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The game $L(d, 1)$-labeling problem of graphs

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ABSTRACT

Let G be a graph and let k be a positive integer. Consider the following two-person game which is played on G: Alice and Bob alternate turns. A move consists of selecting an unlabeled vertex v of G and assigning it a number a from $\{0, 1, 2, \ldots, k\}$ satisfying the condition that, for all $u \in V(G)$, u is labeled by the number b previously, if $d(u, v) = 1$, then $|a - b| \geq d$, and if $d(u, v) = 2$, then $|a - b| \geq 1$. Alice wins if all the vertices of G are successfully labeled. Bob wins if an impasse is reached before all vertices in the graph are labeled. The *game $L(d, 1)$-labeling number* of a graph G is the least k for which Alice has a winning strategy. We use $\tilde{\lambda}_1^d(G)$ to denote the game $L(d, 1)$-labeling number of G in the game Alice plays first, and use $\tilde{\lambda}_2^d(G)$ to denote the game $L(d, 1)$-labeling number of G in the game Bob plays first. In this paper, we study the game $L(d, 1)$-labeling numbers of graphs. We give formulas for $\tilde{\lambda}_1^d(K_n)$ and $\tilde{\lambda}_2^d(K_n)$, and give formulas for $\tilde{\lambda}_1^d(K_{m,n})$ for those d with $d \geq \max\{m, n\}$.

1. Introduction

Let G be a graph and let X be a set of colors. Consider the following two-person game which is played on G: Alice (player 1) and Bob (player 2) alternate turns. A move consists of selecting a previously uncolored vertex v of G and assigning it a color from the color set X distinct from the color assigned previously (by either player) to neighbors of v. If after $n = |V(G)|$ moves, the graph G is colored, Alice is the winner. Bob wins if an impasse is reached before all nodes in the graph are colored, i.e., for every uncolored vertex v and every color α from X, v is adjacent to a vertex having color α. The *game chromatic number* of G is the least cardinality of a color set X for which Alice has a wining strategy. We use $\chi_g^1(G)$ to denote the game chromatic number of G in the game Alice plays first, and use $\chi_g^2(G)$ to denote the game chromatic number of G in the game Bob plays first. Bodlaender [1] first introduced this problem and studied its computational complexity. Faigle et al. [4] showed that the game chromatic number of the class of forests is 4. Kierstead and Trotter [10] showed that the game chromatic number of the class of planar graphs is between 7 and 33. This upper bound was improved by Dinski and Zhu in [3] to 30, then reduced to 19 in [13], and further reduced to 18 in [9]. It was shown in [10] that outerplanar graphs have game chromatic number at most 8, and this upper bound is reduced to 7 in [7].

Given a graph G and nonnegative integers p, q with $p \geq q$, an *$L(p, q)$-labeling* of G is a function f from the vertex set $V(G)$ to the set of all nonnegative integers such that $|f(u) - f(v)| \geq p$ if $d(u, v) = 1$ and $|f(u) - f(v)| \geq q$ if $d(u, v) = 2$. For a nonnegative integer k, a *k–$L(p, q)$-labeling* is an $L(p, q)$-labeling such that no label is greater than k. The *$L(p, q)$-labeling number* of G, denoted by $\lambda_{p,q}(G)$, is the smallest number k such that G has a k–$L(p, q)$-labeling (when $q = 1$, we use $\lambda_p(G)$

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to replace $\lambda_{p,q}(G)$ in short). This problem came from the frequency assignment problem introduced by Hale [8] and was first proposed by Griggs and Yeh [6] and Roberts [11]. Griggs and Yeh [6] showed that the $L(2,1)$-labeling problem is **NP**-complete for general graphs and proved that $\lambda_2(G) \leqslant \Delta^2(G) + 2\Delta(G)$. Chang and Kuo [2] gave a polynomial-time algorithm for the $L(2,1)$-labeling problem on trees. Gonçalves [5] proved that $\lambda_2(G) \leqslant \Delta^2(G) + \Delta(G) - 2$ for any graph G with $\Delta(G) \geq 2$. There are also many other results surrounding this topic; for a recent survey see [12] and the references therein.

We consider a new labeling game which comes from the $L(p,q)$-labeling problem and can be viewed as a generalization of the game coloring problem. Given a graph G and integers k, p, q with $k \geq 0$ and $p > q \geq 0$. Consider the following two-person game which is played on G: Alice (player 1) and Bob (player 2) alternate turns. A move consisting of selecting a previously unlabeled vertex v of G and assigning it a number a from $\{0, 1, 2, \ldots, k\}$ satisfying the condition that, for all $u \in V(G)$, if u is labeled by the number b previously, then $|a-b| \geq p$ if $d(u,v) = 1$, and $|a-b| \geq q$ if $d(u,v) = 2$. Alice wins if all the vertices of G are successfully labeled. Bob wins if an impasse is reached before all vertices in the graph are labeled. The *game $L(p,q)$-labeling number* of a graph G is the least k for which Alice has a winning strategy. We use $\tilde{\lambda}_1^{p,q}(G)$ to denote the game $L(p,q)$-labeling number of G in the game Alice plays first, and use $\tilde{\lambda}_2^{p,q}(G)$ to denote the game $L(p,q)$-labeling number of G in the game Bob plays first. When $q = 1$, we use $\tilde{\lambda}_1^{p}(G)$ (resp. $\tilde{\lambda}_2^{p}(G)$) to replace $\tilde{\lambda}_1^{p,q}(G)$ (resp. $\tilde{\lambda}_2^{p,q}(G)$) for short. Note that from the definition, $\chi_g^1(G) = \tilde{\lambda}_1^{1,0}(G)$ and $\chi_g^2(G) = \tilde{\lambda}_2^{1,0}(G)$.

We study the game $L(d,1)$-labeling number of graphs in this paper. We give formulas for $\tilde{\lambda}_1^d(K_n)$ and $\tilde{\lambda}_2^d(K_n)$, and give formulas for $\tilde{\lambda}_1^d(K_{m,n})$ for those d with $d \geq \max\{m, n\}$.

2. Preliminary

In this section, we first fix some notation and terminology, and then derive an upper bound for the game $L(d,1)$-labeling number of graphs. The following lemma is a direct consequence of the definition.

Lemma 1. *For any graph G, $\tilde{\lambda}_1^d(G) \geq \lambda_d(G)$ and $\tilde{\lambda}_2^d(G) \geq \lambda_d(G)$.*

From now on, for convenience, we use $\tilde{\lambda}^d_{i_{(j,S)}}(G)$ to denote the smallest number needed to complete the $(d,1)$-game on G with player i plays first under a fixed strategy S made by player j. And, to simplify the notations, for a given graph G, we use $C_i^m(j) = (v; l)$ to denote that, under the constraint that the numbers can be used is no more than the number m, player i chooses an unlabeled vertex v and labels it by the number l in the jth move, and use $PC^m(j; v)$ to denote the set of numbers in $\{0, 1, 2, \ldots, m\}$ that can be chosen to label the unlabeled vertex v after the jth move.

From the definitions above, $\tilde{\lambda}_i^d(G) \leq m$ for $i = 1, 2$ if player 1 has a strategy so that $PC^m(j; v) \neq \emptyset$ for all j, $1 \leq j \leq |V(G)| - 1$ and all vertices v that are unlabeled after the jth move. And $\tilde{\lambda}_i^d(G) > m$ for $i = 1, 2$ if player 2 has a strategy so that $PC^m(j; v) = \emptyset$ for some j, $1 \leq j \leq |V(G)| - 1$ and some vertex v that is unlabeled after the jth move.

Given a graph G and $v \in V(G)$, $S \subseteq V(G)$, we use $N_i(v)$ to denote the set $\{u : d_G(u,v) = i\}$, and we let $N_i(S)$ to be the set $\bigcup_{v \in S} N_i(v)$. And, for $a \in \mathbb{N} \cup \{0\}$, we use the notation $n_d(a)$ to denote the set $\{a-d+1, a-d+2, \ldots, a+d-1\} \cap (\mathbb{N} \cup \{0\})$.

Theorem 2. *If G is a graph with maximum degree Δ, then $\tilde{\lambda}_i^d(G) \leq (\Delta + 2d - 2)\Delta$ for $i = 1, 2$ and $d \geq 2$.*

Proof. Let $m = (\Delta + 2d - 2)\Delta$. Consider the following strategy S made by Alice. At the ith step, if there is a vertex v that is still unlabeled, and $PC^m(i-1; v) \neq \emptyset$, then Alice chooses a number a in $PC^m(i-1; v)$ and labels v by a at the ith step. We claim that Alice can complete the $(d,1)$-game played on G under this strategy.

Let w be an unlabeled vertex after the jth step. Define $A_k = \{l : C_i^m(j') = (v; l) \text{ for some } j' \leq j \text{ and } v \in N_k(w)\}$ for $k = 1, 2$. Then, since $PC^m(j; w) = \{0, 1, 2, \ldots, m\} - ((\cup_{l \in A_1} n_d(l)) \cup A_2)$, $|PC^m(j; w)| \geq m + 1 - ((2d-1)\Delta + \Delta(\Delta - 1)) \geq 1$. Hence $PC^m(j; w) \neq \emptyset$. Since $PC^m(j; v) \neq \emptyset$ for all j, $1 \leq j \leq |V(G)| - 1$ and all vertices v that are unlabeled after the jth move, $\tilde{\lambda}^d_{i_{(1,S)}}(G) \leq m = (\Delta + 2d - 2)\Delta$. Thus $\tilde{\lambda}_i^d(G) \leq \tilde{\lambda}^d_{i_{(1,S)}}(G) \leq (\Delta + 2d - 2)\Delta$ for $i = 1, 2$ and $d \geq 2$. □

3. Game $L(d,1)$-labeling number of complete graphs

We study the game $L(d,1)$-labeling number of complete graphs in this section.

Theorem 3. *For all $n \geq 1$, $r \geq 0$, $\tilde{\lambda}_1^d(K_n \cup rK_1) = (4d-1)\left\lceil \frac{n-3}{3} \right\rceil + [(n-1) \bmod 3]d$ and $\tilde{\lambda}_2^d(K_n \cup rK_1) = (4d-1)\left\lceil \frac{n-1}{3} \right\rceil - 2d + [(n-2) \bmod 3]d$.*

Proof. Let m be the largest number that can be used in the $(d,1)$-game on $K_n \cup rK_1$. Consider the following strategy made by Alice. At the ith step, if there is a vertex v of K_n that is still unlabeled, and there exists a number l, $d \leq l \leq m$, such that $\{l-d, l-d+1, \ldots, l\} \subseteq PC^m(i-1; v)$, then Alice labels v with the number j, where $j = \min\{l : d \leq l \leq m, \{l-d, l-d+1, \ldots, l\} \subseteq PC^m(i-1; v)\}$. If no such number exists, and $PC^m(i-1; v) \neq \emptyset$, then Alice chooses an arbitrary number j in $PC^m(i-1; v)$ and labels v with the number j. And if all the vertices of K_n are labeled, choose an unlabeled vertex of rK_1 and label it with 0. We use S_1 to denote this strategy. For Bob, consider the following strategy made by Bob. At the ith step, if there is a vertex v of K_n that is still unlabeled, and there exists a number l, $d-1 \le l \le m$, such that $\{l-d+1, l-d+2, \dots, l\} \subseteq PC^m(i-1; v)$, then Bob labels v with the number j, where $j = \min\{l : d-1 \le l \le m, \{l-d+1, l-d+2, \dots, l\} \subseteq PC^m(i-1; v)\}$. If no such number exists, and $PC^m(i-1; v) \ne \emptyset$, then Bob chooses an arbitrary number j in $PC^m(i-1; v)$ and labels v with the number j. And if all the vertices of K_n are labeled, choose an unlabeled vertex of rK_1 and label it with 0. We use S_2 to denote this strategy.

We prove this theorem by proving a stronger statement. For the strategies S_1 and S_2, $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_n \cup rK_1) \le (4d-1)\left\lceil \frac{n-3}{3} \right\rceil + [(n-1) \bmod 3]d \le \tilde{\lambda}^d_{1_{(2,S_2)}}(K_n \cup rK_1)$ and $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_n \cup rK_1) \le (4d-1)\left\lceil \frac{n-1}{3} \right\rceil - 2d + [(n-2) \bmod 3]d \le \tilde{\lambda}^d_{2_{(2,S_2)}}(K_n \cup rK_1)$. To prove this, we use induction on n. The conclusion clearly holds for $n = 1, 2$. Suppose it holds for all n, $2 \le n < t$.

Claim 1. $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_t \cup rK_1) \le (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$.

Proof of Claim 1. Let $p = \tilde{\lambda}^d_{1_{(1,S_1)}}(K_t \cup rK_1)$. By the definition of S_1, Alice first chooses a vertex of K_t and labels it by d. After the first step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, with Bob plays first. More precisely, if $C_i^p(j) = (v; l)$ for some $j \ge 2$, $l \ge 2d$, and some unlabeled vertex v of K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses a vertex v of K_{t-2} and labels it by $l-2d$ at the $(j-1)$th step. And if $C_i^p(j) = (v; l)$ for some $j \ge 2$, where v is an unlabeled vertex of K_t and $l = 0$, or v is an unlabeled vertex of rK_1, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses an isolated vertex v of $(r+1)K_1$ and labels it by 0 at the $(j-1)$th step. Since

$$\begin{aligned}\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-2} \cup (r+1)K_1) &\le (4d-1)\left\lceil \frac{(t-2)-1}{3} \right\rceil - 2d + [(t-4) \bmod 3]d \\ &= (4d-1)\left\lceil \frac{t-3}{3} \right\rceil - 2d + [(t-1) \bmod 3]d,\end{aligned}$$

we have

$$\tilde{\lambda}^d_{1_{(1,S_1)}}(K_t \cup rK_1) \le \tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-2} \cup (r+1)K_1) + 2d = (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d. \quad \square$$

Claim 2. $\tilde{\lambda}^d_{2_{(2,S_2)}}(K_t \cup rK_1) \ge (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$.

Proof of Claim 2. Let $p = \tilde{\lambda}^d_{2_{(2,S_2)}}(K_t \cup rK_1)$. By the definition of S_2, Bob first selects a vertex of K_t and labels it by $d-1$. After the first step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-1} \cup rK_1$, with Alice plays first. More precisely, if $C_i^p(j) = (v; l)$ for some $j \ge 2$, $l \ge 2d-1$, and some unlabeled vertex v of K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-1} \cup rK_1$, player i chooses a vertex v of K_{t-1} and labels it by $l-2d+1$ at the $(j-1)$th step. And if $C_i^p(j) = (v; l)$ for some $j \ge 2$ and some unlabeled vertex v of rK_1, then we can imagine that on the $(d, 1)$-game played on $K_{t-1} \cup rK_1$, player i chooses an isolated vertex v of rK_1 and labels it by 0 at the $(j-1)$th step. Since

$$\begin{aligned}\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-1} \cup rK_1) &\ge (4d-1)\left\lceil \frac{(t-1)-3}{3} \right\rceil + [(t-2) \bmod 3]d \\ &= (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 4d + 1 + [(t-2) \bmod 3]d,\end{aligned}$$

we have

$$\tilde{\lambda}^d_{2_{(2,S_2)}}(K_t \cup rK_1) \ge \tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-1} \cup rK_1) + 2d - 1 \ge (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d. \quad \square$$

Claim 3. $\tilde{\lambda}^d_{1_{(2,S_2)}}(K_t \cup rK_1) \ge (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$.

Proof of Claim 3. Let $p = \tilde{\lambda}^d_{1_{(2,S_2)}}(K_t \cup rK_1)$. If at the first step, Alice chooses an isolated vertex of rK_1 and labels it by q, then we can imagine that this is just the $(d, 1)$-game played on $K_t \cup (r-1)K_1$, with Bob plays first. Thus by Claim 2 and its proof, the smallest number needed to complete this game in this case is at least $(4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$ and we are done. So, assume that at the first step, Alice chooses a vertex of K_t and labels it by q. Consider the following cases:

Case 1. $q \le d-1$.

In this case, by the definition of S_2, Bob selects a vertex of K_t and labels it by $q + 2d - 1$ at the second step. After this step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-2} \cup rK_1$, with Alice plays first. More precisely, if

$C_i^p(j) = (v; l)$ for some $j \geq 3, l \geq q + 3d - 1$, and some unlabeled vertex v of K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup rK_1$, player i chooses a vertex v of K_{t-2} and labels it by $l - q - 3d + 1$ at the $(j-2)$th step. And if $C_i^p(j) = (v; l)$ for some $j \geq 3$ and some unlabeled vertex v of rK_1, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup rK_1$, player i chooses an isolated vertex v of rK_1 and labels it by 0 at the $(j-2)$th step. Since $\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-2} \cup rK_1) \geq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 4d + 1 + [t \bmod 3]d$, we have

$$\begin{aligned} \tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-2} \cup rK_1) + (q + 3d - 1) &\geq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 4d + 1 + [t \bmod 3]d + (q + 3d - 1) \\ &= (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - d + [t \bmod 3]d + q \\ &\geq (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d, \end{aligned}$$

thus the smallest number needed to complete this game in this case is at least $(4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$.

Case 2. $d \leq q \leq 2d - 2$.

In this case, after the first step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, with Bob plays first. More precisely, if $C_i^p(j) = (v; l)$ for some $j \geq 2$, $l \geq q + d$, and some unlabeled vertex v in K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses a vertex v of K_{t-2} and labels it by $l - q - d$ at the $(j-1)$th step. If $C_i^p(j) = (v; l)$ for some $j \geq 2$, where v is an unlabeled vertex of rK_1, or v is an unlabeled vertex of K_t and $l \leq q - d$, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses an isolated vertex v of $(r+1)K_1$ and labels it by 0 at the $(j-1)$th step. Since $\tilde{\lambda}^d_{2_{(2,S_2)}}(K_{t-2} \cup (r+1)K_1) \geq (4d-1)\left\lceil \frac{t-3}{3} \right\rceil - 2d + [(t-1) \bmod 3]d$, we have

$$\begin{aligned} \tilde{\lambda}^d_{2_{(2,S_2)}}(K_{t-2} \cup (r+1)K_1) + (q + d) &\geq (4d-1)\left\lceil \frac{t-3}{3} \right\rceil - 2d + [(t-1) \bmod 3]d + (q + d) \\ &= (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d + q - d \\ &\geq (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d, \end{aligned}$$

thus the smallest number needed to complete this game in this case is at least $(4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$.

Case 3. $2d - 1 \leq q \leq 3d - 2$.

In this case, by the definition of S_2, Bob selects a vertex of K_t and labels it by $d-1$ at the second step. Since after the second step, two vertices of K_t are labeled and all the numbers in $\{0, 1, \ldots, q + d - 1\}$ cannot be used to label the other vertices of K_t, and $q + d - 1 \leq 4d - 3$, this case is the same as Case 1, since we may imagine that at the first step, Alice chooses a vertex of K_t and labels it with a number $q' = q - 2d + 1$. Hence, by Case 1, the smallest number needed to complete this game in this case is at least $(4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d$.

Case 4. $q \geq 3d - 1$.

In this case, let $\beta = \max\{k : C_2^p(2k) = (v; (k-1)(2d-1) + d - 1), v \text{ is a vertex of } K_t\}$. Clearly, $\beta < t$. By the definition of S_2, there exists $a \leq \beta + 1$, such that $C_1^p(2a-1) = (v; \alpha)$, where v is a vertex of K_t and $\beta(2d-1) \leq \alpha \leq (\beta+1)(2d-1) - 1$. Let $c = \max\{2a - 1, 2\beta\}$:

$$\theta = \begin{cases} 0, & \text{if } \beta(2d-1) \leq \alpha \leq \beta(2d-1) + d - 1, \\ 1, & \text{if } \beta(2d-1) + d \leq \alpha \leq (\beta+1)(2d-1) - 1, \end{cases}$$

$$b = \begin{cases} a, & \text{if } a \leq \beta, \\ a - 1, & \text{if } a = \beta + 1, \end{cases}$$

and let

$$j^* = \begin{cases} j, & \text{if } 1 \leq j \leq \min\{2a-1, 2b\} - 1, \\ j - 2, & \text{if } j \geq \max\{2a-1, 2b\} + 1, \end{cases}$$

for all $j \geq 1, j \notin \{2a - 1, 2b\}$. Note that by the definition of θ, no numbers less than α can be used to label the unlabeled vertices of K_t after the cth step when $\theta = 0$, and when $\theta = 1$, the numbers less than α that can be used to label the unlabeled vertices of K_t after the cth step belong to $\{\beta(2d-1), \beta(2d-1) + 1, \ldots, \alpha - d\}$. Also note that by the definition of b, Bob chooses a vertex of K_t and labels it by $(b-1)(2d-1) + d - 1$ at the $(2b)$th step, where $2b = 2a$ if $a \leq \beta$, and $2b = 2a - 2$ if $a = \beta + 1$. Hence in this case, after the cth step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-2-\theta} \cup (r+\theta)K_1$, with Alice plays first (by ignoring the $(2a-1)$th step made by Alice and the $(2b)$th step made by Bob). More precisely, if $C_i^p(j) = (v; l)$ for some unlabeled vertex v of K_t and some l, $l \leq (b-2)(2d-1) + d - 1$, then we can imagine that on the

$(d,1)$-game played on $K_{t-2-\theta}\cup(r+\theta)K_1$, player i chooses a vertex v of $K_{t-2-\theta}$ and labels it by l at the j^*th step. If $C_i^p(j)=(v;l)$ for some unlabeled vertex v of K_t and some l, $b(2d-1)+d-1\le l\le(\beta-1)(2d-1)+d-1$, then we can imagine that on the $(d,1)$-game played on $K_{t-2-\theta}\cup(r+\theta)K_1$, player i chooses a vertex v of $K_{t-2-\theta}$ and labels it by $l-2d+1$ at the j^*th step. If $C_i^p(j)=(v;l)$ for some unlabeled vertex v of K_t and some l, $l\ge\alpha+d$, then we can imagine that on the $(d,1)$-game played on $K_{t-2-\theta}\cup(r+\theta)K_1$, player i chooses a vertex v of $K_{t-2-\theta}$ and labels it by $l-[\alpha-(\beta-1)(2d-1)+d]$ at the j^*th step. And if $C_i^p(j)=(v;l)$ for some $j\ge 3$, where v is an unlabeled vertex of K_t and $\beta(2d-1)\le l\le\alpha-d$ (when $\theta=1$), or v is a vertex of rK_1, then we can imagine that on the $(d,1)$-game played on $K_{t-2-\theta}\cup(r+\theta)K_1$, player i chooses an isolated vertex v of $(r+\theta)K_1$ and labels it by 0 at the j^*th step. Since $\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-2}\cup rK_1)\ge(4d-1)\left\lceil\frac{t-2}{3}\right\rceil-4d+1+[t \bmod 3]d$ and $\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-3}\cup(r+1)K_1)\ge(4d-1)\left\lceil\frac{t-3}{3}\right\rceil-4d+1+[(t-1) \bmod 3]d$, when $\theta=0$, we have

$$\begin{aligned}
&\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-2}\cup rK_1)+[\alpha-(\beta-1)(2d-1)+d]\\
&\quad\ge(4d-1)\left\lceil\frac{t-2}{3}\right\rceil-4d+1+[t \bmod 3]d+[\alpha-(\beta-1)(2d-1)+d]\\
&\quad=(4d-1)\left\lceil\frac{t-2}{3}\right\rceil+[t \bmod 3]d+[\alpha-(\beta-1)(2d-1)-3d+1]\\
&\quad\ge(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d,
\end{aligned}$$

and when $\theta=1$, we have

$$\begin{aligned}
&\tilde{\lambda}^d_{1_{(2,S_2)}}(K_{t-3}\cup(r+1)K_1)+[\alpha-(\beta-1)(2d-1)+d]\\
&\quad\ge(4d-1)\left\lceil\frac{t-3}{3}\right\rceil-4d+1+[(t-1) \bmod 3]d+[\alpha-(\beta-1)(2d-1)+d]\\
&\quad=(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d+[\alpha-(\beta-1)(2d-1)-3d+1]\\
&\quad\ge(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d.
\end{aligned}$$

Thus the smallest number needed to complete this game in this case is at least $(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d$.

Since for all cases, the smallest number needed to complete this game is at least $(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d$, $\tilde{\lambda}^d_{1_{(2,S_2)}}(K_t\cup rK_1)\ge(4d-1)\left\lceil\frac{t-3}{3}\right\rceil+[(t-1) \bmod 3]d$. □

Claim 4. $\tilde{\lambda}^d_{2_{(1,S_1)}}[K_t\cup rK_1]\le(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-2d+[(t-2) \bmod 3]d$.

Proof of Claim 4. Let $p=\tilde{\lambda}^d_{2_{(1,S_1)}}(K_t\cup rK_1)$. If at the first step, Bob chooses an isolated vertex of rK_1 and labels it by q, then we can imagine that this is just the $(d,1)$-game played on $K_t\cup(r-1)K_1$, with Alice plays first. Thus by Claim 1 and its proof, the smallest number needed to complete this game in this case is no more than $(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-2d+[(t-2) \bmod 3]d$ and we are done. So, assume that at the first step, Bob chooses a vertex of K_t and labels it by q. Consider the following cases:

Case 1. $q\le d-1$.

In this case, after the first step, we can imagine that this is just the $(d,1)$-game played on $K_{t-1}\cup rK_1$, with Alice plays first. More precisely, if $C_i^p(j)=(v;l)$ for some $j\ge 2$, $l\ge q+d$, and some unlabeled vertex v in K_t, then we can imagine that on the $(d,1)$-game played on $K_{t-1}\cup rK_1$, player i chooses a vertex v of K_{t-1} and labels it by $l-q-d$ at the $(j-1)$th step. And if $C_i^p(j)=(v;l)$ for some $j\ge 2$ and some unlabeled vertex v in rK_1, then we can imagine that on the $(d,1)$-game played on $K_{t-1}\cup rK_1$, player i chooses an isolated vertex v of rK_1 and labels it by 0 at the $(j-1)$th step. Since $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_{t-1}\cup rK_1)\le(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-4d+1+[(t-2) \bmod 3]d$, we have

$$\begin{aligned}
\tilde{\lambda}^d_{1_{(1,S_1)}}(K_{t-1}\cup rK_1)+(q+d)&\le(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-4d+1+[(t-2) \bmod 3]d+(q+d)\\
&=(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-2d+[(t-2) \bmod 3]d+q-d+1\\
&\le(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-2d+[(t-2) \bmod 3]d,
\end{aligned}$$

thus the smallest number needed to complete this game in this case is no more than $(4d-1)\left\lceil\frac{t-1}{3}\right\rceil-2d+[(t-2) \bmod 3]d$.

Case 2. $d \leq q \leq 2d-1$.

In this case, after the first step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, with Alice plays first. More precisely, if $C_i^p(j) = (v; l)$ for some $j \geq 2$, $l \geq q+d$, and some unlabeled vertex v in K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses a vertex v of K_{t-2} and labels it by $l-q-d$ at the $(j-1)$th step. If $C_i^p(j) = (v; l)$ for some $j \geq 2$, where v is an unlabeled vertex in rK_1, or v is an unlabeled vertex in K_t and $l \leq q-d$, then we can imagine that on the $(d, 1)$-game played on $K_{t-2} \cup (r+1)K_1$, player i chooses an isolated vertex v of $(r+1)K_1$ and labels it by 0 at the $(j-1)$th step. Since $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_{t-2} \cup (r+1)K_1) \leq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 4d + 1 + [t \bmod 3]d$, we have

$$\begin{aligned}\tilde{\lambda}^d_{1_{(1,S_1)}}(K_{t-2} \cup (r+1)K_1) + (q+d) &\leq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 4d + 1 + [t \bmod 3]d + (q+d)\\ &= (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 2d + [t \bmod 3]d + q - d + 1\\ &\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d,\end{aligned}$$

thus the smallest number needed to complete this game in this case is no more than $(4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$.

Case 3. $2d \leq q \leq 3d-1$.

In this case, by the definition of S_1, Alice selects a vertex of K_t and labels it by d at the second step. After this step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-3} \cup (r+1)K_1$, with Bob plays first. More precisely, if $C_i^p(j) = (v; l)$ for some $j \geq 3$, $l \geq q+d$, and some unlabeled vertex v in K_t, then we can imagine that on the $(d, 1)$-game played on $K_{t-3} \cup (r+1)K_1$, player i chooses a vertex v of K_{t-3} and labels it by $l-q-d$ at the $(j-2)$th step. And if $C_i^p(j) = (v; l)$ for some $j \geq 3$, where v is an unlabeled vertex of K_t and $l = 0$, or v is an unlabeled vertex of rK_1, then we can imagine that on the $(d, 1)$-game played on $K_{t-3} \cup (r+1)K_1$, player i chooses an isolated vertex v of $(r+1)K_1$ and labels it by 0 at the $(j-2)$th step. Since $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-3} \cup (r+1)K_1) \leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 6d + 1 + [(t-2) \bmod 3]d$, we have

$$\begin{aligned}\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-3} \cup (r+1)K_1) + (q+d) &\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 6d + 1 + [(t-2) \bmod 3]d + (q+d)\\ &= (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d + q - 3d + 1\\ &\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d,\end{aligned}$$

thus the smallest number needed to complete this game in this case is no more than $(4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$.

Case 4. $q \geq 3d$.

In this case, let $\beta = \max\{k : C_1^p(2k) = (v; (2k-1)d), v \text{ is a vertex of } K_t\}$. Clearly, $\beta < t$. By the definition of S_1, there exists $b \leq \beta + 1$, such that $C_2^p(2b-1) = (v; \alpha)$, where v is a vertex of K_t and $2\beta d \leq \alpha \leq (2\beta+2)d - 1$. Let $c = \max\{2b-1, 2\beta\}$,

$$\theta = \begin{cases} 0, & \text{if } 2\beta d \leq \alpha \leq (2\beta+1)d - 1,\\ 1, & \text{if } (2\beta+1)d \leq \alpha \leq (2\beta+2)d - 1,\end{cases}$$

$$a = \begin{cases} b, & \text{if } b \leq \beta,\\ b-1, & \text{if } b = \beta + 1.\end{cases}$$

And let

$$j^* = \begin{cases} j, & \text{if } 1 \leq j \leq \min\{2a, 2b-1\} - 1,\\ j-2, & \text{if } j \geq \max\{2a, 2b-1\} + 1\end{cases}$$

for all $j \geq 1, j \notin \{2a, 2b-1\}$. Note that by the definition of θ, no numbers between $(2\beta-1)d$ and α can be used to label the unlabeled vertices of K_t after the cth step when $\theta = 0$, and when $\theta = 1$, the numbers between $(2\beta-1)d$ and α that can be used to label the unlabeled vertices of K_t after the cth step belong to $\{2\beta d, 2\beta d + 1, \ldots, \alpha - d\}$. Also note that by the definition of a, Alice chooses a vertex of K_t and labels it by $(2a-1)d$ at the $(2a)$th step, where $2a = 2b$ if $a \leq \beta$, and $2a = 2b-2$ if $b = \beta + 1$. Hence in this case, after the cth step, we can imagine that this is just the $(d, 1)$-game played on $K_{t-3-\theta} \cup (r+1+\theta)K_1$, with Bob plays first (by ignoring the $(2b-1)$th step made by Bob and the $(2a)$th step made by Alice, and if $C_i^p(j) = (v; (2a-2)d)$ for some unlabeled vertex v of K_t, imagine that on the $(d, 1)$-game played on $K_{t-3-\theta} \cup (r+1+\theta)K_1$, player i chooses an isolated vertex v of $(r+1+\theta)K_1$ and labels it by 0 at the j^*th step). More precisely, if $C_i^p(j) = (v; l)$ for some unlabeled vertex v of K_t and some l, $l \leq (2a-3)d$, then we can imagine that on the $(d, 1)$-game played on $K_{t-3-\theta} \cup (r+1+\theta)K_1$, player i chooses a vertex v of $K_{t-3-\theta}$ and labels it by l at the j^*th step. If $C_i^p(j) = (v; l)$ for some unlabeled vertex v of K_t and some l, $2ad \leq l \leq (2\beta-1)d$, then we can imagine that on the $(d, 1)$-game played on

$K_{t-3-\theta} \cup (r+1+\theta)K_1$, player i choose a vertex v of $K_{t-3-\theta}$ and labels it by $l-2d$ at the j^*th step. If $C_i^p(j) = (v; l)$ for some unlabeled vertex v of K_t and some l, $l \geq \alpha + d$, then we can imagine that on the $(d, 1)$-game played on $K_{t-3-\theta} \cup (r+1+\theta)K_1$, player i chooses a vertex v of $K_{t-3-\theta}$ and labels it by $l - [\alpha - (2\beta - 3)d]$ at the j^*th step. And if $C_i^p(j) = (v; l)$ for some $j \geq 3$, where v is an unlabeled vertex of K_t and $l = (2a-2)d$, or $2\beta d \leq l \leq \alpha - d$ (when $\theta = 1$), or v is a vertex of rK_1, then we can imagine that on the $(d, 1)$-game played on $K_{t-3-\theta} \cup (r+1+\theta)K_1$, player i chooses an isolated vertex v of $(r+1+\theta)K_1$ and labels it by 0 at the j^*th step. Since $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-3} \cup (r+1)K_1) \leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 6d + 1 + [(t-2) \bmod 3]d$ and $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-4} \cup (r+2)K_1) \leq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 6d + 1 + [t \bmod 3]d$, when $\theta = 0$, we have

$$\begin{aligned}
\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-3} \cup (r+1)K_1) + [\alpha - (2\beta - 3)d] &\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 6d + 1 \\
&\quad + [(t-2) \bmod 3]d + [\alpha - (2\beta - 3)d] \\
&\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d,
\end{aligned}$$

and when $\theta = 1$, we have

$$\begin{aligned}
\tilde{\lambda}^d_{2_{(1,S_1)}}(K_{t-4} \cup (r+2)K_1) + [\alpha - (2\beta - 3)d] &\leq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil - 6d + 1 + [t \bmod 3]d + [\alpha - (2\beta - 3)d] \\
&\leq (4d-1)\left\lceil \frac{t-2}{3} \right\rceil + [t \bmod 3]d - d \\
&\leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d.
\end{aligned}$$

Thus the smallest number needed to complete this game in this case is no more than $(4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$.

Since for all cases, the smallest number needed to complete this game is no more than $(4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$, $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_t \cup rK_1) \leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d$. □

By Claims 1 and 3, we have $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_t \cup rK_1) \leq (4d-1)\left\lceil \frac{t-3}{3} \right\rceil + [(t-1) \bmod 3]d \leq \tilde{\lambda}^d_{1_{(2,S_2)}}(K_t \cup rK_1)$. Similarly, by Claims 2 and 4, we have $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_t \cup rK_1) \leq (4d-1)\left\lceil \frac{t-1}{3} \right\rceil - 2d + [(t-2) \bmod 3]d \leq \tilde{\lambda}^d_{2_{(2,S_2)}}(K_t \cup rK_1)$. Thus the conclusion also holds for $n = t$. By the principle of mathematical induction, $\tilde{\lambda}^d_{1_{(1,S_1)}}(K_n \cup rK_1) \leq (4d-1)\left\lceil \frac{n-3}{3} \right\rceil + [(n-1) \bmod 3]d \leq \tilde{\lambda}^d_{1_{(2,S_2)}}(K_n \cup rK_1)$ and $\tilde{\lambda}^d_{2_{(1,S_1)}}(K_n \cup rK_1) \leq (4d-1)\left\lceil \frac{n-1}{3} \right\rceil - 2d + [(n-2) \bmod 3]d \leq \tilde{\lambda}^d_{2_{(2,S_2)}}(K_n \cup rK_1)$ for all $n \geq 1$, $r \geq 0$. Therefore, $\tilde{\lambda}^d_1(K_n \cup rK_1) = (4d-1)\left\lceil \frac{n-3}{3} \right\rceil + [(n-1) \bmod 3]d$ and $\tilde{\lambda}^d_2(K_n \cup rK_1) = (4d-1)\left\lceil \frac{n-1}{3} \right\rceil - 2d + [(n-2) \bmod 3]d$ for all $n \geq 1$, $r \geq 0$. □

By setting $d = 2$ in Theorem 3, we have

Corollary 4. *For all $n \geq 1$, $\tilde{\lambda}^2_1(K_n) = \left\lceil \frac{7n-9}{3} \right\rceil$ and $\tilde{\lambda}^2_2(K_n) = \left\lceil \frac{7n-7}{3} \right\rceil$.*

4. Game $L(d, 1)$-labeling number of complete bipartite graphs

We study the game $L(d, 1)$-labeling number of complete bipartite graphs in this section. For convenience, when consider the graph $K_{m,n}$, we always assume that the partite sets are X_0 and X_1, where $X_0 = \{u_1, u_2, \ldots, u_m\}$, $X_1 = \{v_1, v_2, \ldots, v_n\}$, and $E(K_{m,n}) = \{u_i v_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. And for two integers m, n, we use $\delta_{m,n}$ to denote the number $((m+n) \bmod 2)$.

Lemma 5. *$\tilde{\lambda}^d_1(K_{m,n}) \leq 2d + m + n - 2 - \delta_{m,n}$ for all d, m, n, $d \geq m \geq n \geq 2$.*

Proof. Let $p = 2d + m + n - 2 - \delta_{m,n}$. For an integer l in $A_0 \cup A_1 \cup B_0 \cup B_1$, where $A_0 = \{d, d+1, \ldots, d+n-2\}$, $A_1 = \{p-n+2, p-n+3, \ldots, p\}$, $B_0 = \{p-d-n+2, p-d-n+3, \ldots, p-d\}$, $B_1 = \{0, 1, \ldots, n-2\}$, let

$$l^* = \begin{cases} l + d + m - \delta_{m,n}, & \text{if } l \in A_0 \cup B_1, \\ l - d - m + \delta_{m,n}, & \text{if } l \in A_1 \cup B_0. \end{cases}$$

Consider the following strategy S made by Alice. At the first step, set $\alpha = \beta = 0$, and label u_1 with the number $d + \left\lfloor \frac{m+n-2}{2} \right\rfloor$. For all $j \geq 1$, if $C_2^p(2j) = (v; l)$, and there is still some vertices unlabeled, then Alice decides how to move at the $(2j+1)$th step according to the following rules:

Rule 1. If $\alpha = 0$, $v \in X_0$, and $d + n - 1 \leq l \leq p - d - n + 1$, then choose an unlabeled vertex in X_0 and label it by $p - l$.

Rule 2. If $\alpha = 0$, $v \in X_0$, and $l \leq d - 1$ or $l \geq p - d + 1$, then choose an unlabeled vertex in X_1 and label it by l', where $l' = p - n + 1$ if $l \leq d - 1$ and $l' = n - 1$ if $l \geq p - d + 1$, and set $\alpha = 1$.

Rule 3. If $\alpha = 0$, $v \in X_1$, and $n-1 \le l \le \lfloor \frac{m+n-2}{2} \rfloor$ or $2d + \lfloor \frac{m+n-2}{2} \rfloor \le l \le p-n+1$, then choose an unlabeled vertex in X_0 and label it by l', where $l' = p-d+1$ if $n-1 \le l \le \lfloor \frac{m+n-2}{2} \rfloor$ and $l' = d-1$ if $2d + \lfloor \frac{m+n-2}{2} \rfloor \le l \le p-n+1$, and set $\alpha = 1$.

Rule 4. If $\alpha = 0$, $v \in X_i$, $l \in A_i \cup B_i$, then choose an unlabeled vertex in X_{1-i} and label it by l^*, and increases β by 1. After this, set $\alpha = 1$ if $\beta = n$.

Rule 5. If $\alpha = 1$, choose an unlabeled vertex w in $X_0 \cup X_1$ and label it with an arbitrary number l' in $PC^p(2j; w)$ if $PC^p(2j; w) \neq \emptyset$.

To prove this lemma, we only need to show that Alice can complete the $(d, 1)$-game played on $K_{m,n}$ with the number p by using the strategy S given above. By the definition of S, if $\alpha = 0$ after the jth step, then at least one of the numbers $n-1, p-n+1$ is in $PC^p(j; v)$ for each unlabeled vertex v in X_1, and at least one of the numbers $d-1, p-d+1$ is in $PC^p(j; v)$ for each unlabeled vertex v in X_0. Thus $PC^p(j; v) \neq \emptyset$ for all unlabeled vertices if $\alpha = 0$ after the jth step. Hence we only need to show that if $\alpha = 0$ at the j^*th step, and $\alpha = 1$ after the (j^*+1)th step, then $PC^p(j; v) \neq \emptyset$ for all unlabeled vertices and all $j, j \ge j^*+1$. Consider the following cases:

Case 1. $C_2^p(j^*) = (v; l)$, where $v \in X_0$, and $l \le d-1$ or $l \ge p-d+1$.

We only consider that $l \le d-1$, the case that $l \ge p-d+1$ is similar. Note that in this case, if $C_i^p(j_1) = (w_1; l_1)$ for some $w_1 \in X_0$ and $j_1 < j^*$, then $l_1 \le p-d-n+1$. And if $C_i^p(j_2) = (w_2; l_2)$ for some $w_2 \in X_1$ and $j_2 < j^*$, then $l_2 \ge 2d + \lfloor \frac{m+n-2}{2} \rfloor$. Since at the (j^*+1)th step, Alice chooses an unlabeled vertex in X_1 and labels it by $p-n+1$, after this step, all the numbers in $D_1 = \{p-n+1, p-n+2, \ldots, p\}$ that are not used can be used to label the unlabeled vertices in X_1, and all the numbers in $D_2 = \{0, 1, \ldots, d + \lfloor \frac{m+n-2}{2} \rfloor\}$ that are not used can be used to label the unlabeled vertices in X_0. Therefore, since $|D_1| \ge n$ and $|D_2| \ge m$, Alice can complete the $(d, 1)$-game played on $K_{m,n}$ with the number p in this case.

Case 2. $C_2^p(j^*) = (v; l)$, where $v \in X_1$, and $n-1 \le l \le \lfloor \frac{m+n-2}{2} \rfloor$ or $2d + \lfloor \frac{m+n-2}{2} \rfloor \le l \le p-n+1$.

We only consider that $n-1 \le l \le \lfloor \frac{m+n-2}{2} \rfloor$, the case that $2d + \lfloor \frac{m+n-2}{2} \rfloor \le l \le p-n+1$ is similar. Note that in this case, if $C_i^p(j_1) = (w_1; l_1)$ for some $w_1 \in X_0$ and $j_1 < j^*$, then $l_1 \ge d+l$. Hence if $C_i^p(j_2) = (w_2; l_2)$ for some $w_2 \in X_1$ and $j_2 < j^*$, then $l_2 \le n-2$. Since at the (j^*+1)th step, Alice chooses an unlabeled vertex in X_0 and labels it by $p-d+1$, after this step, all the numbers in $D_1 = \{0, 1, \ldots, l\}$ that are not used can be used to label the unlabeled vertices in X_1, and all the numbers in $D_2 = \{d + \lfloor \frac{m+n-2}{2} \rfloor, d + \lfloor \frac{m+n-2}{2} \rfloor + 1, \ldots, p\}$ that are not used can be used to label the unlabeled vertices in X_0. Therefore, since $|D_1| \ge n$ and $|D_2| \ge m$, Alice can complete the $(d, 1)$-game played on $K_{m,n}$ with the number p in this case.

Case 3. $\beta = n-1$ at the j^*th step and $C_2^p(j^*) = (v; l)$, where $v \in X_i$, $l \in A_i \cup B_i$.

In this case, by the definition of S, at the (j^*+1)th step, Alice chooses an unlabeled vertex in X_{1-i} and labels it by l^*. Note that after this step, all the vertices in X_1 are labeled properly (since β increase by 1 after the jth step only if one of the vertices in X_1 is labeled at the jth step or at the $(j-1)$th step). Let

$$a = \min\{l : C_i^p(j) = (v; l), v \in X_0 \text{ and } j \le j^*+1\}.$$

Since $|B_0| \le n-1, d \le a \le d+n-2$. By the definition of S and a, all the numbers in $\{2d + \lfloor \frac{m+n-2}{2} \rfloor, 2d + \lfloor \frac{m+n-2}{2} \rfloor + 1, \ldots, p\}$ that are used to label some vertex in X_1 are greater than or equal to $a + d + m - \delta_{m,n}$, and all the numbers in $\{0, 1, \ldots, \lfloor \frac{m+n-2}{2} \rfloor\}$ that are used to label some vertex in X_1 are less than or equal to $a - d$. Hence after the (j^*+1)th step, all the numbers in $D = \{a, a+1, \ldots, a+m-\delta_{m,n}\}$ that are not used can be used to label the unlabeled vertices in X_0. Therefore, since $|D| \ge m$, Alice can complete the $(d, 1)$-game played on $K_{m,n}$ with the number p in this case.

From the cases above, $\tilde{\lambda}^d_{1_{(1,S)}}(K_{m,n}) = p \le 2d+m+n-2-\delta_{m,n}$, hence $\tilde{\lambda}_1^d(K_{m,n}) \le \tilde{\lambda}^d_{1_{(1,S)}}(K_{m,n}) = 2d+m+n-2-\delta_{m,n}$ for all $d, m, n, d \ge m \ge n \ge 2$. □

Lemma 6. *$\tilde{\lambda}_1^d(K_{m,n}) \ge 2d + m - 1 - \delta_{m,n}$ for all $d, m, n, d \ge m \ge n \ge 2$.*

Proof. Suppose, to the contrary, $\tilde{\lambda}_1^d(K_{m,n}) \le 2d + m - 2 - \delta_{m,n}$. Let $\tilde{\lambda}_1^d(K_{m,n}) = p$. Since $\lambda_d(K_{m,n}) = d+m+n-1$, by Lemma 1, we have $p \ge d+m+n-1$. Let $D = \{0, 1, \ldots, p\}$. If $C_1^p(1) = (v; l)$, where $v \in X_1$, then Bob chooses an unlabeled vertex in X_1 and labels it by l', where $l' = d$ if $l \le d-1$, and $l' = d-1$ if $l \ge d$. In this case, after the second step, at least $2d$ numbers in D cannot be used to label the vertices in X_0. Since $|D| - 2d \le m-1 < |X_0|$, the $(d, 1)$-game played on $K_{m,n}$ cannot be completed in this case.

If $C_1^p(1) = (v; l)$, where $v \in X_0$, then Bob chooses an unlabeled vertex in X_0 and labels it by l', where $l' = d+m-1$ if $l \le d+n-2$, and $l' = d-1$ if $l \ge d+n-1$. It is easy to verify that in this case, after the second step, at most $n-1$ numbers can be used to label the vertices in X_1. Thus the $(d, 1)$-game played on $K_{m,n}$ also cannot be completed in this case.

From the argument above, $\tilde{\lambda}_1^d(K_{m,n}) = p \ge 2d + m - 1 - \delta_{m,n}$ for all $d, m, n, d \ge m \ge n \ge 2$. □

Theorem 7. *$\tilde{\lambda}_1^d(K_{m,n}) = 2d + m + n - 2 - \delta_{m,n}$ for all $d, m, n, d \ge m \ge n \ge 2$.*

Proof. $\tilde{\lambda}_1^d(K_{m,n}) \le 2d + m + n - 2 - \delta_{m,n}$ follows from Lemma 5, hence we only need to show that for any positive integer $p \le 2d+m+n-3-\delta_{m,n}$, Bob has a strategy to force the $(d, 1)$-game played on $K_{m,n}$ cannot be completed by using numbers in $\{0, 1, \ldots, p\}$. By Lemma 6, we may assume that $p \ge 2d + m - 1 - \delta_{m,n}$. Consider the following strategy S made by Bob. Set $\alpha = 0$. For all $j \ge 1$, if $C_1^p(2j-1) = (v; l)$, and there is still some vertices unlabeled, then Bob decides how to move at the $(2j)$th step according to the following rules:

Rule 1. If $\alpha = 0$, $v \in X_0$, and $l \in A = \{d+n-1, d+n, \ldots, d+m-2-\delta_{m,n}\}$, then choose an unlabeled vertex in X_0 and label it by $2d + m + n - 3 - \delta_{m,n} - l$.

Rule 2. If $\alpha = 0$, $v \in X_0$, and $l \le d+n-2$ or $l \ge d+m-1-\delta_{m,n}$, then choose an unlabeled vertex in X_0 and label it by l', where $l' = d+m+n-2$ if $l \le d+n-2$ and $l' = d-1$ if $l \ge d+m-1-\delta_{m,n}$, and set $\alpha = 1$.

Rule 3. If $\alpha = 0$, $v \in X_1$, then choose an unlabeled vertex in X_1 and label it by l', where $l' = 2d+m-2-\delta_{m,n}$ if $l \le \left\lfloor \frac{p}{2} \right\rfloor$ and $l' = n-1$ if $l \ge \left\lfloor \frac{p}{2} \right\rfloor + 1$, and set $\alpha = 1$.

Rule 4. If $\alpha = 1$, choose an unlabeled vertex w in $X_0 \cup X_1$ and label it with an arbitrary number l' in $PC^p(2j-1; w)$ if $PC^p(2j-1; w) \ne \emptyset$.

By the definition of S, if $\alpha = 0$ after the jth step, then at least one of the numbers $d-1, d+m+n-2$ is in $PC^p(j; v)$ for each unlabeled vertex v in X_0, and at least one of the numbers $0, p$ is in $PC^p(j; v)$ for each unlabeled vertex v in X_1. Thus $PC^p(j; v) \ne \emptyset$ for all unlabeled vertices if $\alpha = 0$ after the jth step. Since $|A| = m - n - \delta_{m,n} < m$, there exists j^*, such that $\alpha = 0$ at the j^*th step, and $\alpha = 1$ after the (j^*+1)th step. Consider the following cases:

Case 1. $C_1^p(j^*) = (v; l)$, where $v \in X_0$, and $l \le d+n-2$ or $l \ge d+m-1-\delta_{m,n}$.

In this case, if $l \le d+n-2$, then Bob chooses an unlabeled vertex in X_0 and labels it by $d+m+n-2$ at the (j^*+1)th step. And if $l \ge d+m-1-\delta_{m,n}$, then Bob chooses an unlabeled vertex in X_0 and labels it by $d-1$ at the (j^*+1)th step. Since $p \le 2d+m+n-3-\delta_{m,n}$, after the (j^*+1)th step, at most $n-1$ numbers can be used to label the vertices in X_1. Thus the $(d, 1)$-game played on $K_{m,n}$ cannot be completed in this case.

Case 2. $C_1^p(j^*) = (v; l)$, where $v \in X_1$.

In this case, if $l \le \left\lfloor \frac{p}{2} \right\rfloor$, then Bob chooses an unlabeled vertex in X_1 and labels it by $2d+m-2-\delta_{m,n}$ at the (j^*+1)th step. And if $l \ge \left\lfloor \frac{p}{2} \right\rfloor + 1$, then Bob chooses an unlabeled vertex in X_1 and labels it by $n-1$ at the (j^*+1)th step. Since $p \le 2d+m+n-3-\delta_{m,n}$, after the (j^*+1)th step, at most $m-1$ numbers can be used to label the vertices in X_0. Thus the $(d, 1)$-game played on $K_{m,n}$ also cannot be completed in this case.

From the argument above, when $p \le 2d+m+n-3-\delta_{m,n}$, Bob has a strategy to force the $(d, 1)$-game played on $K_{m,n}$ cannot be completed by using numbers in $\{0, 1, \ldots, p\}$. Hence $\tilde{\lambda}_1^d(K_{m,n}) \ge 2d+m+n-2-\delta_{m,n}$, and so $\tilde{\lambda}_1^d(K_{m,n}) = 2d+m+n-2-\delta_{m,n}$ for all $d, m, n, d \ge m \ge n \ge 2$. □

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References

[1] H.L. Bodlaender, On the complexity of some coloring games, in: Proceeding of the WG 1990 Workshop on Graph Theoretical Concepts in Computer Science, 1990, pp. 30–40.
[2] G.J. Chang, D. Kuo, The $L(2, 1)$-labeling on graphs, SIAM J. Discrete Math. 9 (1996) 309–316.
[3] T. Dinski, X. Zhu, A bound for the game chromatic number of graphs, Discrete Math. 196 (1999) 109–115.
[4] U. Faigle, U. Kern, H. Kierstead, W.T. Trotter, On the game chromatic number of some classes of graphs, Ars Combin. 35 (1993) 143–150.
[5] D. Gonçalves, On the $L(p, 1)$-labelling of graphs, Discrete Math. Theor. Comput. Sci. (2005) 81–86. AE.
[6] J.R. Griggs, R.K. Yeh, Labeling graphs with a condition at distance two, SIAM J. Discrete Math. 5 (1992) 586–595.
[7] D.J. Guan, X. Zhu, Game chromatic number of outerplanar graphs, J. Graph Theory 30 (1) (1999) 67–70.
[8] W.K. Hale, Frequency assignment: theory and applications, Proc. IEEE 68 (1980) 1497–1514.
[9] H.A. Kierstead, A simple competitive graph coloring algorithm, J. Combin. Theory Ser. B 78 (2000) 57–68.
[10] H.A. Kierstead, W.T. Trotter, Planar graph coloring with an uncooperative partner, J. Graph Theory 18 (6) (1994) 569–584.
[11] F.S. Roberts, Private communication through J. Griggs, 1988.
[12] R.K. Yeh, A survey on labeling graphs with a condition at distance two, Discrete Math. 306 (2006) 1217–1231.
[13] X. Zhu, The game coloring number of planar graphs, J. Combin. Theory Ser. B 75 (1999) 245–258.