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Generalized Halanay inequalities for dissipativity of Volterra functional differential equations ☆

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ABSTRACT

This paper is concerned with the dissipativity of theoretical solutions to nonlinear Volterra functional differential equations (VFDEs). At first, we give some generalizations of Halanay's inequality which play an important role in study of dissipativity and stability of differential equations. Then, by applying the generalization of Halanay's inequality, the dissipativity results of VFDEs are obtained, which provides unified theoretical foundation for the dissipativity analysis of systems in ordinary differential equations (ODEs), delay differential equations (DDEs), integro-differential equations (IDEs), Volterra delay-integro-differential equations (VDIDEs) and VFDEs of other type which appear in practice.

1. Introduction

Let H be a real or complex Hilbert space with the inner product $\langle\cdot,\cdot\rangle$ and the corresponding norm $\|\cdot\|$, X be a dense continuously imbedded subspace of H. For any given closed interval $I \subset R$, let the symbol $C_X(I)$ denote a Banach space consisting of all continuous mappings $x : I \to X$, on which the norm is defined by $\|x\|_\infty = \max_{t\in I} \|x(t)\|$. Consider the initial value problem in nonlinear VFDEs

$$\begin{cases} y'(t) = f\big(t, y(t), y(\cdot)\big), & t \geqslant 0, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{1.1}$$

where τ is a positive constant, $\varphi \in C_X[-\tau, 0]$ is a given initial function, $f : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to H$ is a given locally Lipschitz continuous mapping satisfying

$$\begin{aligned} &2\Re\big\langle u, f\big(t, u, \psi(\cdot)\big)\big\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{t-\mu_2(t)\leqslant\xi\leqslant t-\mu_1(t)} \big\|\psi(\xi)\big\|^2, \\ &\quad u \in X,\ \ \psi \in \mathbf{C}_X[-\tau, +\infty),\ \ t \in [0, +\infty), \end{aligned} \tag{1.2}$$

where the functions $\mu_1(t)$ and $\mu_2(t)$ are assumed to satisfy

$$0 \leqslant \mu_1(t) \leqslant \mu_2(t) \leqslant t + \tau, \quad \forall t \in [0, +\infty), \tag{1.3}$$

$\alpha(t)$ and $\beta(t)$ are continuous functions and $\gamma(t)$ is a bounded continuous functions on the interval $[0, +\infty)$.

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Throughout this paper we always assume that $f(t,\psi(t),\psi(\cdot))$ is independent of the values of the function $\psi(\xi)$ with $t<\xi<+\infty$, i.e. $f(t,\psi(t),\psi(\cdot))$ is a Volterra functional, and that the problem (1.1) has a true solution $y(t)\in C_X[-\tau,+\infty)$. Note that the condition (1.2) arises in the study of several frequently analyzed partial differential equations, including the Navier–Stokes equation whose semi-discrete approximation can be written in the form (1.1) with $\beta(t)\equiv 0$ (cf. [1]).

Many interesting problems in physics and engineering are modelled by dissipative systems in VFDEs (1.1), which are characterized by possessing a bounded absorbing set that all trajectories enter in a finite time and thereafter remain inside (cf. [1,2]). In the study of dissipative systems it is often the asymptotic behavior of the system that is of interest, and so it is highly desirable to have numerical methods that retain the dissipativity of the underlying system.

In 1994, Humphries and Stuart [2] first studied the dissipativity of initial value problems (IVPs) in ODEs. They also investigated the dissipativity of Runge–Kutta methods applying to ODEs. Later, many results on the dissipativity of numerical methods for ODEs have already been found [3,4,17]. For the delay differential equations (DDEs) with constant delay, Huang [5] gave a sufficient condition for the dissipativity of theoretical solution, and investigated the dissipativity of (k,l)-algebraically stable Runge–Kutta methods. Huang [6–8], in addition, obtained some results about the dissipativity of linear θ-methods, $G(c,p,0)$-algebraically stable one-leg methods and multistep Runge–Kutta methods, respectively. In 2004, Tian [9] studied the dissipativity of DDEs with a bounded variable lag and the dissipativity of θ-method. Recently, Gan [13–15] studies the dissipativity of theoretical solution and θ-methods of IDEs, nonlinear VDIDEs and pantograph equations, respectively. Tian [16] give the numerical dissipativity of multistep methods for delay differential equations. Zhen and Huang [19] have given the sufficient conditions for the dissipativity of theoretical solution of nonlinear NDDEs and have obtained the numerical dissipativity for a class of linear multistep methods when they were applied to these problems.

Moreover, Wen [12] discussed the dissipativity of Volterra functional differential equations. In [12], the dissipativity of nonlinear VFDEs (1.1) satisfying (1.2) is investigated, but the studies in [12] were restricted to the case: $\mu_1^{(0)}>0$, where $\mu_1^{(0)}=\inf_{0\leqslant t<+\infty}\mu_1(t)$. This restrictive condition is too strong to some systems which appear frequently in practice belong to the problem class. For example, the DDEs with piecewise delays (cf. [18], also see Example 4.3) and some Volterra delay-integro-differential equations (cf. [14]) do not belong to aforementioned problem class. In order to overcome this objection, in this paper, we first give two generalizations of Halanay's inequality which play an important role in study of dissipativity and stability of differential equations (cf. [10,11]), then obtain the result of dissipativity of VFDEs (1.1) by applying the generalized Halanay's inequality. This process indicates that not only the restrictive condition appear in [12] is removed but also the proof of the result is much simpler than that in [12].

2. A generalized Halanay inequality

In order to discuss the stability of the zero solution of

$$u'(t)=Au(t)+Bu\big(t-\tau^*\big),\quad \tau^*>0, \tag{2.1}$$

Halanay used the inequality as following:

Lemma 2.1 *(Halanay's inequality). (See [10,11].) If*

$$v'(t)\leqslant -Av(t)+B\sup_{t-\tau^*\leqslant s\leqslant t}v(s)\quad \text{for } t\geqslant t_0, \tag{2.2}$$

and if $A>B>0$, then there exist $\gamma>0$ and $k>0$ such that

$$v(t)\leqslant ke^{-\gamma(t-t_0)}\quad \text{for } t\geqslant t_0, \tag{2.3}$$

and hence $v(t)\to 0$ as $t\to\infty$.

In 1996, in order to investigate analytical and numerical stability of an equation of the type

$$u'(t)=f\left(t,u(t),u\big(\alpha(t)\big),\int_{t-\tau(t)}^{t}K\big(t,s,u(s)\big)\,ds\right),\quad t\geqslant t_0, \tag{2.4}$$

$$u(t)=\phi(t),\quad t\leqslant t_0,\ \phi \text{ bounded and continuous for } t\leqslant t_0,$$

under various assumptions on $\tau(t)\geqslant 0$ and $\alpha(t)\leqslant t$, Baker and Tang [11] give one generalization of Halanay inequality as the following Lemma 2.2 and which can be used discussion of the stability of solutions of some general type Volterra functional equations.

Lemma 2.2. *If $v(t)>0$, $t\in(-\infty,+\infty)$, and*

$$v'(t)\leqslant -A(t)v(t)+B(t)\sup_{t-\tau(t)\leqslant s\leqslant t}v(s)\quad (t\geqslant t_0),\qquad v(t)=\big|\psi(t)\big|\quad (t\leqslant t_0), \tag{2.5}$$

where $\psi(t)$ is bounded and continuous for $t \leqslant t_0$, $A(t), B(t) \geqslant 0$ for $t \in [t_0, +\infty)$, $\tau(t) \geqslant 0$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$, and if there exists $\sigma > 0$ such that

$$-A(t) + B(t) \leqslant -\sigma < 0 \quad \text{for } t \geqslant t_0, \tag{2.6}$$

then

$$\text{(i)} \quad v(t) \leqslant \sup_{t \in (-\infty, t_0]} |\psi(t)|, \quad t \geqslant t_0, \tag{2.7}$$

and

$$\text{(ii)} \quad v(t) \to 0 \quad \text{as } t \to +\infty. \tag{2.8}$$

In this paper, we first give one generalization of Lemma 2.2 that can yield a dissipativity result of VFDEs (1.1).

Theorem 2.3 *(A generalized Halanay's inequality). If $u(t) \geqslant 0$, $t \in (-\infty, +\infty)$, and*

$$u'(t) \leqslant \gamma(t) + \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t) \leqslant \xi \leqslant t} u(\xi) \quad (t \geqslant t_0), \qquad u(t) = |\psi(t)| \quad (t \leqslant t_0), \tag{2.9}$$

where $\psi(t)$ is bounded and continuous for $t \leqslant t_0$, continuous functions $\gamma(t) \geqslant 0$, $\beta(t) \geqslant 0$ and $\alpha(t) \leqslant 0$ for $t \in [t_0, +\infty)$, $\tau(t) \geqslant 0$ and $t - \tau(t) \to +\infty$ as $t \to +\infty$, and if there exists $\sigma > 0$ such that

$$\alpha(t) + \beta(t) \leqslant -\sigma < 0 \quad \text{for } t \geqslant t_0, \tag{2.10}$$

then we have

(i)

$$u(t) \leqslant \frac{\gamma^*}{\sigma} + G, \quad t \geqslant t_0. \tag{2.11}$$

If we assume further that there exists $0 < \delta < 1$ such that

$$\delta\alpha(t) + \beta(t) < 0 \quad \text{for } t \geqslant t_0, \tag{2.12}$$

then we have

(ii) *for any given $\epsilon > 0$, there exists $\hat{t} = \hat{t}(G, \epsilon) > t_0$ such that*

$$u(t) \leqslant \frac{\gamma^*}{\sigma} + \epsilon, \quad t \geqslant \hat{t}, \tag{2.13}$$

where

$$G = \sup_{-\infty < \xi \leqslant t_0} |\psi(\xi)| \quad \text{and} \quad \gamma^* = \sup_{t_0 \leqslant t < +\infty} \gamma(t). \tag{2.14}$$

Proof. We now consider the following two cases successively.

Case 1. $\gamma^* > 0$.

First, if $G > 0$, for any $\varepsilon > 1$, we have from (2.9) that $u(t) < \frac{\gamma^*}{\sigma} + \varepsilon G$ for $t \leqslant t_0$; from this we shall deduce that

$$u(t) < \frac{\gamma^*}{\sigma} + \varepsilon G \quad \text{for } t \geqslant t_0. \tag{2.15}$$

To prove (2.15), suppose (2.15) is not true and shall obtain a contradiction. Since $u(t)$ is a continuous function, there exists $t_1 > t_0$ such that $u(t) < \frac{\gamma^*}{\sigma} + \varepsilon G$ $(-\infty < t < t_1)$ and $u(t_1) = \frac{\gamma^*}{\sigma} + \varepsilon G$, which implies that there exist ε_1 $(1 < \varepsilon_1 < \varepsilon)$ and t_2 $(t_0 < t_2 \leqslant t_1)$ such that $u(t) < \frac{\gamma^*}{\sigma} + \varepsilon_1 G$ $(-\infty < t < t_2)$ and $u(t_2) = \frac{\gamma^*}{\sigma} + \varepsilon_1 G$ and we deduce that $u'(t_2) \geqslant 0$. On the other hand, from our assumption, we have

$$u'(t_2) \leqslant \gamma(t_2) + \alpha(t_2)u(t_2) + \beta(t_2) \sup_{t_2-\tau(t_2) \leqslant \xi \leqslant t_2} u(\xi) \leqslant \gamma^* + \big(\alpha(t_2) + \beta(t_2)\big)\left(\frac{\gamma^*}{\sigma} + \varepsilon_1 G\right) \leqslant -\sigma\varepsilon_1 G < 0,$$

in contradiction of our result $u'(t_2) \geqslant 0$. Hence the inequality (2.14) must hold. Since $\varepsilon > 1$ is arbitrary, we let $\varepsilon \to 1$ and obtain

$$u(t) \leqslant \frac{\gamma^*}{\sigma} + G, \quad t \geqslant t_0. \tag{2.16}$$

Secondly, if $G = 0$, applying similar technique as proving (2.15) we can prove $u(t) \leqslant \frac{\gamma^*}{\sigma}$, $t \geqslant t_0$. Therefore (2.11) holds.

We turn our attention to part (ii) of the theorem.

If $G = 0$, it is evident from (2.11) that (2.13) holds. Now we assume that $G > 0$. Let $\limsup_{t\to\infty} u(t) = \eta$, then $0 \leqslant \eta \leqslant \frac{\gamma^*}{\sigma} + G$. Now we prove the $\eta \leqslant \frac{\gamma^*}{\sigma}$. Suppose this is not true, i.e. $\eta > \frac{\gamma^*}{\sigma}$, then we can choose $\varepsilon > 0$ such that $\eta = \frac{\gamma^*}{\sigma} + \varepsilon$. Since

$$\tau(t) \geqslant 0 \quad \text{and} \quad \lim_{t\to\infty}\big(t - \tau(t)\big) = \infty,$$

therefore, we have $\limsup_{t\to\infty} \sup_{t-\tau(t)\leqslant\xi\leqslant t} u(\xi) = \eta$. From (2.10) and $\beta(t) \geqslant 0$ $(t \geqslant t_0)$ we have $\alpha(t) \leqslant -\sigma < 0$. Noting (2.12), thus there exists a sufficiently large $T > 0$ such that $p := \delta + (1-\delta)e^{-\sigma T} < 1$.

For any ξ with $0 < \xi < (1-p)\varepsilon/(p+1)$, using the properties of superior limits we have that there exists $t^* > t_0$ such that

$$u(t) \leqslant \eta + \xi, \qquad \sup_{t-\tau(t)\leqslant\varsigma\leqslant t} u(\varsigma) \leqslant \eta + \xi \quad \text{for } t^* - T \leqslant t \leqslant t^* \quad \text{and} \quad u\big(t^*\big) \geqslant \eta - \xi. \tag{2.17}$$

On the other hand, it follows from (2.9) and (2.10) that

$$\begin{aligned} u'(t) &\leqslant \gamma^* + \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t)\leqslant s\leqslant t} u(s) \leqslant \frac{\alpha(t)+\beta(t)}{-\sigma}\gamma^* + \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t)\leqslant s\leqslant t} u(s) \\ &= \alpha(t)\left(u(t) - \frac{\gamma^*}{\sigma}\right) + \beta(t) \sup_{t-\tau(t)\leqslant s\leqslant t}\left(u(s) - \frac{\gamma^*}{\sigma}\right). \end{aligned} \tag{2.18}$$

Denote $v(t) = u(t) - \frac{\gamma^*}{\sigma}$. Hence (2.18) deduces that

$$v'(t) \leqslant \alpha(t)v(t) + \beta(t) \sup_{t-\tau(t)\leqslant s\leqslant t} v(s) \tag{2.19}$$

and (2.17) deduces that

$$v(t) \leqslant \varepsilon + \xi, \qquad \sup_{t-\tau(t)\leqslant\varsigma\leqslant t} v(\varsigma) \leqslant \varepsilon + \xi \quad \text{for } t^* - T \leqslant t \leqslant t^* \quad \text{and} \quad v\big(t^*\big) \geqslant \varepsilon - \xi. \tag{2.20}$$

Multiplying both sides of (2.19) by $\exp(-\int_{t_0}^{t} \alpha(s)\,ds)$ and then integrating both sides of (2.19) from $t^* - T$ to t^*, we obtain that

$$\begin{aligned} \varepsilon - \xi \leqslant v\big(t^*\big) &\leqslant v\big(t^* - T\big)\exp\left(\int_{t^*-T}^{t^*} \alpha(\eta)\,d\eta\right) + \int_{t^*-T}^{t^*} \beta(\zeta) \sup_{\zeta-\tau(\zeta)\leqslant s\leqslant\zeta} v(s)\exp\left(\int_{\zeta}^{t^*} \alpha(\eta)\,d\eta\right)d\zeta \\ &\leqslant (\varepsilon+\xi)\left(\delta + (1-\delta)\exp\left(\int_{t^*-T}^{t^*} \alpha(\eta)\,d\eta\right)\right) \leqslant (\varepsilon+\xi)\big(\delta + (1-\delta)\exp(-\sigma T)\big) = (\varepsilon+\xi)p \end{aligned}$$

which implies $\xi \geqslant (1-p)\varepsilon/(1+p)$ contrary to the choice of ξ. This contradicts the choice $\eta > \frac{\gamma^*}{\sigma}$. From the properties of superior limits we obtain (2.13) easily.

Case 2. $\gamma^* = 0$.

If only we replace γ^* in the proof of Case 1 by $\gamma^* + \epsilon$ for any given $\epsilon > 0$, then let $\epsilon \to 0$, we can obtain that (2.11) and (2.13) hold.

A combination of Cases 1 and 2 completes the proof of Theorem 2.3. □

On the other hand, we can also give a generalization of Lemma 2.1 which can give a quantitative estimate of solutions of VFDEs (1.1).

Theorem 2.4. *If $u(t) \geqslant 0$, $t \in (-\infty, +\infty)$, and*

$$u'(t) \leqslant \gamma(t) + \alpha(t)u(t) + \beta(t) \sup_{t-\tau(t) \leqslant \xi \leqslant t} u(\xi) \quad (t \geqslant t_0), \qquad u(t) = \left|\psi(t)\right| \quad (t \leqslant t_0), \tag{2.21}$$

where $\psi(t)$ is bounded and continuous for $t \leqslant t_0$, continuous functions $\gamma(t) \geqslant 0$, $\beta(t) \geqslant 0$ and $\alpha(t) \leqslant 0$ for $t \in [t_0, +\infty)$, $\tau(t) \geqslant 0$, and if there exists $\sigma > 0$ such that

$$\alpha(t) + \beta(t) \leqslant -\sigma < 0 \quad \text{for } t \geqslant t_0,$$

then we have

$$u(t) \leqslant \frac{\gamma^*}{\sigma} + Ge^{-\mu^*(t-t_0)}, \quad t \geqslant t_0, \tag{2.22}$$

where G and γ^* are given by (2.14) *and $\mu^* \geqslant 0$ is defined as*

$$\mu^* = \inf_{t \geqslant t_0} \left\{\mu(t)\colon\ \mu(t) + \alpha(t) + \beta(t)e^{\mu(t)\tau(t)} = 0\right\}. \tag{2.23}$$

Proof. If $G = 0$, (2.22) is evident from (2.11). Now we assume that $G > 0$. Denote

$$H(\mu) = \mu + \alpha(t) + \beta(t)e^{\mu\tau(t)}. \tag{2.24}$$

For any given fixed $t \geqslant t_0$, we see that

$$H(0) = \alpha(t) + \beta(t) \leqslant -\sigma < 0, \qquad \lim_{\mu \to +\infty} H(\mu) = +\infty$$

and

$$H'(\mu) = 1 + \tau(t)\beta(t)e^{\mu\tau(t)} > 0.$$

Therefore for any given $t \geqslant t_0$ there is a unique positive μ such that

$$\mu + \alpha(t) + \beta(t)e^{\mu\tau(t)} = 0, \tag{2.25}$$

that means the (2.25) define an implicit function $\mu(t)$ for $t \geqslant t_0$. From that definition, one has $\mu^* \geqslant 0$.

For the case of $\mu^* = 0$, (2.22) follows from (2.11) straightway. Now we prove that (2.22) holds for the case of $\mu^* > 0$.

For any given $\varepsilon > 0$, we can prove that

$$u(t) \leqslant \frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(t-t_0)}, \quad t \geqslant t_0. \tag{2.26}$$

In fact, suppose (2.26) fails, then there is some $t > t_0$ such that

$$u(t) > \frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(t-t_0)}.$$

Let

$$v(t) = \frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(t-t_0)}, \quad t \geqslant t_0,$$

and $w(t) = v(t) - u(t)$. Let $\varsigma = \inf\{t \geqslant t_0\colon\ v(t) - u(t) \leqslant 0\}$. Then we have $w(\varsigma) = v(\varsigma) - u(\varsigma) = 0$, and $w'(\varsigma) = v'(\varsigma) - u'(\varsigma) \leqslant 0$. Hence,

$$\begin{aligned} w'(\varsigma) = v'(\varsigma) - u'(\varsigma) &\geqslant -G\mu^* e^{-\mu^*(\varsigma - t_0)} - \Big[\gamma(\varsigma) + \alpha(\varsigma)u(\varsigma) + \beta(\varsigma) \sup_{\varsigma-\tau(\varsigma) \leqslant \xi \leqslant \varsigma} u(\xi)\Big] \\ &> -G\mu^* e^{-\mu^*(\varsigma - t_0)} - \Big[\gamma^* + \varepsilon + \alpha(\varsigma)u(\varsigma) + \beta(\varsigma) \sup_{\varsigma-\tau(\varsigma) \leqslant \xi \leqslant \varsigma} u(\xi)\Big]. \end{aligned} \tag{2.27}$$

If $\varsigma - \tau(\varsigma) \geqslant t_0$, it follows from (2.27) that

$$\begin{aligned} w'(\varsigma) &> -G\mu^* e^{-\mu^*(\varsigma - t_0)} - \Big[\gamma^* + \varepsilon + \alpha(\varsigma)\Big(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(\varsigma - t_0)}\Big) + \beta(\varsigma)\Big(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(\varsigma - \tau(\varsigma) - t_0)}\Big)\Big] \\ &= -\frac{\gamma^* + \varepsilon}{\sigma}\big(\sigma + \alpha(\varsigma) + \beta(\varsigma)\big) - Ge^{-\mu^*(\varsigma - t_0)}\big[\mu^* + \alpha(\varsigma) + \beta(\varsigma)e^{\mu^*\tau(\varsigma)}\big]. \end{aligned} \tag{2.28}$$

From the define of function $\mu(t)$, we have

$$\mu^* + \alpha(\varsigma) + \beta(\varsigma)e^{\mu^*\tau(\varsigma)} = \mu^* + \alpha(\varsigma) + \beta(\varsigma)e^{\mu^*\tau(\varsigma)} - \mu(\varsigma) - \alpha(\varsigma) - \beta(\varsigma)e^{\mu(\varsigma)\tau(\varsigma)}$$
$$= \big(\mu^* - \mu(\varsigma)\big) + \beta(\varsigma)\big(e^{\mu^*\tau(\varsigma)} - e^{\mu(\varsigma)\tau(\varsigma)}\big) \leqslant 0. \tag{2.29}$$

Noting (2.10), therefore (2.28) yields

$$w'(\varsigma) = v'(\varsigma) - u'(\varsigma) > 0, \tag{2.30}$$

in contradiction of our result $w'(\varsigma) \leqslant 0$.

If $\varsigma - \tau(\varsigma) < t_0$, it follows from (2.27) that

$$\begin{aligned} w'(\varsigma) &> -G\mu^* e^{-\mu^*(\varsigma - t_0)} - \left[\gamma^* + \varepsilon + \alpha(\varsigma)\left(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(\varsigma - t_0)}\right) + \beta(\varsigma)\max\left\{\sup_{\xi \leqslant t_0} u(\xi), \sup_{t_0 \leqslant \xi \leqslant \varsigma} u(\xi)\right\}\right] \\ &\geqslant -G\mu^* e^{-\mu^*(\varsigma - t_0)} - \left[\gamma^* + \varepsilon + \alpha(\varsigma)\left(\frac{\gamma^* + \varepsilon}{\sigma} + Ge^{-\mu^*(\varsigma - t_0)}\right) + \beta(\varsigma)\left(G + \frac{\gamma^* + \varepsilon}{\sigma}\right)\right] \\ &= -\frac{\gamma^* + \varepsilon}{\sigma}\big(\sigma + \alpha(\varsigma) + \beta(\varsigma)\big) - Ge^{-\mu^*(\varsigma - t_0)}\big[\mu^* + \alpha(\varsigma) + \beta(\varsigma)e^{\mu^*(\varsigma - t_0)}\big] \\ &\geqslant -\frac{\gamma^* + \varepsilon}{\sigma}\big(\sigma + \alpha(\varsigma) + \beta(\varsigma)\big) - Ge^{-\mu^*(\varsigma - t_0)}\big[\mu^* + \alpha(\varsigma) + \beta(\varsigma)e^{\mu^*\tau(\varsigma)}\big]. \end{aligned}$$

Here we also obtain that (2.30) holds, which contradicts our result $w'(\varsigma) \leqslant 0$. Hence the inequality (2.26) must hold. Since $\varepsilon > 0$ is arbitrary, we let $\varepsilon \to 0$ and obtain (2.22). The proof of Theorem 2.4 is completed. □

Remark 2.5. In 2004, Tian [9] gives also a generalization of Halanay's inequality similar to that of Theorem 2.4, but he requests that $\tau(t)$ is a constant. In addition, in Theorem 2.4, we do not request that $t - \tau(t) \to +\infty$ as $t \to +\infty$. But in [9] the author proves that $\mu^* > 0$ when $\tau(t) \equiv \tau$, i.e. $\tau(t)$ is a constant. Therefore, we guess that there $\mu^* > 0$ also holds when adding the $t - \tau(t) \to +\infty$ as $t \to +\infty$, but we cannot prove it presently.

3. Dissipativity of VFDEs

Proposition 3.1. *If the mapping f in problem* (1.1) *satisfies the condition* (1.2), *then we have*

$$\gamma(t) \geqslant 0, \qquad \beta(t) \geqslant 0, \quad \forall t \in [0, +\infty). \tag{3.1}$$

Definition 3.2. The problem (1.1) in VFDEs is said to be dissipative in H if there exists a bounded set $B \subset H$, such that for any given bounded set $\Phi \subset H$, there is a time $t^* = t^*(\Phi)$, such that for any given initial function $\varphi \in C_X[-\tau, 0]$ with $\varphi(t)$ contained in Φ for all $t \in [-\tau, 0]$, the values of the corresponding solution $y(t)$ of the problem are contained in B for all $t \geqslant t^*$. Here B is called an absorbing set of the problem.

Theorem 3.3. *Suppose that $y(t)$ is a solution of the problem* (1.1) *satisfying the condition* (1.2), *and there exist constants $\sigma > 0$ and $0 < \delta < 1$ such that*

$$\alpha(t) + \beta(t) \leqslant -\sigma < 0 \quad \textit{and} \quad \delta\alpha(t) + \beta(t) < 0 \quad \textit{for } t \geqslant 0. \tag{3.2}$$

Then,

(i) *for any given $\varepsilon > 0$, there exists a positive number $t^* = t^*(\|\varphi\|_\infty, \varepsilon)$, such that*

$$\|y(t)\|^2 \leqslant \frac{\gamma^*}{\sigma} + \varepsilon, \quad \forall t > t^*;$$

(ii) *for any given $\varepsilon > 0$, the problem* (1.1) *is dissipative with an absorbing set $B = B(0, \sqrt{\gamma^*/\sigma + \varepsilon})$, where $\gamma^* = \sup_{0 \leqslant t < +\infty} \gamma(t)$.*

Proof. To apply the result of Theorem 2.3, we have to extend the define of initial function in (1.1) as $y(t) = \varphi(-\tau)$ for $-\infty < t \leqslant -\tau$.

Let

$$u(t) = \|y(t)\|^2 = \langle y(t), y(t)\rangle.$$

From (1.2), we have

$$\begin{aligned} u'(t) = \frac{d}{dt}\langle y(t), y(t)\rangle = 2\Re\langle y(t), f\big(t, y(t), y(\cdot)\big)\rangle &\leqslant \gamma(t) + \alpha(t)u(t) + \beta(t) \max_{t - \mu_2(t) \leqslant \xi \leqslant t - \mu_1(t)} u(\xi) \\ &\leqslant \gamma(t) + \alpha(t)u(t) + \beta(t) \max_{t - \mu_2(t) \leqslant \xi \leqslant t} u(\xi). \end{aligned}$$

Therefore, for any given $\varepsilon > 0$, it is easy to know from Theorem 2.3 that there exists $t^* > 0$, such that

$$u(t) \leqslant \frac{\gamma^*}{\sigma} + \varepsilon, \quad t \geqslant t^*.$$

This completes the proof of Theorem 3.3. □

Remark 3.4. Theorem 3.3 is different from the dissipativity results obtained by the same author of this paper [12]. In [12], we investigated the dissipativity of nonlinear VFDEs (1.1)–(1.2) with the restrictive condition $\mu_1^{(0)} > 0$, where $\mu_1^{(0)} = \inf_{0 \leqslant t < +\infty} \mu_1(t)$.

Remark 3.5. Many authors have also studied the dissipativity for some special cases of the problem (1.1), such as Humphries and Stuart [2] (for initial value problems in ordinary differential equations), Huang [5] (for system in delay differential equations with constant delay), Tian [9] (for nonlinear system in delay differential equations with a bounded variable lag), Gan [13–15] (for the pantograph delay differential equations, integro-differential equations and nonlinear Volterra delay-integro-differential equations, respectively) and Wen [18] (for nonlinear system in delay differential equations with a piecewise delay).

Because the problem classes which are studied in the present paper contain the problems as special cases which are mentioned above. Therefor, which provides unified theoretical foundation for the dissipativity analysis of systems in ordinary differential equations (ODEs), delay differential equations (DDEs), integro-differential equations (IDEs) and VFDEs of other type which appear in practice.

4. Application to some special cases

(1) *Application to delay differential equations*. Consider the initial value problem in delay differential equations

$$\begin{cases} y'(t) = g\big(t, y(t), y(\eta_1(t)), y(\eta_2(t)), \ldots, y(\eta_r(t))\big), & t \in [0, +\infty), \\ y(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{4.1a}$$

where each $\eta_i \in C_R[0, +\infty)$ satisfies

$$-\tau \leqslant \eta_i(t) \leqslant t, \quad \forall t \in [0, +\infty), \ i = 1, \ldots, r. \tag{4.1b}$$

The mapping $g : [0, +\infty) \times X^{r+1} \to X$ satisfying

$$2\Re\langle u, g(t, u, x_1, x_2, \ldots, x_r)\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{1 \leqslant i \leqslant r} \|x_i\|^2, \quad \forall t \in [0, +\infty), \ u, x_1, x_2, \ldots, x_r \in X, \tag{4.2}$$

the meaning of other symbols are the same as mentioned in Section 2, and we always assume that the problem (4.1) has a true solution $y(t) \in C_X[-\tau, +\infty)$.

Let

$$\hat{f}\big(t, u, \psi(\cdot)\big) = g\big(t, u, \psi(\eta_1(t)), \psi(\eta_2(t)), \ldots, \psi(\eta_r(t))\big), \quad \forall t \in [0, +\infty), \ u \in X, \ \psi \in C_X[-\tau, +\infty).$$

Then the problem (4.1) satisfying the condition (4.2) can be equivalently written in the pattern of the initial value problem (1.1) in VFDEs, i.e.

$$\begin{cases} y'(t) = \hat{f}\big(t, y(t), y(\cdot)\big), & 0 \leqslant t < +\infty, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{4.3}$$

where the mapping $\hat{f} : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to X$ satisfies the condition

$$\begin{aligned} 2\Re\langle u, \hat{f}\big(t, u, \psi(\cdot)\big)\rangle &= 2\Re\langle u, g\big(u, \psi(\eta_1(t)), \ldots, \psi(\eta_r(t))\big)\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{1 \leqslant i \leqslant r} \big\|\psi(\eta_i(t))\big\|^2 \\ &\leqslant \gamma(t) + \alpha(t)\|u\|^2 + \beta(t) \max_{t-\mu_2(t) \leqslant \xi \leqslant t} \big\|\psi(\xi)\big\|^2, \quad \forall t \in [0, +\infty), \ u \in X, \ \psi \in C_X[-\tau, +\infty), \end{aligned}$$

with

$$\mu_1(t) = t - \max_{1 \leqslant i \leqslant r} \eta_i(t), \qquad \mu_2(t) = t - \min_{1 \leqslant i \leqslant r} \eta_i(t).$$

Therefore, Theorem 3.3 in the present paper can be directly applied to this special case, and we thus obtain the following result.

Theorem 4.1. *Suppose that $y(t)$ is a solution of the problem (4.1) satisfying the condition (4.2) and*

$$\lim_{t\to+\infty}\Big(\min_{1\leqslant i\leqslant r}\eta_i(t)\Big)=+\infty, \tag{4.4}$$

and there exist constants $\sigma>0$ and $0<\delta<1$ such that

$$\alpha(t)+\beta(t)\leqslant-\sigma<0 \quad \text{and} \quad \delta\alpha(t)+\beta(t)<0 \quad \text{for } t\geqslant 0.$$

Then,

(i) *for any given $\varepsilon>0$, there exists a positive number $t^*=t^*(\|\varphi\|_\infty,\varepsilon)$, such that*

$$\|y(t)\|^2\leqslant\frac{\gamma^*}{\sigma}+\varepsilon, \quad \forall t>t^*,$$

(ii) *for any given $\varepsilon>0$, the problem (4.1) is dissipative with an absorbing set $B=B(0,\sqrt{\gamma^*/\sigma+\varepsilon})$, where $\gamma^*=\sup_{0\leqslant t<+\infty}\gamma(t)$.*

Example 4.2. The initial value problem of nonlinear pantograph equations

$$\begin{cases} y'(t)=g\big(t,y(t),y(q_1t),y(q_2t),\ldots,y(q_rt)\big), & t\in[0,+\infty),\\ y(t)=\varphi(t), & t\in[-\tau,0]\end{cases} \tag{4.5}$$

(where the constants $q_i\in(0,1]$, $i=1,2,\ldots,r$) fits obviously into the pattern of the problem (4.1) in DDEs with

$$\eta_i(t)=q_it, \quad i=1,\ldots,r.$$

Therefore, for the problem (4.5) satisfying the condition (4.2), all the results of Theorem 4.1 are hold true, and it is easily seen that in this case the equality (4.4) is automatically satisfied. In fact, we have

$$\lim_{t\to+\infty}\Big(\min_{1\leqslant i\leqslant r}\eta_i(t)\Big)=\Big(\min_{1\leqslant i\leqslant r}q_i\Big)\lim_{t\to+\infty}t=+\infty.$$

Note Gan [15] has also studied the dissipativity of the true solution and numerical solution for pantograph equations. However, the results of Example 4.2 of the present paper seem to be more general.

Example 4.3. The initial value problem

$$\begin{cases} y'(t)=g\big(t,y(t),y([t]),y([t-1]),\ldots,y([t-r+1])\big), & t\in[0,+\infty),\\ y(t)=\varphi(t), & t\in[-\tau,0]\end{cases} \tag{4.6}$$

(where the symbol $[t]$ denotes the maximum integer not greater than t) fits obviously into the pattern of the problem (4.1) in DDEs with

$$\eta_i(t)=[t-i+1], \quad i=1,\ldots,r.$$

Therefore, for the problem (4.6) satisfying the condition (4.2), all the results of Theorem 4.1 are hold true, and it is easily seen that in this case the equality (4.4) is automatically satisfied. In fact, we have

$$\lim_{t\to+\infty}\Big(\min_{1\leqslant i\leqslant r}\eta_i(t)\Big)=\lim_{t\to+\infty}[t-r+1]=+\infty.$$

Remark 4.4. We have also studied some special cases of the problem (4.6) (cf. [18]). It is obvious that the problem (4.6) does not belong to the systems which are studied in [12]. In fact, in (4.6),

$$\mu_1(t)=t-[t] \quad \text{and} \quad \mu_1^{(0)}=\inf_{0\leqslant t<+\infty}\mu_1(t)=0.$$

(2) *Application to Volterra delay-integro-differential equations*. Consider the initial value problem in Volterra delay-integro-differential equations

$$\begin{cases} y'(t)=g\left(t,y(t),y\big(\eta(t)\big),\displaystyle\int_{\eta_1(t)}^{\eta_2(t)}K\big(t,\xi,y(\xi)\big)\,d\xi\right), & t\in[0,+\infty),\\ y(t)=\varphi(t), & t\in[-\tau,0],\end{cases} \tag{4.7a}$$

where $\eta,\eta_1,\eta_2\in C_R[0,+\infty)$ with

$$-\tau \leqslant \eta(t) \leqslant t \quad \text{and} \quad -\tau \leqslant \eta_1(t) \leqslant \eta_2(t) \leqslant t, \quad \forall t \in [0, +\infty), \tag{4.7b}$$

where the continuous mapping $K : [0, +\infty) \times [-\tau, +\infty) \times X \to X$ satisfying

$$\|K(t, \xi, x)\| \leqslant L_K \|x\|, \quad \forall t \in [0, +\infty),\ \xi \in [-\tau, +\infty),\ x \in X, \tag{4.8}$$

here $L_K > 0$ denotes a constant, and continuous mapping $g : [0, +\infty) \times X \times X \times X \to X$ satisfying

$$2\Re\langle u, g(t, u, v, x)\rangle \leqslant \gamma(t) + \alpha(t)\|u\|^2 + \tilde{\beta}(t)\|v\|^2 + \omega(t)\|x\|^2, \quad \forall t \in [0, +\infty),\ u, v, x \in X. \tag{4.9}$$

The meanings of other symbols are the same as preceding, and we always assume that the problem (4.7) has a true solution $y(t) \in C_X[-\tau, +\infty)$.

Let

$$\tilde{f}\big(t, u, \psi(\cdot)\big) = g\left(t, u, \psi\big(\eta(t)\big), \int_{\eta_1(t)}^{\eta_2(t)} K\big(t, \xi, \psi(\xi)\big)\, d\xi\right), \quad \forall t \in [0, +\infty),\ u \in X,\ \psi \in C_X[-\tau, +\infty).$$

Then the mapping $\tilde{f} : [0, +\infty) \times X \times C_X[-\tau, +\infty) \to X$ satisfies

$$\begin{aligned} 2\Re\langle u, \tilde{f}\big(t, u, \psi(\cdot)\big)\rangle &= 2\Re\left\langle u, g\left(t, u, \psi\big(\eta(t)\big), \int_{\eta_1(t)}^{\eta_2(t)} K\big(t, \xi, \psi(\xi)\big)\, d\xi\right)\right\rangle \\ &\leqslant \gamma(t) + \alpha(t)\|u\|^2 + \tilde{\beta}(t)\big\|\psi\big(\eta(t)\big)\big\|^2 + \omega(t)\left\| \int_{\eta_1(t)}^{\eta_2(t)} K\big(t, \xi, \psi(\xi)\big)\, d\xi \right\|^2. \end{aligned}$$

Because

$$\begin{aligned} \left\| \int_{\eta_1(t)}^{\eta_2(t)} K\big(t, \xi, \psi(\xi)\big)\, d\xi \right\| &\leqslant \int_{\eta_1(t)}^{\eta_2(t)} \big\|K\big(t, \xi, \psi(\xi)\big)\big\|\, d\xi \leqslant L_K \int_{\eta_1(t)}^{\eta_2(t)} \big\|\psi(\xi)\big\|\, d\xi \\ &\leqslant L_K\big(\eta_2(t) - \eta_1(t)\big) \max_{\eta_1(t) \leqslant \xi \leqslant \eta_2(t)} \big\|\psi(\xi)\big\|, \quad \forall t \in [0, +\infty),\ u \in X,\ \psi \in C_X[-\tau, +\infty). \end{aligned}$$

Therefore, the problem (4.7) satisfying the conditions (4.8) and (4.9) can be equivalently written in the pattern of the initial value problem (1.1) in VFDEs, i.e.

$$\begin{cases} y'(t) = \tilde{f}\big(t, y(t), y(\cdot)\big), & 0 \leqslant t < +\infty, \\ y(t) = \varphi(t), & -\tau \leqslant t \leqslant 0, \end{cases} \tag{4.10}$$

with

$$\mu_1(t) = t - \max\{\eta(t), \eta_2(t)\}, \qquad \mu_2(t) = t - \min\{\eta(t), \eta_1(t)\}, \qquad \beta(t) = \tilde{\beta}(t) + \omega(t)\big(L_K\big(\eta_2(t) - \eta_1(t)\big)\big)^2.$$

Thus the result for VFDEs obtained in the present paper can be directly applied to this special case, and we thus obtain the following dissipativity result.

Theorem 4.5. *Suppose that $y(t)$ is a solution of the problem* (4.7) *satisfying the conditions* (4.8), (4.9) *and*

$$\lim_{t \to +\infty} \min\{\eta(t), \eta_1(t)\} = +\infty,$$

and there exist constants $\sigma > 0$ and $0 < \delta < 1$ such that

$$\alpha(t) + \tilde{\beta}(t) + \omega(t)\big(L_K\big(\eta_2(t) - \eta_1(t)\big)\big)^2 \leqslant -\sigma < 0 \quad \text{and} \quad \delta\alpha(t) + \tilde{\beta}(t) + \omega(t)\big(L_K\big(\eta_2(t) - \eta_1(t)\big)\big)^2 < 0 \quad \text{for } t \geqslant 0.$$

Then,

(i) *for any given $\varepsilon > 0$, there exists a positive number $t^* = t^*(\|\varphi\|_\infty, \varepsilon)$, such that*

$$\big\|y(t)\big\|^2 < \frac{\gamma^*}{\sigma} + \varepsilon, \quad \forall t > t^*,$$

(ii) *for any given $\varepsilon > 0$, the problem* (4.7) *is dissipative with an absorbing set $B = B(0, \sqrt{\gamma^*/\sigma + \varepsilon})$.*

Remark 4.6. In the case where

$$\eta_2(t) = t, \qquad \eta_1(t) = \eta(t) = t - \tau$$

with constant $\tau > 0$ and $\alpha, \tilde{\beta}, \gamma$ are constants, (4.7) is just the problem which is studied by Gan in [14]. However, the results of Theorem 4.5 of the present paper seem to be more general.

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