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New additive results for the generalized Drazin inverse

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ABSTRACT

In this paper, we investigate additive properties of generalized Drazin inverse of two Drazin invertible linear operators in Banach spaces. Under the commutative condition of $PQ = QP$, we give explicit representations of the generalized Drazin inverse $(P+Q)^d$ in term of P, P^d, Q and Q^d. We consider some applications of our results to the perturbation of the Drazin inverse and analyze a number of special cases.

1. Introduction

Let X be a complex Banach space. Denote by $\mathcal{B}(X)$ the Banach algebra of all bounded linear operators on X. For an operator $T \in \mathcal{B}(X)$, the symbols $\mathcal{N}(T)$, $\mathcal{R}(T)$ and $\sigma(T)$ will denote the null space, the range and the spectrum of T, respectively. For $T \in \mathcal{B}(X)$, if there exists an operator $T^D \in \mathcal{B}(X)$ satisfying the following three operator equations [17]

$$TT^D = T^D T, \qquad T^D T T^D = T^D, \qquad T^{k+1}T^D = T^k, \tag{1}$$

then T^D is called a Drazin inverse of T. The smallest k such that (1) holds, is called the index of T, denoted by $\operatorname{ind}(T)$. Notice also that $\operatorname{ind}(T)$ (if it is finite) is the smallest non-negative integer k such that $\mathcal{R}(T^{k+1}) = \mathcal{R}(T^k)$ and $\mathcal{N}(T^{k+1}) = \mathcal{N}(T^k)$ hold. The conditions (1) are equivalent to

$$TT^D = T^D T, \qquad T^D T T^D = T^D, \qquad T - T^2 T^D \text{ is nilpotent.} \tag{2}$$

The concept of the generalized Drazin inverse (GD-inverse) on an infinite-dimensional Banach space was introduced by Koliha [23], which is the element $T^d \in \mathcal{B}(X)$ such that

$$TT^d = T^d T, \qquad T^d T T^d = T^d, \qquad T - T^2 T^d \text{ is quasi-nilpotent.} \tag{3}$$

If T is GD-invertible, then the spectral idempotent P of T corresponding to $\{0\}$ is given by $P = I - TT^d$. The operator matrix form of T with respect to the space decomposition $X = \mathcal{N}(P) \oplus \mathcal{R}(P)$ is given by

$$T = T_1 \oplus T_2,$$

where T_1 is invertible and T_2 is quasi-nilpotent [1,2].

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In recent years, the characterizations of the Drazin inverses of matrices or operators on an infinite-dimensional space have been considered by many authors (cf. [4–38]). Castro-González et al. [4–10], Djordjević and Wei [15], Hartwig et al. [20,21] and Koliha [23–26] have studied the generalized Drazin inverse on a Banach space. Some additive properties and the explicit expressions for the GD-inverse of the sum are obtained in [4,8,11,15,21,37].

In this paper, using the technique of block operator matrices, we will investigate explicit representations of the generalized Drazin inverse $(P+Q)^d$ in term of P, P^d, Q and Q^d under the condition of $PQ = QP$. Our results are improvement over the main results of [11,15]. Indeed, a totally new approach is provided to express the GD-inverse.

This paper is organized as follows. An explicit formula for $(P+Q)^d$ is presented in Section 2. This is a key step of the paper. In Section 3, special cases are given to indicate the various applications of our main results.

2. Main results

In the first part of this section, we give a new expression for the GD-inverse of $P+Q$ in term of P, P^d, Q and Q^d. It is interesting to note that our result is quite different from the expression for $(P+Q)^d$ in [15].

Theorem 1. *Let $P, Q \in \mathcal{B}(X)$ be GD-invertible and $PQ = QP$. Then $P+Q$ is GD-invertible if and only if $I + P^d Q$ is GD-invertible. In this case we have*

$$(P+Q)^d = P^d\big(I+P^dQ\big)^d QQ^d + \big(I-QQ^d\big)\left[\sum_{n=0}^{\infty}(-Q)^n\big(P^d\big)^n\right]P^d + Q^d\left[\sum_{n=0}^{\infty}\big(Q^d\big)^n(-P)^n\right]\big(I-PP^d\big)$$

and

$$(P+Q)(P+Q)^d = \big(PP^d+QP^d\big)\big(I+P^dQ\big)^d QQ^d + \big(I-QQ^d\big)PP^d + QQ^d\big(I-PP^d\big).$$

Proof. Since P is GD-invertible, we can write

$$P = P_0 \oplus P_{00}, \tag{4}$$

where P_0 is invertible, P_{00} is quasi-nilpotent. Since $PQ = QP$ we can decompose

$$Q = Q_0 \oplus Q_{00}, \tag{5}$$

where Q_0 and Q_{00} are GD-invertible such that $P_0Q_0 = Q_0P_0$ and $P_{00}Q_{00} = Q_{00}P_{00}$. In a similar way we conclude that

$$P_0 = P_1 \oplus P_2, \qquad P_{00} = P_3 \oplus P_4$$

and

$$Q_0 = Q_1 \oplus Q_2, \qquad Q_{00} = Q_3 \oplus Q_4,$$

where P_i $(i=1,2)$, Q_j $(j=1,3)$ are invertible, P_m $(m=3,4)$, Q_n $(n=2,4)$ are quasi-nilpotent and $P_iQ_i = Q_iP_i$ $(i = 1,2,3,4)$. Now we have

$$P+Q = (P_1+Q_1)\oplus(P_2+Q_2)\oplus(P_3+Q_3)\oplus(P_4+Q_4).$$

Since P_2 is invertible, Q_2 is quasi-nilpotent and $P_2Q_2 = Q_2P_2$, we have $\sigma(P_2^{-1}Q_2) \subset \sigma(P_2^{-1})\sigma(Q_2) = \{0\}$. Thus $P_2^{-1}Q_2$ is quasi-nilpotent and $I + P_2^{-1}Q_2$ is invertible and

$$\begin{aligned}
0\oplus(P_2+Q_2)^{-1}\oplus 0\oplus 0 &= 0\oplus\big(I+P_2^{-1}Q_2\big)^{-1}P_2^{-1}\oplus 0\oplus 0\\
&= 0\oplus\left[\sum_{n=0}^{\infty}P_2^{-n}(-Q_2)^n\right]P_2^{-1}\oplus 0\oplus 0\\
&= \big(I-QQ^d\big)\left[\sum_{n=0}^{\infty}\big(P^d\big)^{n+1}(-Q)^n\right].
\end{aligned}$$

Similarly, we see that $P_3Q_3^{-1}$ is quasi-nilpotent and $I + P_3Q_3^{-1}$ is invertible with

$$\begin{aligned}
0\oplus 0\oplus(P_3+Q_3)^{-1}\oplus 0 &= 0\oplus 0\oplus\big(I+Q_3^{-1}P_3\big)^{-1}Q_3^{-1}\oplus 0\\
&= 0\oplus 0\oplus Q_3^{-1}\left[\sum_{n=0}^{\infty}Q_3^{-n}(-P_3)^n\right]\oplus 0\\
&= \left[\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n\right]\big(I-PP^d\big).
\end{aligned}$$

Since P_4 and Q_4 are quasi-nilpotent, and $P_4Q_4 = Q_4P_4$, we get that $P_4 + Q_4$ is GD-invertible and $(P_4 + Q_4)^d = 0$.

Hence, $P + Q$ is GD-invertible if and only if $P_1 + Q_1$ is GD-invertible. Note that P_1, P_1^{-1}, Q_1, Q_1^{-1}, $P_1 + Q_1$ and $(P_1 + Q_1)^d$ commute. It is easy to know $(P_1 + Q_1)^d$ is GD-invertible if and only if $I + P^dQ$ is GD-invertible, thus

$$\begin{aligned}(P_1+Q_1)^d \oplus 0 \oplus 0 \oplus 0 &= P_1^{-1}\big(I + P_1^{-1}Q_1\big)^d \oplus 0 \oplus 0 \oplus 0\\ &= P^d\big(I + P^dQ\big)^d Q Q^d\end{aligned}$$

and

$$\begin{aligned}(P_1+Q_1)(P_1+Q_1)^d \oplus 0 \oplus 0 \oplus 0 &= (P+Q)P^d\big(I + P^dQ\big)^d Q Q^d\\ &= \big(PP^d + QP^d\big)\big(I + P^dQ\big)^d Q Q^d.\end{aligned}$$

Now, we arrive at

$$\begin{aligned}(P+Q)^d &= (P_1+Q_1)^d \oplus (P_2+Q_2)^{-1} \oplus (P_3+Q_3)^{-1} \oplus (P_4+Q_4)^d\\ &= P^d\big(I+P^dQ\big)^d QQ^d + \big(I - QQ^d\big)\left[\sum_{n=0}^{\infty}(-Q)^n\big(P^d\big)^{n+1}\right] + \left[\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n\right]\big(I - PP^d\big)\end{aligned}$$

and

$$\begin{aligned}(P+Q)(P+Q)^d &= (P_1+Q_1)(P_1+Q_1)^d \oplus I \oplus I \oplus 0\\ &= \big(PP^d + QP^d\big)\big(I+P^dQ\big)^d QQ^d + \big(I - QQ^d\big)PP^d + QQ^d\big(I - PP^d\big). \qquad \square\end{aligned}$$

Remark. Djordjević and Wei [15] showed that

(1) If $P, Q, P+Q \in \mathcal{B}(X)$ are GD-invertible and $PQ = QP$, then $P + Q$ is GD-invertible and

$$(P+Q)^d = (C_P + C_Q)^d\big[I + (C_P + C_Q)^d(Q_P + Q_Q)\big]^{-1},$$

where $P = C_P + Q_P$ and $Q = C_Q + Q_Q$ are known as the core-quasi-nilpotent decomposition of P and Q, respectively.

(2) If P and Q are GD-invertible and $PQ = 0$, then $P + Q$ is GD-invertible and

$$(P+Q)^d = \big(I - QQ^d\big)\left[\sum_{n=0}^{\infty}Q^n\big(P^d\big)^n\right]P^d + Q^d\left[\sum_{n=0}^{\infty}\big(Q^d\big)^n P^n\right]\big(I - PP^d\big).$$

(3) Remark (1) shows that core-quasi-nilpotent decomposition of P and Q are the key results which allow us to derive an expression for the GD-inverse of $(P+Q)^d$. Compare Theorem 1 with the results of [15], we can see that our result gives some analogue to the expression $(P+Q)^d$ with the condition $PQ = 0$ in [15].

From Theorem 1, some special expressions for $(P+Q)^d$ can be obtained immediately.

Corollary 2. *Let $P, Q \in \mathcal{B}(X)$ be GD-invertible such that $PQ = QP$ and $I + P^dQ$ is GD-invertible.*

(1) *If $PQ = QP = 0$, then $P^dQ = Q^dP = 0$ and $(P+Q)^d = P^d + Q^d$.*
(2) *If P and Q are quasi-nilpotent, then $(P+Q)^d = 0$.*
(3) *If Q is quasi-nilpotent, then*

$$(P+Q)^d = \sum_{n=0}^{\infty}\big(P^d\big)^{n+1}(-Q)^n = \big(I + P^dQ\big)^{-1}P^d.$$

(4) *If $Q^k = 0$, then $(P+Q)^d = \sum_{n=0}^{k-1}(P^d)^{n+1}(-Q)^n = (I + P^dQ)^{-1}P^d$.*
(5) *If $Q^2 = 0$, then $(P+Q)^d = P^d - (P^d)^2Q = (I + P^dQ)^{-1}P^d$.*
(6) *If $Q^k = Q$ $(k \geqslant 3)$, then $Q^d = Q^{k-2}$ and*

$$\begin{aligned}(P+Q)^d &= P^d\big(I+P^dQ\big)^d Q^{k-1} + \big(I - Q^{k-1}\big)P^d + Q^{k-2}\left[\sum_{n=0}^{\infty}\big(Q^d\big)^n(-P)^n\right]\big(I - PP^d\big)\\ &= P^d\big(I+P^dQ\big)^d Q^{k-1} + \big(I - Q^{k-1}\big)P^d + Q^{k-2}\big(I + PQ^{k-2}\big)^d\big(I - PP^d\big).\end{aligned}$$

(7) *If $Q^2 = Q$, then $Q^d = Q$ and*

$$(P+Q)^d = P^d\big(I+P^dQ\big)^dQ + (I-Q)P^d + Q\Bigg[\sum_{n=0}^{\infty}(-P)^n\Bigg]\big(I-PP^d\big)$$
$$= P^d\big(I+P^dQ\big)^dQ + (I-Q)P^d + Q(I+P)^d\big(I-PP^d\big).$$

(8) *If $P^2 = P$ and $Q^2 = Q$, then $I + PQ$ is invertible and $P(I+PQ)^{-1}Q = \frac{1}{2}PQ$. In this case,*

$$(P+Q)^d = P(I+PQ)^{-1}Q + Q(I-P) + (I-Q)P$$
$$= P + Q - \frac{3}{2}PQ.$$

In the next part, we will be mainly concerned with upper block triangular operator matrices. However, every result that we obtain will have an analogue for lower block triangular operator matrices, the statement and the proofs are left to the reader with interest. However, we need following result which is proved in [21] for matrices, has been extended to a bounded linear operator [16] and for arbitrary elements in a Banach algebra [8].

Theorem 3. *If $A \in \mathcal{B}(X)$ and $B \in \mathcal{B}(Y)$ are GD-invertible, $C \in \mathcal{B}(Y,X)$, then $M = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix}$ are GD-invertible and $M^d = \begin{pmatrix} A^d & X \\ 0 & B^d \end{pmatrix}$, where*

$$X = \big(A^d\big)^2\Bigg[\sum_{n=0}^{\infty}\big(A^d\big)^nCB^n\Bigg]\big(I-BB^d\big) + \big(I-AA^d\big)\Bigg[\sum_{n=0}^{\infty}A^nC\big(B^d\big)^n\Bigg]\big(B^d\big)^2 - A^dCB^d.$$

In [11] it is presented an expression of $(P+Q)^d$ under the condition that Q is quasi-nilpotent such that $P^\pi PQ = P^\pi QP$ and $Q = QP^\pi$. In fact, if we assume that $P^\pi Q(I-P^\pi) = 0$ instead of Q quasi-nilpotent with $Q = QP^\pi$, we will get a general result.

Theorem 4. *Let $P \in \mathcal{B}(X)$ be GD-invertible and $Q \in \mathcal{B}(X)$ such that $\|QP^d\| < 1$, $P^\pi Q(I-P^\pi) = 0$ and $P^\pi PQ = P^\pi QP$. If $P^\pi Q$ is GD-invertible, then $P+Q$ is GD-invertible. In this case,*

$$(P+Q)^d = \big(I+P^dQ\big)^{-1}P^d + \big(I+P^dQ\big)^{-1}\big(I-PP^d\big)\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n$$
$$+ \Bigg[\sum_{n=0}^{\infty}\big(\big(I+P^dQ\big)^{-1}P^d\big)^{n+2}Q\big(I-PP^d\big)(P+Q)^n\Bigg]\big(I-PP^d\big)$$
$$\times\Bigg[I-(P+Q)\big(I-PP^d\big)\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n\Bigg].$$

Proof. Since P is GD-invertible and $P^\pi Q(I-P^\pi) = 0$, P and Q have the form

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & Q_3 \\ 0 & Q_2 \end{pmatrix} \tag{6}$$

with respect to the space decomposition $X = \mathcal{N}(P^\pi) \oplus \mathcal{R}(P^\pi)$, where P_1 is invertible and P_2 is quasi-nilpotent. $\|QP^d\| < 1$ implies that $I + P^dQ$ is invertible. $P^\pi PQ = P^\pi QP$ implies that $P_2Q_2 = Q_2P_2$. Since $P^\pi Q$ is GD-invertible, Q_2 is GD-invertible. It follows from Theorem 1 that

$$(P_2+Q_2)^d = \sum_{n=0}^{\infty}\big(Q_2^d\big)^{n+1}(-P_2)^n.$$

By Theorem 3, $(P+Q)^d$ has the form

$$(P+Q)^d = \begin{pmatrix} (P_1+Q_1)^{-1} & S \\ 0 & \sum_{n=0}^{\infty}(Q_2^d)^{n+1}(-P_2)^n \end{pmatrix},$$

where

$$S=\left[\sum_{n=0}^{\infty}(P_1+Q_1)^{-n-2}Q_3(P_2+Q_2)^n\right]\left[I-(P_2+Q_2)\sum_{n=0}^{\infty}\left(Q_2^d\right)^{n+1}(-P_2)^n\right]$$
$$-(P_1+Q_1)^{-1}Q_3\sum_{n=0}^{\infty}\left(Q_2^d\right)^{n+1}(-P_2)^n.$$

Note that the product and the sum of P, Q, P^d and Q^d are still the upper triangular operator matrices. Thus

$$(P_1+Q_1)^{-1}\oplus 0=\left(I+P^dQ\right)^{-1}P^d$$

and

$$0\oplus\sum_{n=0}^{\infty}\left(Q_2^d\right)^{n+1}(-P_2)^n=P^\pi\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n.$$

A straightforward computation gives

$$\begin{pmatrix}0 & S\\ 0 & 0\end{pmatrix}=\left[\sum_{n=0}^{\infty}\left(\left(I+P^dQ\right)^{-1}P^d\right)^{n+2}QP^\pi(P+Q)^n\right]P^\pi\left[I-(P+Q)P^\pi\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n\right]$$
$$-\left(I+P^dQ\right)^{-1}P^dQP^\pi\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n.$$

Hence

$$(P+Q)^d=\left(I+P^dQ\right)^{-1}P^d+\left(I+P^dQ\right)^{-1}P^\pi\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n$$
$$+\left[\sum_{n=0}^{\infty}\left(\left(I+P^dQ\right)^{-1}P^d\right)^{n+2}QP^\pi(P+Q)^n\right]P^\pi\left[I-(P+Q)P^\pi\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n\right].$$

The proof is complete. □

Now it is a position to discuss some special cases of Theorem 4.

Corollary 5. *Let $P\in\mathcal{B}(X)$ be GD-invertible and $Q\in\mathcal{B}(X)$ such that $\|QP^d\|<1$, $P^\pi Q(I-P^\pi)=0$ and $P^\pi PQ=P^\pi QP$,*

(1) (*Theorem* 2.3 *in* [11].) *If $QPP^d=0$ and Q is quasi-nilpotent, then Theorem* 4 *is simplified to*

$$(P+Q)^d=\sum_{n=0}^{\infty}\left(P^d\right)^{n+2}Q(P+Q)^n+P^d.$$

(2) (*Theorem* 2.3 *in* [6].) *If $P^\pi Q=QP^\pi$, $\sigma(P^\pi Q)=0$, then Theorem* 4 *becomes*

$$(P+Q)^d=\left(I+P^dQ\right)^{-1}P^d=P^d\left(I+QP^d\right)^{-1}.$$

We can also prove the following result, which is a generalization of Theorem 3 and Theorem 4.

Theorem 6. *Let P and $Q\in\mathcal{B}(X)$ be GD-invertible. Let F be an idempotent such that $FP=PF$, $(I-F)QF=0$, $(PQ-QP)F=0$ and $(I-F)(PQ-QP)=0$. If $(P+Q)F$ and $(I-F)(P+Q)$ are GD-invertible, then $(P+Q)$ is GD-invertible and*

$$(P+Q)^d=\sum_{n=0}^{\infty}\triangle^{n+2}FQ(I-F)(P+Q)^n(I-F)\left[I-(P+Q)\triangle\right]$$
$$+\left[I-(P+Q)\triangle\right]F\sum_{n=0}^{\infty}(P+Q)^nFQ(I-F)\triangle^{n+2}+(I-\triangle FQ)(I-F)\triangle+\triangle F,$$

where

$$\triangle=P^d\left(I+P^dQ\right)^dQQ^d+\left(I-QQ^d\right)\left[\sum_{n=0}^{\infty}(-Q)^n\left(P^d\right)^{n+1}\right]+\left[\sum_{n=0}^{\infty}\left(Q^d\right)^{n+1}(-P)^n\right]\left(I-PP^d\right).\tag{7}$$

Proof. Since $F^2 = F$, we have $F = I \oplus 0$ with respect to space decomposition $X = \mathcal{R}(F) \oplus \mathcal{N}(F)$. From $FP = PF$ and $(I - F)QF = 0$, we know that

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} Q_1 & Q_3 \\ 0 & Q_2 \end{pmatrix}. \tag{8}$$

Since $(P + Q)F$ and $(I - F)(P + Q)$ are GD-invertible, $P_i + Q_i$ $(i = 1, 2)$ are GD-invertible. Hence

$$(P + Q)^d = \begin{pmatrix} (P_1 + Q_1)^d & X \\ 0 & (P_2 + Q_2)^d \end{pmatrix},$$

where

$$\begin{aligned} X = {} & \left[\sum_{n=0}^{\infty} \bigl((P_1 + Q_1)^d\bigr)^{n+2} Q_3 (P_2 + Q_2)^n \right] \times \bigl[I - (P_2 + Q_2)(P_2 + Q_2)^d\bigr] \\ & + \bigl[I - (P_1 + Q_1)(P_1 + Q_1)^d\bigr] \times \left[\sum_{n=0}^{\infty} (P_1 + Q_1)^n Q_3 \bigl((P_2 + Q_2)^d\bigr)^{n+2} \right] \\ & - (P_1 + Q_1)^d Q_3 (P_2 + Q_2)^d \end{aligned}$$

by Theorem 3.

From $(PQ - QP)F = 0$ and $(I - F)(PQ - QP) = 0$ we know that $P_i Q_i = Q_i P_i$ $(i = 1, 2)$. Note that P, Q, P^d and Q^d are all the upper triangular operator matrices. By Theorem 1, a straightforward computation shows that

$$\begin{aligned} (P_1 + Q_1)^d \oplus 0 &= P_1^d \bigl(I + P_1^d Q_1\bigr)^d Q_1 Q_1^d \oplus 0 + \bigl(I - Q_1 Q_1^d\bigr) \left[\sum_{n=0}^{\infty} Q_1^n \bigl(-P_1^d\bigr)^n \right] P_1^d \oplus 0 \\ &\quad + Q_1^d \left[\sum_{n=0}^{\infty} \bigl(-Q_1^d\bigr)^n P_1^n \right] \bigl(I - P_1 P_1^d\bigr) \oplus 0 \\ &= P^d \bigl(I + P^d Q\bigr)^d Q Q^d F + \bigl(I - Q Q^d\bigr) \left[\sum_{n=0}^{\infty} \bigl(-P^d\bigr)^n Q^n \right] P^d F \\ &\quad + Q^d \left[\sum_{n=0}^{\infty} \bigl(-Q^d\bigr)^n P^n \right] \bigl(I - P P^d\bigr) F \\ &= \triangle F, \end{aligned} \tag{9}$$

where $\triangle$ is defined as Eq. (7). Similarly, we can prove that

$$0 \oplus (P_2 + Q_2)^d = (I - F)\triangle. \tag{10}$$

Observe that

$$\begin{aligned} &\begin{pmatrix} 0 & [((P_1 + Q_1)^d)^{n+2} Q_3 (P_2 + Q_2)^n] \times [I - (P_2 + Q_2)(P_2 + Q_2)^d] \\ 0 & 0 \end{pmatrix} \\ &\quad = \bigl[(\triangle F)^{n+2} F Q (I - F)(P + Q)^n\bigr] \times \bigl[I - (P + Q)(I - F)\triangle\bigr] \\ &\quad = \triangle^{n+2} F Q (I - F)(P + Q)^n \bigl[I - (P + Q)\triangle\bigr], \end{aligned}$$

with

$$\begin{aligned} &\begin{pmatrix} 0 & [I - (P_1 + Q_1)(P_1 + Q_1)^d] \times [(P_1 + Q_1)^n Q_3 ((P_2 + Q_2)^d)^{n+2}] \\ 0 & 0 \end{pmatrix} \\ &\quad = \bigl[I - (P + Q)\triangle F\bigr] \times \bigl[(P + Q)^n F Q (I - F)\bigl((I - F)\triangle\bigr)^{n+2}\bigr] \\ &\quad = \bigl[I - (P + Q)\triangle\bigr](P + Q)^n F Q (I - F)\triangle^{n+2} \end{aligned}$$

and

$$\begin{pmatrix} 0 & -(P_1 + Q_1)^d Q_3 (P_2 + Q_2)^d \\ 0 & (P_2 + Q_2)^d \end{pmatrix} = (I - \triangle F Q)(I - F)\triangle.$$

Hence

$$(P+Q)^d = \sum_{n=0}^{\infty} \triangle^{n+2} FQ(I-F)(P+Q)^n\big[I-(P+Q)\triangle\big] + \big[I-(P+Q)\triangle\big]\sum_{n=0}^{\infty}(P+Q)^n FQ(I-F)\triangle^{n+2} + (I-\triangle FQ)(I-F)\triangle + \triangle F.$$

The proof is complete. □

3. Special cases

Let us use Theorem 6 to analyze some interesting special perturbations of linear operators. We thereby extend earlier work by several authors [6,8,11,17,21] and partially solve a problem posed in 1975 by Campbell and Meyer [3], who consider it difficult to establish the norm estimates for the perturbation of the Drazin inverse.

Case (1). If $QF=0$, then $\triangle F = P^d F$, $Q\triangle F = 0$. Thus Theorem 6 reduces to

$$(P+Q)^d = \sum_{n=0}^{\infty} \big(P^d\big)^{n+2} FQ(P+Q)^n\big[I-(P+Q)\triangle\big] + \big(I-PP^d\big)\sum_{n=0}^{\infty}(P+Q)^n FQ\triangle^{n+2} - P^d FQ\triangle + (I-F)\triangle + P^d F.$$

Case (1.a). If $QF=0$ and $F = I - PP^d$, then $P^dF=0$, $(P+Q)^nF = P^nF$. Case (1) just is

$$(P+Q)^d = \sum_{n=0}^{\infty} P^n\big(I-PP^d\big)Q\triangle^{n+2} + PP^d\triangle.$$

Case (1.a.1). If $QF=0$, $F=I-PP^d$ and Q is quasi-nilpotent, then, by Corollary 2(3),

$$PP^d\triangle = PP^d\sum_{n=0}^{\infty}\big(P^d\big)^{n+1}(-Q)^n = PP^d\big(I+P^dQ\big)^{-1}P^d.$$

Case (1.a) becomes

$$(P+Q)^d = \sum_{n=0}^{\infty} P^n\big(I-PP^d\big)Q\big(I+P^dQ\big)^{-(n+2)}\big(P^d\big)^{n+2} + PP^d\big(I+P^dQ\big)^{-1}P^d.$$

Case (1.a.2). If $QF=FQ=0$, $F=I-PP^d$ and Q is quasi-nilpotent, then Case (1.a.1) turns into

$$(P+Q)^d = P^d\big(I+QP^d\big)^{-1}.$$

Case (1.b). If $QF=0$ and $F=PP^d$, then

$$(P+Q)^nF = P^nF = FP^nF = F(P+Q)^nF.$$

So we have

$$\big(I-PP^d\big)\sum_{n=0}^{\infty}(P+Q)^nFQ\triangle^{n+2} = 0.$$

Since

$$\begin{aligned} -P^dFQ\triangle + (I-F)\triangle + P^dF &= \big(I-P^dQ\big)(I-F)\triangle + P^dF \\ &= \big(I+P^dQ\big)^{-1}P^\pi\triangle + P^dF \end{aligned}$$

and $P^\pi\triangle = P^\pi\sum_{n=0}^{\infty}(Q^d)^{n+1}(-P)^n$ by Corollary 2(3) and Eq. (10), Case (1) becomes

$$(P+Q)^d = \sum_{n=0}^{\infty}\big(P^d\big)^{n+2}Q(P+Q)^n\left[I-(P+Q)\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n\right] + \big(I+P^dQ\big)^{-1}P^\pi\sum_{n=0}^{\infty}\big(Q^d\big)^{n+1}(-P)^n + P^d.$$

Case (1.b.1). (See Theorem 2.3 in [11].) If $QF = 0$, $F = PP^d$ and Q is quasi-nilpotent, then Case (1.b) can be simplified as

$$(P+Q)^d = \sum_{n=0}^{\infty} \left(P^d\right)^{n+2} Q (P+Q)^n + P^d.$$

Case (1.b.2). If $QF = FQ = 0$, $F = PP^d$ and Q is quasi-nilpotent, then Case (1.b.1) becomes

$$(P+Q)^d = P^d.$$

Case (2). (See [8].) If $QF = (I-F)Q = 0$, then $\triangle = P^d - P^d Q P^d$, $(I-F)(P+Q)^n = (I-F)P^n$, $(P+Q)^n F = P^n F$, $(I-F)[I-(P+Q)\triangle] = (I-F)(I-PP^d)$ and $(I-F)\triangle = (I-F)P^d$. Theorem 6 reduces to

$$(P+Q)^d = \sum_{n=0}^{\infty} \left(P^d\right)^{n+2} Q P^n \left(I - PP^d\right) + \left(I - PP^d\right) \sum_{n=0}^{\infty} P^n Q \left(P^d\right)^{n+2} + P^d - P^d Q P^d.$$

Case (2.a). If $QF = (I-F)Q = P(I-F) = 0$, then $QP = QP^d = 0$ and Case (2) is

$$(P+Q)^d = \left(P^d\right)^2 Q + P^d.$$

Case (2.b). If $QF = (I-F)Q = FP = 0$, then $PQ = P^d Q = 0$ and Case (2) turns to be

$$(P+Q)^d = Q \left(P^d\right)^2 + P^d.$$

Remark. Case (2) shows that results of this paper are more general than the corresponding results in [8,16,21]. Cases (1.b), (1.b.1) and (1.b.2) are also the special case of Theorem 4. Analogous results are proved in complex Banach algebras in [11].

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