Some $L^p$ Inequalities for the Polar Derivative of a Polynomial

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Let $p_n(z)$ be a polynomial of degree $n$ and $D_n\{p_n(z)\}$ its polar derivative. It has been proved that if $p_n(z)$ has no zeros in $|z| < 1$, then for $\delta \geq 1$ and $|\alpha| \geq 1$,

$$
\left( \int_0^{2\pi} |D_n\{p_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1)F_\delta\left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
$$

where $F_\delta = (2\pi/\int_0^{2\pi} |1 + e^{i\theta}|^\delta d\theta)^{1/\delta}$. We also obtain analogous inequalities for the class of polynomials having all their zeros in $|z| \leq 1$ and for the class of polynomials satisfying $p_n(z) \equiv z^n p_n(1/2)$. © 2001 Academic Press

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1. INTRODUCTION

Let \( \mathcal{P}_n \) denote the set of polynomials (over the complex field) of degree less than or equal to \( n \). If \( p_n \in \mathcal{P}_n \), then according to a well known result known as Bernstein’s inequality (see [12])

\[
\max_{|z|=1} |p_n'(z)| \leq n \max_{|z|=1} |p_n(z)|. \tag{1}
\]

Here the equality holds if and only if \( p_n(z) \) has all its zeros at the origin. For polynomials \( p_n(z) \) not vanishing in \( |z| < 1 \), it was conjectured by Erdős and proved later by Lax [7] that (1) can be replaced by

\[
\max_{|z|=1} |p_n'(z)| \leq n \max_{|z|=1} |p_n(z)| / |\alpha| \tag{2}
\]

The above inequality is sharp with equality holding for polynomials of the form \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

Let \( p_n(z) \) be a complex number. We define \( D_{\alpha}\{p_n\} \), the polar derivative of \( p_n \), by

\[
D_{\alpha}\{p_n(z)\} = np_n(z) + (\alpha - z)p_n'(z). \tag{3}
\]

It is easy to see that \( D_{\alpha}\{p_n\} \in \mathcal{P}_{n-1} \) and that \( D_{\alpha}\{p_n\} \) generalizes the ordinary derivative in the sense that

\[
\lim_{\alpha \to \infty} \frac{D_{\alpha}\{p_n(z)\}}{\alpha} = p_n'(z) \tag{4}
\]

uniformly with respect to \( z \) for \( |z| \leq R \), \( R > 0 \).

The polynomial \( D_{\alpha}\{p_n\} \) has been called by Laguerre [6] the “émanant” of \( p_n \), by Pólya and Szegő [9] the “derivative of \( p_n \) with respect to the point \( \alpha \)” and by Marden [8] simply “the polar derivative of \( p_n \).” It is obviously of interest to obtain estimates concerning growth of \( D_{\alpha}\{p_n(z)\} \) and one such estimate is due to Aziz [1], who extended the inequality (2) due to Lax [7] for \( D_{\alpha}\{p_n\} \) by proving

**Theorem A.** If \( p_n(z) \) is a polynomial of degree \( n \) having no zeros in the disk \( |z| < 1 \), then for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \), we have

\[
\max_{|z|=1} |D_{\alpha}\{p_n(z)\}| \leq \frac{n}{2} (|\alpha| + 1) \max_{|z|=1} |p_n(z)|. \tag{5}
\]

The result is best possible and equality in (5) holds for \( p_n(z) = \lambda + \mu z^n \), where \( |\mu| = |\lambda| \) and \( \alpha \geq 1 \).

**Remark 1.1.** If we divide both sides of (5) by \( |\alpha| \) and make \( |\alpha| \to \infty \), we get inequality (2) due to Lax [7].
The following theorem, which is an \( L^p \) analogue of (2), is due to de Bruijn [2] (for an alternate proof, see Rahman [10]).

**Theorem B.** If \( p_n(z) \) is a polynomial of degree \( n \) having no zeros in \( |z| < 1 \), then for \( \delta \geq 1 \)

\[
\left( \int_0^{2\pi} |p'_n(e^{i\theta})|^{\delta} d\theta \right)^{1/\delta} \leq n C_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^{\delta} d\theta \right)^{1/\delta},
\]

where \( C_\delta = (2\pi/ \int_0^{2\pi} |1 + e^{i\theta}|^{\delta} d\theta)^{1/\delta} \). This inequality is sharp and equality holds for \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

The above result of de Bruijn was extended for \( \delta \geq 0 \) by Rahman and Schmeisser [11].

In this paper we will obtain \( L^p \) inequalities for the polar derivative of \( p_n \in \mathcal{P}_n \). As we will see our results generalize both Theorem A due to Aziz [1] and Theorem B of de Bruijn [2]. We will prove

**Theorem 1.1.** If \( p_n \in \mathcal{P}_n \) and \( p_n(z) \) has no zeros in \( |z| < 1 \), then for \( \delta \geq 1 \) and for every real or complex number \( \alpha \) with \( |\alpha| \geq 1 \),

\[
\left( \int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^{\delta} d\theta \right)^{1/\delta} \leq n (|\alpha| + 1) F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^{\delta} d\theta \right)^{1/\delta},
\]

where \( F_\delta = (2\pi/ \int_0^{2\pi} |1 + e^{i\theta}|^{\delta} d\theta)^{1/\delta} \). In the limiting case, when \( \delta \to \infty \), the above inequality is sharp and equality holds for the polynomial \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

**Theorem 1.2.** If \( p_n \in \mathcal{P}_n \) and \( p_n(z) \) has all its zeros in \( |z| \leq 1 \), then for \( \delta \geq 1 \) and for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \),

\[
\left( \int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^{\delta} d\theta \right)^{1/\delta} \leq n (|\alpha| + 1) F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^{\delta} d\theta \right)^{1/\delta},
\]

where \( F_\delta \) is the same as in Theorem 1.1. Again in the limiting case, when \( \delta \to \infty \), the above inequality (8) is sharp and equality holds for \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

If in Theorem 1.1, we make \( \delta \to \infty \), we get Theorem A, due to Aziz [1]. Further if we divide both sides of inequality (7) in Theorem 1.1 by \( |\alpha| \) and make \( |\alpha| \to \infty \), we get Theorem B.

Note that if \( p_n \in \mathcal{P}_n \), so does the polynomial \( q_n(z) = z^n p_n(1/2) \). Further, if \( p_n(z) \) has no zeros in \( |z| < 1 \) then \( q_n(z) \) has all its zeros in \( |z| \leq 1 \) and therefore applying Theorem 1.2 to \( q_n(z) \) we get
Corollary 1.1. If \( p_n \in \mathcal{P}_n \) and \( p_n(z) \) has no zeros in \( |z| < 1 \), then for \( \delta \geq 1 \) and for every real or complex number \( \alpha \) with \( |\alpha| \leq 1 \),
\[
\left( \int_0^{2\pi} |D_\alpha \{q_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1)F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
\]
where \( q_n(z) = z^n p_n(1/\sqrt{z}) \) and \( F_\delta \) is the same as in Theorem 1.1. Again in the limiting case, when \( \delta \to \infty \), the above inequality is sharp and equality holds for the polynomial \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

Note that by (3), for \( 0 \leq \theta < 2\pi \),
\[
|\left[D_\alpha \{q_n(e^{i\theta})\}\right]_{\alpha=0}| = |nq_n(e^{i\theta}) - e^{i\theta} q'_n(e^{i\theta})| = |p'_n(e^{i\theta})|.
\]

Since \( |p_n(e^{i\theta})| = |q_n(e^{i\theta})| \), \( 0 \leq \theta < 2\pi \), if we take \( \alpha = 0 \) in the above corollary we get Theorem B, which is due to de Bruijn [2]. If we make \( \delta \to \infty \), in Corollary 1.1 we get

Corollary 1.2. If \( p_n \in \mathcal{P}_n \) and \( p_n(z) \) has no zeros in \( |z| < 1 \), then for every real or complex \( \alpha \) with \( |\alpha| \leq 1 \),
\[
\max_{|z|=1} |D_\alpha \{q_n(z)\}| \leq \frac{n}{2} (|\alpha| + 1) \max_{|z|=1} |p_n(z)|,
\]
where \( q_n(z) = z^n p_n(1/\sqrt{z}) \). The result is best possible and equality holds for \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

Since \( \max_{|z|=1} |D_\alpha \{q_n(z)\}| = \max_{|z|=1} |p_n(z)| \) by (10), if we take \( \alpha = 0 \) in Corollary 1.2 we get inequality (2) which is due to Lax [7].

It may be remarked that the arguments used in the proofs of Theorems 1.1 and 1.2 also yield

Theorem 1.3. If \( p_n \in \mathcal{P}_n \) and satisfies \( p_n(z) \equiv q_n(z) \), where \( q_n(z) = z^n p_n(1/\sqrt{z}) \), then for \( \delta \geq 1 \) and for every real or complex \( \alpha \),
\[
\left( \int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1)F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
\]
where \( F_\delta \) is the same as in Theorem 1.1. In the limiting case, when \( \delta \to \infty \), the above inequality is sharp and equality holds for the polynomial \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).

If we divide both sides of inequality (12) by \( |\alpha| \) and make \( |\alpha| \to \infty \), we get the following result which is due to Dewan and Govil [3].

Corollary 1.3. If \( p_n \in \mathcal{P}_n \) and satisfies \( p_n(z) \equiv q_n(z) \), where \( q_n(z) = z^n p_n(1/\sqrt{z}) \), then for \( \delta \geq 1 \)
\[
\left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta} \leq nF_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
\]
where \( F_\delta = (2\pi/\int_0^{2\pi} [1 + e^{i\theta}]^\delta d\theta)^{1/\delta} \). The result is best possible and equality holds for the polynomial \( p_n(z) = \lambda + \mu z^n \), where \( |\lambda| = |\mu| \).
2. LEMMAS

For the proofs of our theorems we need the following lemmas.

**Lemma 2.1** [1]. If \( p_n \in \mathcal{P}_n \) and \( \alpha \) is a complex number with \( |\alpha| \geq 1 \), then for \( |z| = 1 \),

\[
|D_\alpha \{ p_n(z) \}| + |D_\alpha \{ q_n(z) \}| \leq n(|\alpha| + 1) \max_{|z|=1} |p_n(z)|, \tag{14}
\]

where \( q_n(z) = z^n \overline{p_n(1/z)} \).

The above lemma is a special case of a result due to Aziz [1, Lemma 2].

**Lemma 2.2** [1]. If \( p_n \in \mathcal{P}_n \) and \( \alpha \) is a complex number with \( |\alpha| \geq 1 \), then for \( |z| \geq 1 \),

\[
|D_\alpha \{ q_n(z) \}| \geq |D_\alpha \{ p_n(z) \}|. \tag{15}
\]

The above lemma is also due to Aziz [1, p. 190].

**Lemma 2.3** [10]. Let \( \mathcal{P}_n \) denote the linear space of polynomials \( p_n(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_n z^n \) of degree at most \( n \) with complex coefficients normed by \( \| p_n \| = \max_{0 \leq \theta < 2\pi} |p_n(e^{i\theta})| \). Define the linear functional \( L \) on \( \mathcal{P}_n \) as

\[
L : p_n \rightarrow l_0 a_0 + l_1 a_1 + \cdots + l_n a_n, \tag{16}
\]

where the \( l_j \)'s, are complex numbers. If the norm of the functional \( L \) is \( N \), then

\[
\int_0^{2\pi} \Theta \left( \frac{1}{N} \sum_{k=0}^n l_k a_k e^{ik\theta} \right) d\theta \leq \int_0^{2\pi} \Theta \left( \left| \sum_{k=0}^n a_k e^{ik\theta} \right| \right) d\theta, \tag{17}
\]

where \( \Theta(t) \) is a non-decreasing convex function of \( t \).

The above lemma is due to Rahman [10, Lemma 3].

**Lemma 2.4** [1]. Let \( p_n \in \mathcal{P}_n \). Then for every real or complex \( \alpha \) with \( |\alpha| \neq 0 \)

\[
|D_\alpha \{ q_n(e^{i\theta}) \}| = |\alpha||D_{1/\pi} \{ p_n(e^{i\theta}) \}|, \tag{18}
\]

where \( q_n(z) = z^n \overline{p_n(1/z)} \).
The above lemma is due to Aziz [1]. However, for the sake of completeness, we present brief outlines of its proof. Since \( q'_n(e^{i\theta}) = ne^{i\theta(n-1)} \frac{p_n'(e^{i\theta})}{p_n(e^{i\theta})} - e^{i\theta(n-2)} \frac{p_n'(e^{i\theta})}{p_n(e^{i\theta})} \), for \( 0 \leq \theta < 2\pi \), we get

\[
D\alpha \{ q_n(e^{i\theta}) \} = nq_n(e^{i\theta}) + (\alpha - e^{i\theta})q'_n(e^{i\theta})
\]

\[
= ne^{i\theta}p_n(e^{i\theta}) + n(\alpha - e^{i\theta})e^{i\theta(n-1)}p_n(e^{i\theta})
\]

\[
- (\alpha - e^{i\theta})e^{i\theta(n-2)}p_n'(e^{i\theta})
\]

\[
= e^{i\theta(n-1)} \left[ n\alpha p_n(e^{i\theta}) + (1 - \alpha e^{i\theta})p_n'(e^{i\theta}) \right],
\]

(19)

which gives

\[
|D\alpha \{ q_n(e^{i\theta}) \}| = |\alpha| \left| np_n(e^{i\theta}) + \frac{1}{\alpha} - e^{i\theta} \right| p_n'(e^{i\theta})
\]

\[
= |\alpha| |D_{1/\alpha} \{ p_n(e^{i\theta}) \}|.
\]

3. PROOFS OF THE THEOREMS

**Proof of Theorem 1.1.** Let \( M = \max_{0 \leq \theta < 2\pi} |p_n(e^{i\theta})| \). Then by Lemma 2.1, for every real or complex \( \alpha \),

\[
|D\alpha \{ p_n(e^{i\theta}) \}| + |D\alpha \{ q_n(e^{i\theta}) \}| \leq nM(|\alpha| + 1).
\]

(20)

Note that by Lemma 2.4,

\[
|D\alpha \{ q_n(e^{i\theta}) \}| = |\alpha| |D_{1/\alpha} \{ p_n(e^{i\theta}) \}| = |n\alpha p_n(e^{i\theta}) + (1 - \alpha e^{i\theta})p_n'(e^{i\theta})|,
\]

which is equivalent to

\[
|D\alpha \{ q_n(e^{i\theta}) \}| = |n\alpha e^{i\theta}p_n(e^{i\theta}) + (1 - \alpha e^{i\theta})e^{i\theta}p_n'(e^{i\theta})|.
\]

(21)

It follows from (20) and (21) that, for every \( \gamma \), \( 0 \leq \gamma < 2\pi \),

\[
|np_n(e^{i\theta}) + (\alpha - e^{i\theta})p_n'(e^{i\theta}) + e^{i\gamma} \{ n\alpha e^{i\theta}p_n(e^{i\theta}) + (1 - \alpha e^{i\theta})e^{i\theta}p_n'(e^{i\theta}) \}|
\]

\[
\leq nM(|\alpha| + 1),
\]

which is equivalent to

\[
|n(1 + \alpha e^{i\gamma}e^{i\theta})p_n(e^{i\theta}) + [(\alpha - e^{i\theta}) + e^{i\gamma}(1 - \alpha e^{i\theta})e^{i\theta}]p_n'(e^{i\theta})|
\]

\[
\leq nM(|\alpha| + 1).
\]

(22)

Since \( p_n'(e^{i\theta}) = -ie^{-i\theta} \frac{d}{d\theta} p_n(e^{i\theta}) \), Eq. (22) is equivalent to

\[
|n(1 + \alpha e^{i\gamma}e^{i\theta})p_n(e^{i\theta}) - i(\alpha e^{-i\theta} - 1) + e^{i\gamma}(1 - \alpha e^{i\theta})] \frac{d}{d\theta} p_n(e^{i\theta})|
\]

\[
\leq nM(|\alpha| + 1),
\]
which gives

\[ |\{n(1 + \bar{\alpha}e^{i\gamma}e^{i\theta})I - i[(\alpha e^{-i\theta} - 1) + e^{i\gamma}(1 - \bar{\alpha}e^{i\theta})]D\}p_n(e^{i\theta})| \]

\[ \leq nM(\vert \alpha \vert + 1), \quad (23) \]

where \( I(p_n(e^{i\theta})) = p_n(e^{i\theta}) \) is the identity operator and \( D(p_n(e^{i\theta})) = \frac{d}{d\theta}p_n(e^{i\theta}) \) is the differentiation operator. Since \( I \) and \( D \) are linear operators, it follows from (23) that the operator

\[ \Lambda = n(1 + \bar{\alpha}e^{i\gamma}e^{i\theta})I - i[(\alpha e^{-i\theta} - 1) + e^{i\gamma}(1 - \bar{\alpha}e^{i\theta})]D \]

(24)
is a bounded linear operator on \( \mathcal{P}_n \). In particular,

\[ L(p_n(e^{i\theta})) = [\Lambda p_n(e^{i\theta})]_{\theta=0} = n(1 + \bar{\alpha}e^{i\gamma})p_n(1) - i\left\{ (\alpha - 1) + e^{i\gamma}(1 - \bar{\alpha}) \left[ \frac{dp_n(e^{i\theta})}{d\theta} \right] \right\}_{\theta=0} \]

is a bounded linear functional which, in view of (23), has norm \( N \leq n(\vert \alpha \vert + 1) \). Therefore by Lemma 2.3, and noting that \( D_\alpha \{ p_n(e^{i\theta}) \} = np_n(e^{i\theta} - i(\alpha e^{-i\theta} - 1)\frac{d}{d\theta}p_n(e^{i\theta}) \), we get for all \( \delta \geq 1 \)

\[ \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} e^{i\gamma}n\bar{\alpha}e^{i\theta}p_n(e^{i\theta}) + (1 - \bar{\alpha}e^{i\theta})e^{i\theta}p_n'(e^{i\theta}) \right|^\delta d\theta \]

\[ \leq n^\delta(\vert \alpha \vert + 1)^\delta \int_0^{2\pi} \left| p_n(e^{i\theta}) \right|^\delta d\theta, \]

which gives

\[ \int_0^{2\pi} \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} e^{i\gamma}n\bar{\alpha}e^{i\theta}p_n(e^{i\theta}) + (1 - \bar{\alpha}e^{i\theta})e^{i\theta}p_n'(e^{i\theta}) \right|^\delta d\theta d\gamma \]

\[ \leq 2\pi n^\delta(\vert \alpha \vert + 1)^\delta \int_0^{2\pi} \left| p_n(e^{i\theta}) \right|^\delta d\theta. \quad (25) \]

Note that \( D_\alpha \{ p_n(e^{i\theta}) \} \) being a polynomial has a finite number of zeros. Besides, we can clearly interchange the order of integration. Hence,

\[ \int_0^{2\pi} \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} e^{i\gamma}n\bar{\alpha}e^{i\theta}p_n(e^{i\theta}) + (1 - \bar{\alpha}e^{i\theta})e^{i\theta}p_n'(e^{i\theta}) \right|^\delta d\theta d\gamma \]

\[ = \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} \right|^\delta \int_0^{2\pi} \left| 1 + e^{i\gamma}n\bar{\alpha}e^{i\theta}p_n(e^{i\theta}) - i(1 - \bar{\alpha}e^{i\theta})\frac{d}{d\theta}p_n(e^{i\theta}) \right|^\delta d\gamma d\theta \]

\[ = \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} \right|^\delta \int_0^{2\pi} \left| e^{i\gamma} + \frac{|D_\alpha \{ q_n(e^{i\theta}) \}|}{|D_\alpha \{ p_n(e^{i\theta}) \}|} \right|^\delta d\gamma d\theta \]

\[ \geq \int_0^{2\pi} \left| D_\alpha \{ p_n(e^{i\theta}) \} \right|^\delta d\theta \int_0^{2\pi} \left| e^{i\gamma} + 1 \right|^\delta d\gamma, \quad \text{by Lemma 2.2.} \quad (26) \]
Thus on combining (25) and (26), we get
\[
\int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^\delta d\theta \int_0^{2\pi} |e^{i\gamma} + 1|^\delta d\gamma \leq 2\pi n^\delta (|\alpha| + 1)^\delta 
\]
\[
\times \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta,
\]
from which the theorem follows.

**Proof of Theorem 1.2.** Since \( p_n(z) \) has all its zeros in \(|z| \leq 1\), the polynomial \( q_n(z) = e^\delta p_n(1/z) \) has no zeros in \(|z| < 1\) and therefore applying Theorem 1.1 to the polynomial \( q_n(z) \), we get for \(|\alpha| \geq 1\)
\[
\left( \int_0^{2\pi} |D_\alpha \{q_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1) F_\delta \left( \int_0^{2\pi} |q_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta}.
\]
If \(|\alpha| \leq 1\) then \( \frac{1}{|\alpha|} \geq 1\); hence if we apply Lemma 2.4 to (27) we get for \(|\alpha| \leq 1\)
\[
\left( \int_0^{2\pi} |D_\alpha \{q_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n \left( \frac{1}{|\alpha|} + 1 \right) F_\delta \left( \int_0^{2\pi} |q_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta},
\]
which by Lemma 2.4 is clearly equivalent to
\[
\left( \int_0^{2\pi} |D_\alpha \{q_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1) F_\delta \left( \int_0^{2\pi} |q_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta}.
\]
Since \( \int_0^{2\pi} |q_n(e^{i\theta})|^\delta d\theta = \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \), inequality (29) is clearly equivalent to (8), and the proof of Theorem 1.2 is thus complete.

**Proof of Theorem 1.3.** The proof of (12) in the case \(|\alpha| \geq 1\) follows on the same lines as that of Theorem 1.1 except that in the proof instead of applying Lemma 2.2, we use the fact that the hypothesis \( p_n(z) \equiv q_n(z) \) implies \( |D_\alpha \{q_n(e^{i\theta})\}| = |D_\alpha \{p_n(e^{i\theta})\}| \), \( 0 < \theta \leq 2\pi \). Thus if \( p_n \in \mathcal{F}_n \) and satisfies \( p_n(z) \equiv q_n(z) \), then for \( \delta \geq 1 \) and \(|\alpha| \geq 1\), we have
\[
\left( \int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n(|\alpha| + 1) F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta}.
\]
To prove (12) when \(|\alpha| \leq 1\), note that \(|\alpha| \leq 1\) implies \( \frac{1}{|\alpha|} \geq 1\), and therefore applying (30) to \( p_n(z) \) we get for \(|\alpha| \leq 1\),
\[
\left( \int_0^{2\pi} |D_\alpha \{p_n(e^{i\theta})\}|^\delta d\theta \right)^{1/\delta} \leq n \left( \frac{1}{|\alpha|} + 1 \right) F_\delta \left( \int_0^{2\pi} |p_n(e^{i\theta})|^\delta d\theta \right)^{1/\delta}
\]
which in view of Lemma 2.4 gives, for $|\alpha| \leq 1$,

$$\frac{1}{|\alpha|} \left( \int_{0}^{2\pi} |D_{\alpha} \{ q_{n}(e^{i\theta}) \} |^\delta d\theta \right)^{1/\delta} \leq \frac{1}{|\alpha| + 1} F_{\delta} \left( \int_{0}^{2\pi} |p_{n}(e^{i\theta}) |^\delta d\theta \right)^{1/\delta} .$$  \hspace{1cm} (32)

Since $q_{n}(z) \equiv p_{n}(z)$, the above inequality is clearly equivalent to (12) and the proof of Theorem 1.3 is thus complete.

**REFERENCES**