certainly a restriction to $A(t)$, although the general case may be treated at the expense of reducing $A(t)$ to the block form in (10). Unfortunately, this reduction is rarely explicit (and often time-dependent). We refer to Section 2.2 for a related discussion.

6.2. Construction of stable manifolds

We now present our results on the existence of stable manifolds. These are Lipschitz manifolds $\mathcal{W} \subset \mathbb{R}^{n+1}$ over the space $\mathbb{R}_0^+ \times E = \mathbb{R}_0^+ \times E_k$ (see (58) and (59)), where E_k is the linear space corresponding to the negative Lyapunov exponents of (5). Furthermore, $\mathcal{W}$ is required to be forward invariant under solutions of the equation in (9). We continue to assume that the matrix function A and the perturbation f satisfy the conditions H1–H2 in Section 3. We also consider the conditions

$$q\chi_k + \gamma_U < 0 \quad \text{and} \quad \chi_k + \gamma_V < \chi_{k+1}. \tag{63}$$

Note that both inequalities in (63) are automatically satisfied when the regularity coefficients γ_U and γ_V are sufficiently small, and that the first inequality is satisfied for a given γ_U provided that q is sufficiently large.

In view of (63) we can choose $\varepsilon > 0$ such that

$$T_1 = T_1(\varepsilon) = q\overline{a} + a < 0 \quad \text{and} \quad T_2 = T_2(\varepsilon) = \overline{a} - \underline{b} + b < 0, \tag{64}$$

where the constants in (15) take the values given by (61)–(62). Note that in view of the first inequality in (64) we have indeed $\overline{a} < 0$ as required in (15). We also consider the constants β and γ given by (20) and (22) with the values of a and b in (61)–(62).

The following is now an immediate consequence of Theorem 3.

Theorem 7. *Assume that* H1–H2 *hold. If the conditions in* (63) *hold, then for each $\varepsilon > 0$ satisfying* (64), *there exist $\delta > 0$ and a unique function $\varphi \in \mathcal{X}_\beta$ such that the set $\mathcal{W}$ in* (21) *has the properties in Theorem* 3, *with the constants in* (15) *given by* (61)–(62).

When Eq. (5) is Lyapunov regular we obtain the following statement as a consequence of Theorem 7 (or of Theorem 3). The advantage is that in this case we do not need the conditions in (64) (or in (63)).

Theorem 8. *Assume that* H1–H2 *hold. If Eq.* (5) *is Lyapunov regular, then for each* $\varepsilon > 0$, *there exist* $\delta > 0$ *and a unique function* $\varphi \in \mathcal{X}_\beta$ *such that the set* $\mathcal{W}$ *in* (21) *has the properties in Theorem* 3, *with* $a = b = 2\varepsilon$.

Proof. We first observe that any initial value problem as in (5) can be reduced to one with an upper triangular $A(t)$ for every $t \geqslant 0$. More precisely, let $v(t)$ be the solution of the initial value problem in (5). By the discussion in [1, Section 1.3] (see in particular Lemma 1.3.3 in [1]), we have the following statement.

Lemma 6. *There are continuous functions* $D\colon \mathbb{R}_0^+ \to M_n(\mathbb{R})$ *and* $U\colon \mathbb{R}_0^+ \to M_n(\mathbb{R})$ *with* $U(0) = I$ *such that*:

1. $D(t)$ *is upper triangular for every* $t \geqslant 0$, *and*
$$\limsup_{t\to+\infty} \frac{1}{t} \log^+ \|D(t)\| = 0;$$
2. $t \mapsto U(t)$ *is differentiable, and* $U(t)$ *is unitary for each* $t \geqslant 0$;
3. *setting* $x(t) = U(t)^{-1}v(t)$ *for each* $t \geqslant 0$, *we have* $x'(t) = D(t)x(t)$.

In addition, due to the block form of $A(t)$ in (10) the matrix $U(t)$ can also be chosen in block form with blocks of the same dimension. Although the function $U(t)$ need not be known explicitly, Lemma 6 shows that without loss of generality we can simply consider upper triangular systems.

We have the following criterion of regularity, which can be obtained from the discussion in [1, Section 1.3] (see in particular Lemma 1.3.5 in [1]). Alternatively, it can be obtained from more general results in [4].

Lemma 7. *Assume that* $A(t)$ *is upper triangular for every* $t \geqslant 0$. *Eq.* (5) *is Lyapunov regular if and only if for* $i = 1, \ldots, n$ *there exist the limits*
$$\alpha_i = \lim_{t\to+\infty} \frac{1}{t} \int_0^t a_i(s)\, ds,$$
where $a_1(t), \ldots, a_n(t)$ *are the entries in the diagonal of* $A(t)$.

By Lemma 6, the initial value problem in (5) is equivalent to
$$x' = D(t)x, \quad x(0) = v_0, \tag{65}$$
with
$$D(t) = \begin{pmatrix} E(t) & 0 \\ 0 & F(t) \end{pmatrix}, \tag{66}$$

where

$$E(t) = D(t)|E \quad \text{and} \quad F(t) = D(t)|F$$

are upper triangular for every $t \geqslant 0$. Similarly, we can show that the initial value problem in (60) is equivalent to

$$y' = -D(t)^* y, \quad y(0) = w_0, \tag{67}$$

with the solutions $w(t)$ of (60) and $y(t)$ of (67) related by $w(t) = U(t)y(t)$ using the same operator $U(t)$ as in Lemma 6. Since $U(t)$ is unitary for each $t \geqslant 0$, the Lyapunov exponents for the equations in (65) and (67) coincide, respectively, with the Lyapunov exponents χ and $\widetilde{\chi}$ for the equations in (5) and (60). Therefore, the regularity coefficient of the new pair of equations ((65) and (67)) is the same as the regularity coefficient for Eqs. (5) and (60).

In particular, since Eq. (5) is regular, the same happens with Eq. (60). It follows from Lemma 7 that there exist the limits

$$\lim_{t \to +\infty} \frac{1}{t} \int_0^t d_i(s)\, ds,$$

where $d_1(t), \dots, d_n(t)$ are entries in the diagonal of $D(t)$. Considering now separately the upper triangular blocks $E(t)$ and $F(t)$ of the matrix $D(t)$ (see (66)), applying once more Lemma 7 but now in the opposite direction, we conclude that both systems $x' = E(t)x$ in E and $x' = F(t)x$ in F are also Lyapunov regular. Since the Lyapunov exponents χ and $\widetilde{\chi}$ agree with those for Eqs. (65) and (67), the systems $x' = B(t)x$ in E and $x' = C(t)x$ in F are also regular. In particular, $\gamma_U = \gamma_V = 0$. Therefore, condition (63) is automatically satisfied and we can apply Theorem 7 to obtain the desired result. In particular, it follows from (61)–(62) that $a = b = 2\varepsilon$. $\square$

We could also formulate corresponding statements concerning the existence of unstable manifolds. These are entirely analogous to the former ones, and correspond to consider the Lyapunov exponent $\chi^-\colon \mathbb{R}^n \to \mathbb{R} \cup \{-\infty\}$ defined by

$$\chi^-(v_0) = \limsup_{t \to -\infty} \frac{1}{|t|} \log \|v(t)\|, \tag{68}$$

where $v(t)$ is the solution of (5) with $s = 0$. It follows from the abstract theory of Lyapunov exponents that the function χ^- takes at most $r^- \leqslant n$ distinct values on $\mathbb{R}^n \setminus \{0\}$, say

$$-\infty \leqslant \chi^-_{r^-} < \cdots < \chi^-_{k^-+1} < 0 \leqslant \chi^-_{k^-} < \cdots < \chi^-_1$$

for some $1 \leqslant k^- \leqslant r^-$, assuming that there is at least one negative Lyapunov exponent, with respect to the Lyapunov exponent χ^- in (68). We can then proceed in a similar manner to obtain the existence of unstable manifolds as an application of Theorem 5.

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