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Effects of variable viscosity and thermal conductivity on MHD flow and heat transfer of a dusty fluid

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Abstract The problem of magnetohydrodynamic flow and heat transfer of a viscous, incompressible and electrically conducting dusty fluid over an unsteady stretching sheet is analyzed numerically. The fluid viscosity and thermal conductivity are assumed to vary as an exponential function of temperature. The governing fundamental equations are approximated by a system of nonlinear ordinary differential equations using similarity transformations. The obtained similarity equations are solved numerically using RKF-45 method. Numerical computation has been carried out for horizontal velocity profiles, temperature, Nusselt number and skin friction coefficient for various values of the flow parameters that are presented for both VWT and VHF respectively. A comparison with previously published work is performed and the results are found to be in good agreement.

1. Introduction

During past few decades, considerable interest has been given to study the flow of an incompressible viscous fluid over a continuous moving solid surface which has important applications in a variety of manufacturing processes. For instance, a number of technical processes concerning polymers involve the cooling of continuous strips extruded from a die by drawing them through a cooling bath and in the process of drawing, these strips are subsequently stretched to achieve the desired thickness. The quality of final product depends on the rate of heat transfer and therefore cooling procedure has to be controlled effectively. The MHD flow in electrically conducting fluid can control the rate of cooling and the desired quality of product can be achieved [1]. The steady flow on a moving continuous flat surface was first considered by Sakiadis [2] who has developed a numerical solution using a similarity transformation. Chiam [3] reported solutions for steady hydro-magnetic flow over a surface stretching with a power law velocity with the distance along the surface. Tsou et al. [4] studied a wide ranging analytical and experimental investigation of the flow and heat transfer characteristics of the boundary layer on a continuous moving surface. The two-dimensional flow
caused solely by a linearly stretching sheet in an incompressible fluid which has a very simple closed from exponential solution and was established by Crane [5]. The temperature field in the flow over stretching surface subject to a uniform heat flux was studied by Grubka and Bobba [6], while Elbashbashy et al. [7,8] considered the case of stretching surface with a variable surface heat flux. Further they have presented similarity solutions of the boundary layer equations, which describe the unsteady flow and heat transfer over a stretching sheet. Sharidan et al. [9] investigated an unsteady flow and heat transfer of a viscous and incompressible fluid over a stretching sheet.

All the abovementioned studies continued their discussions by assuming the physical properties of the ambient fluid were to be constants. However, it is known that the physical properties of fluid may change significantly with temperature, especially for fluid viscosity (see [10–13]). The increase of temperature leads to the increase in the transport phenomena by reducing the viscosity across the momentum boundary layer and due to which the heat transfer rate at the wall is also affected. Therefore, to predict the flow and heat transfer rates it is necessary to take into account the temperature dependent viscosity of the fluid. For lubricating fluids heat generated by internal friction and the corresponding rise in the temperature affects the viscosity of the fluid and so that the fluid viscosity no longer be assumed constant. McCormack and Crane [14] have studied and gave comprehensive discussion on boundary layer flow and due to which the heat transfer rate at the wall is also increased. Consideration of temperature-dependent viscosity, thermal conductivity and magnetic parameter, variable fluid viscosity on free convective and heat transfer of a viscous and incompressible fluid over a stretching sheet.

Recently Gireesha et al. [20,21] have studied hydromagnetic flow of a dusty fluid over a stretching sheet. The above literature describes the boundary layer flow of fluid phase only. Vajravelu and Nayfeh [19] discussed hydro-magnetic flow of a dusty fluid over a stretching sheet. Recently Gireesha et al. [20,21] have studied hydromagnetic boundary layer flow and heat transfer of dusty fluid over a stretching sheet for both steady and unsteady flow.

In view of these, the present problem is concentrated on the fluid properties which depend on high temperature. Thus the main motivation of the present paper was to study the effects of variable fluid viscosity and thermal conductivity on MHD boundary layer flow and heat transfer of dusty fluid past a stretching sheet. The fluid is assumed to be viscous and incompressible. Consideration of temperature-dependent viscosity, thermal conductivity and magnetic parameter yields a highly non-linear coupled system of partial differential equations. The coupled non-linear partial differential equations governing the problem are reduced to a system of coupled highly non-linear higher-order ordinary differential equations by applying suitable similarity transformations. The system of higher order ordinary differential equations is then solved by employing Runge–Kutta–Fehlberg-45 method. Attempts have been made to investigate the effect of variable fluid viscosity, magnetic parameter, variable thermal conductivity parameter, fluid interaction parameter, Prandtl number and Eckert number on the flow behavior and heat transfer process.

2. Mathematical formulation of the problem

Consider an unsteady boundary layer flow and heat transfer of a viscous incompressible and electrically conducting dusty fluid past a stretching surface coinciding with the plane \( y = 0 \) and the flow being confined to \( y > 0 \). The \( x \)-axis is taken in the direction of the main flow along the sheet and \( y \)-axis is normal to it. Keeping the origin fixed, two equal and opposite forces are applied along the in the \( x \)-axis which results in stretching of the sheet and hence, the flow is generated as shown in Fig. 1. The continuous sheet moves in its own plane with velocity \( u = \frac{v}{C_0} \) and the temperature distribution varies along both the sheet and time. The fluid moves in the \( x \)-direction with a velocity (\( u \)-component) equal to the velocity of the solid surface and the velocity of the fluid in the \( x \)-direction approaches to zero asymptotically. The uniform transverse magnetic field \( B_0 \) is imposed along the positive \( y \)-axis. The magnetic Reynolds number is assumed to be small so that the induced magnetic field is neglected in comparison with the applied magnetic field. Both the fluid and dust particle clouds are supposed to be static at the beginning and the dust particles are assumed to be spherical in shape and uniform in size. The number density of these is taken to be a constant throughout the flow. Under the foregoing assumptions and invoking the usual boundary layer approximation, the governing equations for momentum and heat transfer of dusty fluid in the presence of variable fluid properties (fluid viscosity and thermal conductivity) take the following form:

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (2.1)
\]

\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{1}{\rho} \frac{\partial}{\partial y} \left( \mu \frac{\partial u}{\partial y} \right) + \frac{k}{\rho} (u_p - u) - \frac{\sigma B_0^2}{\rho} u, \quad (2.2)
\]

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \frac{k}{m} (u - u_p), \quad (2.3)
\]

\[
\frac{\partial u_p}{\partial t} + u \frac{\partial u_p}{\partial x} + v \frac{\partial u_p}{\partial y} = \frac{k}{m} (v - v_p), \quad (2.4)
\]

\[
\frac{\partial (\rho_p u_p)}{\partial x} + \frac{\partial (\rho_p v_p)}{\partial y} = 0, \quad (2.5)
\]

We have the following nomenclature:

\((u, v)\) and \((u_p, v_p)\) denote the velocity components of the fluid and particle phase along the \( x \) and \( y \)-axis, respectively.

![Figure 1 Schematic diagram of the flow.](image)
Further, \( \rho, \; \vec{B}_s, \; \rho_p, \; k, \; N \) and \( m \) are density of the fluid, induced magnetic field, density of particle phase, Stokes constant, number of dust particles and mass of dust particles respectively, and \( \mu \) is the variable dynamic coefficient of viscosity. In deriving these equations, the Stokesian drag force is considered for the interaction between the fluid and particle phase and the induced magnetic field is neglected. It is also assumed that the external electric field is zero and the electric field due to polarization of charges is negligible.

The boundary conditions applicable to the above problem are

\[
u = u_s(x, t) \quad v = v_s(x, t) \quad \text{at} \quad y = 0
\]

\[u \rightarrow 0, \; v_p \rightarrow 0, \; v_p \rightarrow v, \; \rho_p \rightarrow \rho, \; \text{as} \; y \rightarrow \infty, \quad (0 < E < 1)
\]

(2.6)

where \( u_s(x, t) = \frac{cx}{1 - xt} \) is velocity of sheet, \( v_s(x, t) = -\frac{c}{\sqrt{1 - xt}} \) is suction velocity, \( v_0 \) is prescribed constant, \( E \) is density ratio and \( c \) is stretching rate being a positive constant, \( x \) is positive constant which measures the unsteadiness.

Eqs. (2.1)-(2.5) subjected to boundary conditions (2.6), admit self-similar solution in terms of the similarity function \( f \) and the similarity variable \( \eta \) as

\[u = \frac{cx}{1 - 2t} f'\left(\eta \right), \quad v = -\sqrt{\frac{c}{1 - 2t}} f(\eta), \quad v_p = \frac{cx}{1 - 2t} F(\eta),
\]

\[v_p = \sqrt{\frac{c}{1 - 2t}} G(\eta), \quad \eta = \sqrt{\frac{c}{1 - 2t}} \gamma, \quad \rho_p = H(\eta),
\]

\[B_0 = B_0 \left(1 - xt\right)^{-\frac{1}{2}}, \quad v = \frac{c_0}{\rho},
\]

(2.7)

where prime denotes the differentiation with respect to \( \eta \).

The variation of the viscosity with temperature is assumed as

\[\mu = \mu_0 e^{-\beta_p \theta(\eta)}
\]

(2.8)

where \( \mu_0 \) is the viscosity at temperature \( T_p \) and \( \beta_p \) is variable viscosity parameter. In general \( \beta_p > 0 \) for liquids and \( \beta_p < 0 \) for gases.

Substituting set of Eqs. (2.7) and (2.8) into the Eqs. (2.1)-(2.5) and equating the co-efficient of \( t_0 \) on both sides of the equations, one can get

\[f''(\eta) + f'(\eta) [f'(\eta)]^\theta = \beta_p \theta(\eta) - A \theta(\eta)
\]

(2.9)

\[\left[f''(\eta) + \theta f'(\eta)ight]^\theta - \theta f(\eta)^2 + \beta \theta(\eta) = 0
\]

(2.10)

\[A f'(\eta) + f(\eta) + G(\eta) + F(\eta) + f'(\eta) = 0
\]

(2.11)

\[H(\eta) G(\eta) + H(\eta) = G(\eta) H(\eta) = 0
\]

(2.12)

where \( \rho_p = \frac{c}{\sqrt{1 - 2t}} \) is relative density, \( A = \frac{2}{\alpha} \) is an unsteady parameter which measures unsteadiness, \( l = mn/\rho_p \) is mass concentration, \( M = \sigma B_0^2/\rho c \) is magnetic parameter and \( \beta = \frac{1}{\alpha} \) is fluid particle interaction parameter.

The corresponding boundary conditions are transformed to

\[f'(\eta) = 1, \quad f(\eta) = R \quad \text{at} \quad \eta = 0,
\]

\[f'(\eta) = 0, \quad F(\eta) = 0, \quad G(\eta) = -f(\eta), \quad H(\eta) = E \quad \text{as} \quad \eta \rightarrow \infty.
\]

(2.13)

where \( R = \frac{c_0}{\sqrt{1 - 2t}} \), is suction parameter.

### 3. Heat transfer analysis

The governing two dimensional boundary layer equations of energy with variable thermal conductivity are given by

\[\rho c_0 \left[\frac{\partial T}{\partial t} + \frac{\partial T}{\partial x} v + \frac{\partial T}{\partial y} v_p \right] = \frac{\partial}{\partial x} \left[ \eta^2 \frac{\partial T}{\partial x} \right] + \frac{\partial}{\partial y} \left[ \eta^2 \frac{\partial T}{\partial y} \right] - \frac{\eta^2}{\eta^2} \left[ \eta^2 \frac{\partial T}{\partial y} \right]
\]

(3.1)

\[N_{Ec} \left[ \frac{\partial T_p}{\partial t} + u_p \frac{\partial T_p}{\partial x} + v_p \frac{\partial T_p}{\partial y} \right] = N_{Ec} \left[ \frac{\partial T}{\partial t} - T \right]
\]

(3.2)

where \( T \) and \( T_p \) are the temperature of the fluid and dust particle respectively, \( c_p \) and \( c_m \) are the specific heat of fluid and dust particles respectively, \( \tau \) is the thermal equilibrium time i.e., time taken by the dust cloud to adjust its temperature to that of fluid, \( \tau \) is the relaxation time of the dust particle i.e., the time taken by the dust particle to adjust its velocity to that of fluid. The variation of thermal diffusivity with the dimensionless temperature \( \kappa^* \) is assumed to vary as a linear function of temperature in the form

\[\kappa^* = \kappa^*_0 (1 + \beta_2 \theta(\eta)),
\]

(3.3)

where \( \beta_2 \) is a parameter which depends on the nature of the fluid and \( \kappa^*_0 \) is the value of thermal diffusivity at the temperature \( T_p \).

The solution of Eqs. (3.1) and (3.2) depends on the nature of the prescribed boundary condition. Here, the two types of heating processes are discussed.

#### 3.1. Case-1: variable wall temperature (VWT-Case)

For this heating process, the variable wall temperature is assumed to be a quadratic function of \( x \) and is given by

\[T = T_w = T_{in} + T_0 \left(\frac{x^2}{b^2}\right) \left(1 - xt\right)^{-2} \quad \text{at} \quad y = 0,
\]

\[T \rightarrow T_{in}, \quad T_p \rightarrow T_{in} \quad \text{as} \quad y \rightarrow \infty.
\]

(3.4)

In order to obtain similarity solution for the temperatures \( \theta(\eta) \) and \( \theta_p(\eta) \) define dimensionless temperature variables as

\[\theta(\eta) = \frac{T - T_{in}}{T_s - T_{in}}, \quad \theta_p(\eta) = \frac{T_p - T_{in}}{T_s - T_{in}},
\]

(3.5)

where \( T_{in} = T_0 \left(\frac{x^2}{b^2}\right) \left(1 - xt\right)^{-2} \theta(\eta) \).

Using Eqs. (3.3)-(3.5) in the Eqs. (3.1) and (3.2), we get

\[\theta(\eta) + \left[ \frac{Pr \theta(\eta) + \beta_p \theta(\eta)}{1 + \beta_2 \theta(\eta)} \right] \theta(\eta) = \frac{2Pr \theta(\eta) \theta(\eta)}{1 + \beta_2 \theta(\eta)}
\]

(3.6)

\[\left[ \frac{Pr \theta(\eta) + \beta_p \theta(\eta)}{1 + \beta_2 \theta(\eta)} \right] \theta(\eta) = \frac{N \theta(\eta)}{1 + \beta_2 \theta(\eta)}
\]

(3.7)

where \( Pr = \frac{v^2}{\kappa} \) is Prandtl number, \( Ec = \frac{v^2}{\kappa} \) is the Eckert number, \( a_1 = \frac{1}{\sigma} \) is local fluid particle interaction parameters for temperature \( \gamma = \frac{a_1}{\sigma} \).

Using the Eqs. (3.4) and (3.5), the corresponding boundary conditions for \( \theta(\eta) \) and \( \theta_p(\eta) \) reduce to the following form:
In order to obtain similarity solution for temperature, define the dimensionless temperature variables in VHF case as same in Eq. (3.5) whereas $T_w - T_\infty = \frac{q_w}{k} \left( \frac{x}{y^*} \right)^{1/2} \theta(\eta)$.

Table 1 Comparison of numerical solutions of $f(\eta)$, $f'(\eta)$, $f''(\eta)$ with $A = 0$, $M = 3$, $R = 2$, $\beta = 0$, $\beta_1 = 0$.

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<th>$f'(\eta)$</th>
<th>$f''(\eta)$</th>
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Table 2 Skin friction coefficient $e^{-\beta}f''(0)$, wall temperature gradient $\theta'(\eta)$ and temperature function $\theta(\eta)$ for different values $A$, $M$, $\beta_1$, and $\beta_2$.

For this heating process, employ the following variable heat flux boundary conditions:

\[
\begin{align*}
\frac{\partial T}{\partial y} &= \frac{q_w(x,t)}{k^2}, & \text{at} \ y = 0, \\
T &\to T_\infty, \quad T_p &\to T_\infty & \text{as} \ y \to \infty,
\end{align*}
\]

where $q_w(x,t) = q_w(x)\left(1 - \frac{x}{y^*}\right)^{1/2}$.

In order to obtain similarity solution for temperature, define the dimensionless temperature variables in VHF case as same in Eq. (3.5) whereas $T_w - T_\infty = \frac{q_w}{k} \left( \frac{x}{y^*} \right)^{1/2} \theta(\eta)$.

\[
\theta(\eta) = 1 \quad \text{at} \ \eta = 0
\]

\[
\theta(\eta) \to 0, \quad \theta_p(\eta) \to 0 \quad \text{as} \ \eta \to \infty
\]

3.2. Case-2: variable heat flux (VHF-Case)

\[\begin{align*}
\text{Figure 2} \quad \text{Effect of unsteady parameter } A \text{ and Magnetic parameter } M \text{ for fluid and dust phase velocity.}
\end{align*}\]
\[ \theta''(\eta) + \frac{\left[ Pr\theta(\eta) + \beta_2 \theta'(\eta) \right]}{1 + \beta_2 \theta(\eta)} \theta'(\eta) = -2 \frac{Pr \theta(\eta) \theta'(\eta)}{1 + \beta_2 \theta(\eta)} \\
- \frac{A Pr}{2} \left[ \frac{1}{1 + \beta_2 \theta(\eta)} \right] + \frac{N}{\rho} \left[ \frac{Pr \theta(\eta)}{1 + \beta_2 \theta(\eta)} \theta'(\eta) \right] \left[ \theta_\infty(\eta) - \theta(\eta) \right] \\
+ \frac{N}{\rho} \left[ \frac{Pr \nabla c \beta}{1 + \beta_2 \theta(\eta)} \right] \left[ F(\eta) - f(\eta) \right]^2 = 0. \tag{3.10} \]

\[ G(\eta) \theta'_\infty(\eta) + 2F(\eta) \theta'_\infty(\eta) = \frac{A}{2} \left[ \theta_\infty(\eta) + \eta \theta'(\eta) \right] + \alpha_f [ \theta_\infty(\eta) - \theta(\eta) ] = 0. \tag{3.11} \]

The corresponding boundary conditions becomes
\[ \theta(\eta) = -1 \text{ at } \eta = 0, \tag{3.12} \]
\[ \theta(\eta) \to 0, \quad \theta'_\infty(\eta) \to 0 \text{ as } \eta \to \infty. \tag{3.13} \]

The physical quantities of interest in this problem are the skin friction coefficient \( C_f \) and the local Nusselt number \( Nu_x \), which are defined as
\[ C_f = \frac{\tau_w}{\rho U_w}, \quad Nu_x = \frac{\nu_w}{k(T_w - T_\infty)}. \tag{3.14} \]

where the skin friction \( \tau_w \) and the heat transfer from the sheet \( q_w \) are given by
\[ \tau_w = \mu \left( \frac{\partial \theta}{\partial y} \right)_{y=0}, \quad q_w = -k \left( \frac{\partial T}{\partial y} \right)_{y=0}. \tag{3.15} \]

Using the non-dimensional variables, we obtain
\[ C_f Re_s^{1/2} = e^{-h} f''(0), \quad Nu_x / Re_s^{1/2} = -\theta'(0) (\text{VWT}), \quad Nu_x / Re_s^{1/2} = -\frac{1}{\theta(0)} (\text{VHF}). \]

4. Numerical solution

The governing partial differential Eqs. (2.1)–(2.5) of continuity and momentum are reduced into non-linear ordinary differential equations using the set of similarity variable (2.7). The obtained equations constitute a highly non-linear coupled boundary value problem of third, first and second order, respectively and are solved numerically by using well known RKF-45 method with the help of symbolic software Maple [22]. To assess the validity and accuracy of the present numerical scheme, computed values of \( f(\eta), f'(\eta), f''(\eta) \) for \( (A = 0, M = 3, R = 2, \beta_1 = 0) \) are compared with those reported by Vajravelu and Nayfeh [19] and are shown in Table 1. It is observed that a very good agreement has been achieved with their results.
5. Results and discussion

By applying similarity analysis to the governing equations and the boundary conditions, the two independent variables are reduced by one; consequently, the governing equations reduce to a system of non-linear ordinary differential equations with the appropriate boundary conditions. The obtained non-linear ordinary differential equations possess no closed form solutions. Therefore, a numerical solution of the problem under consideration is needed. Finally, the system of similarity Eqs. (2.9)–(2.12), and either Eqs. (3.6) and (3.7) for VWT case or (3.10) and (3.11) for VHF case are solved numerically. The numerical solutions are obtained for various values of the parameters such as unsteadiness parameter $A$, magnetic parameter $M$, and variable viscosity parameter $b_1$. For positive values of $b_1$, the viscosity of the fluid decreases with an increase in the temperature and this is the case for fluids such as water and lubrication oils, while for negative values of $b_1$, the viscosity of the fluid increases with an increase in the temperature. On the other hand, the positive values of $b_2$ mean that the thermal diffusivity increases with an increase in temperature and this is the case for fluids such as water and air, whereas increasing effect of thermal diffusivity parameter $b_2$ is to increase the skin friction coefficient. Nusselt number decreases with increase in unsteady parameter $A$, whereas it increases with increase in the magnetic parameter $M$ for both VWT and VHF cases. The effect of variable viscosity parameter and thermal diffusivity parameter is to increase the Nusselt number in both the cases.

It is observed from Table 2 that the skin friction coefficient decreases with increasing values of unsteady parameter $A$, magnetic parameter $M$, and variable viscosity parameter $b_1$. Here the negative value of $e^{-b_1}f'(0)$ means the solid surface exerts a drag force on the fluid. This is due to the development of the velocity boundary layer is caused solely on the stretching plate, whereas increasing effect of thermal diffusivity parameter $b_2$ is to increase the skin friction coefficient. Nusselt number decreases with increase in unsteady parameter $A$, whereas it increases with increase in the magnetic parameter $M$ for both VWT and VHF cases. The effect of variable viscosity parameter and thermal diffusivity parameter is to increase the Nusselt number in both the cases.

Figure 5 Effect of fluid interaction parameter $\beta$ on temperature distribution for fluid and dust phase.

Figure 6 Effect of unsteady parameter $A$ on temperature distribution for fluid and dust phase.
Fig. 2(a) represents horizontal velocity profile of both fluid and dust particles for various values of $A$ when $Pr = 0.72$, $R = 2$, $M = 0.5$, $N = 0.2$, and $\beta = 0.5$. From this one can observe that the velocity decreases with the increase of the unsteady parameter $A$. It is interesting to note that the thickness of boundary decreases with increasing values of $A$. This is due to the fluid flow caused solely by the stretching sheet.

From Fig. 2(b) it is observed that the velocity decreases with increase of magnetic parameter $M$. This is because of the fact that the introduction of transverse magnetic field to an electrically conducting fluid gives rise to a resistive type of force, known as Lorentz force. This force has tendency to slow down the motion of fluid in the boundary layer and hence it leads to enhanced deceleration of the flow.

The effect of fluid viscosity parameter $\beta_1$ on horizontal velocity profiles $f(\eta)$ verses $\eta$ is depicted in Fig. 3(a). It is observed that the horizontal velocity profiles $f(\eta)$ decrease with an increasing values of fluid viscosity parameter $\beta_1$. This is because of the fact that with an increase in the value of fluid viscosity parameter $\beta_1$ decreases the horizontal velocity boundary layer thickness. Physically, this is because a given larger $\beta_1$ implies higher temperature difference between the surface and the ambient fluid. The effect of variable viscosity parameter on temperature is depicted in Fig. 3(b), and the increase in the value of $\beta_1$ has tendency to increase the thermal boundary layer thickness. This causes to increase the values of $\theta(\eta)$; thus, increase in the $\beta_1$ increases the temperature of the fluid and dust phase.

Fig. 4(a) and (b) shows the effect of fluid particle interaction parameter $\beta$ on velocity components of the fluid phase velocity $f(\eta)$ and dust phase velocity $F(\eta)$ for both VWT and VHF cases. It is observed that if $\beta$ increases we can find the decrease in the fluid phase velocity and increase in the dust phase velocity. Also it reveals that for large values of $\beta$, the relaxation time of the dust particle decreases then the velocities of both fluid and dust particle will be the same.

Fig. 5(a) and (b) depicts the effect of fluid particle interaction parameter $\beta$ on temperature profile for both VWT and VHF cases. The temperature of the fluid and dust phase increases with the increase in the value of $\beta$ and it indicates that the fluid particles temperature is parallel to the dust particles temperature. Also one can observe that fluid phase temperature is higher than that of dust phase.

Fig. 6(a) and (b) is the graphical representation for temperature distribution of VWT and VHF cases, for different values of unsteady parameter $A$. It is evident that temperature of fluid and dust particle is found to be decrease with increase of

Figure 7 Effect of Prandtl number $Pr$ on temperature distribution for fluid and dust phase.

Figure 8 Effect of Eckert number $Ec$ on temperature distribution for fluid and dust phase.
Temperature at a point of surface decreases significantly with the increase of unsteady parameter $A$, i.e., rate of heat transfer increases with increasing values of unsteady parameter $A$. Physically, it means that the temperature gradient at the surface increases as $A$ increases, which implies the increase of heat transfer rate $h_0(0)$. We have used the value $a_1 = 2$ throughout our thermal analysis.

Fig. 7(a) and (b) depicts temperature profiles of $\theta(\eta)$ and $\theta_p(\eta)$ for different values of $Pr$. We infer from these figures that temperature of fluid and dust particles decreases with the increase in $Pr$, and this is because of the increase in Prandtl number $Pr$ indicates the increase of the fluid heat capacity or the decrease of the thermal diffusivity, hence causes a diminution of the influence of the thermal expansion to the flow, physically which means that the momentum boundary layer is thicker than the thermal boundary layer. The temperature in both VWT and VHF cases asymptotically approaches to zero in the free stream region.

Fig. 8(a) and (b) indicates the temperature profile of $\theta(\eta)$ and $\theta_p(\eta)$ for VWT and VHF cases respectively. Here the effect of increase in the Eckert number values greatly affects the temperature of the fluid and dust phase. The temperature of the fluid and dust phase increases as the Eckert number $Ec$ increases for both the cases VWT and VHF. This is due to fact that the heat energy is stored in the liquid due to frictional heating.

Fig. 9(a) and (b) shows the temperature profiles of $\theta(\eta)$ and $\theta_p(\eta)$ for different values of Number density $N$ for VWT and VHF cases respectively. From these figures, it is observed that the temperature of fluid and dust phases decreases with the increase of $N$.

Fig. 10(a) and (b) shows that the variation of thermal conductivity affects temperature profiles. The increase in the value of $\beta_2$ increases the temperature of fluid and dust phase; hence, the variation of thermal conductivity cannot be neglected. Moreover, the rise in the magnitude of the temperature is quite significant, showing that the volume rate of flow at a section perpendicular to the plate increases with the increase in $\beta_2$.

Fig. 11(a) and (b) reveals the effect of magnetic parameter $M$ and variable viscosity parameter $\beta_1$ on transverse velocity $G(\eta)$. From these figures, it is noticed that the increase in magnetic parameter and viscosity parameter decreases the velocity profile $G(\eta)$. Since the numerical values of the transverse velocity at different values of parameters are negative, the arrow mark is shown upwards in the graph.

Fig. 12(a) and (b) displays the effect of magnetic parameter $M$ and variable viscosity parameter $\beta_1$ on particle density $N$.
These figures infer that particle density $H(\eta)$ increases as magnetic and variable viscosity parameters increase, i.e. the particle density is maximum at the surface of the stretching sheet.

6. Conclusions

The effect of variable viscosity and thermal diffusivity on MHD boundary layer flow of dusty fluid flow over a stretching sheet has been investigated. The highly non-linear momentum Eqs. (2.1)–(2.5) and heat transfer boundary layer Eqs. (3.1) and (3.2) are reduced into coupled ordinary differential equations using similarity transformations. Resultant coupled ordinary differential Eqs. (2.9)–(2.12) and (3.6) and (3.7) for VWT case and (3.10) and (3.11) for VHF case have been solved numerically by RKF45 method [22]. The results are analyzed for the situation when stretching boundary is prescribed by non-isothermal variable wall temperature (VWT) and variable heat flux (VHF) which varies quadratically with the flow directional coordinate $x$. The effect of various physical parameters such as unsteady parameter $A$, Prandtl number $Pr$, Eckert number $Ec$, magnetic parameter $M$, variable viscosity parameter $\beta_1$ and thermal diffusivity parameter $\beta_2$ on heat transfer characteristics is discussed for different numerical values with the help of Table 3. Some of the interesting observations of the study are listed as follows:

- Increase in unsteady parameter and magnetic parameter, decreases the velocity components of both fluid and dust phases.
- The increasing effect of fluid interaction parameter decreases the fluid phase velocity and increases the dust phase velocity.
- Increase in the variable viscosity parameter decreases the fluid and dust phase velocities whereas it increases the temperature profiles of both the phases.
- Effect of unsteady parameter decreases the temperature profiles of fluid and dust phase for both the cases of VWT and VHF.
- Effect of thermal diffusivity parameter increases the temperature profiles of fluid and dust phase for both the cases of VWT and VHF.
- Always the rate of heat transfer $\theta'(0)$ is negative and $\theta(0)$ is positive.
- $\theta'(0)$ and $\theta(0)$ increase with the increasing values of fluid interaction parameter, variable viscosity parameter, thermal diffusivity parameter and Eckert number.

Figure 11  Effect of Magnetic parameter $M$ on $H(\eta)$ and $G(\eta)$.

Figure 12  Effect of variable viscosity parameter $\beta_1$ on $H(\eta)$ and $G(\eta)$.
Table 3 Values of wall temperature gradient $\theta'(0)$ for (VWT Case) and wall temperature function $\theta(0)$ for different values of $\beta$, $A$, $Pr$, $Ec$, $N$, $\beta_1$ and $\beta_2$

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- The variable thermal conductivity has an impact in enhancing the skin friction coefficient; hence, fluids with less thermal conductivity may be opted for effective cooling.
- The increase in Prandt number decreases the thermal boundary layer thickness.
- If $M = 3$, $R = 2$, $\beta = 0$, $\beta_1 = 0$ and $A \rightarrow 0$, then our results coincide with the results reported by Vajravelu and Nayfeh [19].

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References


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