Scheme of Constructing Soliton-Type Solutions of the (2+1)-Dimensional Harry Dym Equation

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Abstract—The scheme of constructing multisoliton solutions of the Harry Dym equation in (2+1) dimensions is presented. The general formula for N-soliton solution is obtained. The case of N = 1 is considered in detail. The analysis of the family of three-parameter solutions is performed. The plots of these solutions are presented. The auto-Bäcklund transformations for the Kadomtsev-Petviashvili, modified Kadomtsev-Petviashvili and Harry Dym-(2+1) equations are shown to be reduced to the phase shift of the corresponding soliton solutions.

Keywords—Completely integrable nonlinear equations, Multisoliton solutions, Auto-Bäcklund transformations.

1. INTRODUCTION

The main object of the present paper is the (2+1)-dimensional Harry Dym equation (HDE-(2+1)). First, it was introduced by Konopelchenko and Dubrovsky [1] and to our knowledge now is the only one known two-dimensional generalization of the so-called WKI (Wadati, Konno and Ichikawa) equations [2]. Among them one can mention the Harry Dym equation (HDE) (see [3–8] and references therein), the loop-soliton equation [9–13] and the nonlinear Schrödinger-type equation [14,15]. All the above-mentioned equations have wide physical applications (see the above references) and possess a lot of exotic mathematical properties which distinguish them from all other known completely integrable nonlinear equations.

The distinctive feature of WKI equations is that the associated linear problem differs from the Zakharov-Shabat spectral problem that generates the 'usual' AKNS equations [16]. The crucial point of the WKI spectral problem is the dependence of the wave function’s asymptotics (as the spectral parameter tends to infinity) on the coefficients of the system. As a consequence, all known solutions of the WKI equations are intrinsically implicit. This means that they depend on space and time variables through some phase function, the latter being defined implicitly by a functional equation. That is why, in spite of the fact that each WKI equation may be linked by a sequence of transformations [11,17] with an AKNS equation, WKI equations possess many unusual features, and indeed form a new class of integrable equations deserving special study.

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The above-mentioned peculiarities of the WKI equations lead to the necessity of elaborating new methods for constructing exact solutions of these equations. Among these methods we mention the 'higher-times approach' to constructing the multisoliton solutions of the HDE [7], its generalization for finite-gap solutions [8] and the further application to multisoliton solutions of the loop-soliton equation [13]. This method allows us to construct intrinsically by its nature solutions on the 'almost explicit' level. Namely, the phase function mentioned above (the main point of implicitness) is defined as a solution of the functional equation, the latter being known explicitly. The solution itself then is described explicitly in terms of this phase function.

One could expect that all the exotic features of one-dimensional WKI equations are inherited by their generalizations to (2+1)-dimensions. This can be seen in the example of the HDE(2+1). It has the form

$$r_t = r^3 r_{xxx} - 3r^{-1} \left( r^2 \partial_x^{-1} \left( \frac{1}{r} \right)_y \right)_y,$$

where $\partial_x^{-1}$ is the integral operator

$$\partial_x^{-1} g = \int_{-\infty}^{x} g(x', y, t) \, dx'$$

and generalizes the HDE

$$r_t = r^3 r_{xxx}.$$

Among the mentioned inherited peculiarities is the dependence of the high energy asymptotics of wave functions of the associated spectral problem on the solution $r(x, y, t)$ itself. This prevents the direct applications of traditional methods of studying (2+1)-dimensional equations (such as the Kadomtsev-Petviashvili (KP) equation, the modified Kadomtsev-Petviashvili (mKP) equation, the Davey-Stewartson (DS) equation) to the HDE-(2+1) [18,19]. However the $\tilde{\partial}$-method elaborated for the mKP-II equation in [20] and links between the latter and the HDE-(2+1) [21] allowed us to construct a class of exact implicit solutions of the HDE-(2+1).

In the present paper we propose the scheme of constructing soliton-type solutions for the HDE-(2+1) which do not belong to the above-mentioned class. These solutions turn out to be implicit as in the case of (1+1) dimensions. Unfortunately, the present scheme does not allow us to achieve the above described 'one-dimensional' level of explicitness. Nevertheless, the collision of a small number of solitons can be studied 'almost explicitly.' To make this independent of the number of solitons one has to improve the scheme by a more delicate generalization of the 'higher-times approach' developed in (1+1)-dimensions.

The organization of the paper is as follows. In Section 2, we refer to the preliminary facts concerning the HD-(2+1) hierarchy on the basis of the paper [21]. In Section 3, we describe the general scheme of constructing multisoliton solutions of the equation (1). To this end, we consider the HD-(2+1) hierarchy as well as the equation itself. An essential point in the scheme is the link of this hierarchy with the mKP hierarchy [21]. Section 4 deals with the auto-Bäcklund transformations for the KP, mKP and HD-(2+1) equations. They are shown to be reduced to the constant phase shift in the multisoliton case. In Section 5, we construct the general formula for $N$-soliton solution of the HDE-(2+1) for $N \geq 2$. The case $N = 1$ is considered in detail in the other sections. The explicit expressions for one-soliton (three-parametric) solutions and their phase functions are obtained in Section 6. In Section 7 of the paper, we perform the analytical study of the latter. Namely, the smoothness of the solutions, the behavior at infinity and other properties are analyzed. Section 8 deals with the numerical analysis. The plots of the solutions are presented. The links by the auto-Bäcklund transformation are also considered.

### 2. PRELIMINARIES

Along with the HDE-(2+1) we consider the whole "two-dimensional" Harry Dym (HD-(2+1)) hierarchy and its links with the KP and mKP hierarchies. All these hierarchies are known [21].
to be represented in the Lax form

\[ L_{t_q} = [P_{\geq k}(L^q), L], \quad q = 1, 2, 3, \ldots, \quad k = 0, 1, 2, \quad (3) \]

where by \([\cdot, \cdot]\) we denote the commutator of the pair of operators and \(P_{\geq k}\) is the projector onto the differential part of the operator of the order \(\geq k\); i.e., if \(A\) is the pseudodifferential operator defined by

\[ A = \sum_i u_i(x) \partial_x^i \]

then

\[ P_{\geq k} A = \sum_{i \geq k} u_i(x) \partial_x^i. \]

Herein, the power \(q\) labels the hierarchy of equations associated with a given Lax operator \(L\).

We use the more convenient notations \(t_2 \equiv y, t_3 \equiv t\).

We choose the following operators:

\[ k = 0: \quad L_{\text{KP}} = \partial_x + \frac{1}{2} u \partial_x^{-1} + u_2 \partial_x^{-2} + u_3 \partial_x^{-3} + \ldots \quad (4) \]

\[ k = 1: \quad L_{\text{mKP}} = \partial_x + v_1 \partial_x^{-1} + v_2 \partial_x^{-2} + v_3 \partial_x^{-3} + \ldots \quad (5) \]

\[ k = 2: \quad L_{\text{HD}} = r \partial_x + r_0 + r_1 \partial_x^{-1} + r_2 \partial_x^{-2} + r_3 \partial_x^{-3} + \ldots \quad (6) \]

Here the "higher coefficients" \(u, v\) and \(r\) carry no index, since they will be taken as distinguished fields which satisfy identifiable integrable equations. These are the KP hierarchy for \(k = 0\), the mKP hierarchy for \(k = 1\) and the HD-(2+1) hierarchy for \(k = 2\) as indicated by the choice of subscripts for the Lax operators. The remaining fields \(u_2, u_3, \ldots, v_1, v_2, \ldots, r_0, r_1, \ldots\) may be regarded as auxiliary fields to be eliminated in the construction.

The eigenfunction for the hierarchy (3) is defined as follows [21]. For given \(k = 0, 1\) or 2, a function \(\Phi = \Phi(x, \tau)\) satisfying the linear equation

\[ \Phi_{t_q} = P_{\geq k}(L^q)\Phi, \quad q \in \mathbb{N} \quad (7) \]

shall be called an eigenfunction for the hierarchy of the Lax equations (3). By \(\tau\) we denote the set of variables

\[ \tau = (t_2, t_3, t_4, \ldots). \]

For the cases \(k = 0, 1, 2\) under consideration here these equations are compatible and can be considered simultaneously for different \(q\)'s.

Here we introduce more convenient notations for the Lax operators

\[ L_{\text{HD}} \equiv L, \quad L_{\text{mKP}} \equiv M, \quad L_{\text{KP}} \equiv N. \]

In what follows, we need the statements [21] dealing with the links between the hierarchies.

**STATEMENT 1.** (Link between mKP and HD-(2+1) hierarchies.) Let \(M = M(x', \tau')\) satisfy the hierarchy \(M_{t_q} = [P_{\geq 1}(M^q), M], q \in \mathbb{N}\). Let \(\Theta_1(x', \tau')\) and \(\Theta_2(x', \tau')\) be two eigenfunctions of this hierarchy. Then the operator-valued function

\[ L(x, \tau) = M(x', \tau'), \quad (8a) \]

under the transformation

\[ x = \Theta_1(x', \tau'), \quad \tau = \tau', \quad (8b) \]

satisfies the hierarchy

\[ L_{t_q} = [P_{\geq 2}(L^q), L], \quad (8c) \]
and the function

\[ \psi(x, \tau) = \Theta_2(x', \tau') \]  

is an eigenfunction for (8c), i.e.,

\[ \psi_t = P_{\geq 2}(L^q)\psi. \]

**STATEMENT 2.** (Auto-Bäcklund transformation for mKP hierarchy.) Let \( M = M(x', \tau') \) satisfy the hierarchy \( M_{t_q} = [P_{\geq 1}(M^q), M], \) \( q \in \mathbb{N}. \) Let \( \Theta_1(x', \tau') \neq 0 \) and \( \Theta_2(x', \tau') \) be two eigenfunctions of this hierarchy. Then,

\[ M' = \Theta_1^{-1}(x')M\Theta_1(x') \]

also satisfies the hierarchy

\[ M_{t_q}' = [P_{\geq 1}(M^q), M'], \]

and

\[ \Theta' = \Theta_1^{-1}(x')\Theta_2(x'). \]

is an eigenfunction for (9b), that is

\[ \Theta_{t_q}' = P_{\geq 1}(M^q)\Theta'. \]

The higher coefficients \( v \) and \( v' \) of the operators \( M \) and \( M' \), respectively, satisfy the auto-Bäcklund transformation for the mKP equation

\[ \partial_{x'}^{-1}((v' - v)x') = (v' + v)_x + (v'^2 - v^2). \]  

**STATEMENT 3.** (Link between KP and mKP hierarchies.) Let \( N = N(x', \tau') \) satisfy the hierarchy of Lax equations \( N_{t_q} = [P_{\geq 0}(N^q), N], \) \( q \in \mathbb{N}. \) Let \( \Phi_1(x', \tau') \neq 0 \) and \( \Phi_2(x', \tau') \) be two eigenfunctions of this hierarchy. Then

\[ N = \Phi_1^{-1}(x')N\Phi_1 \]

satisfies the hierarchy

\[ N_{t_q}' = [P_{\geq 1}(N^q), N'], \]

and

\[ \Theta = \Phi_1^{-1}(x')\Phi_2(x'). \]

is an eigenfunction for (10b), i.e.,

\[ \Theta_{t_q}' = P_{\geq 1}(N^q)\Theta. \]

The higher coefficients \( u \) and \( u' \) of the operators \( N \) and \( M', \) respectively, are linked by the Miura transformation

\[ u = -v^2 - v_{x'} + \partial_{x'}^{-1}v_{x'}. \]  

**STATEMENT 4.** (Auto-Bäcklund transformation for KP hierarchy.) Let \( N = N(x', \tau') \) satisfy the hierarchy \( N_{t_q}' = [P_{\geq 0}(N^q), N], q \in \mathbb{N}. \) Let \( \Phi_1(x', \tau') \neq 0 \) and \( \Phi_2(x', \tau') \) be two eigenfunctions of this hierarchy. Then,

\[ N' = \Phi_1(x')\Phi_1^{-1}(x')N\Phi_1^{-1}(x')\Phi_1^{-1}(x') \]

also satisfies the hierarchy

\[ N_{t_q}' = [P_{\geq 0}(N^q), N'], \]

and

\[ \Phi' = \Phi_1(x')\Phi_1^{-1}(x')\Phi_2(x'). \]

is an eigenfunction for (11b), i.e.,

\[ \Phi_{t_q}' = P_{\geq 0}(N^q)\Phi'. \]

The higher coefficients \( u \) and \( u' \) of the operators \( N \) and \( N' \), respectively, satisfy the auto-Bäcklund transformation for the KP equation

\[ u + u' + \frac{1}{2}(w' - w)^2 = \partial_{x'}^{-1}(w' - w)_{x'} \]

where \( w = \partial_{x'}^{-1}u \) and \( w' = \partial_{x'}^{-1}u'. \)
3. THE GENERAL SCHEME

An essential point in the scheme of constructing multisoliton solutions of the HDE-(2+1) is the link between the HD-(2+1) and mKP hierarchies. Under the transformation (8) the higher coefficient \( \tau(x, \tau) \) of the operator \( L(x, \tau) \) is expressed in terms of eigenfunction \( \Theta_1 \) of the mKP hierarchy as follows [21]:

\[
\tau(x, \tau) = \Theta_1(x', \tau').
\] (12)

Let us rewrite this link in another way. Generalizing the form for the link between the mKP and the HD-(2+1) equations [22] to the case of the whole hierarchies, we propose the following one:

\[
x' - x + \varepsilon(x, \tau) \quad \Rightarrow \quad x' = x,
\] (13)

where \( \varepsilon(x, \tau) \) is some phase function. Then by using the representations (8), (12) one can obtain the relation between the solution and its phase function

\[
\varepsilon(x, \tau) = \frac{1}{\tau(x, \tau)} - 1
\] (14)

that coincide with the corresponding relation for the HD-(2+1) equation. Thus the proposed form (13)-(14) is valid for the transformation between the whole hierarchies as well as between the equations.

To complete the link between the hierarchies, we present the expression for the phase function's derivatives with respect to \( t_q \) in terms of mKP eigenfunction. It can be obtained by the differentiation of (13) with respect to \( t_q \) by using (8b) and (14). It reads

\[
\varepsilon_{t_q}(x, \tau) = -\frac{1}{\tau} \Theta_{1t_q}, \quad q = 2, 3, \ldots.
\] (15)

Therefore the link between the HD-(2+1) and mKP hierarchies includes the change of the independent variables (13)-(14) and the relations connecting HD-(2+1) solution and its phase function with the mKP eigenfunction (12), (15). Thus, with knowing any eigenfunction of the mKP hierarchy one can construct the solution of the HDE-(2+1).

To obtain an expression for the eigenfunction we use the auto-Bäcklund transformation (ABT) for the mKP hierarchy (9). Under this transformation the higher coefficient \( v'(x', \tau') \) of the operator \( M' \) is related to that \( v \) of the operator \( M \) by the formula

\[
v' = v + \frac{\Theta_{1x'}x'}{\Theta_{1x}}
\] (16)

that may be rewritten in the form

\[
(\log \Theta_{1x'})_{x'} = v' - v.
\] (17)

As it follows from [20] any soliton solution \( w(x', y', t') \) of the mKP equation can be represented in the form

\[
w(x', y', t') = \partial_{x'} \log W(x', y', t').
\]

Therefore soliton solutions \( v \) and \( v' \) related by (17) read as

\[
v = \partial_{x'} \log V
\]

\[
v' = \partial_{x'} \log V'.
\] (18)

After the inserting these expressions into (17) one can see that the function \( \Theta_{1x'} \) may be chosen as follows:

\[
\Theta_{1x'}(x', \tau') = \frac{V'(x', \tau')}{V(x', \tau')}
\] (19)
Reducing the obtained results only to the one equation from the HD-(2+1) hierarchy, namely to the HDE-(2+1) itself, we can formulate the following.

**Theorem 1.** The soliton solutions of the HDE-(2+1) have the implicit nature and are described in a closed form as follows:

\[ r(x, y, t) = R(x + \varepsilon(x, y, t), y, t), \]  

where the phase function \( \varepsilon(x, y, t) \) is defined implicitly as a solution of the functional equation

\[ \varepsilon(x, y, t) = E(x + \varepsilon(x, y, t), y, t). \]

The functions \( R \) and \( E \) are given explicitly and read

\[ R(x', y', t') = \frac{V'(x', y', t')}{V(x', y', t')}, \]  

\[ E(x', y', t') = \int_{-\infty}^{t'} (1 - R(x, y', t')) \, dz. \]

Here the functions \( V, V' \) are generated through (18) by the pair of the mKP soliton solutions linked by the ABT (9) and \( y' = y, t' = t \).

In Section 5, we write down the functions \( R \) and \( E \) explicitly.

**Remark.** In (1+1)-dimensional case the ABT (16) is reduced to the change of the sign of solution, that is

\[ u' = -u, \]  

such that the formula (20) for HDE-solution transforms into the following one:

\[ r(x, t) \equiv R(x', t') = \frac{1}{V^2(x', t')}. \]

### 4. AUTO-BÄCKLUND TRANSFORMATIONS FOR SOLITON-TYPE SOLUTIONS

From the previous section, it follows that in order to construct solution of the HDE-(2+1) it is necessary to know two solutions of the mKP equation linked by the ABT (9). In this section, we study the ABT for all hierarchies under consideration.

First, let us derive a relation including both the ABT for the KP hierarchy and the link between the KP and mKP hierarchies.

Under the transformation (10) the higher coefficient \( v \) of the operator \( M(x', \tau') \) is expressed in terms of the eigenfunction \( \Phi_1 \) of the KP hierarchy by the formula

\[ v = \Phi_1 x'. \]  

From the representation for \( v(x', \tau') \) (18), it follows that the eigenfunction \( \Phi_1 \) may be chosen as

\[ \Phi_1(x', \tau') = V(x', \tau'). \]

Then we consider the auto-Bäcklund transformation (11) for the KP hierarchy. Under this transformation the higher coefficients \( u \) and \( u' \) of the operators \( N \) and \( N' \), respectively, are linked by the relation

\[ u' = u + 2(\log V)_{x' \tau'}. \]
Taking into account the fact that any soliton solution \(q(x', \tau')\) of the KP equation can be represented in the form \([23]\),
\[ q = 2 \partial_x^2 \log Q \]
and applying this to the solutions \(u\) and \(u'\) in (24), i.e.,
\[ u = 2 \partial_x^2 \log U \]
\[ u' = 2 \partial_x^2 \log U' \]
one can obtain the following relation:
\[ V = \frac{U'}{U}. \quad (25) \]

The \(\tau\)-function \(U\) for \(N\)-soliton KP solution is known to be represented in the form \([16]\),
\[ U(x', \tau') = \sum_{\mu} \exp \left\{ \sum_{1 \leq i < j}^N A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i (\zeta_i - \omega_i) \right\} \quad (26) \]
where by \(\sum_{\mu}^\prime\) we denote the summation over the vectors \(\mu = (\mu_1, \mu_2, \ldots, \mu_N) \in \mathbb{Z}^N\) whose components are equal to 0 or 1,
\[ \zeta_i = (p_i + q_i) x' + (q_i^2 - p_i^2) y' + 4 (p_i^3 + q_i^3) t' + \log \frac{2p_i q_i R_i}{p_i + q_i}, \]
\[ t = 1, \ldots, N, \quad \omega_i = \log \left( \frac{-q_i}{p_i} \right), \]
p_i, q_i, R_i are arbitrary real constants and
\[ \exp A_{ij} = \frac{(p_i - p_j)(q_i - q_j)}{(p_i + q_j)(q_i + p_j)}. \quad (27) \]

The \(\tau\)-function \(V(x', \tau')\) for \(N\)-soliton mKP solution may be shown to have the form
\[ V(x', \tau') = \frac{\sum_{\mu}^\prime \exp \left\{ \sum_{1 \leq i < j}^N A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i (\zeta_i - \omega_i) \right\}}{\sum_{\mu}^\prime \exp \left\{ \sum_{1 \leq i < j}^N A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i (\zeta_i - \omega_i) \right\}}, \quad N \geq 2. \quad (28) \]

From the relation (25) and representations (26),(28) one may conclude that
\[ U'(x', \tau') = \sum_{\mu}^\prime \exp \left\{ \sum_{1 \leq i < j}^N A_{ij} \mu_i \mu_j + \sum_{i=1}^N \mu_i \zeta_i \right\}. \quad (29) \]

Since the functions \(U\), \(V\) and \(V'\) depend on their arguments \(x'\) and \(t'\) only by means of the linear combinations of the form \(\zeta_i, i = 1, \ldots, N\) it is convenient to introduce the notations
\[ \zeta = (\zeta_1, \ldots, \zeta_N), \quad \omega = (\omega_1, \ldots, \omega_N) \quad W(x', \tau') \equiv W(\zeta) \quad (30) \]
where \(W(x', \tau')\) is any function depending on its arguments \(x'\) and \(t'\) by means of their linear combinations \(\zeta_i, i = 1, \ldots, N\).

Then, from the expressions for \(U\) (26) and \(U'\) (29) one can conclude that
\[ U'(\zeta) = U(\zeta + \omega). \quad (31) \]
Thus, in order to obtain a solution of the KP equation linked with the initial one by the auto-Bäcklund transformation (11d) it is necessary to shift the variable in the latter in accordance with the formula

\[ u'(\zeta) = u(\zeta + \omega). \]  

(32)

Let us consider now the auto-Bäcklund transformation for mKP hierarchy. Analogously, the using of the relation (25) for the function \( V' \) defined by (18) gives

\[ V' = \frac{U''}{U'} \]  

(33)

where \( U'' \) is the \( \tau \)-function for the KP solution \( u'' \) linked with \( u' \) by the ABT (11d), which due to (31) reads

\[ U''(\zeta) = U'(\zeta + \omega). \]  

(34)

Now the relation (33) may be rewritten in the form

\[ V'(\zeta) = \frac{U'(\zeta + \omega)}{U(\zeta + \omega)} = V(\zeta + \omega). \]

Thus, the ABT (9d) for the mKP equation in the multisoliton case is also reduced to the shift of variable on the constant vector

\[ v'(\zeta) = v(\zeta + \omega). \]  

(35)

The case of the HDE-(2+1) can be considered in the same way. In fact, from the relation (20) one can observe that the HDE-(2+1) solution \( r'(x,\tau) \equiv R'(\zeta) \), constructed on the basis of two solutions \( v' \) and \( v'' \) linked by the ABT (9d), may be obtained from solution \( r(x,t) \equiv R(\zeta) \) by means of the shift of variable on the constant vector, that is

\[ R'(\zeta) = R(\zeta + \omega). \]  

(36)

Therefore we may formulate the following.

**Proposition.** In multisoliton case the auto-Bäcklund transformations for all KP, mKP and HD-(2+1) equations are reduced to the shift of variable

\[ \zeta \rightarrow \zeta + \omega \]

where \( \omega \) is constant vector defined as

\[ \omega = \left( \log \left( -\frac{q_1}{p_1} \right), \log \left( -\frac{q_2}{p_2} \right), \ldots, \log \left( -\frac{q_N}{p_N} \right) \right) \]

5. **N-Soliton Solution of the HDE-(2+1)**

As it has been shown in Section 3, the \( N \)-soliton solution of the HDE-(2+1) can be constructed on the basis of two \( N \)-soliton solutions of mKP equation linked by the auto-Bäcklund transformation (9d) by means of the relation

\[ R(\zeta) = \frac{V'(\zeta)}{V(\zeta)} \]  

(37)

and subsequent change of the variables (13)

\[ r(x,y,t) = R(\zeta). \]
The expression for the function $V'(\zeta)$ is given by (28), the function $V'(\zeta)$ is constructed in accordance with the Proposition and reads

$$V'(\zeta) = \frac{\sum' \mu \exp \left\{ \sum_{1 \leq i < j} N A_{ij} \mu_j + \sum_{i=1}^N \mu_i (\zeta_i + \omega_i) \right\}}{\sum' \mu \exp \left\{ \sum_{1 \leq i < j} N A_{ij} \mu_j + \sum_{i=1}^N \mu_i \zeta_i \right\}}, \quad N \geq 2. \quad (38)$$

Then we have the solution of the HDE$(2+1)$ in the following form:

$$r(x, y, t) \equiv R(\zeta) = \frac{\sum' \mu \exp \left\{ \sum_{1 \leq i < j} N A_{ij} \mu_j + \sum_{i=1}^N \mu_i (\zeta_i - \omega_i) \right\}}{\left[ \sum' \mu \exp \left\{ \sum_{1 \leq i < j} N A_{ij} \mu_j + \sum_{i=1}^N \mu_i \zeta_i \right\} \right]^2} \times \sum' \mu \exp \left\{ \sum_{1 \leq i < j} N A_{ij} \mu_j + \sum_{i=1}^N \mu_i (\zeta_i + \omega_i) \right\}, \quad N \geq 2 \quad (39)$$

where

$$\zeta_i = (p_i + q_i) x' + (q_i^2 - p_i^2) y' + 4 \left( p_i^3 + q_i^3 \right) t' + \log \frac{2p_i q_i R_i}{p_i + q_i}.$$

According to Theorem 1, the phase function $e(x, y, t)$ is defined implicitly as a solution of the functional equation (21a), where the function $E$ is calculated on use if (21b) with $R(x', y, t) = R(\zeta)$.

The obtained functions $r(x, y, t)$ and $e(x, y, t)$ describe $N$-soliton solution completely.

6. THE CASE $N = 1$. THREE-PARAMETER SOLUTIONS

One-soliton solution of the HDE$(2+1)$ is obtained from the relation (37) with $N = 1$ where

$$\zeta = (p + q) x' + \left(q^2 - p^2\right) y' + 4 \left(p^3 + q^3\right) t' + \log \frac{2pq R}{p + q},$$

$p, q$ and $R$ are three arbitrary constants and $V, V'$ are the $\tau$-functions for two one-soliton mKP solutions linked by the ABT (9d). These $\tau$-functions read as

$$V = \left(1 + e^\xi\right) \left(1 - \frac{p}{q} e^\xi\right)^{-1} = \left(1 - \alpha e^\xi\right) \left(1 + \frac{p}{q} e^\xi\right)^{-1} \quad (40)$$

$$V' = \left(1 + \frac{q}{p} e^\xi\right) \left(1 - \alpha e^\xi\right) \quad (41)$$

where

$$\alpha = \text{sgn} \left( -\frac{2pq R}{p + q} \right), \quad \text{and}$$

$$\xi = (p + q) x' + \left(q^2 - p^2\right) y' + 4 \left(p^3 + q^3\right) t' + \log \left| \frac{2pq R}{p + q} \right|. \quad (42)$$

Then the relation (37) gives the following expression for one-soliton solution of the HDE$(2+1)$:

$$r(x, y, t) \equiv R(\xi) = \left(\frac{p}{q} + \alpha e^\xi\right) \left(\frac{q}{p} + \alpha e^\xi\right) \left(1 - \alpha e^\xi\right)^{-2} \quad (43)$$

where the variable $\xi$ is rewritten in terms of the arguments $x, y$ and $t$ as follows

$$\xi = (p + q) \left(x + e(x, y, t)\right) + \left(q^2 - p^2\right) y + 4 \left(p^3 + q^3\right) t + \log \left| \frac{2pq R}{p + q} \right|. \quad (44)$$

The phase function $e(x, y, t)$ after reconstructing according to the equation (21) takes the form

$$e(x, y, t) \equiv E(\xi) = \left(\frac{1}{p} + \frac{1}{q}\right) \alpha e^\xi \left(\alpha e^\xi - 1\right)^{-1}. \quad (45)$$

Equations (43)-(45) coincide with the corresponding ones obtained by slightly different method in [22]. They describe a class of three-parametrical solutions of HDE$(2+1)$ completely.
7. ANALYTICAL STUDY OF THE SOLUTIONS

In the previous section, three-parametric solutions of equation (1) depending on their arguments by means of the linear combination of \( x, y, t \) and some phase function \( \varepsilon(x, y, t) \) were constructed. The explicit expressions for the function \( R(\xi) = r(x, y, t) \) describing the solution itself and the function \( E(\xi) \) defining its phase function \( \varepsilon(x, y, t) \) are given by (43) and (45), respectively.

It is clear that at any fixed time the obtained solutions depend on \( x \) and \( y \) through the linear combination

\[
z = (q + p) x + (q^2 - p^2) y + \xi_0.
\]

So, one may conclude that in the direction \((q + p)x + (q^2 - p^2)y = \text{const}\), the solutions are constant and consequently "nonvanishing" at the infinity. Therefore it is sufficient to investigate the profiles of solutions \( r(x, y, t), \varepsilon(x, y, t) \) along the line \( y = (q - p)x \).

Let us consider the dependence of solutions (43)-(45) on the parameters \( \alpha, p, \) and \( q \). One can see that the character of solutions crucially depends on the signs of \( \alpha \) or \( p/q \). Therefore one can distinguish the following types of solutions:

- \( \alpha = -1, q/p > 0 \),
- \( \alpha = 1, q/p > 0 \),
- \( \alpha = -1, q/p < 0 \),
- \( \alpha = 1, q/p < 0 \).

In the next section, we perform the computer analysis for all these types of solutions and present the plots of their profiles against \( z \) along the line \( y = (q - p)x \). Three-dimensional plots of these solutions are obtained by translation of the above profiles along the direction \((p + q)x + (q^2 - p^2)y = \text{const}\). In all the above cases, three-dimensional plots of the functions \( r(x, y, t) \) and \( \varepsilon(x, y, t) \) describing the solution on the plane \((x, y)\) are given.

8. NUMERICAL ANALYSIS OF THE SOLUTIONS

This section deals with the discussion of plots of the solutions of HD-(2+1) defined by (43)-(45). One can observe that the functions \( R(\xi) \) and \( E(\xi) \) are symmetrical on the parameters \( q \) and \( p \). Therefore, it is sufficient to consider the case \(|q| \leq |p|\).

(A) **The Case of \( \alpha = -1 \) and \( q/p > 0 \).**

The solutions of this type are described by the functions

\[
R(\xi) = \left( \frac{q}{p} - e^{\xi} \right) \left( \frac{p}{q} - e^{\xi} \right) (1 + e^{\xi})^{-2},
\]

\[
E(\xi) = \left( \frac{1}{q} + \frac{1}{p} \right) e^{\xi} (e^{\xi} + 1)^{-1}.
\]

One can see that solutions are nonsingular. The plots of \( a \)-type solutions' profiles with different values of the parameter \( m = q/p \) are performed in Figure 1a. At \( m = 1 \) the solution has the form which is typical for the cusp-soliton solution of the Harry Dym equation in (1+1)-dimensions (HD-(1+1)). Really, at \( q = p \) the formulae (47) and (48) are reduced to the following ones

\[
R(\zeta) = \tanh^2(\zeta),
\]

\[
E(\zeta) = \frac{1}{q} (1 + \tanh \zeta)
\]

and \( \zeta = q(x + \varepsilon(x, t)) + 4q^2 t + \zeta_0 \). They coincide with the analytical expressions for nonsingular solution of HD-(1+1) [4,6,7].

As it follows from Figure 1a with \( m \) increasing, the cusp-soliton solution turns into loop which is further inflated so that its height and width are extended simultaneously. The loop height
changing can be also observed from Figure 2. The latter represents the dependence of the functions $r$ and $\varepsilon$ values at the point $z^*$ on the parameter $m$. Here by $z^*$ we denote the abscissa of the center of symmetry of the loop. From Figure 2 one can also see that the function $r$ is two-valued and the function $\varepsilon$ is three-valued at the point $z^*$. This property essentially distinguishes the solutions of HD-(2+1) from the corresponding ones of HD-(1+1). At $m = 1$ the functions $r$ and $\varepsilon$ are single-valued at all $z$ since at this point one obtains the above-mentioned one-dimensional reduction.

The plots of the a-type solution $T$ and its phase function $E$ for $p = 1$ and $q = 0.3$ on the $(x, y)$-plane are performed in Figure 3a.

(B) The Case of $\alpha = 1$ and $q/p > 0$.

With such choice of the parameters, the solutions are described by the functions

$$R(\xi) = \left(\frac{q}{p} + e^{\xi}\right) \left(\frac{p}{q} + e^{\xi}\right) (1 - e^{\xi})^{-2},$$

$$E(\xi) = \frac{1}{q} \left(\frac{1}{p} + 1\right) e^{\xi} (e^{\xi} - 1)^{-1}. \quad (50)$$

One can see that solutions of this type are singular. They have a singularity at the point $\xi = 0$. The plots of b-type solutions' profiles with different values of the parameter $m = q/p$ are
performed in Figure 1b. It is obvious that solutions have two branches. At $m = 1$ the solution of HD-(2+1) is reduced to that of HD-(1+1). With $m$ increasing the cross-point of the branches approaches to 1, and the branches diverge. It is seen that no qualitative change has happened.

The plots of the b-type solution $r$ and its phase function $\varepsilon$ for $p = 1$ and $q = 0.3$ on the $(x, y)$-plane are performed in Figure 3b.

Remark. As it can be observed from Section 4, solutions of this type are connected with solutions of a-type by the auto-Bäcklund transformation for HD-(2+1) [21]. Its "spatial" part reads as follows:

$$dx' = \frac{r'}{r} dx + (rr')_z dy + 2 \left( r' r_x + 2 r^2 r'_x - 3 r r' \partial_x \left( \frac{1}{r} \right)_y \right)_z dt,$$

$$y' = y, \quad t' = t,$$

$$\left( \frac{r'}{r} \right)_y = (rr')_{xx}.$$

Indeed, this transformation converts the solution $\{R(\xi), E(\xi)\}$ into the solution $\{R(\varphi), E(\varphi)\}$ where $\varphi = \xi + \log q/p + i \pi$. So let us denote objects corresponding to the a-type solutions by the subscript $a$, and those corresponding to the b-type solutions by the subscript $b$. Then, it can be
written that
\[ TR_a(\xi) = R_a(\varphi), \quad TR_b(\xi) = R_b(\varphi), \]
or,
\[ TR_b(\xi) = R_b \left( \xi + \log \frac{q}{p} \right), \]
\[ TR_b(\xi) = R_a \left( \xi + \log \frac{q}{p} \right), \tag{52} \]
where the operator \( T \) transforms the solution to its auto-Bäcklund transformation. Analogous relations take place for the function \( E(\xi) \).

(C) THE CASE OF \( \alpha = -1 \) AND \( q/p < 0 \). The solutions of this type are described by the functions (47) and (48). It is seen that solutions are nonsingular. The plots of c-type solutions' profiles with different values of the parameter \( m = q/p \) are presented in Figure 1c. At \( m = 1 \) the solution is reduced to the trivial one so that it has no (1+1)-dimensional analogues. It is a new type of solution which appears only in (2+1)-dimensional case. With \( m \neq 1 \), the solution represents a smooth single-valued function. With \( m \) increasing the maximum of function \( r \) grows. Such behavior of the solutions can be also observed from Figure 2 at \( m < 0 \). For such values of the parameter \( m \) \( z^* \) is abscissa of the maximum of the function \( r \).
The plots of the c-type solution \( r \) and its phase function \( \varepsilon \) for \( p = 1 \) and \( q = -0.3 \) on the \((x, y)\)-plane are presented in Figure 3c.

**Remark.** As it can be observed from Section 4, solutions of c-type which are different by translation on the real constant phase are connected to each other by the auto-Bäcklund transformation.
for HD-(2+1) (51). If we denote objects corresponding to the c-type solutions by the subscript $c$ then we have

$$TR_c(\xi) = R_c(\varphi),$$

(53)

where $\varphi = \xi + i\pi + \log q/p$ and operator $T$ is defined in Case (B). Noticing that at $q/p < 0$ we may rewrite the definition of the phase $\varphi$ as follows:

$$\varphi = \xi + 2i\pi + \log \left| \frac{q}{p} \right|,$$

(54)
then the link (53) takes the form

$$TR_c(\xi) = R_c \left( \xi + \log \left| \frac{q}{p} \right| \right).$$

(55)

Analogous relation takes place for the function $E(\xi)$.

(D) The Case of $a = -1$ and $q/p < 0$. The solutions of this type are described by the functions (49) and (50). It is seen that solutions have a singularity at the point $\xi = 0$. The plots of d-type solutions' profiles with different values of the parameter $m = q/p$ are presented in Figure 1d. At $m = 1$ the solution is reduced to the rational one [24]. It also has no (1+1)-dimensional analogues. It is another new type of solution appearing only in (2+1)-dimensional
case. With $m \neq 1$ the solution consists of two curves. Each curve represents two-valued function. One can observe that it is not defined on the whole $z$-axis. There exists some vicinity where the solution is undefined. With $m$ increasing the above curves diverge and the vicinity of the solution nonexisting is extended.

The plots of the d-type solution $r$ and its phase function $e$ for $p = 1$ and $q = -0.3$ on the $(x, y)$-plane are performed in Figure 3d.

**Remark.** Similar to Case (C), two solutions of d-type which are different by translation on the real constant phase are connected between each other by the auto-Bäcklund transformation for HD-(2+1) (51). Let us denote objects related to the d-type solutions by the subscript $d$. Then
(d) The plots of the d-type solution $r$ and its phase function $\varepsilon$ for $p = 1$ and $q = -0.3$.

Figure 3. (cont.)

it follows that

$$TR_d(\xi) = R_d(\varphi),$$  \hspace{1cm} (56)

where $\varphi = \xi + i\pi + \log q/p$. Taking into account the observation (54), the link (56) may be rewritten in the form

$$TR_d(\xi) = R_d \left( \xi + \log \left| \frac{q}{p} \right| \right).$$  \hspace{1cm} (57)

The same relation is valid for the function $E(\xi)$. To the end, we present the results in Table 1.
Table 1. Here $\zeta = q(x + c(x, t)) + 4q^3t + \zeta_0$.

<table>
<thead>
<tr>
<th>Type</th>
<th>Parameters</th>
<th>Smoothness</th>
<th>$R(\xi)$, (1+1)-limit</th>
<th>$E(\xi)$, (1+1)-limit</th>
<th>ABT</th>
</tr>
</thead>
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<tr>
<td>a</td>
<td>$\alpha = -1$</td>
<td>nonsingular</td>
<td>$\tanh^2 \xi$</td>
<td>$\frac{1}{q}(1 + \tanh \xi)$</td>
<td>$bl_{\xi + \log q/p}$</td>
</tr>
<tr>
<td>b</td>
<td>$\alpha = 1$</td>
<td>singular</td>
<td>1</td>
<td>0</td>
<td>$al_{\xi + \log q/p}$</td>
</tr>
<tr>
<td>c</td>
<td>$\alpha = -1$</td>
<td>nonsingular</td>
<td>$\coth^2 \xi$</td>
<td>$\frac{1}{q}(1 + \coth \xi)$</td>
<td>$cl_{\xi + \log</td>
</tr>
<tr>
<td>d</td>
<td>$\alpha = 1$</td>
<td>singular</td>
<td>---</td>
<td>---</td>
<td>$dl_{\xi + \log</td>
</tr>
</tbody>
</table>

9. CONCLUSIONS

The scheme of constructing multisoliton solutions of the HDE-(2+1) was developed. The general formula for $N$-soliton solution was presented. The explicit expressions for one-soliton solutions and their phase functions were obtained. The detailed analysis of the obtained solutions was performed. It was shown that they can be divided into four different types in dependence on their parameters' values. Each type of solution was investigated analytically. Furthermore, two-dimensional plots of the profiles of solutions along the line $y = (q-p)x$ and three-dimensional plots of solutions were constructed numerically for all above types. Transformations of the solutions with the parameter changing were analyzed. It was shown that the solutions of different types are connected between each other or with themselves by the auto-Bäcklund transformation for HDE-(2+1).

The auto-Bäcklund transformations for all KP, mKP and HD-(2+1) equations was also studied in multisoliton case.

REFERENCES

22. L.A. Dmitrieva and M.A. Khlabystova, Three-parametric solutions of (2+1)-dimensional Harry Dym equation, IPRT Preprint, #34-94.