MATHEMATICS

# THE UNITARY ANALOGUES OF SOME IDENTITIES FOR CERTAIN ARITHMETICAL FUNCTIONS

BY

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## § 1. *Introduction*

If $d$ is a divisor of the positive integer $n$ (notation: $d/n$) and $(d, n/d)=1$ we say that $d$ is a *unitary divisor* of $n$; notation: $d//n$; for instance we have always $1//n$ and $n//n$.

Definition. The *unitary convolution* $h(n)$ of two arithmetical functions $f(n)$ and $g(n)$ is defined by

$$h(n)=\sum_{d//n} f(d)\, g\left(\frac{n}{d}\right);$$

notation: $h(n)=(f \circ g)(n)$.

Now it is easy to introduce by means of this unitary convolution new arithmetical functions which may be regarded as the *unitary analogues* of well-known functions such as Möbius' function $\mu(n)$, Euler's function $\varphi(n)$ and others. For example, it is possible to define the functions $\mu(n)$ and $\varphi(n)$ by means of the formulas:

$$\sum_{d/n} \mu(d)=\begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n>1 \end{cases},$$

$$\varphi(n)=\sum_{d/n} \mu(d)\frac{n}{d}.$$

Similarly, we now define their unitary analogues $\mu^*(n)$ and $\varphi^*(n)$ by

$$\sum_{d//n} \mu^*(d)=\begin{cases} 1 \text{ if } n=1 \\ 0 \text{ if } n>1 \end{cases},$$

$$\varphi^*(n)=\sum_{d//n} \mu^*(d)\frac{n}{d}.$$

In § 3 we shall derive some relations between these unitary functions, while in § 2 we shall develop the theory for the corresponding ordinary arithmetical functions.

§ 2. *Relations, derived by means of the Dirichlet-product*

If we introduce in the set $\Omega$ of all arithmetical functions the operations $+$, defined by

$$(f+g)(n)=f(n)+g(n)$$

and $*$, defined by

$$(f * g)(n)=\sum_{d/n} f(d)\, g\left(\frac{n}{d}\right),$$

(the latter being the so-called convolution or Dirichlet-product), we get a domain of integrity in which the law of the unique factorization into primes holds [2], p. 982–985. The multiplicative group of units is formed by the functions $f(n)$ with $f(1)\neq 0$; of this group the multiplicative functions (except the function which is identical zero) form a subgroup.

This algebraic structure in $\Omega$ furnishes us a method to introduce new arithmetical functions as generalizations of classical functions [2], p. 978–979 and of proving known formulas in a very simple way. We may give a few examples.

Defining the functions $e(n)$ and $I_\alpha(n)$ by

$$e(n)=\begin{cases}1 & \text{if } n=1\\ 0 & \text{if } n>1\end{cases}$$

(i.e. $e(n)$ is the unit-element in $\Omega$),

$$I_\alpha(n)=n^\alpha,\ \alpha \text{ real}$$

and writing $\dfrac{e}{f}(n)$ for the inverse of the (unit-)function $f(n)$, we have the well-known formulas (compare § 1)

$$\mu(n)=\frac{e}{I_0}(n),$$

$$\varphi(n)=(\mu * I_1)(n).$$

We now define the function $\varphi_\alpha(n)$ by

$$\varphi_\alpha(n)=(\mu * I_\alpha)(n),$$

hence

$$(I_0 * \varphi_\alpha)(n)=I_\alpha(n).$$

Since $\mu(n)$ and $I_\alpha(n)$ are both multiplicative, $\varphi_\alpha(n)$ is multiplicative [1]). From the definition of $\varphi_\alpha(n)$ we see that

$$\varphi_\alpha(p^k)=\mu(1)p^{\alpha k}+\mu(p)p^{\alpha(k-1)}+\mu(p^2)p^{\alpha(k-2)}+\ldots+\mu(p^k)=p^{\alpha k}-p^{\alpha(k-1)}$$

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1) More precisely: $\varphi_\alpha(mn)=\varphi_\alpha(m)\,\varphi_\alpha(n)\dfrac{(m,n)^\alpha}{\varphi_\alpha((m,n))}$, $\alpha\neq 0$ which follows from [1], theorem 3.

and hence

$$\varphi_\alpha(n) = n^\alpha \prod_{p/n} \left(1 - \frac{1}{p^\alpha}\right).$$

The functions $\varphi_\alpha(n)$ occur in the literature for various values of $\alpha$. If we put $\alpha = 2, 3, 4, \ldots$ we have $\varphi_\alpha(n) = J_\alpha(n)$, where $J_\alpha(n)$ denotes Jordan's totient-function of order $\alpha$. As an example of a negative value of $\alpha$ we mention a result of CESÀRO, see e.g. [4], p. 128:

$$\det |\{\mu, \nu\}|_{\substack{\mu=1,2,\ldots,n \\ \nu=1,2,\ldots,n}} = (n!)^2 \varphi_{-1}(1)\varphi_{-1}(2)\ldots\varphi_{-1}(n),$$

where $\{\mu, \nu\}$ = l.c.m. of $\mu$ and $\nu$.

In the sequel $\dfrac{g}{\overline{f}}(n)$ denotes the convolutionproduct of $g(n)$ and the inverse of an unit $f(n)$; hence $\left(f * \dfrac{g}{\overline{f}}\right)(n) = g(n)$.

Theorem 1. Let $\{N_\alpha(n)\}$, $\alpha$ real, be a set of completely multiplicative arithmetical functions not containing the zero-function, such that $N_\alpha(n) \cdot N_\beta(n) = N_{\alpha+\beta}(n)$ and let $\psi_\alpha(n) = (\mu * N_\alpha)(n)$.
Then

$$\frac{N_\alpha}{\overline{N_\beta}}(n) = \psi_{\alpha-\beta}(n) N_\beta(n).$$

Proof.

$$(N_\beta * \psi_{\alpha-\beta} N_\beta)(n) = \sum_{d/n} \psi_{\alpha-\beta}\left(\frac{n}{d}\right) N_\beta\left(\frac{n}{d}\right) N_\beta(d) =$$

$$N_\beta(n)\,(I_0 * \psi_{\alpha-\beta})\,(n) = N_\beta(n) \cdot N_{\alpha-\beta}(n) = N_\alpha(n).$$

From the definition of $\varphi_\alpha(n)$ it follows that $\dfrac{\varphi_\alpha}{\overline{\varphi_\beta}}(n) = \dfrac{I_\alpha}{\overline{I_\beta}}(n)$; if we now take $N_\alpha(n) = I_\alpha(n)$ in theorem 1, we get:

Corollary 1.

$$\frac{\varphi_\alpha}{\overline{\varphi_\beta}}(n) = \varphi_{\alpha-\beta}(n) \cdot I_\beta(n).$$

For $\alpha, \beta, \alpha-\beta = 1, 2, 3, \ldots$, this is a well-known result of Gegenbauer for the Jordan's functions, see e.g. [4], p. 149.

Corollary 2. If we define as usual $\sigma_\alpha(n) = \sum_{d/n} d^\alpha = (I_0 * I_\alpha)(n)$ we have

$$\frac{\sigma_\alpha}{\overline{\sigma_\beta}}(n) = \varphi_{\alpha-\beta}(n) \cdot I_\beta(n).$$

For $\alpha = 2, 3, \ldots$ and $\beta = \alpha - 1$ this was proved by NEGOES, see [5]. The functions $\varphi_\alpha(n)$ occur also in the following result of THACKER, see e.g. [4], p. 140:

Theorem 2. Let $r_1, r_2, \ldots, r_{\varphi(n)}$ denote the $\varphi(n)$ positive integers $\leqslant n$ and prime to $n$; then for $k=0, 1, 2, \ldots$

$$r_1{}^k + r_2{}^k + \ldots + r^k_{\varphi(n)} =$$
$$n^k \left\{ \frac{1}{k+1} \varphi(n) - c_{0,k} e(n) + c_{1,k} \varphi_{-1}(n) + c_{3,k} \varphi_{-3}(n) + c_{5,k} \varphi_{-5}(n) + \ldots \right\}$$

where

$$c_{l,k} = \frac{1}{l+1} \binom{k}{l} B_{l+1} \text{ if } k > l, \ c_{l,k} = 0 \text{ if } k \leqslant l;$$

as usual $B_1, B_2, B_3, B_4, B_5, B_6, \ldots$ denote the Bernoullian numbers $-\frac{1}{2}, \frac{1}{6}, 0, -\frac{1}{30}, 0, \frac{1}{42}, \ldots\ldots$ [2]).

Proof. Defining $\Phi_k(n)$ by $\Phi_k(n) = r_1{}^k + r_2{}^k + \ldots + r^k_{\varphi(n)}$, $k = 0, 1, \ldots,$ we have

$$(\Phi_k * I_k)(n) = \sum_{d/n} d^k \Phi_k \left(\frac{n}{d}\right) = \sum_{d/n} \sum_{(\nu, n/d)=1} \nu^k = \sum_{\nu=1}^{n} \nu^k =$$
$$\frac{1}{k+1} I_{k+1}(n) - c_{0,k} I_k(n) + c_{1,k} I_{k-1}(n) + c_{3,k} I_{k-3}(n) + \ldots .$$

Thus

$$\Phi_k(n) = \frac{1}{k+1} \frac{I_{k+1}}{I_k}(n) - c_{0,k} \frac{I_k}{I_k}(n) + c_{1,k} \frac{I_{k-1}}{I_k}(n) + c_{3,k} \frac{I_{k-3}}{I_k}(n) + \ldots$$

and hence, it follows from theorem 1, putting $N_\alpha(n) = I_\alpha(n)$:

$$\Phi_k(n) = n^k \left\{ \frac{1}{k+1} \varphi(n) - c_{0,k} e(n) + c_{1,k} \varphi_{-1}(n) + c_{3,k} \varphi_{-3}(n) + \ldots \right\}.$$

## § 3. *Some unitary analogues*

Lemma 1. If $b/\!/a$ and $c/\!/b$ then $c/\!/a$.

Proof. $a = b\beta$, $(b, \beta) = 1$ and $b = c\gamma$, $(c, \gamma) = 1$. Since $c$ is a divisor of $b$ we have $(c, \beta) = 1$ and thus $(c, \beta\gamma) = 1$; hence, putting $\beta\gamma = \alpha$, we have $a = c\alpha$, $(c, \alpha) = 1$, which proves the lemma.

Definition. Let $n$ and $m$ be two positive integers,

$$n = \prod_{\varrho=1}^{r} p_\varrho^{\alpha_\varrho} \text{ and } m = \prod_{\varrho=1}^{r} p_\varrho^{\beta_\varrho}, \ \alpha_\varrho, \beta_\varrho \geqslant 0,$$

with $p_1, p_2, \ldots, p_r$ different prime-numbers; then we define

$$[n, m] = \prod_{\varrho=1}^{r} p_\varrho^{\delta(\alpha_\varrho, \beta_\varrho)},$$

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[2]) In this form the Bernoullian numbers can be defined symbolically by $B_0 = 1$, $B_n = (1 + B)^n$, $n > 1$, if we put $B^n = B_n$.

where

$$\delta(\alpha, \beta) = \begin{cases} 0 \text{ if } \alpha \neq \beta \\ \alpha \text{ if } \alpha = \beta \end{cases}.$$

It is clear that $[n, m]$ is a unitary divisor of $n$ and of $m$ and that $[n, m]$ is the largest number with this property.

Lemma 2. $a /\!/ [b, c]$ if and only if $a /\!/ b$ and $a /\!/ c$.

Proof. Let $a /\!/ b$ and $a /\!/ c$; then $a\beta = b$, $(a, \beta) = 1$ and $a\gamma = c$, $(a, \gamma) = 1$. From this it follows that $[b, c] = a[\beta, \gamma]$, $(a, [\beta, \gamma]) = 1$ and thus $a /\!/ [b, c]$.
Conversely, $a /\!/ [b, c] /\!/ b$ implies $a /\!/ b$; similarly, $a /\!/ c$.

Theorem 3. With the ordinary addition $+$ as additive operation and the unitary convolution $\circ$ as multiplicative operation, the set $\Omega$ of all arithmetical functions forms a commutative ring with unit-element.
The unit-element is again the function

$$e(n) = \begin{cases} 1 \text{ if } n = 1 \\ 0 \text{ if } n > 1 \end{cases}.$$

The proof is very simple. We only check the associative law for the multiplicative operation:

$$((f \circ g) \circ h)(n) = \sum_{\substack{dd''' = n \\ (d, d''') = 1}} \sum_{\substack{d'd'' = d \\ (d', d'') = 1}} f(d')g(d'')h(d''') = \sum_{d'd''d''' = n}' f(d')g(d'')h(d'''),$$

where the dash in the summation-sign indicates that we sum only over the triples $(d', d'', d''')$ with $(d', d'') = 1$, $(d'd'', d''') = 1$. This condition, however, is equivalent to $(d', d'') = (d'', d''') = (d''', d') = 1$ and since this is symmetric in $d', d'', d'''$, the associative law follows.

Again the group of units is formed by the functions $f(n)$ with $f(1) \neq 0$. *Of this group the multiplicative functions* (except the zero-function) *form a subgroup* ([3], lemma 6.1).
If $f(n)$ is a unit, its inverse is denoted by $\frac{e}{f}(n)$.

It is easily seen that $\Omega$ contains zero-divisors, hence with the above operations $\Omega$ does not form a domain of integrity.
We now introduce the unitary analogues of some ordinary arithmetical functions:

$$\mu^*(n) = \frac{e}{I_0}(n),$$

$$\varphi_\alpha^*(n) = (\mu^* \circ I_\alpha)(n) \text{ [3]}$$

$$\sigma_\alpha^*(n) = (I_0 \circ I_\alpha)(n), \text{ especially}$$

$$\sigma_0^*(n) = \tau^*(n), \text{ the number of unitary divisors of } n.$$

[3]) In the sequel $\varphi_1^*(n)$ will be denoted by $\varphi^*(n)$.

Theorem 4. If $n=\prod_{\varrho=1}^{r} p_\varrho^{l_\varrho}$, $l_\varrho>0$, then

$$\mu^*(1)=1,\ \mu^*(n)=(-1)^r,\ n>1;$$

$$\varphi_\alpha^*(1)=1,\ \varphi_\alpha^*(n)=n^\alpha \prod_{\varrho=1}^{r}\left(1-\frac{1}{p_\varrho^{\alpha l_\varrho}}\right),\ n>1;$$

$$\sigma_\alpha^*(1)=1,\ \sigma_\alpha^*(n)=n^\alpha \prod_{\varrho=1}^{r}\left(1+\frac{1}{p_\varrho^{\alpha l_\varrho}}\right),\ n>1;$$

especially

$$\tau^*(n)=2^r,\ n>1.$$

Proof. All the functions are multiplicative, so we have only to evaluate $\mu^*(p^k)$, $\varphi_\alpha^*(p^k)$, $\sigma_\alpha^*(p^k)$, $k>0$.

$$\mu^*(p^k)\,I_0(1)+\mu^*(1)\,I_0(p^k)=0, \text{ hence } \mu^*(p^k)=-1.$$

$$\varphi_\alpha^*(p^k)=\mu^*(p^k)\,I_\alpha(1)+\mu^*(1)\,I_\alpha(p^k)=-1+p^{\alpha k}.$$

$$\sigma_\alpha^*(p^k)=I_0(p^k)\,I_\alpha(1)+I_0(1)\,I_\alpha(p^k)=1+p^{\alpha k}.$$

Theorem 5. Let $\{a_n\}_{n=1}^\infty$ be an arbitrary sequence of real or complex numbers and let $A_n$ be defined by

$$A_n=\sum_{d//n} a_d,\ n=1,2,\ldots.$$

Then

$$\det|A_{[\mu,\nu]}|_{\substack{\mu=1,2,\ldots,n\\ \nu=1,2,\ldots,n}}=a_1\cdot a_2\cdot\ldots\cdot a_n.$$

Proof. If $X$ is the matrix $\{b_{\mu\nu}\}_{\substack{\mu=1,\ldots,n\\ \nu=1,\ldots,n}}$ with

$$b_{\mu\nu}=\begin{cases}1 \text{ if } \nu//\mu\\ 0 \text{ otherwise}\end{cases},$$

and $Y$ is the matrix $\{a_\nu b_{\mu\nu}\}_{\substack{\mu=1,\ldots,n\\ \nu=1,\ldots,n}}$, then clearly $\det X=1$ and $\det Y=a_1a_2\ldots a_n$. For the general element $c_{\mu\nu}$ of the matrix $XY^T$ we have on the other hand

$$c_{\mu\nu}=\sum_{\lambda=1}^{n} b_{\mu\lambda}\,a_\lambda\,b_{\nu\lambda}=\sum_{\substack{\lambda=1\\ \lambda//\mu\\ \lambda//\nu}}^{n} a_\lambda=\sum_{\lambda//[\mu,\nu]} a_\lambda,$$

in view of lemma 2. Hence $c_{\mu\nu}=A_{[\mu,\nu]}$ and therefore

$$\det|A_{[\mu,\nu]}|=\det X\cdot\det Y=a_1a_2\ldots a_n.$$

From the definition of $\varphi^*(n)$ it follows immediately

$$(I_0\circ\varphi^*)(n)=(I_0\circ\mu^*\circ I_1)(n)=I_1(n),$$

hence $\sum_{d//n}\varphi^*(d)=n$. In this way we obtain the following

Corollary of theorem 5.

$$\begin{vmatrix} [1,1], & [1,2], & \dots, & [1,n] \\ [2,1], & [2,2], & \dots, & [2,n] \\ \cdot & \cdot & \cdot & \cdot \\ [n,1], & [n,2], & \dots, & [n,n] \end{vmatrix} = \varphi^*(1)\,\varphi^*(2)\,\dots\,\varphi^*(n).$$

This is the unitary analogue of the determinant of SMITH:

$$\begin{vmatrix} (1,1), & (1,2), & \dots, & (1,n) \\ (2,1), & (2,2), & \dots, & (2,n) \\ \cdot & \cdot & \cdot & \cdot \\ (n,1), & (n,2), & \dots, & (n,n) \end{vmatrix} = \varphi(1)\,\varphi(2)\,\dots\,\varphi(n),$$

see e.g. [6], VIII, Aufgabe 57.

Theorem 6. Let $\{N_\alpha(n)\}$, $\alpha$ real, be a set of multiplicative functions, not containing the zero-function, such that

$$N_\alpha(n)\cdot N_\beta(n) = N_{\alpha+\beta}(n), \text{ and let } \psi_\alpha(n) = (\mu^* \circ N_\alpha)(n).$$

Then

$$\frac{N_\alpha}{N_\beta}(n) = \psi_{\alpha-\beta}(n)\cdot N_\beta(n).$$

The proof is similar to the one of theorem 1 and is omitted. Note that here the function $N_\alpha(n)$ need not be completely multiplicative.

Corollary.

$$\frac{\varphi_\alpha{}^*}{\varphi_\beta{}^*}(n) = \varphi^*_{\alpha-\beta}(n)\cdot I_\beta(n),$$

the analogue of the extension of Gegenbauer's relation.

Following COHEN [3], we denote by $(a, n)_*$ the largest divisor of $a$ which is unitary divisor of $n$. The positive integers $a \leqslant n$ with $(a, n)_* = 1$ form the so-called semi-reduced system mod $n$. Now by a well-known argument of SYLVESTER, see e.g. [6], VIII, Aufgabe 21, we see that if $n = \prod_{\varrho=1}^{r} p_\varrho^{l_\varrho}$, the number of elements of the semi-reduced system is exactly

$$n - \frac{n}{p_1^{l_1}} - \frac{n}{p_2^{l_2}} - \dots + \frac{n}{p_1^{l_1}p_2^{l_2}} + \dots = n\prod_{\varrho=1}^{r}\left(1 - \frac{1}{p_\varrho^{l_\varrho}}\right) = \varphi^*(n).$$

Theorem 7. Let $r_1, r_2, \dots, r_{\varphi^*(n)}$ denote the elements of the semi-reduced system mod $n$.
Then

$$r_1{}^k + r_2{}^k + \dots + r^k_{\varphi^*(n)} = n^k\left\{\frac{1}{k+1}\varphi^*(n) + \tfrac{1}{2}e(n) + c_{1,k}\,\varphi^*_{-1}(n) + c_{3,k}\,\varphi^*_{-3}(n) + \dots\right\},$$
$$k = 0, 1, 2, \dots.$$

Again the proof can be omitted, since it runs similarly to that of theorem 2.

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