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On the Frame–Stewart algorithm for the multi-peg Tower of Hanoi problem

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Abstract

It is proved that seven different approaches to the multi-peg Tower of Hanoi problem are all equivalent. Among them the classical approaches of Stewart and Frame from 1941 can be found.

Keywords: Multi-peg Tower of Hanoi problem; Frame–Stewart algorithm; Recursion; Dynamic programming

1. Introduction

The Tower of Hanoi problem(s) [2] presents an unusual phenomenon in the mathematical and computer science literature. On one side it was extensively studied by many authors, Stockmeyer's survey [14] lists 207 relevant references, not including in psychological journals and textbooks in discrete mathematics. On the other hand, many papers only rediscover known results, "prove" results under wrong assumptions, or claim to solve the general multi-peg problem by solving a particular recursion for which it is not known whether it is really an optimal one. For instance, Stockmeyer [13] lists 13 (!) references which more or less only rediscover contributions of Frame [6] and Stewart [12] from 1941. Another concrete example of several wrong approaches is the false assumption that the largest disk moves at most once in a shortest path between two regular states of the three peg problem. This confusion is nicely described and clarified by Hinz [8]. We also refer to a nice semi-survey of Poole [10].

Probably, the greatest challenge related to the Towers of Hanoi is given by increasing the number of pegs to more than three. In 1908, Dudeney [4] posed the problem

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for four pegs and called it *the Reve's Puzzle*. Then, in 1941, two currently famous "solutions" for any number of pegs followed, one due to Frame [6] and the other to Stewart [12]. Most researchers believe that the solutions proposed by Frame and Stewart are optimal. Interestingly, many also feel no need to prove this fact, even if it was pointed out already in 1941 by the problems editor of the American Mathematical Monthly [5] that their proofs of optimality only apply to algorithms of a certain scheme! For instance, in the 1970s additional "solutions" followed [11,1]. Then, in the early 1980s Wood [15] and Cull and Ecklund [3] correctly pointed out that the problem is still open. However, even this was not enough to prevent new "solutions". For instance, very recently [9] one can read in the abstract: "This paper solves completely the generalized p-peg Tower of Hanoi problem when $p \geqslant 4$". In reality the author restricts himself to Stewart's recursive scheme and on this particular scheme applies the dynamic programming.

In literature, the solutions of Frame and Stewart are presented in different forms. In fact, even if the solutions of Frame and Stewart are "essentially the same", cf. [13], we could find no published rigorous proof of this fact. (It is true, though, that for the Reve's Puzzle the solutions are easily seen to be the same.) Since in the context of the Tower of Hanoi problems there has already been several obvious (but wrong) facts around, we feel justified to clarify this point of view. More precisely, we present seven[2] different approaches and prove that all of them are equivalent, i.e. yield the same number of moves. Thus, after proving this result one can really speak about *the* presumed optimal solution.

The rest of the paper is organized as follows. In the next section, we introduce the following functions (here and later n stands for the number of disks, and p for the number of pegs):

$F(n,p)$ — reflects Frame's algorithm;
$F'(n,p)$ — same as $F(n,p)$, but no monotonicity is required;
$S(n,p)$ — reflects Stewart's algorithm;
$A'(n,p)$ — reflects an algorithm taking into account all partitions;
$A(n,p)$ — same as $A'(n,p)$, but monotonicity is required.

In Section 3 we continue with a proof that $A' = F' = S$. Then, we introduce an explicit formula

$X(n,p)$ — which is (essentially) a sum of powers of 2

and prove an equivalent expression for it. In Section 5 we demonstrate that $X = S$. In the final section we show that F and A coincide with all the rest which enables us to state the result of this paper:

$$A(n,p) = A'(n,p) = F(n,p) = F'(n,p) = S(n,p) = X(n,p).$$

[2] Or maybe six — it depends on whether two different presentations of $X(n,p)$ are counted separately or not.

2. Recursive definitions

Let $M(n,p)$ be the minimum number of moves required to solve the Tower of Hanoi problem with n disks and p pegs.

Only for $p \geqslant 4$ pegs and $n \geqslant p$ disks the problem considered is nontrivial, i.e. it has been proved that $M(n,3)=2^n-1$, and that $M(n,p)=2n-1$ for $n<p$. Therefore, in the trivial case $p=3$ or $n<p$ we shall define all the functions treated here to be defined as $M(n,p)$. For technical reasons it is useful to note that $M(1,2)=1$. We shall not define $M(n,2)$ for $n \geqslant 2$.

Definition 2.1. If $p=2,3$ or $n<p$

$$F(n,p)=F'(n,p)=A(n,p)=A'(n,p)=S(n,p)=M(n,p).$$

All five different definitions for recursive calculations of presumed optimal solution will now be given only outside the trivial range, for $n,p \in \mathbb{Z}^+$ satisfying $p \geqslant 4$ pegs and $n \geqslant p$, taking the values from the trivial range as initial values, when required. Also, when proving results about these functions and about explicit formulas, discussed in Sections 4 and 5, some care will be taken about the trivial range.

The following definition reflects the **F**rame's algorithm for the multi-peg Tower of Hanoi problem, but it differs from the original definition, since it does not require partitions of n to be monotone.

Definition 2.2. For $p \geqslant 4$ and $n \geqslant p$ let $F'(n,p)$ be the minimum of

$$\{2F'(n_1,p)+2F'(n_2,p-1)+\cdots+2F'(n_{p-2},3)+1 \mid n_1+\cdots+n_{p-2}+1=n\},$$

where $n_i \in \mathbb{Z}^+$.

Note that 1 in the sum above may be interpreted as $F'(1,2)$.

The original **F**rame's definition was essentially the following:

Definition 2.3. For $p \geqslant 4$ and $n \geqslant p$ let $F(n,p)$ be the minimum of

$$\{2F(n_1,p)+2F(n_2,p-1)+\cdots+2F(n_{p-2},3)+1 \mid$$
$$n_1+\cdots+n_{p-2}+1=n,\ n_1 \geqslant n_2 \geqslant \cdots \geqslant n_{p-2}\},$$

where $n_i \in \mathbb{Z}^+$.

Note that 1 in the sum above may be interpreted as $F(1,2)$.

In Fig. 1, we present strategy for disk moves corresponding to $F(n,p)$ or $F'(n,p)$. First we have all n disks on the source peg 0. We divide the $n-1$ smallest disks into $p-2$ groups of disks of consecutive size. The top n_1 smallest disks are moved from the source peg 0 to the auxiliary peg 1 using all p pegs. The next n_2 smallest disks are moved from the source peg 0 to the auxiliary peg 2 using $p-1$ pegs. We

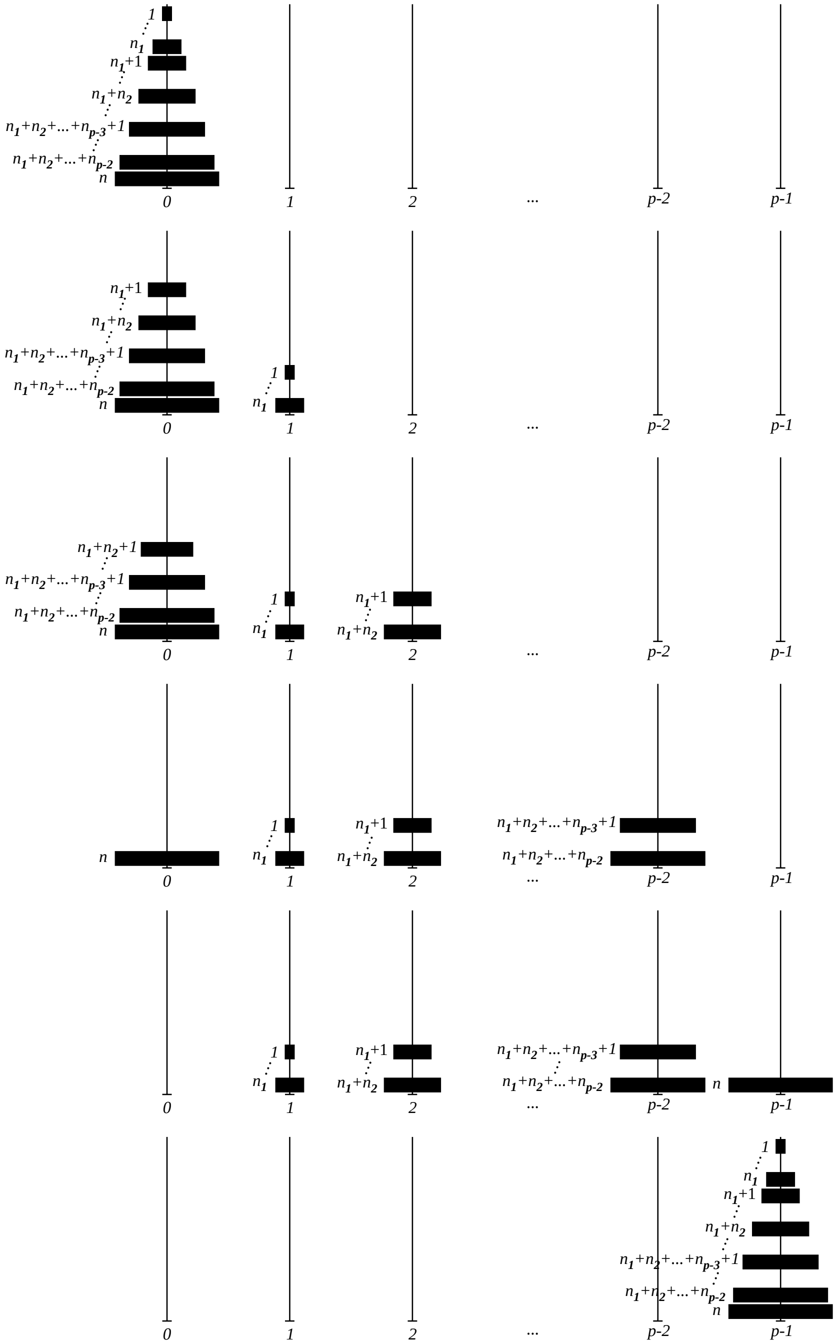

Fig. 1. Disk moves corresponding to $F(n,p)$ or $F'(n,p)$.

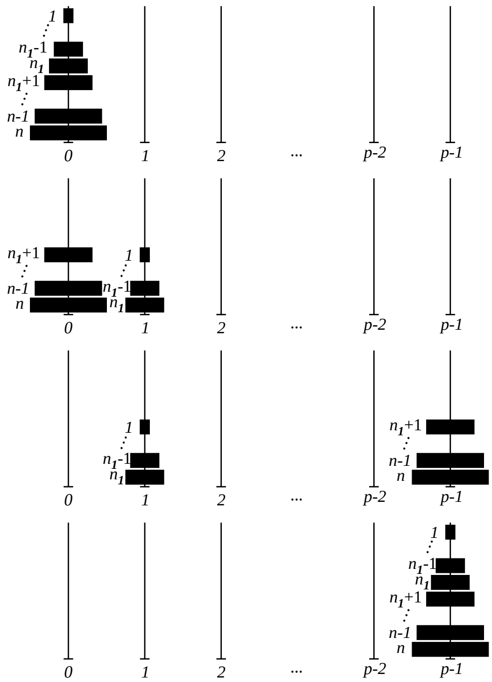

Fig. 2. Disk moves corresponding to $S(n, p)$.

continue removing groups of disks from the source peg to the auxiliary pegs, in each step using one peg less than that in the previous one. At the end the last group of disks is moved from the source peg 0 to the auxiliary peg $p-2$ using the remaining 3 admissible pegs. The last (nth) disk is then moved from the source peg 0 to the destination peg $p-1$. After that we move all groups of disks to the destination peg, taking them in the reverse order.

We now introduce the function $S(n, p)$ which corresponds to the **S**tewart's algorithm for the multi-peg Tower of Hanoi problem.

Definition 2.4. For $p \geqslant 4$ and $n \geqslant p$ let

$$S(n, p) = \min\{2S(n_1, p) + S(n_2, p-1) \mid n_1 + n_2 = n\},$$

where $n_1, n_2 \in \mathbb{Z}^+$.

In Fig. 2, we present a strategy for disk moves corresponding to $S(n, p)$.

First all n disks on the source peg 0. We divide the disks into two groups, the first consisting of the smallest n_1 disks and the second consisting of the remaining $n - n_1$ disks. The top n_1 disks are moved from the source peg 0 to the auxiliary peg 1 using all p pegs. The remaining $n - n_1$ disks are then moved from the source peg 0 to the destination peg $p-1$ using $p-1$ pegs (all of them but peg 1). Finally, disks from peg 1 are moved to the destination peg $p-1$ using all p pegs.

By simple examples it can be shown that $S(n,p)$ does not have a monotone counterpart (i.e. it gives a different function). For example, the minimum of $2S(n_1,5)+S(n_2,4)$ taken over all pairs n_1,n_2 with $n_1+n_2=9$ is 27, while the minimum taken only over pairs with $n_1 \geqslant n_2$ is 31 (note that it suffices to compare minima of sums involving only function S, without introducing the monotone version of it explicitly).

Next, we introduce an algorithm that takes into account **a**ll partitions:

Definition 2.5. For $p \geqslant 4$ and $n \geqslant p$ let $A'(n,p)$ be defined as the minimum of

$$\{2A'(n_1,p)+2A'(n_2,p-1)+\cdots+2A'(n_{p-2},3)+1 \mid n_1+\cdots+n_{p-2}+1=n\} \cup \{2A'(n_1,p)+2A'(n_2,p-1) +\cdots+2A'(n_{p-3},4)+A'(n_{p-2},3) \mid n_1+\cdots+n_{p-2}=n\} \cup\cdots\cup \{2A'(n_1,p)+A'(n_2,p-1) \mid n_1+n_2=n\},$$

where $n_i \in \mathbb{Z}^+$.

Note that 1 in the sum above may be interpreted as $A'(1,2)$.

The monotone version of $A'(n,p)$ is as follows:

Definition 2.6. For $p \geqslant 4$ and $n \geqslant p$ let $A(n,p)$ be defined as the minimum of

$$\{2A(n_1,p)+2A(n_2,p-1)+\cdots+2A(n_{p-2},3)+1 \mid n_1+\cdots+n_{p-2}+1=n,\ n_1 \geqslant n_2 \geqslant \cdots\} \cup\{2A(n_1,p)+2A(n_2,p-1)+\cdots+2A(n_{p-3},4)+A(n_{p-2},3) \mid n_1+\cdots+n_{p-2}=n,\ n_1 \geqslant n_2 \geqslant \cdots\} \cup\cdots\cup \{2A(n_1,p)+A(n_2,p-1) \mid n_1+n_2=n,\ n_1 \geqslant n_2\},$$

where $n_i \in \mathbb{Z}^+$.

Note that 1 in the sum above may be interpreted as $A(1,2)$.

3. Proof of $A' = F' = S$

We first state the following two lemmas on the function $A'(n,p)$. Even if the first one is more or less obvious (from the fact that $A'(n,p) \geqslant M(n,p) \geqslant 2n-1$) and the second deals basically only with the trivial cases, we include their formal proofs, the reason being that several "obvious facts" related to the Tower of Hanoi problem turned out to be false.

Lemma 3.1. *For any* $n \geqslant 1,\ p \geqslant 3$ *we have* $A'(n,p) \geqslant 2n-1$.

Proof. For $n < p$, the claim holds by the definition of $A'(n,p)$. For $p=3$ and $n \geqslant 3$ we have $A'(n,3)=2^n-1 \geqslant 2n-1$. Finally, for $p \geqslant 4$ and $n \geqslant p$ we have

$$A'(n,p)=2A'(n_1,p)+2A'(n_2,p-1)+\cdots+2A'(n_{p-i},i+1)+A'(n_{p-i+1},i),$$

where $n_1+n_2+\cdots+n_{p-i+1}=n$ and $2 \leqslant i \leqslant p-1$. By induction we have

$$\begin{aligned} A'(n,p) &\geqslant 2[(2n_1-1)+\cdots+(2n_{p-i}-1)]+2n_{p-i+1}-1 \\ &= 2(n_1+\cdots+n_{p-i}+n_{p-i+1})+2(n_1+\cdots+n_{p-i})-2(p-i)-1 \\ &= 2n+2(n_1+\cdots+n_{p-i})-2(p-i)-1 \geqslant 2n-1 \end{aligned}$$

and we are done. □

Lemma 3.2. *For any $n \geqslant 2$, $p \geqslant 3$, and i, $n_1,\ldots,n_{p-i+1}$ with $2 \leqslant i \leqslant p-1$ and $n=n_1+n_2+\cdots+n_{p-i+1}$ we have*

$$A'(n,p) \leqslant 2A'(n_1,p)+2A'(n_2,p-1)+\cdots+2A'(n_{p-i},i+1)+A'(n_{p-i+1},i).$$

Proof. If $p \geqslant 4$ and $n \geqslant p$ the statement follows directly from the definition.

For $p=3$ it follows $i=2$ and we have to prove only $A'(n,3) \leqslant 2A'(n_1,3)+A'(n_2,2)$, which is nondegenerate only for $n_2=1$. However, then we have to prove $A'(n,3) \leqslant 2A'(n-1,3)+A'(1,2)$, which is trivial (since it says $2^n-1 \leqslant 2(2^{n-1}-1)+1$).

The final case when $n < p$, $p \geqslant 4$, is done by induction on n.

If $n=2$ then $2=n=n_1+n_2+\cdots+n_{p-i+1} \geqslant p-i+1$. Thus, $p-i \leqslant 1$ and as on the other hand we have $p-i \geqslant 1$ it follows that $p-i=1$. Then $n_1=n_2=1$. Therefore, one only has to show that $A'(2,p) \leqslant 2A'(1,p)+A'(1,p-1)$, which is indeed the case $(3 \leqslant 2\cdot 1+1)$.

The inductive step is as follows:

$$\begin{aligned} &2A'(n_1,p)+(2A'(n_2,p-1)+\cdots+2A'(n_{p-i},i+1)+A'(n_{p-i+1},i)) \\ &\quad \geqslant 2A'(n_1,p)+A'(n-n_1,p-1). \end{aligned}$$

Therefore, our claim will be proved by showing that $2A'(n_1,p)+A'(n-n_1,p-1) \geqslant A'(n,p)$. By Lemma 3.1 we have

$$\begin{aligned} &2A'(n_1,p)+A'(n-n_1,p-1) \geqslant 2(2n_1-1)+2(n-n_1)-1 \\ &\quad = 2(n_1+n-n_1)+(2n_1-3) \geqslant 2n-1 = A'(n,p), \end{aligned}$$

which completes the argument. □

Proposition 3.3. *For any $n \geqslant 1$, $p \geqslant 3$ we have $A'(n,p)=S(n,p)$.*

Proof. The proof is by induction on n. More precisely, we are going to prove that the claim holds for all $n \geqslant 4$, and, for a fixed n, that it is true for all p with $4 \leqslant p \leqslant n$.

For $n=4$ we only need to consider $p=4$. A direct simple computation shows that in this case we have

$$A'(4,4)=S(4,4)=9.$$

Assume now that the induction hypothesis is true for all m with $4 \leqslant m < n$. Let p be an arbitrary integer satisfying $4 \leqslant p \leqslant n$.

The induction hypothesis implies that for all $n_1,\ n_2 \in \mathbb{Z}^+$ for which $n_1+n_2=n$, we have

$$2S(n_1,p)+S(n_2,p-1)=2A'(n_1,p)+A'(n_2,p-1).$$

Thus, the set by which we define $S(n,p)$ (as its minimum) is a subset of the set by which we define $A'(n,p)$ (as its minimum). It follows that $A'(n,p) \leqslant S(n,p)$.

It remains to prove that $A'(n,p) \geqslant S(n,p)$. If for some n_1 and n_2 with $n_1+n_2=n$ we have $A'(n,p)=2A'(n_1,p)+A'(n_2,p-1)$, then we are done, since in this case $A'(n,p)$ is in the set of which $S(n,p)$ is the minimum. Suppose therefore that

$$A'(n,p)=2A'(n_1,p)+2A'(n_2,p-1)+\cdots+2A'(n_{p-i},i+1)+A'(n_{p-i+1},i),$$

where $n_1+n_2+\cdots+n_{p-i+1}=n$ and $2 \leqslant i \leqslant p-2$. (Note that in the particular case of $i=2$ we have $n_{p-i+1}=n_{p-1}=1$ and $A'(n_{p-1},2)=1$). Then, $n_2+n_3+\cdots+n_{p-i+1}=n-n_1$. By Lemma 3.2 we have

$$A'(n-n_1,p-1) \leqslant 2A'(n_2,p-1)+\cdots+2A'(n_{p-i},i+1)+A'(n_{p-i+1},i)$$

and therefore, using the induction hypothesis, we conclude

$$\begin{aligned} A'(n,p) &\geqslant 2A'(n_1,p)+A'(n-n_1,p-1) \\ &= 2S(n_1,p)+S(n-n_1,p-1) \geqslant S(n,p). \qquad \square \end{aligned}$$

Proposition 3.4. *For any $n \geqslant 1$ and any $p \geqslant 3$ we have $A'(n,p)=F'(n,p)$.*

Proof. As in the proof of Proposition 3.3 we proceed by induction on n. For $n=4$ we have

$$\begin{aligned} F'(4,4) &= \min\{2F'(n_1,4)+2F'(n_2,3)+1 \mid n_1+n_2+1=4\} \\ &= \min\{2A'(n_1,4)+2A'(n_2,3)+1 \mid n_1+n_2=3\} \\ &= \min\{2A'(1,4)+2A'(2,3)+1, 2A'(2,4)+2A'(1,3)+1\} \\ &= \min\{2(1+3)+1, 2(3+1)+1\}=9=A'(4,4). \end{aligned}$$

Assume now that $F'(m,p)=A'(m,p)$ for all m with $4 \leqslant m < n$. The set by which we define $F'(n,p)$ (as its minimum) is a subset of the set by which we define $A'(n,p)$ (as its minimum), because for all $n_1,\ldots,n_{p-2}$ for which $n_1+\cdots+n_{p-2}+1=n$, we have

$$2F'(n_1,p)+\cdots+2F'(n_{p-2},3)+1=2A'(n_1,p)+\cdots+2A'(n_{p-2},3)+1.$$

Therefore, $F'(n,p)\geqslant A'(n,p)$.

To prove the other inequality, we use the following dynamic programming argument. By Proposition 3.3 we can write

$$A'(n,p)=2A'(n_1,p)+A'(\widehat{n}_2,p-1),$$

where $n_1+\widehat{n}_2=n$. Applying this argument again we have $A'(\widehat{n}_2,p-1)=2A'(n_2,p-1)+A'(\widehat{n}_3,p-2)$, where $n_2+\widehat{n}_3=\widehat{n}_2$ and so

$$A'(n,p)=2A'(n_1,p)+2A'(n_2,p-1)+A'(\widehat{n}_3,p-2).$$

Now, how far can this procedure be executed? We distinguish two cases.

Case 1: $A'(n,p)=2A'(n_1,p)+\cdots+2A'(n_{p-i},i+1)+A'(\widehat{n}_{p-i+1},i)$, *with* $i>2$ *and* $\widehat{n}_{p-i+1}=1$. Set $r=\max\{k\mid n_k>1\}$ and note that $r\leqslant p-i$. Then, we have

$$\begin{aligned}A'(n,p)&=2A'(n_1,p)+\cdots+2A'(n_r,p-r+1)+2A'(1,p-r)\\&\quad+\cdots+2A'(1,i+1)+A'(1,i)\\&=2A'(n_1,p)+\cdots+2(2A'(n'_r,p-r+1)+A'(n'_{r+1},p-r))\\&\quad+2A'(1,p-r-1)+\cdots+2A'(1,i)+A'(1,i-1)\\&>2A'(n_1,p)+\cdots+2A'(n'_r,p-r+1)+2A'(n'_{r+1},p-r)\\&\quad+2A'(1,p-r-1)+\cdots+2A'(1,i)+A'(1,i-1)\geqslant A'(n,p).\end{aligned}$$

Note that in the above computation the change of variables in the second equality is possible since all the values of $A'(1,p-r),\dots,A'(1,i)$ as well of $A'(1,p-r-1),\dots,A'(1,i-1)$ are equal to 1 and $i-1\geqslant 2$. Thus, we have a contradiction which shows that this case is not possible.

Case 2: $A'(n,p)=2A'(n_1,p)+\cdots+2A'(n_{p-i},i+1)+A'(\widehat{n}_{p-i+1},i)$, *with* $i=2$ *and* $\widehat{n}_{p-i+1}=1$. Now, we have

$$\begin{aligned}A'(n,p)&=2A'(n_1,p)+\cdots+2A'(n_{p-2},3)+1\\&=2F'(n_1,p)+\cdots+2F'(n_{p-2},3)+1\\&\geqslant F'(n,p)\end{aligned}$$

and we are done. □

Combining Propositions 3.3 and 3.4 we can state:

Corollary 3.5. *For any* $n\geqslant 1$ *and any* $p\geqslant 3$ *we have*

$$A'(n,p)=F'(n,p)=S(n,p)\geqslant M(n,p).$$

4. Explicit formulas

Explicit formulas we are going to discuss in this section had appeared already in Frame [6], but have been treated rather heuristically, and — which seems to be the major deficiency of Frame's approach — they appeared as statements about $M(n,p)$. Therefore, we believe it is a necessity to give an independent treatment of them, and using only their properties as well as properties of A', F', S, to prove that they really represent these functions (that shall be done in the following section, and it will turn out to be rather nontrivial).

Definition 4.1. Let $n \geqslant 2$ and $p \geqslant 3$. Then, $X(n,p)$ is defined as

$$X(n,p) = \sum_{t=0}^{s-1} 2^t \frac{(p-3+t)!}{(p-3)!t!} + 2^s \left[n - \frac{(p-3+s)!}{(p-2)!(s-1)!} \right],$$

where s is the largest integer for which the last term on the right side is positive, i.e.

$$s = \max \left\{ k \in \mathbb{Z}^+ \mid n - \frac{(p-3+k)!}{(p-2)!(k-1)!} > 0 \right\}.$$

$X(1,p)$ is defined as

$$X(1,p) = 1.$$

Using binomial coefficients notation, this definition (in case $n \geqslant 2$) can be reformulated in the following way:

$$X(n,p) = \sum_{t=0}^{s-1} 2^t \binom{p-3+t}{p-3} + 2^s \left[n - \binom{p-3+s}{p-2} \right],$$

where

$$s = \max \left\{ k \in \mathbb{Z}^+ \mid n - \binom{p-3+k}{p-2} > 0 \right\}.$$

Already this is a sufficient motivation for the introduction of functions $h_p(x)$, $p \in \mathbb{Z}^+$:

Definition 4.2. For $p \geqslant 3$, let

$$h_p(x) = \binom{p-3+x}{p-2}, \quad x \in \mathbb{R}, \quad x \geqslant 0.$$

From their direct presentation in the form

$$h_p(x) = \frac{x(x+1)\cdots(x+p-4)(x+p-3)}{(p-2)!},$$

it is obvious that (for $p \geqslant 3$) these functions are strictly increasing polynomials (on nonnegative reals).

Since they are strictly increasing, they have inverses

$$g_p = h_p^{-1},$$

which are strictly increasing on nonnegative reals as well.

Definition 4.3. Define (for $p \geqslant 3$)

$$f_p(x) = \lceil g_p(x) \rceil,$$

where $g_p = h_p^{-1}$.

Lemma 4.4. *The number s from Definition* 4.1 (*for* $n \geqslant 2$) *can be written as*

$$s = f_p(n) - 1.$$

Proof. Obviously,

$$s = \max\{k \in \mathbb{Z}^+ \mid n - h_p(k) > 0\}.$$

Since h_p is strictly increasing, this means that s is determined as the unique integer satisfying

$$h_p(s) < n, \quad h_p(s+1) \geqslant n$$

or equivalently

$$s < g_p(n), \quad s + 1 \geqslant g_p(n),$$

i.e.

$$g_p(n) \leqslant s + 1 < g_p(n) + 1$$

and this proves the lemma. □

By Lemma 4.4 $X(n, p)$ (for $n \geqslant 2$) may be described by a single formula as follows:

$$X(n,p) = \sum_{t=0}^{f_p(n)-2} 2^t \binom{p-3+t}{p-3} + 2^{f_p(n)-1}\left[n - \binom{p+f_p(n)-4}{p-2}\right].$$

Lemma 4.5. $X(n,p) - X(n-1,p) = 2^{f_p(n)-1}$, *for any* $n \geqslant 2$ *and* $p \geqslant 3$.

Proof. $X(2,p) - X(1,p) = 2^{f_p(2)-1}$ is true, because $f_p(2) = 2$. For $n \geqslant 3$ we distinguish two cases.

Case 1: $f_p(n-1) = f_p(n)$. In this case, $X(n,p) - X(n-1,p)$ equals

$$2^{f_p(n)-1}\left[n - \binom{p+f_p(n)-4}{p-2}\right] - 2^{f_p(n)-1}\left[n - 1 - \binom{p+f_p(n)-4}{p-2}\right],$$

which in turn equals $2^{f_p(n)-1}(n-(n-1)) = 2^{f_p(n)-1}$.

Case 2: $f_p(n-1) \neq f_p(n)$. If we denote $f_p(n)-1$ by s, then $s = g_p(n-1)$. Suppose not. Then, it would be $g_p(n-1) < s$, hence $n-1 < h_p(s)$. If $h_p(s) \geqslant n$, then $s \geqslant g_p(n)$, and finally $s \geqslant f_p(n)$, which contradicts $f_p(n) = s+1$. Therefore $h_p(s) < n$. It would follow that the integer $h_p(s)$ is strictly between $n-1$ and n, which is a contradiction.

Obviously, it follows also that $f_p(n-1) = s$.

Therefore, in this case $X(n,p) - X(n-1,p)$ can be written as

$$2^{f_p(n)-2}\binom{p+f_p(n)-5}{p-3} + 2^{f_p(n)-1}\left[n - \binom{p+f_p(n)-4}{p-2}\right]$$

$$-2^{f_p(n-1)-1}\left[n-1-\binom{p+f_p(n-1)-4}{p-2}\right].$$

This simplifies to

$$2^{s-1}\binom{p+s-4}{p-3} + 2^{s}\left[n-\binom{p+s-3}{p-2}\right] - 2^{s-1}\left[n-1-\binom{p+s-4}{p-2}\right]$$

$$= 2^{s-1}\left[\binom{p+s-4}{p-3} - 2\binom{p+s-3}{p-2} + \binom{p+s-4}{p-2}\right]$$

$$+2^{s-1}[2n-n+1] = 2^{s-1}\left[n+1-\binom{p+s-3}{p-2}\right] = 2^{s},$$

since

$$\binom{p+s-3}{p-2} = n-1. \qquad \Box$$

Summing up these equalities, together with $X(1,p) = 1 = 2^{f_p(1)-1}$, we obtain

Theorem 4.6. *For any* $n \geqslant 1$ *and* $p \geqslant 3$

$$X(n,p) = \sum_{k=1}^{n} 2^{f_p(k)-1}.$$

5. Proof of $X = S$

In this section we prove the following theorem:

Theorem 5.1. *For all* $p \geqslant 3$, $n \geqslant 1$, *it holds true that*

$$X(n,p) = S(n,p).$$

Proof. We shall prove the claim inductively, together with the following statement:

$$S(h_p(k), p) = 2S(h_p(k-1), p) + S(h_p(k) - h_p(k-1), p-1)$$

for $p \geqslant 4$ and $k \geqslant 2$.

This additional claim forms an important ingredient of our inductive proof of the original statement, but also, together with presentations of $S(n, p)$ in the form $2S(n_1, p) + S(n_2, p-1)$ given in the proof, it can be used for computational purposes.

Note that $h_p(k) - h_p(k-1) = h_{p-1}(k)$ — what we have here are binomial coefficients in disguise!

The trivial range is settled easily.

$S(n,3) = 2^n - 1 = 1 + 2 + \cdots + 2^{n-1} = X(n,3)$, since $f_3(k) = k$.

Also, for $p \geqslant 4$ and $n < p$, we have $X(n, p) = 1 + (n-1)2 = 2n - 1 = S(n, p)$.

In addition to that, we see that $2S(h_p(1), p) + S(h_{p-1}(2), p-1) = 2 \cdot 1 + 2(p-2) - 1 = 2p - 3 = 2(p-1) - 1 = S(h_p(2), p)$.

Now, let n satisfy the inequality

$$h_p(k) + 1 \leqslant n \leqslant h_p(k+1)$$

for some $k \geqslant 2$. Our inductive assumption is that $S(m, p) = X(m, p)$ for all $m, 1 \leqslant m < n$, and that $S(h_p(k), p) = 2S(h_p(k-1), p) + S(h_{p-1}(k), p-1)$.

Then, $f_p(n) = k+1$, $f_p(h_p(k)) = k$, and $f_p(\ell) \geqslant k+2$, for all $\ell > h_p(k+1)$.

If $n_1 > h_p(k)$, then $f_p(n_1) > k$ (in fact $f_p(n_1) = k+1$), and therefore

$$\begin{aligned} &2S(n_1, p) + S(n - n_1, p-1) = 2X(n_1, p) + X(n - n_1, p-1) \\ &\quad = 2(X(n_1 - 1, p) + 2^{f_p(n_1)-1}) + X(n - n_1, p-1) \\ &\quad\quad > 2S(n_1 - 1, p) + S(n - n_1, p-1) + 2^k \geqslant S(n-1, p) + 2^k \\ &\quad = S(n-1, p) + 2^{f_p(n)-1} = X(n, p). \end{aligned}$$

If $n_1 \leqslant h_p(k-1)$, then

$$n - n_1 \geqslant n - h_p(k-1) > h_p(k) - h_p(k-1) = h_{p-1}(k)$$

and therefore $f_{p-1}(n - n_1) \geqslant k+1$. It follows

$$\begin{aligned} &2S(n_1, p) + S(n - n_1, p-1) = 2X(n_1, p) + X(n - n_1, p-1) \\ &\quad = 2X(n_1, p) + X(n - n_1 - 1, p-1) + 2^{f_{p-1}(n-n_1)-1} \\ &\quad \geqslant S(n-1, p) + 2^k = X(n, p). \end{aligned}$$

In the remaining case $h_p(k-1) < n_1 \leqslant h_p(k)$, since $f_p(n_1) = k$, we have

$$\begin{aligned} &2S(n_1, p) + S(n - n_1, p-1) = 2X(n_1, p) + X(n - n_1, p-1) \\ &\quad = 2(X(n_1 - 1, p) + 2^{f_p(n_1)-1}) + X(n - n_1, p-1) \\ &\quad = 2S(n_1 - 1, p) + S(n - n_1, p-1) + 2^k \geqslant S(n-1, p) + 2^k = X(n, p). \end{aligned}$$

Thus, since these three cases cover all the possibilities, it follows that $S(n,p) \geqslant X(n,p)$. Also, if some n_1 satisfies $2S(n_1,p)+S(n-n_1,p-1)=X(n,p)$, it means that $S(n,p)=X(n,p)=2S(n_1,p)+S(n-n_1,p-1)$.

Suppose $S(n-1,p)=X(n-1,p)=2S(m_1,p)+S(n-1-m_1,p-1)$, where $h_p(k-1) \leqslant m_1, m_1+1 \leqslant h_p(k)$. In that case $f_p(m_1+1)=k$ and

$$\begin{aligned}
&2S(m_1+1,p)+S(n-(m_1+1),p-1)\\
&\quad = 2(S(m_1,p)+2^{f_p(m_1+1)-1})+S(n-(m_1+1),p-1)\\
&\quad = 2S(m_1,p)+S(n-1-m_1,p-1)+2^k = S(n-1,p)+2^k\\
&\quad = X(n-1,p)+2^{f_p(n)-1} = X(n,p)
\end{aligned}$$

and it follows that $S(n,p)=X(n,p)=2S(m_1+1,p)+S(n-1-m_1,p-1)$.

Using this result, starting from inductive assumption

$$\begin{aligned}
S(h_p(k),p) &= X(h_p(k),p)\\
&= 2S(h_p(k-1),p)+S(h_p(k)-h_p(k-1),p-1)
\end{aligned}$$

we get [3]

$$\begin{aligned}
S(h_p(k)+1,p) &= X(h_p(k)+1,p)\\
&= 2S(h_p(k-1)+1,p)+S(h_p(k)-h_p(k-1),p-1),\\
S(h_p(k)+2,p) &= X(h_p(k)+2,p)\\
&= 2S(h_p(k-1)+2,p)+S(h_p(k)-h_p(k-1),p-1),\\
&\;\vdots\\
S(m,p) = X(m,p) &= 2S(h_p(k),p)+S(h_p(k)-h_p(k-1),p-1),
\end{aligned}$$

where $m=h_p(k)+(h_p(k)-(h_p(k-1))$.

It is obvious that $m < h_p(k+1)$ (since it is equivalent to $h_p(k)-h_p(k-1) < h_p(k+1)-h_p(k)$, i.e. to $h_{p-1}(k) < h_{p-1}(k+1)$).

Hence, it remains to fulfill the inductive step in the remaining case when n satisfies $m < n \leqslant h_p(k+1)$. "Inductive proof" now means that inductively we prove $S(t,p)=X(t,p)=2S(h_p(k),p)+S(t-h_p(k))$ for all t satisfying $m \leqslant t \leqslant h_p(k+1)$. Note that by the choice of m that is true for $t=m$.

Let n satisfy $m < n \leqslant h_p(k+1)$ and let the assumption hold true for $n-1$.

In that case $h_p(k)-h_p(k-1)=m-h_p(k) < n-h_p(k) \leqslant h_p(k+1)-h_p(k)$. It means $h_{p-1}(k) < n-h_p(k) \leqslant h_{p-1}(k+1)$, and therefore it follows that $f_{p-1}(n-h_p(k))=k+1$.

[3] In [7] Hinz uses a similar approach for the proof of $X(n,4)=S(n,4)$.

Now, we have

$$
\begin{aligned}
&2S(h_p(k), p) + S(n - h_p(k), p - 1)\\
&\quad = 2X(h_p(k), p) + X(n - h_p(k), p - 1)\\
&\quad = 2X(h_p(k), p) + X(n - 1 - h_p(k), p - 1) + 2^{f_{p-1}(n-h_p(k))-1}\\
&\quad = X(n - 1, p) + 2^k = X(n, p).
\end{aligned}
$$

Note that as a special case (for $n = h_p(k + 1)$) we get

$$2S(h_p(k), p) + S(h_p(k + 1) - h_p(k), p - 1) = S(h_p(k + 1), p),$$

thus finishing also the inductive proof of the additional claim, introduced at the beginning of the proof. □

6. And finally: all together now

Finally, we are able to show that the original Frame's function F, as well as our function A, both coincide with all the rest.

Lemma 6.1. $X(n, p) - X(n - 1, p) \leqslant X(n, p - 1) - X(n - 1, p - 1)$.

Proof. It means $2^{f_p(n)-1} \leqslant 2^{f_{p-1}(n)-1}$, hence is equivalent to $f_p(n) \leqslant f_{p-1}(n)$, which is easily seen to be true. □

Corollary 6.2. $X(m, p) + X(n, p - 1) \geqslant X(n, p) + X(m, p - 1)$, *if* $m < n$.

Proof. The inequality is equivalent to

$$X(n, p) - X(m, p) \leqslant X(n, p - 1) - X(m, p - 1)$$

and this follows by summing up inequalities

$$
\begin{aligned}
X(n, p) - X(n - 1, p) &\leqslant X(n, p - 1) - X(n - 1, p - 1),\\
X(n - 1, p) - X(n - 2, p) &\leqslant X(n - 1, p - 1) - X(n - 2, p - 1),\\
&\vdots\\
X(m + 2, p) - X(m + 1, p) &\leqslant X(m + 2, p - 1) - X(m + 1, p - 1),\\
X(m + 1, p) - X(m, p) &\leqslant X(m + 1, p - 1) - X(m, p - 1),
\end{aligned}
$$

which hold by Lemma 6.1. □

Proposition 6.3. *For any* $n \geqslant 1$ *and any* $p \geqslant 3$, *we have* $F'(n, p) = F(n, p)$.

Proof. For the trivial range it is so by definition. Therefore, we shall prove the equality by induction on n for $p \geqslant 4$ and $n \geqslant p$.

Obviously, $F'(n,p) \leqslant F(n,p)$, since more partitions are taken into account on the left hand side.

Let $F'(n,p) = 2F'(n_1,p) + \cdots + 2F'(n_{p-2},3) + 1$, where $n_1 + \cdots + n_{p-2} + 1 = n$ and $n_i < n_{i+1}$, for some i.

Then, using Corollary 6.2 and inductive assumptions, we get

$$\begin{aligned} F'(n,p) &= 2(F(n_1,p) + \cdots + F(n_{i-1}, p-i+2) \\ &\quad + F(n_i, p-i+1) + F(n_{i+1}, p-i) + \cdots) + 1 \\ &= 2(X(n_1,p) + \cdots + X(n_{i-1}, p-i+2) \\ &\quad + X(n_i, p-i+1) + X(n_{i+1}, p-i) + \cdots) + 1 \\ &\geqslant 2(X(n_1,p) + \cdots + X(n_{i-1}, p-i+2) \\ &\quad + X(n_{i+1}, p-i+1) + X(n_i, p-i) + \cdots) + 1 \\ &\geqslant 2(F(n_1,p) + \cdots + F(n_{i-1}, p-i+2) \\ &\quad + F(n_{i+1}, p-i+1) + F(n_i, p-i) + \cdots) + 1. \end{aligned}$$

After finitely many repetitions of this procedure, we get that $F'(n,p)$ is at least as large as one of the defining sums for $F(n,p)$, hence $F'(n,p) \geqslant F(n,p)$. □

Proposition 6.4. *For any* $n \geqslant 1$ *and any* $p \geqslant 3$, *we have* $A'(n,p) = A(n,p)$.

Proof. The trivial range is once again trivial. In the nontrivial range we have $A'(n,p) \leqslant A(n,p)$, since the set defining the right-hand side is a subset of the set defining the left-hand side. Then, we have $A(n,p) \leqslant F(n,p)$, for the same reason, and we already know that $A'(n,p) = F(n,p)$. □

Hence, we have completed the proof of

Theorem 6.5. *For any* $n \geqslant 1$ *and any* $p \geqslant 3$ *we have*

$$A(n,p) = A'(n,p) = F(n,p) = F'(n,p) = S(n,p) = X(n,p) \geqslant M(n,p).$$

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