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On Strong Mori domains

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Abstract

In this note we investigate properties of Strong Mori domains, which form a proper subclass of Mori domains. In particular, we show that Strong Mori domains satisfy the Principal Ideal Theorem, the Hilbert Basis Theorem and the Krull Intersection Theorem. We also provide some new characterizations of Krull domains and show that the complete integral closure of a Strong Mori domain is a Krull domain. © 1999 Elsevier Science B.V. All rights reserved.

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0. Introduction

The subject of Mori domains continues to generate considerable interest. Mori domains are known to possess many properties regarding v-ideals that one finds in a Noetherian domain, regarding all ideals. However, some rather desirable properties do not carry over; most notably Mori domains need not permit primary decomposition of v-ideals and need not satisfy the Principal Ideal Theorem (PIT).

In this paper we continue our investigation of Strong Mori (SM) domains, which form a proper subclass of Mori domains. In [17] we showed that every w-ideal of an SM domain has a primary decomposition consisting of w-ideals. In Section 1 of this paper we prove not only that SM domains satisfy the PIT, but also satisfy the Hilbert Basis Theorem and the Krull Intersection Theorem for finite-type w-modules.

In Section 2 we discuss w-multiplication domains and provide some new characterizations of Krull domains. Finally, in Section 3 we define the w-dimension of a domain R, and show that the complete integral closure of an SM domain is a Krull domain.

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Throughout this paper R denotes an integral domain with field of quotients K, and M represents an R-module. An ideal J of R is called a *GV-ideal* (denoted $J \in GV(R)$) if J is finitely generated and $J^{-1} = R$. A torsion-free module M is called a *w-module* if whenever $Jx \subseteq M$ ($J \in GV(R)$ and $x \in M \otimes K$), then $x \in M$. An ideal I of R is a *w-ideal* if I is a w-module over R. The *w-envelope* of the torsion-free module M is given by $M_w = \{x \in M \otimes K : Jx \subseteq M$ for some $J \in GV(R)\}$, and is the minimal w-module containing M. A w-module M is said to be of *finite type* if $M = B_w$ for some finitely generated submodule B of M. A *Strong Mori module* (SM module) is a w-module which satisfies the ascending chain condition on w-submodules. Finally, R is an *SM domain* if R is an SM module over R.

1. SM Domains

For the convenience of the reader we begin by recording four facts previously established in [17].

Proposition 1.1. *Let P be a primary ideal of R. Then P is a w-ideal if and only if $P_w \neq R$.*

Not every primary ideal (nor for that matter, every prime ideal) is a w-ideal. For example, consider $R = \mathbb{Z}[x]$ and $P = R2 + Rx$. Observe that $P_w = R$.

Recall that a proper submodule A of M is called *P-primary* if $P = \sqrt{(A:M)}$ is a prime ideal and $rx \in A$ ($r \in R$, $x \in M$) implies $r \in P$ or $x \in A$. If in addition $P = (A:M)$, we say that A is *P-prime*.

Proposition 1.2. *Let M be a w-module and let A be a primary submodule of M. Then A is a w-module if and only if $(A:M)_w \neq R$. Hence if $(A:M)_w \neq R$, then $(A:M)$ is a w-ideal.*

Proposition 1.3. *If I is any ideal of R and M is torsion-free, then $(IM)_w = (I_w M_w)_w$. Consequently, for each positive integer n, $(I^n)_w = ((I_w)^n)_w$.*

Proposition 1.4. *Let I be a proper w-ideal of R and let M be a finite-type w-module. If $(IM)_w = M$, then $M = 0$.*

In [17] we showed that for submodules of a torsion-free module, the w-envelope of the intersection is the intersection of the w-envelopes. We now demonstrate a somewhat similar statement regarding sums.

Lemma 1.5. *Let M be torsion-free and let A, B be submodules of M. Then $(A+B)_w = (A_w + B_w)_w$.*

Proof. Clearly $(A+B)_w \subseteq (A_w+B_w)_w$. It suffices then to show that $A_w+B_w \subseteq (A+B)_w$. If $x=u+v$, where $u \in A_w$ and $v \in B_w$, then there exist $J_1, J_2 \in GV(R)$ such that $J_1 u \subseteq A$ and $J_2 v \subseteq B$. Thus, $J_1 J_2 x \subseteq A+B$, and note that $J_1 J_2 \in GV(R)$. □

Let P be a prime ideal of R, and let M be any R-module. There need not be any P-prime submodules of M, even if P contains the annihilator of M. However, if M is finitely generated, then for every submodule A of M and every prime ideal P of R such that $(A:M) \subseteq P$, A is contained in some P-prime submodule of M. Furthermore, in this case $((A+PM):M)=P$. We now prove similar results for the case of finite-type w-modules and w-submodules.

Theorem 1.6. *Let M be a finite-type w-module and let A be a w-submodule of M. Let P be a prime ideal of R minimal over $(A:M)$. Then*

(1) $((A+PM)_w : M)=P$.

(2) *There exists a P-prime submodule Q of M which is minimal among P-prime submodules over A.*

Proof. (1) Clearly, $P \subseteq ((A+PM)_w : M)$. Now, M is of finite type, so that $M=B_w$ for some finitely generated submodule B of M. We also have that P is a w-ideal [7, Proposition 1.1]. Thus, if $r \in ((A+PM)_w : M)$ then $rB \subseteq rM \subseteq (A+PM)_w = (A+PB_w)_w = (A+(PB_w)_w)_w = (A+(PB)_w)_w = (A+PB)_w$. Hence, for some $J \in GV(R)$ we have $JrB \subseteq A+PB$. Choose $x_1, \ldots, x_n \in B$ such that $B=Rx_1+\cdots+Rx_n$. Now for each $\lambda \in J$ we have $\lambda r x_i = y_i + \sum_{j=1}^{n} a_{ij} x_j$, for some $y_i \in M$, $a_{ij} \in P$ $(1 \le i,j \le n)$. By the standard determinant argument, there exists $a \in P$ such that $(\lambda^n r^n - a)B \subseteq A$. Thus, $(\lambda^n r^n - a)M = (\lambda^n r^n - a)B_w = ((\lambda^n r^n - a)B)_w \subseteq A_w = A$. It follows that $\lambda^n r^n - a \in (A:M) \subseteq P$, and hence, $\lambda r \in P$. Consequently, $Jr \subseteq P$, so that $r \in P_w = P$.

(2) Let $Q=\{x \in M : sx \in (A+PM)_w$ for some $s \in R \backslash P\}$. It is routine to show that $P \subseteq (Q:M)$. If $r \in (Q:M)$ then $rB \subseteq rM \subseteq Q$. So there exists $s \in R\backslash P$ such that $srB \subseteq (A+PM)_w$. Thus, $srM = srB_w = (srB)_w \subseteq (A+PM)_w$, and we have $sr \in ((A+PM)_w : M) = P$. It follows that $r \in P$.

Now, if $tx \in Q$, $x \in M$ and $t \in R$, then $stx \in (A+PM)_w$ for some $s \in R \backslash P$. Consequently, $t \in P$ or $x \in Q$, so that Q is prime. Now, let C be any P-prime submodule of M containing A. Then C is a w-module [17, Theorem 3.1], so that $(A+PM)_w \subseteq C_w = C$. It follows that $Q \subseteq C$. □

Let A be a submodule of the torsion-free module M, and let $r \in R$. In keeping with the usual notation we set $(A:_M r)=\{x \in M : rx \in A\}$. Note that if A and M are both w-modules, then $(A:_M r)$ is likewise a w-module.

Lemma 1.7. *Let I be an ideal of an SM domain R, and let B be a w-submodule of a finite type w-module M. If C is a w-submodule of M which is maximal with respect to the property $C \cap B = (IB)_w$, then $I^n M \subseteq C$ for some positive integer n.*

Proof. Since R is an SM domain, I_w is of finite type. Hence, $I_w=(a_1,\dots,a_m)$ for some $a_1,\dots,a_m\in I$. Choose i, $1\le i\le m$. Now, for each positive integer k, let $B_k=(C:_M a_i^k)$. Note that $B_1\subseteq B_2\subseteq\cdots$ is an ascending chain of w-submodules of M. Since M is an SM module [17, Theorem 4.5], there exists a positive integer t such that $B_j=B_t$ for all $j\ge t$. We claim that $(a_i^tM+C)_w\cap B=(IB)_w$. Note that $(IB)_w=C\cap B\subseteq(a_i^tM+C)_w\cap B$. Now if $x\in(a_i^tM+C)\cap B$, then $x=a_i^ty+u$ for some $y\in M$, $u\in C$. Thus, $a_ix=a_i^{t+1}y+a_iu\in a_iB\subseteq IB\subseteq(IB)_w\subseteq C$. Hence, $a_i^{t+1}y\in C$, so that $y\in B_{t+1}=B_t$. Therefore, $a_i^ty\in C$ and thus, $x\in C\cap B=(IB)_w$. Now, $(a_i^tM+C)_w\cap B=((a_i^tM+C)\cap B)_w\subseteq((IB)_w)_w=(IB)_w$.

By the maximality of C, we see that $a_i^tM\subseteq a_i^tM+C\subseteq C$. Since this holds for each i $(1\le i\le m)$, we can choose a positive integer n such that $(a_1,\dots,a_m)^nM\subseteq C$. By Proposition 1.3 we have $I^nM\subseteq((I_w)^n)_wM=(((a_1,\dots,a_m)_w)^n)_wM=((a_1,\dots,a_m)^n)_wM\subseteq((a_1,\dots,a_m)^nM)_w\subseteq C_w=C$. □

Theorem 1.8 (Krull Intersection Theorem for SM domains). *Let R be an SM domain and let M be a finite type w-module. If $B=\bigcap_{k=1}^{\infty}(I^kM)_w$, where I is an ideal of R, then $B=(IB)_w$. If in addition $I_w\ne R$, then $B=0$.*

Proof. Let Γ be the collection of all w-submodules C of M such that $C\cap B=(IB)_w$. Note that Γ is nonempty since $(IB)_w\in\Gamma$. M is an SM module, whence Γ has a maximal element, say C. By the previous result, $I^nM\subseteq C$ for some positive integer n. Hence, $B=\bigcap_{k=1}^{\infty}(I^kM)_w\subseteq(I^nM)_w\subseteq C_w=C$, and so $B=B\cap C=(IB)_w$.

Now, B is of finite type, and since $B=(IB)_w=(I_wB)_w$ and $I_w\ne R$, we see that $B=0$ as a result of Proposition 1.4. □

Let $w\operatorname{Max}R$ denote the set of ideals of R which are maximal in the set of all w-ideals of R. We will refer to these ideals as *maximal w-ideals*. In an SM domain, every w-ideal is contained in a maximal w-ideal.

Theorem 1.9. *R is an SM domain if and only if R_P is Noetherian for every $P\in w\operatorname{Max}R$ and each non-zero element of R lies in only finitely many maximal w-ideals. Furthermore, if R is an SM domain, then $R=\bigcap_{P\in w\operatorname{Max}R}R_P$.*

Proof. Suppose that R is an SM domain. That R_P is Noetherian was shown in [17, Proposition 4.6]. The second part follows from [3, Proposition 2.2(b)] and [17, Proposition 5.7].

Conversely, let $x\ne 0$ be an element of a w-ideal I of R, and let $P_1,\dots,P_n$ be the maximal w-ideals of R which contain x. Now for each i, $1\le i\le n$, R_{P_i} is Noetherian and, hence, there exist a positive integer $k(i)$ and $x_{i1},\dots,x_{ik(i)}\in I$ such that $I_{P_i}=(x_{i1},\dots,x_{ik(i)})_{P_i}$. Let B be the ideal of R generated by x and the collection of all such x_{ij}. So B is finitely generated and is contained in I. Also, for each i $(1\le i\le n)$, $I_{P_i}\subseteq B_{P_i}$. Furthermore, for each $P\in w\operatorname{Max}R$ where $P\ne P_i$ for all $i=1,\dots,n$, we have $x\notin P$; hence, $B_P=R_P=I_P$. Thus, $B_P=I_P$ for all $P\in w\operatorname{Max}R$. It follows that $B_w=I_w=I$

[16, Lemma 1.5]. In particular, I is of finite type, and thus, R is an SM domain [17, Theorem 4.3]. Finally, if R is an SM domain, then $R = \bigcap_{P \in w\,\mathrm{Max}\,R} R_P$ follows immediately from [17, Proposition 3.4]. □

Corollary 1.10. *Let R be an SM domain with $\dim R = 1$. Then R is Noetherian.*

Proof. Every non-zero prime ideal P is a minimal prime to some non-zero element of R, whence a w-ideal [7, Proposition 1.1]. Clearly, then P belongs to $w\,\mathrm{Max}\,R$. The result follows from Theorem 1.9 and [12, p.73, ex. 10]. □

It is worth noting that not every SM domain is Noetherian. For example, let F be any field and consider the polynomial ring $R = F[x_1, x_2, \ldots]$ in countably many indeterminates. Since R is a UFD, it is, therefore, an SM domain, but of course R is not Noetherian. One should also note that a Mori domain need not satisfy the Principal Ideal Theorem (see, e.g., [2]). However, we are now able to show that every SM domain does satisfy the PIT.

Corollary 1.11 (PIT for SM domains). *Let R be an SM domain and let x be a non-zero non-unit element of R. If P is a prime ideal of R minimal over (x), then $\mathrm{ht}\,P = 1$.*

Proof. As before, P is necessarily a w-ideal, hence, P is contained in some maximal w-ideal Q of R. By Theorem 1.9, R_Q is Noetherian and since P_Q is a minimal prime over x in R_Q, we have $\mathrm{ht}\,P_Q = 1$. It follows that $\mathrm{ht}\,P = 1$. □

More generally, we have

Corollary 1.12 (Generalized PIT for SM domains). *Let R be an SM domain and let $I = (a_1, \ldots, a_n)_w$ be a w-ideal of R. If P is a prime ideal of R minimal over I, then $\mathrm{ht}\,P \le n$.*

We now present the other main result of this section.

Theorem 1.13 (The Hilbert Basis Theorem for SM domains). *If R is an SM domain, then $R[x]$ is likewise an SM domain.*

Proof. Let A be a w-ideal of $R[x]$ and let I be the set of leading coefficients of the polynomials in A. Then I_w is a finite-type ideal of R, hence, there exist a positive integer m and $a_1, \ldots, a_m \in I$ such that $I_w = (a_1, \ldots, a_m)_w$. Let $f_1, \ldots, f_m$ be polynomials in A whose leading coefficients are $a_1, \ldots, a_m$, respectively, and whose degrees are $n_1, \ldots, n_m$, respectively. Now let $n = \max\{n_1, \ldots, n_m\}$ and let $B = R[x]f_1 + \cdots + R[x]f_m$. Then for any $f \in A$, with leading coefficient a and degree k, there exists $J \in GV(R)$ such that $Ja \subseteq (a_1, \ldots, a_m)$. Let $J = (\lambda_1, \ldots, \lambda_t)$ and note that for each i, $1 \le i \le t$, there exist $r_{ij} \in R$, $1 \le j \le m$, such that $\lambda_i a = \sum_{j=1}^{m} r_{ij} a_j$. If $k \ge n$, then for each i $(1 \le i \le t)$ let $g_i = \lambda_i f - \sum_{j=1}^{m} r_{ij} f_j x^{k-n_j}$, and note that g_i belongs to A and has degree strictly less than k. If we still have $\deg g_i \geq n$ for some i $(1 \leq i \leq t)$, we continue this process. After finitely many steps we have some $J' \in GV(R)$ such that $J'f \subseteq (A \cap M) + B$, where $M = R \oplus Rx \oplus \cdots \oplus Rx^{n-1}$. Note that M is an SM module, whence $(A \cap M)_w = (Rh_1 + \cdots + Rh_s)_w$ for some $h_1, \ldots, h_s \in A \cap M$. Letting $C = R[x]h_1 + \cdots + R[x]h_s$ we see that $J'f \subseteq (A \cap M) + B \subseteq C_w + B$ and thus $J'[x]f \subseteq C_w + B$ (as ideals of $R[x]$). Note that the choice of C is independent of the choice of f. It follows then that $A \subseteq (C_w + B)_w = (C_w + B_w)_w = (C + B)_w$ by Lemma 1.5. But $C, B \subseteq A$, so that $(C + B)_w \subseteq A_w = A$. Hence, A is of finite-type and thus $R[x]$ is an SM domain. □

It is well known that an intersection with finiteness condition of Mori domains is itself a Mori domain. It is reasonable to ask whether a similar result holds regarding SM domains. The authors at present are unable to answer this question, however.

2. w-Multiplication domains and Krull domains

A fractional ideal I of R is said to be *w-invertible* if $(II^{-1})_w = R$. It is known that I is w-invertible if and only if I_w is of finite type and I_P is principal for all $P \in w\operatorname{Max} R$ (see, e.g., [1, 17]). R is called a *w-multiplication domain* if the set of w-fractional ideals of finite type forms a group under w-multiplication. It is clear that a w-multiplication domain is a PVMD, where a PVMD is a domain in which each non-zero ideal is t-invertible; equivalently, the v-ideals of finite type form a group under v-multiplication. One should note that an analogue of the following theorem holds for PVMDs.

Theorem 2.1. *The following conditions are equivalent for a domain R.*

(1) *Every non-zero finite type w-ideal of R is w-invertible.*

(2) *Every finitely generated fractional ideal of R is w-invertible.*

(3) *Every 2-generated fractional ideal of R is w-invertible.*

(4) *For each non-zero prime w-ideal P of R, R_P is a valuation domain.*

(5) *For each maximal w-ideal Q of R, R_Q is a valuation domain.*

(6) *R is integrally closed and for each non-zero $a, b \in R$, the prime ideal $(ax + b)K[x] \cap R[x]$ of $R[x]$ contains an element $f = \sum_{i=0}^{n} a_i x^i$ such that $(a_0, \ldots, a_n)_w = R$.*

(7) *R is a w-multiplication domain.*

Proof. (1) ⇒ (2). If I is a finitely generated fractional ideal of R we can choose a non-zero $r \in R$ such that $rI \subseteq R$. Since $rI_w = (rI)_w$ is w-invertible, so is I.

(2) ⇒ (3) and (4) ⇒ (5) are trivial. The proofs of (3) ⇒ (5), (5) ⇒ (6) and (6) ⇒ (7) are provided in [7, Theorem 1.1]. For (7) ⇒ (1), see [6, Theorem 5].

(5) ⇒ (4). Let P be a non-zero prime w-ideal of R and let Q be a maximal w-ideal of R containing P. Since R_Q is a valuation domain, then $R_P = (R_Q)_{PR_Q}$ is also a valuation domain. □

By [13, Corollary 4.2] we have

Corollary 2.2. *In a w-multiplication domain, every prime w-ideal is a t-ideal.*

Corollary 2.3. *Let R be a w-multiplication domain and let P be a prime ideal of R. Then R_P is a valuation domain if and only if P is a w-ideal.*

Corollary 2.4. *Let R be a w-multiplication domain and let a,b be any non-zero elements of R. Then $(a)\cap(b)$ is w-invertible, and hence, is a finite type v-ideal.*

Before leaving the topic of valuation domains, we remark that it is well known that every Mori valuation domain is a DVR (see, e.g., [15, Corollary 2]).

Let a,b be non-zero elements of R. We will slightly abuse the standard notation and let $(a:b)=\{x\in R: bx\in(a)\}$. Clearly $(a:b)=R\cap R\frac{a}{b}$.

Lemma 2.5. *Let R be any domain and let P be a w-invertible prime w-ideal of R. Then the following hold:*

(1) P is a minimal prime of a principal ideal and $P=(a:b)$ for some non-zero $a,b\in R$.

(2) P is a maximal w-ideal and P is also a v-ideal.

Proof. Since P is w-invertible, then $P^{-1}P\not\subseteq P$, and (1) follows from [11, Lemma 1.2]. Now suppose that Q is any ideal of R which properly contains P. P is w-invertible, whence P is of finite type. Thus, there exists a finitely generated ideal B of R such that $P=B_w$. Let $J=B+Rc$, where c is any element of $Q\backslash P$. We will show that $J^{-1}=R$. This will imply that J belongs to $GV(R)$, and since $J\subseteq Q$, we will have $Q_w=R$, and the proof will be complete. So suppose that $x\in J^{-1}$. Then $xcB\subseteq B\subseteq P$, which implies $xB\subseteq P$. Thus, $xP=xB_w=(xB)_w\subseteq P_w=P$, so that $x(PP^{-1})\subseteq PP^{-1}\subseteq R$. It follows that $x\in(PP^{-1})^{-1}=((PP^{-1})_w)^{-1}=R^{-1}=R$. □

Corollary 2.6. *Let P be a w-invertible prime ideal of R. Then either $P_w=R$ or P is a maximal w-ideal.*

Proof. If $P_w\neq R$ then P is a w-ideal by Proposition 1.1. Apply Lemma 2.5. □

Proposition 2.7. *Let P be a prime ideal of a w-multiplication domain R. Then P is a w-invertible w-ideal if and only if $P=(a:b)$ for some non-zero $a,b\in R$.*

Proof. If $P=(a:b)$ for some non-zero $a,b\in R$, then $P=R\cap R\frac{a}{b}\subseteq a(a,b)^{-1}\subseteq(a:b)=P$. Since R is a w-multiplication domain, then (a,b) is w-invertible, and thus P is w-invertible as well. It is clear that P is a v-ideal, and therefore a w-ideal. The converse follows from Lemma 2.5. □

Let I be an ideal of R. We say that I is the *w-product* of the prime ideals $P_1,\dots,P_n$ if $I_w=(P_1\cdots P_n)_w$. We are now ready to give some new characterizations of Krull domains.

Theorem 2.8. *The following are equivalent for a domain R:*

(1) *R is a Krull domain.*

(2) *Every non-zero ideal of R is w-invertible.*

(3) *Every non-zero prime w-ideal of R is w-invertible.*

(4) *Every non-zero ideal I with $I_w \neq R$ can be written uniquely as a w-product of a finite number of prime w-ideals of R.*

(5) *R is an integrally closed SM domain.*

(6) *R is a w-multiplication SM domain.*

(7) *R is an SM domain and R_P is a DVR for each $P \in w\,\mathrm{Max}\,R$.*

(8) *Every maximal w-ideal of R is w-invertible and has height one.*

(9) *R is an SM domain and every maximal w-ideal of R is w-invertible.*

Proof. For (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3), see [17, Theorem 5.4].

(1) and (2) $\Rightarrow$ (4). Let I be a non-zero ideal of R with $I_w \neq R$. Then there is a prime w-ideal P_1 which contains I, and note that $(P_1^{-1}P_1)_w = R$. Thus $I = I(P_1^{-1}P_1)_w$, so that $I_w = (I(P_1^{-1}P_1)_w)_w = (I(P_1^{-1}P_1))_w = ((IP_1^{-1})_w P_1)_w$. By Lemma 2.5, P_1 is a v-ideal, whence $P_1^{-1} \neq R$. Note that $I_w \neq (IP_1^{-1})_w$, for otherwise, we would have $P_1^{-1} = (P_1^{-1})_w = ((I^{-1}I)P_1^{-1})_w = (I^{-1}(IP_1^{-1})_w)_w = (I^{-1}I_w)_w = (I^{-1}I)_w = R$. Now if $(IP_1^{-1})_w = R$, then $I_w = (P_1)_w = P_1$. Otherwise, we can choose a prime w-ideal P_2 such that $(IP_1^{-1})_w = ((IP_1^{-1}P_2^{-1})P_2)_w = ((IP_1^{-1}P_2^{-1})_w P_2)_w$. By the above argument, we see that $I_w = ((IP_1^{-1}P_2^{-1})_w P_1 P_2)_w$. If $(IP_1^{-1}P_2^{-1})_w \neq R$, continue the process. Now R is an SM domain [17, Theorem 5.4], and thus, this process terminates, say $(IP_1^{-1} \cdots P_n^{-1})_w = R$ for some prime w-ideals $P_1, \ldots, P_n$. It follows that $I_w = (P_1 \cdots P_n)_w$.

To see that the above w-product is unique, suppose there exist prime w-ideals $Q_1, \ldots, Q_m$ such that $I_w = (Q_1 \cdots Q_m)_w$. Then $Q_1 \cdots Q_m \subseteq (P_1 \cdots P_n)_w \subseteq (P_1)_w = P_1$, and hence, without loss of generality, we may suppose $Q_1 \subseteq P_1$. By Lemma 2.5, Q_1 is a maximal w-ideal, so that $Q_1 = P_1$. The rest is now routine.

(4) $\Rightarrow$ (3). Let P be a non-zero prime w-ideal of R, and let c be a non-zero element in P. Then $(c)_w = (P_1 \cdots P_n)_w$ for some prime w-ideals $P_1, \ldots, P_n$ of R. Because (c) is w-invertible, then for each $i = 1, \ldots, n$, P_i is w-invertible, and thus is a maximal w-ideal, by Lemma 2.5. Now since $P_k \subseteq P$ for some $1 \leq k \leq n$, then $P = P_k$ is w-invertible.

(1) $\Rightarrow$ (5). By [17, Theorem 5.4].

(5) $\Rightarrow$ (7). Observe that for each $P \in w\,\mathrm{Max}\,R$, R_P is an integrally closed Noetherian domain by Theorem 1.9, whence R_P is a DVR.

(7) $\Rightarrow$ (6) $\Rightarrow$ (2). By Theorem 2.1.

(3) and (7) $\Rightarrow$ (8). Note that every maximal w-ideal is prime. The result follows easily.

(8) $\Rightarrow$ (9). By [17, Theorem 4.3], it suffices to show that every prime w-ideal of R is of finite type. Now if P is a non-zero prime w-ideal, then by hypothesis P is a maximal w-ideal, and hence is w-invertible. Thus $P = P_w$ is of finite type.

(9) $\Rightarrow$ (7). Let $P \in w\,\mathrm{Max}\,R$. Note that R_P is Noetherian, and the maximal ideal PR_P is principal. Thus R_P is a DVR. $\square$

3. *w*-Overdomains

Let T be an overdomain of R. If T is a w-module (as an R-module) then for convenience we say that T is a *w-overdomain* of R. It is clear that for any overdomain T of R, T_w is a w-overdomain of R.

Lemma 3.1. *Let T be a w-overdomain of R. Then*

(1) *If $J \in GV(R)$ then $JT \in GV(T)$.*

(2) *If Q is a w-ideal of T, then Q is a w-module over R, and thus $Q \cap R$ is a w-ideal of R.*

Proof. (1) First note that since J is a finitely generated ideal of R, then JT is a finitely generated ideal of T. Now, let L be the quotient field of T, and let $x \in L$ such that $JTx \subseteq T$. Since T is a w-module over R, then $Tx \subseteq T$ and hence $x \in T$. Thus, $(JT)^{-1} = T$ and so $JT \in GV(T)$.

(2) Let $J \in GV(R)$ and suppose $Jx \subseteq Q$ for some $x \in Q \otimes K \subseteq T \otimes K$. Then $JTx \subseteq Q$ and thus $x \in Q$, by (1). □

We define the *w-dimension* of R as $w\text{-}\dim R = \sup\{\operatorname{ht} P : P \in w\operatorname{Max} R\}$. From Theorem 2.8 we see that the w-dimension of a Krull domain is at most 1.

Theorem 3.2. *For any domain R, the following are equivalent:*

(1) *R is an SM domain with w-$\dim R \le 1$.*

(2) *For any non-zero w-ideal I of R, every descending chain of w-ideals of R containing I stabilizes.*

(3) *For every positive integer n and every w-submodule M of R^n such that rank $M = n$, every descending chain of w-submodules of R^n containing M stabilizes.*

(4) *For every finite type w-module M and every w-submodule N of M such that* $\operatorname{rank} N = \operatorname{rank} M$, *every descending chain of w-submodules of M containing N stabilizes.*

Proof. (1)⇒(2). Let $I_1 \supseteq I_2 \supseteq \cdots$ be a descending chain of w-ideals of R containing I. By Theorem 1.9 there are only finitely many maximal w-ideals containing I, say $P_1, \ldots, P_n$. For each i ($1 \le i \le n$), R_{P_i}/I_{P_i} is Artinian [12, Theorem 90], hence, there is a positive integer k_i such that $(I_t)_{P_i} = (I_{k_i})_{P_i}$ for all $t \ge k_i$. Letting $k = \max\{k_1, \ldots, k_n\}$, we have $(I_t)_P = (I_k)_P$ for all $t \ge k$ and for each $P \in \{P_1, \ldots, P_n\}$. On the other hand, for each $P \in w\operatorname{Max} R \backslash \{P_1, \ldots, P_n\}$, since $I \nsubseteq P$ we have $(I_t)_P = R_P$ for every positive integer t. Hence, $I_t = I_k$ for all $t \ge k$ [16, Lemma 1.5].

(2)⇒(1). For each non-zero $x \in R$, there are only finitely many maximal w-ideals of R which contain x. This is so, because if $P_1, P_2, \ldots$ are distinct maximal w-ideals, each containing x, then $P_1 \supseteq P_1 \cap P_2 \supseteq \cdots$ is a descending chain which must stabilize, a contradiction. We now claim that for any $P \in w\operatorname{Max} R$ and any non-zero w-ideal I of R, R_P/I_P is Artinian and R_P is Noetherian. It will then follow by Theorem 1.9

that R is an SM domain. Suppose $N_1 \supseteq N_2 \supseteq \cdots$ is a chain of ideals of R_P which contain I_P. For each positive integer i, let $I_i = N_i \cap R$, and note that $(I_1)_w \supseteq (I_2)_w \supseteq \cdots$ is a descending chain of w-ideals of R containing $I_P \cap R \supseteq I$. Hence, there exists a positive integer k such that $(I_i)_w = (I_k)_w$ for all $i \geq k$. Since $(J_w)_P = J_P$ for any ideal J of R, it follows then that $N_i = (I_i)_P = (I_k)_P = N_k$ for all $i \geq k$, i.e., R_P/I_P is Artinian. Now suppose $0 \neq A \subseteq A_1 \subseteq A_2 \subseteq \cdots$ is a chain of ideals of R_P. Then, $I = A \cap R \neq 0$ and $(I_w)_P = I_P = A$. Thus R_P/A is Artinian, whence R_P/A is Noetherian. It follows that R_P is Noetherian.

To see that w-$\dim R \leq 1$, let $P \in w\operatorname{Max} R$ and let $0 \neq x \in P$. Then R_P/xR_P is Artinian, whence, $\dim(R_P/xR_P) = 0$. Thus, PR_P is a minimal prime of xR_P and so $\operatorname{ht}(PR_P) \leq 1$. It follows that $\operatorname{ht} P \leq 1$.

(2) $\Rightarrow$ (3). Let $N_1 \supseteq N_2 \supseteq \cdots$ be a chain of w-submodules of R^n containing M. For each positive integer i, let $C_i = \pi(N_i)$, where $\pi : R^n \to R$ is the nth projection. Since $\operatorname{rank} M = n$ we have $\pi(M) \neq 0$. Then $(C_1)_w \supseteq (C_2)_w \supseteq \cdots$ is a chain of w-ideals of R and hence must stabilize, say at the positive integer k. Now, let $\{e_1, \ldots, e_n\}$ be the standard basis of R^n and let $F = Re_1 \oplus \cdots \oplus Re_{n-1}$. Then $N_1 \cap F \supseteq N_2 \cap F \supseteq \cdots$ is a chain of w-submodules of F containing $M \cap F$. Since $\operatorname{rank}(M \cap F) = n - 1$, by induction this chain must stabilize, without loss of generality, at k as well. Now let $x \in N_k$ and let i be any positive integer such that $i \geq k$. Then $\pi(x) \in C_k \subseteq (C_i)_w$, and thus $\pi(Jx) = J\pi(x) \subseteq C_i$ for some $J \in GV(R)$. Since J is finitely generated, there exists a finitely generated submodule B of N_i such that $\pi(B) = \pi(Jx)$. Thus, $Jx \subseteq (F + B) \cap N_k = (F \cap N_k) + B \subseteq N_i$, and since N_i is a w-module, we have $x \in N_i$.

(3) $\Rightarrow$ (4). This follows easily from the fact that M can be embedded in R^n, where $n = \operatorname{rank} M$.

(4) $\Rightarrow$ (2). This is trivial. □

From the above proof we have

Corollary 3.3. *Let R be an SM domain with w-$\dim R \leq 1$, and let I be a non-zero w-ideal of R. Then R_P/I_P is Artinian for all maximal w-ideals P which contain I.*

Theorem 3.4. *Let R be an SM domain with w-$\dim R = 1$, and let T be a w-overdomain of R such that $T \subseteq K$. Then T is an SM domain with w-$\dim T \leq 1$.*

Proof. Let Q be a non-zero w-ideal of T. Since $Q \cap R \neq 0$ we can choose a non-zero $x \in R$ such that $Tx \subseteq Q$. By Theorem 1.9 there are only finitely many maximal w-ideals of R containing x, say $P_1, \ldots, P_n$. For any $P \in \{P_1, \ldots, P_n\}$, R_P is Noetherian with $\dim R_P = 1$, hence, T_P/xT_P is a finitely generated R_P-module by the proof of [12, Theorem 93]. Let $N_1 \supseteq N_2 \supseteq \cdots$ be a chain of w-ideals of T containing Q. Then there exists a positive integer k such that for each $P \in \{P_1, \ldots, P_n\}$ we have $(N_t)_P = (N_k)_P$ for all $t \geq k$. Note that for any $P \in w\max R \backslash \{P_1, \ldots, P_n\}$, since $x \notin P$, we have $xT_P = T_P$ and thus $(N_t)_P = (N_k)_P$ for all positive integer t. Thus, $N_t = N_k$ for all $t \geq k$ ([16, Lemma 1.5] and Lemma 3.1). Now apply Theorem 3.2. □

Finally, we make one further observation relating SM domains and Krull domains.

Theorem 3.5. *If R is an SM domain, then the complete integral closure of R is a Krull domain.*

Proof. This follows easily from Theorem 1.9, [8, Lemma 2.2], and the Mori Nagata Theorem. □

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