Semigroup Kernels, Poisson Bounds, and Holomorphic Functional Calculus

Xuan T. Duong

School of Mathematics, Physics, Computing and Electronics,
Macquarie University, Sydney, Australia

and

Derek W. Robinson

Centre for Mathematics and its Applications, Australian National University,
Canberra, ACT 0200, Australia

Received February 24, 1995

Let $L$ be the generator of a continuous holomorphic semigroup $S$ whose action is determined by an integral kernel $K$ on a scale of spaces $L_p(X;\rho)$. Under mild geometric assumptions on $(X,\rho)$, we prove that if $L$ has a bounded $H\alpha$-functional calculus on $L_2(X,\rho)$ and $K$ satisfies bounds typical for the Poisson kernel, then $L$ has a bounded $H\alpha$-functional calculus on $L_p(X,\rho)$ for each $p \in (1,\infty)$. Moreover, if $(X,\rho)$ is of polynomial type and $K$ satisfies second-order Gaussian bounds, we establish criteria for $L$ to have a bounded Hörmander functional calculus or a bounded Davies-Heffler-Sjöstrand functional calculus.

1. INTRODUCTION

Spectral theory provides a method of defining the bounded operator

$$\xi(L) = \int_{\mathbb{R}} dE_L(\lambda) \xi(\lambda)$$

for each self-adjoint operator $L$ on a Hilbert space and for each bounded measurable Borel function $\xi$ on the real line $\mathbb{R}$. The spectral representation then gives the bound

$$\|\xi(L)\| \leq \|\xi\|_{\infty},$$

where $\|\cdot\|$ denotes the Hilbert space norm and $\|\cdot\|_{\infty}$ the $L_\infty$-norm. Alternatively the abstract theory of continuous semigroups developed by 89

0022-1236/96 $18.00$

Copyright © 1996 by Academic Press, Inc.
All rights of reproduction in any form reserved.
Hille, Yosida and others allows the exponentiation of the unbounded operator $L$ by a variety of algorithms involving the resolvents $(\lambda I - L)^{-1}$. These algorithms led to the development of notions of functional calculus broader than that provided by spectral theory. In particular one can use complex analysis and the Cauchy integral formula to define quite general functions of $L$ from the resolvent.

Our aim is to analyze the existence of a functional calculus for unbounded operators acting on certain function spaces. We assume the unbounded linear operator $L$ generates a continuous holomorphic semigroup on a Hilbert space $L_2(X; \rho)$ with a kernel satisfying appropriate bounds. The bounds include the usual Gaussians but also cover kernels with slow decrease. In addition we assume that $L$ possesses a bounded functional calculus on $L_2(X; \rho)$ and conclude that $L$ has a bounded functional calculus on the spaces $L_p(X; \rho)$ with $p \in (1, \infty)$.

The space $(X, \rho)$ is assumed to have the usual doubling property and the remaining assumptions are sufficiently mild that they are satisfied by large classes of strongly elliptic and subelliptic operators or pseudodifferential operators. Compared to previously known results which rely on good estimates on the kernels the present results are more general and depend on general boundedness and dispersivity properties of the semigroup kernel. Our arguments do not use translation or dilation invariance nor do we need any smoothness of the heat kernels. Although our method uses techniques of singular integration theory we do not need Hölder estimates and the proofs do not depend upon any delicate cancellations. The method relies upon $L_2$-estimates to obtain weak type $(1, 1)$ bounds. A similar reasoning has previously been given by Hebisch [Heb90].

The general approach we adopt toward functional calculus is based on McIntosh’s notion [McI86] of a bounded $H^f$-functional calculus and we first recall the basic definitions of this theory. Further details can be found in the recent paper of Cowling et al. [CDMY96].

A closed operator $L$ in a Banach space $X$ is said to be of type $\omega$, $0 \leq \omega < \pi$, if its spectrum is a subset of the closed sector $S_\omega = \{ z \in \mathbb{C} : |\arg z| \leq \omega \} \cup \{0\}$, and the resolvents $(\lambda I - L)^{-1}$ satisfy bounds

$$
\| (\lambda I - L)^{-1} \| \leq c_\omega |\lambda|^{-1}
$$

whenever $|\arg \lambda| \geq \mu > \omega$. In particular, $L$ is of type $\omega$ with $\omega \in [0, \pi/2]$ if and only if it generates a bounded holomorphic semigroup with sector of holomorphy $A(\pi/2 - \omega) = \{ z \in \mathbb{C} : |\arg z| < \pi/2 - \omega \}$.

New let $H^f_\omega(S^0_\mu)$, $\mu > \omega$, be the usual space of bounded holomorphic functions in the open sector $S^0_\mu = A(\mu)$, the interior of $S^0_\mu$, and define

$$
\Psi(S^0_\mu) = \{ \xi \in H^f_\omega(S^0_\mu) : |\xi(z)| \leq c |z|^s/(1 + |z|^2) \text{ for some } s > 0, c > 0 \}.
$$
Suppose that $\omega < \theta < \mu$. Let $\gamma$ be the contour defined by the function
\[
\gamma(t) = \begin{cases} 
  te^{it} & \text{if } t \in (0, \infty), \\
  -te^{-it} & \text{if } t \in (-\infty, 0).
\end{cases}
\]
Then for $\xi \in \mathcal{P}(S_0^\mu)$ let
\[
\xi(L) = (2\pi i)^{-1} \int_\gamma d\lambda \xi(\lambda)(L - \lambda I)^{-1}.
\]

The above integral is absolutely convergent in the norm topology and $\xi(L)$ is a bounded linear operator which is independent of the choice of $\theta$. Furthermore, for $\xi \in H_\infty(S_0^\mu)$ we define
\[
\xi(L) = (2\pi i)^{-1} (I + L)^{-1} \int_\gamma d\lambda \xi(\lambda)(1 + \lambda)^{-2} \lambda(L - \lambda I)^{-1}
\]
when $L$ is a one-to-one operator of type $\omega$ with dense domain and dense range. This definition is consistent with the previous one for $\xi \in \mathcal{P}(S_0^\mu)$ and for general $\xi \in H_\infty(S_0^\mu)$ the resulting operator $\xi(L)$ is defined on $D(L)$ and is relatively bounded by $L$. Nevertheless, $\xi(L)$ is not necessarily bounded. Therefore $L$ is defined to have a bounded $H_\infty$-functional calculus on $\mathcal{X}$ whenever the operators $\xi(L)$ are bounded and, in addition, there is $C_\mu > 0$ such that
\[
\|\xi(L)\| \leq C_\mu \|\xi\|_{\infty}
\]
for all $\xi \in H_\infty(S_0^\mu)$ and some $\mu > \omega$ where the $L_\infty$-norm is now defined by $\|\xi\|_{\infty} = \sup \{ |\xi(z)| : z \in S_0^\mu \}$.

The generators of bounded holomorphic semigroups are of type $\omega$ if $\omega < \pi/2$ and the notion of a bounded $H_\infty$-functional calculus is well defined for such generators. But not all of these generators have a bounded calculus. Criteria for a bounded calculus are given in [McI86], [CDMY96] and [BdL92]. For example, all maximal accretive operators on a Hilbert space have a bounded $H_\infty$-functional calculus. More generally, an operator $L$ of type $\omega$ on a Hilbert space has a bounded $H_\infty$-functional calculus if and only if the imaginary powers $t \in \mathbb{R} \mapsto L^t$ are bounded operators. Since the imaginary powers form a one-parameter group it then follows by a standard argument that one has bounds
\[
\|L^t\| \leq Me^{\mu t}
\]
for some $M \geq 1$, $\mu > 0$ and all $t \in \mathbb{R}$. Note that if an operator $L$ of type $\omega$ on a Banach space has a bounded $H_\infty$-functional calculus then the imaginary powers $t \in \mathbb{R} \mapsto L^t$ are bounded operators but the converse is
not always valid [CDMY96]. Nevertheless this is of interest for several reasons. For example, the existence of the imaginary powers as bounded operators ensures that the domains of fractional powers \( L^s \) form a family of interpolation spaces with respect to the complex method [Tri78]. This property is in turn useful in the solution of inhomogeneous hyperbolic evolution equations and related nonlinear equations [DoV90], [PS93], [AHS94] or in the derivation of regularity and universality properties for the domains of elliptic operators [ER93a].

In Section 2 we consider a semigroup \( S \) with generator \( L \) and a kernel acting on a space \( L_s(X; \rho) \). Using weak kernel bounds we derive three basic estimates together with interpolation properties. Subsequently, in Section 3, this preliminary material is applied to holomorphic semigroups and used to establish the existence of a bounded \( H^\alpha \)-functional calculus for \( L \) on the spaces \( L_p(X; \rho) \), \( p \in (1, \infty) \). In Section 4 we discuss several variations of these results and in Section 5 we describe extensions to spaces of exponential growth. Finally in Section 6 we discuss alternative functional calculi.

2. KERNEL BOUNDS AND HOMOGENEITY

Let \( X \) denote a topological space which is of homogeneous type, in the sense of Christ [Chr90], Definition VI.1, with respect to a metric \( d \). Further let \( \rho \) denote a positive measure on \( X \) such that the balls \( B(x; r) = \{ y \in X : d(y; x) < r \} \) are \( \rho \)-measurable, with finite measure, for all \( x \in X \) and \( r > 0 \). Throughout we assume that \( X \) has the doubling property,

\[
|B(x; 2r)| \leq c_2 |B(x; r)|
\]

for some \( c_2 \geq 1 \) uniformly for all \( x \in X \) where \( |A| = \rho(A) \) denotes the volume of the set \( A \).

The doubling property, which implies the homogeneity of \( X \) by the argument on pages 67–68 of [CoW71], also implies the strong homogeneity property,

\[
|B(x; \lambda r)| \leq c \lambda^n |B(x; r)|
\]

for some \( c, n > 0 \) uniformly for all \( \lambda \geq 1 \) and \( x \in X \). The parameter \( n \) is a measure of the “dimension” of the space.

Subsequently, we will restrict attention to spaces for which there is some uniformity in volume growth. The essential uniformity is expressed by bounds

\[
\text{ess sup}_{x \in X} |B(x; r)| \leq b \text{ ess inf}_{x \in X} |B(x; r)|
\]

for some \( b > 0 \) and all \( r > 0 \).
A space is of polynomial type if there are two integers \( D' \) and \( D \), the local dimension and the dimension at infinity, such that
\[
e^{-1}t^{D'} \leq |B(x; r)| \leq t^{D}
\]
for all \( r \leq 1 \) and
\[
\leq t^{D} \leq |B(x; r)| \leq C t^{D}
\]
for all \( r \geq 1 \) uniformly for \( x \in X \). It can then be verified that the strong homogeneity property is valid with \( n \geq D' + D \). The uniformity property is evident.

Next let \( S \) denote a strongly continuous semigroup acting on \( L_2(X; \rho) \) with the action of \( S \) determined by an integral kernel \( K \), i.e.,
\[
(S \varphi)(x) = \int_X \rho(y) K(x; y) \varphi(y),
\]
where \( K \) is a bounded measurable function over \( X \times X \) which is integrable with respect to the first variable uniformly with respect to the second. The semigroup property of \( S \) requires that \( K \) is a convolution semigroup in the sense
\[
K_{s+t}(x; y) = (K_s * K_t)(x; y) = \int_X \rho(z) K_s(x; z) K_t(z; y)
\]
for all \( s, t > 0 \) and the continuity requires that \( K_t \to \delta \) as \( t \to 0 \) in the weak* sense. The adjoint semigroup \( S^* \) also has a kernel \( K^* \) and it follows by duality that
\[
K_{s}^*(x; y) = K^*_s(y; x).
\]
The kernel \( K \) is defined to satisfy Poisson bounds of order \( m > 0 \) if
\[
|K_s(x; y)| \leq G_s(x; y) \wedge G_s(y; x)
\]
for all \( x, y \in X \) and \( t > 0 \), where
\[
G_s(x; y) = |B(x; t^m)|^{-1} g(d(x; y)^m t^{-1}),
\]
and \( g \) is a bounded decreasing function satisfying
\[
\lim_{r \to \infty} r^{m+\epsilon} g(r^m) = 0
\]
for some $\delta > 0$ with $n$ the “dimension” entering in the strong homogeneity property. We can and do assume that $g$ is continuous and strictly positive.

The kernel bounds are assumed in this symmetric form as $K$ and $K^*$ should satisfy similar bounds. In most applications to differential operators one would expect the kernels to be at least Hölder continuous functions over $X \times X$ but we do not assume any continuity. Therefore the foregoing bounds have to be interpreted in the sense of the natural order on the space $L^{\rho}(X \times X; \rho \times \rho)$.

The above bounds with $g(x) = a \exp(-b x^{1/(m-1)})$, where $a, b > 0$, are typical of the bounds on heat kernels associated with strongly elliptic or subelliptic operators of order $m$ on $\mathbb{R}^d$, or on a general Lie group (see, for example, [Rob91]). Moreover, the Poisson semigroup on $\mathbb{R}^d$ has a kernel of the form $\exp(-|x-y|^2 t + 1)/2$ and hence the bounds are satisfied for $m = 1$ with $g(x) = a(x^2 + 1)^{-(d+1)/2}$. Alternatively, if $m = 2$, and $g(x) = a \exp(-b x^2)$, the bounds are satisfied by the heat kernel associated with the Laplace–Beltrami operator on a complete Riemannian manifold with non-negative Ricci curvature (see [Dav89], Theorem 5.5.6).

We need three general properties of the kernel bounds.

**Proposition 2.1.** Let $(X, \rho, d)$ be a space of homogeneous type with the doubling property and $G$, the function defined by (7) and (8). Then for each $\gamma \in [0, \delta)$ there is a $c > 0$ such that

$$\text{ess sup}_{x \in X} \int_{d(x, y) \geq r} dp(y) G(x; y) \leq c(1 + rt^{-1})^{-\gamma} \quad (9)$$

for all $r \geq 0$ and $t > 0$.

**Proof.** First remark that

$$\int_{d(x, y) \geq r} dp(y) G(x; y) \leq g(r^\gamma t^{-1})^{1-\varepsilon} \int_X dp(y) |B(x; t^{1/m})|^{-1} g(d(x, y)^m t^{-1})^{1-\varepsilon}$$

for each $\varepsilon \in (0, 1]$. Next for $\varepsilon < \delta(n+\delta)^{-1}$ we derive a bound, uniform in $x$ and $t$, on the integral $I$ on the right hand side of the last inequality.

Splitting the region of integration into a sequence of annular regions $kt^{1/m} \leq d(x, y) < (k+1)t^{1/m}$ one immediately obtains an estimate

$$I(x; t) \leq \sum_{k \geq 0} |B(t^{1/m})|^{-1} (|B((k+1)t^{1/m})| - |B(kt^{1/m})|) g(k^{m/(1-\varepsilon)})$$
where for simplicity we have suppressed the argument \(x\) in the expression for the ball, e.g., \(B(t^{1/m}) = B(x; t^{1/m})\). It then follows by use of the doubling property, expressed as strong homogeneity, that

\[
I(x; t) \leq \sum_{k \geq 0} |B(t^{1/m})|^{1-\epsilon} |B((k+1)^{1/m}) - g((k+1)^{1/m})|^{1-\epsilon} \\
\leq c \sum_{k \geq 0} (k+1)^n (g(k^{1/m}) - g((k+1)^{1/m})|^{1-\epsilon} \\
\leq c \sum_{k \geq 0} g(k^{1/m})^{1-\epsilon} ((k+1)^n - k^n).
\]

But it follows from the decrease property (8) of \(g\) that the last sum is bounded whenever \(\epsilon < \delta(n + \delta)^{-1}\). Therefore \(I\) is uniformly bounded and one has a bound

\[
\int_{d(x; y) \geq r} dp(y) G_t(x; y) \leq ag(rt^{-1})^\gamma.
\]

But, by assumption, \(g\) satisfies a bound \(g(r^{m-1}) \leq b(1 + r^{m-1})^{-\gamma(n + \delta)}\). Hence combination of these bounds gives the desired result with \(c = ab\) and \(\gamma(\delta(n + \delta)) < \delta\).

**Corollary 2.2.** If the kernel \(K\) satisfies Poisson bounds of order \(m > 0\) then for each \(\gamma \in [0, \delta)\) there is a \(c > 0\) such that

\[
\int_{d(x; y) \geq r} dp(y) (|K_t(x; y)| + |K_t(y; x)|) \leq c(1 + r^{m-1})^{-\gamma} 
\]

for all \(r \geq 0\) and \(t > 0\).

This is a direct consequence of the definition of the bounds and Proposition 2.1.

We refer to the estimates (9) as brownian estimates. They immediately imply that the semigroup extends to all the \(L_p\)-spaces and this does not require any further geometric assumptions on the underlying space.

**Proposition 2.3.** Let \((X, \rho)\) be a \(\sigma\)-finite measure space with a metric \(d\) and \(S\) a strongly continuous semigroup on \(L_2(X, \rho)\) with a kernel \(K\) satisfying the brownian estimates

\[
\text{ess sup}_{x \in X} \int_{d(x; y) \geq r} dp(y) (|K_t(x; y)| + |K_t(y; x)|) \leq b(rt^{-1}),
\]

where \(b\) is a bounded function which tends to zero at infinity.
It follows that $S$ extends to a uniformly bounded semigroup on each of the spaces $L_p(X; \rho)$, $p \in [1, \infty]$, which is strongly continuous if $p \in [1, \infty)$ and weakly* continuous if $p = \infty$.

Proof. Since $(X, \rho)$ is $\sigma$-finite $L_2 \cap L_p$ is norm-dense in $L_p$ for $p \in [1, \infty)$ and weakly* dense in $L_\infty$. It follows that $S_t$ extends to a densely-defined operator on each $L_p$-space, also denoted by $S_t$, and the brownian estimates imply that the $S_t$ are bounded on $L_1$ and $L_\infty$ with norms satisfying bounds

$$
\|S_t\|_{1 \rightarrow 1} \leq \text{ess sup}_{y \in X} \int_X dp(x) |K_t(x; y)| < \infty
$$

and

$$
\|S_t\|_{\infty \rightarrow \infty} = \text{ess sup}_{x \in X} \int_X dp(y) |K_t(x; y)| < \infty
$$

uniformly for $t > 0$. Therefore, by interpolation, $S_t$ extends to a bounded operator on each $L_p$-space and $\|S_t\|_{p \rightarrow p} \leq M < \infty$ for a suitable $M > 0$ uniformly for $p \in [1, \infty]$ and $t \geq 0$. It then follows straightforwardly that the $S_t$ satisfy the semigroup property on each of the $L_p$-spaces.

If $p \in (1, \infty)$ the semigroup $S$ on $L_p$ is weakly, hence strongly, continuous by another density argument. Therefore it remains to consider the cases $p = 1$ and $p = \infty$.

If $\varphi \in L_1 \cap L_2$ has support in the ball $B(x; r)$ then

$$
\int_{d(x, y) \leq r + s} dp(y) \|(S_t \varphi)(y) - \varphi(y)\| \leq |B(x; r + s)|^{1/2} \|S_t \varphi - \varphi\|_2.
$$

But

$$
\int_{d(x, y) > r + s} dp(y) \|(S_t \varphi)(y) - \varphi(y)\|
$$

$$
= \int_{d(x, y) > r + s} dp(y) \|(S_t \varphi)(y)\|
$$

$$
\leq \text{ess sup}_{y \in X} \int_{d(x, y) > r} dp(x) |K_t(x; y)| \|\varphi\|_1
$$

$$
\leq h(s^m t^{-1}) \|\varphi\|_1
$$

by the brownian estimates. Therefore $\|S_t \varphi - \varphi\|_1 \to 0$ as $t \to 0$ and strong continuity on $L_1$ follows by a density argument. Since the adjoint $S^*$ of $S$
The kernel \( K^*(x; y) = \overline{K(y; x)} \) it follows by the same argument that \( S^* \) extends to a strongly continuous semigroup on \( L_1 \). Therefore the dual of \( S^* \) is a weakly* continuous extension of \( S \) to \( L_\infty \). 

The next two properties of the kernel bounds involve pointwise estimates on the function \( G \) defined by (7) and (8). The first estimate involves the Hardy–Littlewood maximal function \( Mf \) which is defined by

\[
(Mf)(x) = \sup_{r > 0} \frac{1}{|B(x; r)|} \int_{B(x; r)} dp(y) |f(y)|
\]

for \( f \in L_\rho(X; \rho), \rho \in [1, \infty] \), and which is bounded on \( L_\rho(X; \rho) \) for all \( \rho \in (1, \infty) \) with a norm which diverges as \( \rho \to 1 \) (see, for example, [Chr90] Theorem VI.4).

**Proposition 2.4.** For each \( \rho \in [1, \infty] \) and \( r > 0 \) there is a \( c > 0 \) such that

\[
|(S_t f)(x)| \leq \int_X dp(y) G_t(x; y) |f(y)| \leq c(Mf)(x)
\]

for all \( f \in L_\rho(X; \rho) \), \( \rho \)-almost everywhere.

**Proof.** The first inequality is a direct consequence of the kernel bounds. For the second we use the estimates of the proof of Proposition 2.1. These rely upon the observation that the function \( y \mapsto G_t(x; y) \) is pointwise bounded by the function \( \chi_{t, \epsilon} \), where

\[
\chi_{t, \epsilon} (y) = |B(x; t^{1/m})|^{-1} g(k^m),
\]

on the annulus \( \{ y: kt^{1/m} \leq d(x; y) < (k+1)t^{1/m} \} \). Moreover,

\[
\int_X dp(y) \chi_{t, \epsilon} (y) \leq c < \infty
\]

uniformly for \( x \) and \( t \). Now if \( \chi_k \) denotes the characteristic function of the ball \( B(x; kt^{1/m}) \) then one has

\[
\chi_{t, \epsilon} = \sum_{k \geq 1} \lambda_k |B(x; kt^{1/m})|^{-1} \chi_k
\]

with

\[
\lambda_k = |B(x; t^{1/m})|^{-1} (B(x; kt^{1/m}) \ (g(k^m) - g((k+1)^m))).
\]
But the estimates of the proof of Proposition 2.1 establish that $\sum_{k \geq 1} \hat{\lambda}_k < \infty$. Therefore one has an estimate

$$\int_X dp(y) \, G_t(x; y) \, |f(y)| \leq C(Mf)(x)$$  \hspace{1cm} (10)

$\rho$-almost everywhere. \hfill \Box

The third estimate is a Harnack-type inequality.

**Proposition 2.5.** For each $\nu > 0$ there are $c, \mu \geq 1$ such that

$$\sup_{z \in B(y; r)} G_t(x; z) \leq c \inf_{z \in B(y; r)} G_{\nu t}(x; z)$$

uniformly for $x, y \in X$ and $r > 0$ with $r^\mu \leq \nu t$.

**Proof.** If $x \notin B(y; 3r)$ one has

$$d(x; z_1) \leq d(x; z_2) + d(z_1; z_2) \leq d(x; z_2) + 2r \leq 2d(x; z_2)$$

for all $z_1, z_2 \in B(y; r)$. Hence

$$G_t(x; z_2) \leq |B(x; 2 \nu t^{1/\mu})| \, |B(x; t^{1/\mu})|^{-1} \, G_{\nu t}(x; z_1)$$

and the desired estimate is valid with $c = c_2$ and $\mu = 2^m$ uniformly for $r, t > 0$. If, however, $x \in B(y; 3r)$ then

$$|B(x; t^{1/\mu})|^{-1} \, g(2^{2m \nu t^{-1}}) \leq G_t(x; z) \leq |B(x; t^{1/\mu})|^{-1} \, g(0)$$

for all $z \in B(y; r)$. Therefore, since we have assumed that $g$ is strictly positive,

$$\sup_{z \in B(y; r)} G_t(x; z) \leq g(0) \, g(2^{2m \nu t^{-1}}) \, G_t(x; z)$$

for all $z \in B(y; r)$. But

$$G_t(x; z) \leq |B(x; 2 \nu t^{1/\mu})| \, |B(x; t^{1/\mu})|^{-1} \, G_{\nu t}(x; z) \leq c_2 G_{2\nu t}(x; z)$$

and the estimate is now valid with $c = c_2 g(0) \, g(2^{2m \nu t^{-1}}) \, G_t(x; z)$ uniformly for $r, t > 0$ satisfying $r^\mu \leq \nu t$. \hfill \Box

3. $H_\sigma$-FUNCTIONAL CALCULUS ON $L_p$

Let $S$ be an interpolating semigroup on the $L_p$-spaces, $L_p(X; \rho)$, with generator $L$. Assuming $S$ has a bounded $H_\sigma$-functional calculus on $L_2$ we...
aim to establish that its extension to the $L_p$-spaces, $p \in (1, \infty)$, has a bounded $H_\omega$-functional calculus. The essential ingredients are the strong homogeneity property and bounds on the semigroup kernel, of the type discussed in the previous section, in a sector of the complex plane.

First, recall that if $L$ generates a bounded holomorphic semigroup with sector of holomorphy $A(\theta) = \{ z \in \mathbb{C} : |\arg z| < \theta \leq \pi/2 \}$ then $L$ is of type $\omega$ for all $\omega > \pi/2 - \theta$. Secondly, recall that if $L$ has a bounded $H_\omega(S_0^\rho)$-functional calculus on $L_\rho(X; \rho)$ for some $\mu > \omega$ then it has a bounded functional calculus for all $\mu > \omega$ (see [McI86], Section 8).

**Theorem 3.1.** Let $L$ be the generator of a strongly continuous semigroup $S$ on $L_\rho(X; \rho)$ where $(X, \rho, d)$ is a space of homogeneous type with the doubling property and $S$ is bounded and holomorphic in a sector $\{ z : |\arg z| < \theta \leq \pi/2 \}$.

Assume that

(i) $L$ has a bounded $H_\omega(S_0^\rho)$-functional calculus on $L_\rho(X; \rho)$ for some $\mu > \pi/2 - \theta$,

(ii) $S_\rho$ has a kernel $K_\rho$ satisfying the bounds

$$|K_\rho(x; y)| \leq G_{\Re z}(x; y) \wedge G_{\Re z}(y; x)$$

for all $x, y \in X$ and $z \in \{ z : |\arg z| < \theta \}$ where $\phi \in (0, \theta)$ and $G_\phi$ is defined by (7) and (8).

It follows that $L$ has a bounded $H_\omega(S_0^\rho)$-functional calculus on $L_\rho(X; \rho)$ for each $p \in (1, \infty)$ and each $\mu > \pi/2 - \phi$. Moreover, $\zeta(L)$ is of weak type $(1, 1)$ for each $\zeta \in H_\omega(S_0^\rho)$ with $\mu > \pi/2 - \phi$.

**Proof.** The proof relies on a Calderón–Zygmund decomposition of a function on a space of homogeneous type in combination with a separation of each of the components $f_i$ of the decomposition into a smooth part $S_\rho f_i$ and a residue $(I - S_\rho) f_i$. The smooth parts are estimated with the aid of Propositions 2.4 and 2.5 and this only requires the kernel bounds for real $t$. The residues are handled with the brownian estimates for complex $t$. This argument is based on a comparable analysis given by Hebisch [Heb90] whose result was more restricted as it only aimed to establish a Hörmander multiplier theorem for Schrödinger operators on $\mathbb{R}^d$. In contrast to the usual applications of Calderón–Zygmund theory this method does not need any smoothness on the semigroup kernel and there is no apparent utilization of cancellations.

Let $\zeta \in \mathcal{P}_\omega(S_0^\rho)$ with $\mu > \pi/2 - \phi$. We may assume that $\|\zeta\|_\omega = 1$ and $\|\zeta(L)\|_{L^2, -2} \leq c$. Since the adjoint semigroup $S^*$ has a kernel $K^*$ such that $K^*(x; y) = K(y; x)$ the assumptions of the theorem are invariant under duality. We will prove that $\zeta(L)$ is of weak type $(1, 1)$ and then one can use
interpolation theory in combination with duality to deduce that \( \zeta(L) \) is bounded on \( L_p(X; \rho) \) for \( p \in \langle 1, \infty \rangle \). Finally, if \( \zeta \in \mathcal{P}_m(S^n) \) then \( \zeta(L) \) is bounded on \( L_p(X; \rho) \) for \( p \in \langle 1, \infty \rangle \) by McIntosh's convergence lemma [McI86] and weak \((1, 1)\) by Exercise 6.L of [ADM96].

Let \( f \) be an integrable function. We fix the Calderón–Zygmund decomposition of \( f \) at height \( \lambda \) (\( > \| f \|_1/\rho(X) \)) (see [CoW71] Section 3.2), i.e., we choose functions \( g \) and \( f_i \) and balls \( B_j = B(x_j, r_j) \) such that

\[
\begin{align*}
(a_1) & \quad f = g + h \text{ with } h = \sum_i f_i, \\
(a_2) & \quad |g(x)| \leq c \lambda, \\
(a_3) & \quad \text{supp } f_i \subset B_i \text{ and each point of } X \text{ is contained in at most } M \\
& \quad \text{balls,} \\
(a_4) & \quad \|f_i\|_1 \leq c \lambda |B_i|, \\
(a_5) & \quad \sum_i |B_i| \leq c \lambda^{-1} \|f\|_1.
\end{align*}
\]

Note that \( (a_4) \) and \( (a_5) \) imply that \( \|h\|_1 \leq c \|f\|_1 \). Hence \( \|g\|_1 \leq (1+c) \|f\|_1 \).

Next we decompose \( h \) as the sum of two functions

\[
\begin{align*}
h_1 &= \sum_i S_i f_i, \quad h_2 = \sum_i (I - S_i) f_i,
\end{align*}
\]

where \( t_i = r_i^n \). Then

\[
\rho(\{ x \in X : |(\zeta(L) f_i)(x)| > \lambda \}) \leq \rho(\{ x \in X : |(\zeta(L) g)(x)| > \lambda/3 \}) + \sum_{i=1}^2 \rho(\{ x \in X : |(\zeta(L) h_i)(x)| > \lambda/3 \}).
\]

(11)

But for the “good” part \( g \), one has the following estimate:

\[
\begin{align*}
\rho(\{ x \in X : |(\zeta(L) g)(x)| > \lambda \}) & \leq \lambda^{-1} \int_X d\mu(x) |(\zeta(L) g)(x)|^2 \\
& \leq c \lambda^{-2} \int_X d\mu(x) |g(x)|^2 \\
& \leq c \lambda^{-1} \int_X d\mu(x) |g(x)| \leq c \lambda^{-1} \|f\|_1.
\end{align*}
\]

(12)

(Here and in the following we use the convention that \( c \) denotes a positive constant whose value can vary from line to line.) Therefore the first term on the right hand side of (11) has a bound \( c \lambda^{-1} \|f\|_1 \).
Next we consider the $h_1$-term in (11). This satisfies the estimate

$$\rho(\{x \in X : |\xi(L) h_1(x)| > \lambda_1\}) \leq \lambda_1^{-2} \|\xi(L) h_1\|_2^2 \leq c \lambda_1^{-2} \|\sum_j S_{j,i} f_i\|_2^2$$

and we first use the Harnack estimates of Proposition 2.5 to bound the individual terms in the sum. One has

$$|(S_{j,i} f_i)(x)| \leq \int_X dp(y) G_{m_i}(x ; y) |f_i(y)|$$

$$\leq \|f_i\|_1 \sup_{y \in B_i} G_{m_i}(x ; y)$$

$$\leq c \lambda |B_i| \inf_{y \in B_i} G_{m_i}(x ; y) \leq c \lambda \int_X dp(y) G_{m_i}(x ; y) \chi_i(y),$$

where $\chi_i$ denotes the characteristic function of the ball $B_i$. Note that the values of $c$ and $\mu$ can be chosen independent of $i$ because $r_i^\mu / r_i = 1$.

Secondly, it follows from the foregoing estimate and Proposition 2.4 that

$$|(h, S_{j,i} f_i)| \leq c \lambda (G_{m_i} * |h|, \chi_i) \leq c \lambda (Mh, \chi_i)$$

for all $h \in L_2(X; \rho)$ where $M$ is the Hardy–Littlewood maximal function. Since $M$ is bounded on $L_2$ it then follows that

$$\|\sum_j S_{j,i} f_i\|_2 = \sup \{||(h, \sum_j S_{j,i} f_i)| : \|h\|_2 \leq 1\}$$

$$\leq c \lambda \sup \{\sum_i (M_{|B_i|}, \chi_i) : \|h\|_2 \leq 1\} \leq c \lambda \|\sum_i \chi_i\|_2.$$

But this together with property (a$_3$) of the Calderón–Zygmund decomposition immediately yields the estimate

$$\|\sum_j S_{j,i} f_i\|_2 \leq c \lambda (\sum_j |B_i|)^{1/2} \leq c \lambda^{1/2} \|f\|_2^{1/2}.$$ 

Therefore,

$$\rho(\{x \in X : |\xi(L) h_1(x)| > \lambda_1\}) \leq c \lambda_1^{-1} \|f\|_1. \quad (13)$$

Thus the $h_1$-term on the right hand side of (11) has a bound $c \lambda_1^{-1} \|f\|_1$. Next we consider the $h_2$-term in (11).
Let $B_{2,i}$ be the ball with the same centre as $B_i$ but double the radius. Then

$$
\rho(\{ x \in X : |(\tilde{\zeta}(L) h_2)(x)| > \lambda \}) 
\leq \sum_i |B_{2,i}| + \rho(\{ x \in X \setminus \bigcup_i B_{2,i} : |(\tilde{\zeta}(L) h_2)(x)| > \lambda \}) \quad (14)
$$

and (a), together with the doubling property, imply

$$
\sum_i |B_{2,i}| \leq c\lambda^{-1} \| f \|_1. \quad (15)
$$

Thus the first term on the right hand side of (14) has the desired bound and we use the brownian estimates to control the second term.

As a preliminary remark that if $\varphi \in \langle -\theta, 0 \rangle$ then $t > 0 \mapsto K_{\varphi,t}$ is the kernel of the continuous semigroup $\{S_{\varphi,t}\}_{t > 0}$ and since the kernel satisfies the bounds defined by (7) and (8) one can apply Proposition 2.1 to deduce that it satisfies the brownian estimates

$$
\text{ess sup}_{x \in X} \int_{d(x,y) \geq r} dp(y) G_t(x;y) \leq c(1 + r^m t^{-1})^{-\gamma}
$$

for some $\gamma \in \langle 0, \delta \rangle$ and for all $r \geq 0$ and $t > 0$. Now we return to the examination of the second term on the right hand side of (13).

First define $\xi_i$ by $\xi_i(x) = \tilde{\zeta}(x) (1 - e^{-r_i \gamma})$. Then $\tilde{\zeta}(L) h_2 = \sum_i \xi_i(L) f_j$. Secondly, decompose each $\xi_j(L)$ as a sum $\xi_+(L) + \xi_-(L)$, where

$$
\xi_\pm(L) = (2\pi i)^{-1} \int_{\gamma \pm} dv \xi_j(v)(L - vI)^{-1}
$$

and $\gamma_+$ correspond to the two line segments in the contour $\gamma$ of (3). Both components $\xi_\pm$ are estimated in a similar manner. Consider $\xi_+$.

One has

$$
\xi_+(L) = \int_{\gamma_+} dz S_{\pm} \eta_+(z) \quad (16)
$$

with

$$
\eta_+(z) = -(2\pi i)^{-1} \int_{\gamma_+} dv e^{\gamma v} \xi_j(v),
$$
where \( \Gamma_+(t) = t e^{i \pi/2 - \pi}, \ t \in (0, \infty) \), and \( \alpha \) is chosen such that \( \text{Re} \, \varepsilon = -|z| \, |v| \, \sigma < 0 \) if \( z \in \Gamma_+ \) and \( v \in \gamma_+ \). Then, since \( \| \xi \|_\omega = 1 \),

\[
\int_{X \times B_2} d\mu(x) \left| (\xi_+(L) f_j)(x) \right|
\leq \int_0^\infty d\varepsilon(1 - e^{-\varepsilon}) \int_{B_j} dp(y) \left| f_j(y) \right|
\times \int_{d(x, y) \geq r_j} dp(x) \left| K_\varepsilon(x; y) \right|.
\]

Now \( |(1 - e^{-\varepsilon})| \leq c \) since \( \text{Re} \, \varepsilon \geq 0 \) and in addition \( |(1 - e^{-\varepsilon})| \leq ct \, \varepsilon \) if \( t \, \varepsilon \leq 1 \). Therefore we split the integral \( I \) occurring in the last estimate into two parts \( I_1, I_2 \) corresponding to integration over \( t \, \varepsilon > 1 \) and \( t \, \varepsilon \leq 1 \), respectively. Next using the brownian estimates

\[
I_1 \leq c \| f_j \|_1 \int_0^\infty dt \int_0^\infty du e^{-\sigma u(1 + r_j^m t^{-1})^{-\gamma}}
\]

with \( b, \sigma > 0 \). Then changing variables \( t, u = ur_j^m \rightarrow u \) and \( t/r_j^m \rightarrow t \) one has

\[
I_1 \leq c \| f_j \|_1 \int_0^\infty dt \int_1^{\infty} du e^{-\sigma u(1 + t^{-1})^{-\gamma}}
\leq c \| f_j \|_1 \int_0^\infty dt \int_1^{\infty} du e^{-\sigma u(1 + t^{-1})^{-\gamma}} \leq c \| f_j \|_1.
\]

The last conclusion follows because \( \gamma > 0 \). Alternatively,

\[
I_2 \leq c \| f_j \|_1 \int_0^\infty dt \int_0^{1/r_j} du \ t^m e^{-\sigma u(1 + r_j^m t^{-1})^{-\gamma}}
\leq c \| f_j \|_1 \int_0^\infty dt \int_0^{1/r_j} du \ e^{-\sigma u(1 + t^{-1})^{-\gamma}}
\leq c \| f_j \|_1 \int_0^\infty dt \ (t^{-1} + t^{-2})(1 + t^{-1})^{-\gamma} \leq c \| f_j \|_1,
\]

where we have again used \( \gamma > 0 \). Since similar estimates are valid for \( \xi_-(L) \), one deduces that
\[ \rho(\{ x \in X \cup \bigcup_{j} B_{2,j} : \| (\xi(L) h_{2}(x)) > \lambda \}) \]

\[ \leq \lambda^{-1} \int_{X \cup \bigcup_{j} B_{2,j}} d\mu(x) \left( \sum_{j} (\xi_{j}(L) f_{j})(x) \right) \]

\[ \leq \lambda^{-1} \sum_{j} \int_{X \setminus B_{2,j}} d\mu(x) \left( (\xi_{j}(L) f_{j})(x) \right) \]

\[ \leq c\lambda^{-1} \sum_{j} \| f_{j} \| _{1} \leq c\lambda^{-1} \| f \| _{1} . \]

Therefore the \( h_{2} \)-term on the right hand side of (11) has a bound \( c\lambda^{-1} \| f \| _{1} \).

Since the other two terms have already been bounded in this form one deduces that

\[ \rho(\{ x \in X : \| (\xi(L) f)(x) \| > \lambda \}) \leq c\lambda^{-1} \| f \| _{1} \]

and \( \xi(L) \) is of weak type \((1,1)\).

**Remark 3.2.** If the generator \( L \) is maximal accretive then it automatically has a bounded \( H_{\infty} \)-functional calculus on \( L_{2} \). Therefore the existence of a bounded functional calculus on the \( L_{p} \)-spaces, \( p \not\in (1, \infty) \), follows from the kernel bounds (7). In particular this is the case if \( L \) is self-adjoint.

One disadvantage of Theorem 3.1 is that one needs the kernel bounds for complex time and we next examine variations of the result that only require bounds for real time. First we remark that the kernel bounds for real \( t \) suffice to prove the existence of a bounded \( H_{\infty}(S_{\mu}) \)-functional calculus for \( \mu \in \langle \pi/2, \pi \rangle \). For example, adopt the assumptions of Theorem 3.1 but with (ii) replaced by

(ii) ‘ \( S_{t} \) has a kernel \( K_{t} \) satisfying the bounds

\[ |K_{t}(x; y)| \leq G(x; y) \wedge G(y; x) \]

for all \( x, y \in X \) and \( t > 0 \).

Then if \( \xi \in H_{\infty}(S_{\mu}) \) with \( \mu \in \langle \pi/2, \pi \rangle \) one has the representation (3) but the contour \( \gamma \) is now in the open left half plane, i.e., one has \( \Re > \pi/2 \). Since the resolvent \((L - \lambda I)^{-1}\) can be represented as the Laplace transform of \( S_{t} \), \( t > 0 \) one has

\[ \xi(L) = (2\pi i)^{-1} \int_{\gamma} d\lambda \xi(\lambda)(L - \lambda I)^{-1} = \int_{0}^{\infty} dt \int_{\gamma} d\lambda \xi(\lambda) e^{\lambda t}S_{t} \]

and the important point is that for \( \lambda \in \gamma \) and \( t > 0 \) one has \( \Re \lambda t = -t |\lambda| \sigma < 0 \) with \( \sigma = |\cos \gamma| \) where \( \gamma \in \langle \pi/2, \pi \rangle \).
The proof of boundedness of $\xi(L)$ then proceeds as previously up to bounding the second terms on the right hand side of (14). Now, following the foregoing observations, the representation (16) is replaced by

$$\xi(L) = \int_0^\infty dt \int_{\mathbb{R}_+} dv \, \xi(v) \, e^{vt}$$

where $\gamma(t) = e^{it}$ with $\chi \in \langle \pi/2, \pi \rangle$. Then, since $\|\xi\|_{\infty} = 1$,

$$\int_{\mathcal{X} \setminus \mathcal{B}_2} dp(x) \, |(\xi(L) f)(x)| \leq \int_0^\infty dt \int_0^\infty d |y| \, e^{-|y|^2 \sigma |1 - e^{-t}y|}$$

$$\times \int_{\mathcal{B}_2} dp(y) \, |f(y)|$$

$$\times \int_{d(x, y) \geq t} dp(x) \, |K(x; y)|$$

with $\sigma = |\cos \lambda|$. Therefore the required bound follows by the previous estimates.

Our next aim is to establish a version of the theorem which is valid for all $\mu > 0$ and only requires kernel bounds for real $t > 0$. As a preliminary we examine the problem of extending kernel bounds from real to complex values of $t$ (see, for example, [Dav89], Section 3.4, and [Ouh93], [Dav93b]). A variation of a recent argument of Davies [Dav94] establishes that this is possible if one already has an appropriate uniform bound for complex $t$. The argument is independent of the semigroup property and applies to a general holomorphic family of bounded operators.

**Proposition 3.3.** Let $(\mathcal{X}, \rho, d)$ be a homogeneous space with the doubling property and $z \in A(\rho) \mapsto K_z \in C$ the kernel of a holomorphic family of bounded operators on $L^2(\mathcal{X}; \rho)$. Assume that $K$ satisfies the following bounds for some $m > 0$:

(i) there is a $c_1 > 0$ such that

$$|K_z(x; y)| \leq c_1 \, |B(x; (\text{Re } z)^{i\infty})|^{-1}$$

for all $x, y \in \mathcal{X}$ and $z \in A(\rho)$.

(ii) there is a bounded decreasing function $b$ such that

$$|K_z(x; y)| \leq |B(x; t^{i\infty})|^{-1} \, b(d(x, y)^m \, t^{-1})$$

for all $x, y \in \mathcal{X}$ and $t > 0$. 


It follows that for each $\varepsilon \in (0, 1]$ and $\theta \in \langle 0, \varepsilon \varphi \rangle$ there is a $c > 0$ such that
\[ |K_d(x; y)| \leq c \left[ |B(x; (\Re z)^{1/m})|^{-1} b(d(x; y)^m |z|^{-1}) \right]^{1-\varepsilon} \] (17)
for all $x, y \in X$ and all $z \in \mathcal{A}(\theta)$.

Proof. Define $F$ on the sector $\mathcal{A}(\varphi)$ by setting
\[ F(w) = c \left[ |B(x; t^{1/m})| \right] K_t(w) \left[ K_t(w) \right]^{1-\varphi}. \]
Since $\Re w \geq 0$ one has $\Re(t + w) \geq t$ and then by choosing $c$ sufficiently small one may ensure that $|F(w)| \leq 1$, by assumption (i). Moreover, if $w \geq 0$ then assumption (ii) implies that
\[ |F(w)| \leq cb(d^{m_1} - 1), \]
where $d = d(x; y)$. Now we can bound $F$ by the argument Davies gives in his proof of [Dav94], Theorem 4.

Set $H(z) = F(z^*)$ where $z = \varphi/\pi$ and $z^* = \exp(\alpha \log z)$ for $\arg z \in [-\pi, \pi]$. Then $H$ has a cut along the negative real axis but is analytic elsewhere. Moreover, $|H(z)| \leq 1$ for all $z \in C$. Next define $J$ on $C \setminus (-\infty, 0]$ by $J(z) = \log |H(z)|$. Then $J$ is harmonic where it is finite. It is subharmonic on its domain and takes values in $[-\infty, 0]$. Using the Poisson formula on the half-plane one then has
\[ J(u + iv) \leq v \pi \int_{-\infty}^{\infty} ds \frac{J(s)}{(u-s)^2 + v^2} \leq v \pi \int_0^{\infty} ds \frac{J(s)}{(u-s)^2 + v^2} \]
since $J$ is negative. But for $s > 0$
\[ J(s) = \log |F(s^*)| \leq \log \{ 1 \wedge cb(d^{m_1} - 1) \} \]
by the earlier estimates on $F$ and the decrease property of $g$. Therefore
\[ J(u + iv) \leq \log \{ 1 \wedge cb(d^{m_1} - 1) \} \]
for all $u \in \mathbb{R}$ and $v > 0$. But, by explicit integration, one has
\[ J(u + iv) \leq \pi^{-1}(\pi/2 + \tan^{-1} u/v) \log \{ 1 \wedge cb(d^{m_1} - 1) \} \]
\[ = \left( 1 - \pi^{-1} \tan^{-1} u/v \right) \log \{ 1 \wedge cb(d^{m_1} - 1) \} \]
for $u \in \mathbb{R}$ and $v > 0$. A similar estimate for $v < 0$ then leads to the conclusion
\[ J(u + iv) \leq \log \{ 1 \wedge cb(d^{m_1} - 1)^{1-\varepsilon} \} \]
for all $u, v \in \mathbb{R}$ such that $|\tan^{-1} v/u| \leq \varepsilon \pi$. Consequently, one has bounds

$$|K_{r \to w}(x; y)| \leq c |B(x; t^{1/2})|^{-1} h(d(x; y))^{m/2} t^{-1 - \varepsilon}$$

for all $w \in \mathbb{C}$ with $|\arg w| \leq \varepsilon \pi$.

Finally, if $|\arg w| = \varepsilon \pi$ and $z = t + w^* t$ then $z = t(1 + R \cos(\varepsilon \pi)) + itR \sin(\varepsilon \pi)$ for some $R > 0$. Hence

$$|B(x; t^{1/2})|^{-1} \leq c(1 + R \cos(\varepsilon \pi/k))^{m/2} |B(x; (\Re z)^{1/2})|^{-1}$$

by the strong homogeneity property. In addition $(\Re z^{-1}) \leq t^{-1}$. Moreover, for $\varepsilon$ and $\pi$ fixed one can arrange that $|\arg z|$ is arbitrarily close to $\varepsilon \pi$ by choosing $R$ large. The statement of the proposition follows immediately.

In order to exploit Proposition 3.3 and obtain a version of Theorem 3.1 which only involves kernel bounds for real $t$ it is necessary to derive the uniform bounds for complex $t$. But these follow from uniform bounds on the semigroup whenever the earlier kernel bounds are satisfied. First, the semigroup $S$ and the kernel $K$ satisfy

$$\|S_t\|_{1 \to 2} = \|S_t^*\|_{2 \to \infty} \leq \text{ess sup}_{y \in X} \left( \int_X dp(x) |K_t(x; y)|^2 \right)^{1/2}.$$

Hence if $K$ satisfies the bounds (7) then

$$\|S_t\|_{1 \to 2} \wedge \|S_t\|_{2 \to \infty} \leq c \text{ess sup}_{x \in X} |B(x; t^{1/2})|^{-1/2}$$

for all $t > 0$. Now suppose that $S$ on $L^2(X; \rho)$ is holomorphic in the sector $A(\varphi)$ and in addition uniformly bounded, i.e.,

$$\|S_z\|_{2 \to 2} \leq M$$

for some $M \geq 1$ and all $z \in A(\varphi)$. But if $\theta \in \langle 0, \varphi \rangle$ there is a $\delta \in \langle 0, 1 \rangle$ such that $\delta t + is \in A(\theta)$. Then

$$\|S_z\|_{1 \to \infty} \leq \|S_{(1 - \delta)/2}\|_{1 \to 2} \|S_{\delta t + is}\|_{2 \to \infty} \|S_{(1 - \delta)/2}\|_{2 \to \infty}$$

$$\leq M \|S_{(1 - \delta)/2}\|_{1 \to 2} \|S_{(1 - \delta)/2}\|_{2 \to \infty}$$

$$\leq c \text{ess sup}_{x \in X} |B(x; ((1 - \delta) t/2)^{1/2})|^{-1}$$

$$\leq c((1 - \delta)/2)^{-nm} \text{ess sup}_{x \in X} |B(x; t^{1/2})|^{-1},$$
where the last estimate uses the strong homogeneity property (4). Consequently
\[
|K_z(x; y)| \leq \|S_z\|_1 \leq c \sup_{x \in X} |B(x; (\text{Re } z)^{1/m})|^{-1}
\]
for all \( z \in \mathcal{A}(\theta) \). Finally if \((X, \rho, d)\) has the uniformity property (6) then one has the bounds
\[
|K_z(x; y)| \leq c_1 |B(x; (\text{Re } z)^{1/m})|^{-1}
\]
of assumptions (i) of Proposition 3.3 for all \( x, y \in X \) and all \( z \in \mathcal{A}(\theta) \).

In combination with Theorem 3.1 one then has the following conclusion.

**Theorem 3.4.** Let \( L \) be the generator of a strongly continuous semigroup \( S \) on \( L_2(X; \rho) \) where \((X, \rho, d)\) is a space of homogeneous type with the doubling property (4) and the uniformity property (6). Assume \( S \) is holomorphic and uniformly bounded in the sector \( \{ z : |\text{arg } z| < \phi \leq \pi/2 \} \) and in addition that

(i) \( L \) has a bounded \( H_\infty(\mathcal{S}_\mu) \)-functional calculus on \( L_2(X; \rho) \) for some \( \mu > \pi/2 - \phi \).

(ii) \( S \) has a kernel \( K \) satisfying the bounds
\[
|K_z(x; y)| \leq G(x; y) \wedge G(y; x)
\]
for all \( x, y \in X \) and \( t > 0 \) where \( G_t \) is defined by (7) and (8).

It follows that \( L \) has a bounded \( H_\infty(\mathcal{S}_\mu) \)-functional calculus on \( L_p(X; \rho) \) for each \( p \in (1, \infty) \) and \( \xi(L), \xi \in H_\infty(\mathcal{S}_\mu) \), is of weak type \((1, 1)\) for \( \mu \) sufficiently close to \( \pi/2 \).

**Proof.** First, it follows from the preceding discussion that \( K_z \) satisfies the uniform bound
\[
|K_z(x; y)| \leq c |B(x; (\text{Re } z)^{1/m})|^{-1}
\]
for all \( x, y \in X \) and all \( z \) in each smaller subsector \( \mathcal{A}(\theta) \), \( \theta \in (0, \phi) \).

Secondly, Proposition 3.3 implies that for each \( c \in (0, 1] \) and \( \theta \in (0, \epsilon \phi) \) one has bounds
\[
|K_z(x; y)| \leq c |B(x; (\text{Re } z)^{1/m})|^{-1} g(d(x; y)^m |z|^{-1})^{1-c}
\]
for all \( x, y \in X \) and all \( z \in \mathcal{A}(\theta) \). But similar arguments can be applied to the adjoint semigroup and these lead to similar bounds with \( x \) and \( y \) interchanged. Therefore
\[
|K_z(x; y)| \leq c G_{\text{Re } z}(x; y) \wedge G_{\text{Re } z}(y; x)
\]
for all \( x, y \in X \) and all \( z \in \mathcal{A}(\theta) \) but now

\[
G_r(x; y) = |B(x; t^{1/m})|^{-1} g(d(x; y)^m t^{-1})^{1-\varepsilon}.
\]

Hence the hypotheses of Theorem 3.1 are satisfied for all \( z \) in a sufficiently small sector but with \( g \) replace by \( g^{1-\varepsilon} \). Now \( g(r^m) \sim r^{-n-\gamma} \) as \( r \to \infty \) and hence \( g(r^m) \sim r^{-n-\gamma} \) with \( \gamma = \delta(1-\varepsilon) - \nu \). Since \( \gamma > 0 \) whenever \( \varepsilon < \delta(n + \delta) \) Theorem 3.1 can be applied for this range of \( \varepsilon \) and in particular for sufficiently small \( \theta \). Thus the stated result is a direct corollary of the earlier theorem.

4. VARIATIONS

The conclusions of Theorem 3.1 are stable under several variations of the basic hypotheses. In this section we examine two such alternatives.

The first assumption of Theorem 3.1 specifies that \( L \) has a bounded \( H_\sigma \)-functional calculus on \( L_2(X; \rho) \). The emphasis on the \( L_2 \)-space is natural because the bounded functional calculus is a consequence of more easily verifiable properties of \( L \) such as self-adjointness or maximal accretiveness. Nevertheless it is not essential that the assumption be expressed in terms of the \( L_2 \)-space.

**Theorem 4.1.** Let \( L \) be the generator of a strongly continuous semigroup \( S \) on \( L_{p_0}(X; \rho) \) where \( (X, \rho, d) \) is a space of homogeneous type with the doubling property, \( p_0 \in (1, \infty) \) and \( S \) is bounded and holomorphic in a sector \( \mathcal{A}(\theta) = \{ z: |\arg z| < \theta \leq \pi/2 \} \).

Assume that

(i) \( L \) has a bounded \( H_{\sigma_1}(S_{\mu_1}) \)-functional calculus on \( L_{p_1}(X; \rho) \) for some \( \mu_1 > \pi/2 - \theta \),

(ii) \( S \) has a kernel \( K \) satisfying the bounds

\[
|K_r(x; y)| \leq |G_{\text{Re} z}(x; y) \wedge G_{\text{Re} z}(y; x)|
\]

for all \( x, y \in X \) and \( z \in \mathcal{A}(\varphi) \) where \( \varphi \in (0, \theta) \) and \( G_r \) is defined by (7) and (8).

It follows that \( L \) has a bounded \( H_{\sigma_1}(S_{\mu_1}) \)-functional calculus on \( L_{p_1}(X; \rho) \), \( p \in (1, \infty) \), for all \( \mu > \pi/2 - \varphi \). Moreover, \( \zeta(L) \) is of weak type \((1,1)\).

**Remark 4.2.** Although this theorem is based upon kernel bounds for complex \( t \) one can obtain variants which only require kernel bounds for real \( t \). In particular if the space \((X, \rho, d)\) has the uniformity property (6)
the real $t$ bounds extend to complex $t$ bounds by the arguments used in the previous section.

**Proof.** The proof is similar to the proof of Theorem 3.1 but some details need modification.

First, we can assume $p_0 \geq 2$ because if $p_0 < 2$ we just consider the adjoint operator $L^*$ and the adjoint semigroup.

Secondly, if $f = g + h$, $h = \sum f_i$ is the Calderón–Zygmund decomposition of $f$ at height $\lambda$ used previously then we prove that $\hat{\xi}(L) f$ satisfies weak type $(1, 1)$ estimates for all $\hat{\xi} \in H_\omega(S^\mu)$, where $\mu_2 = \mu_1 \vee \mu$, by examining the individual terms as before.

The estimate for the “good” part $g$ is virtually unchanged; one uses $L^\#_p$-estimates in an obvious way in place of $L_2$-estimates.

The “bad” part $h$ is again decomposed as the sum of two functions

$$h_1 = \sum_i S_{\lambda_i} f_i, \quad h_2 = \sum_i (I - S_{\lambda_i}) f_i,$$

where $t_i = r_i^\mu$. Then the estimation of $\hat{\xi}(L) h_2$ is exactly as before. It just depends on the decay properties of the function $x \mapsto (1 - e^{-\lambda x})$ and the kernel bounds. It does not rely upon any assumption $p_0 = 2$.

The estimation of $\hat{\xi}(L) h_1$ does, however, require slight modification. Arguing as previously

$$\|h, S_{\lambda_i} f_i\|_p \leq c \lambda \|Mh, \lambda_i\|_p \leq c \lambda \|X_i\|_p \|X_i\|_p$$

for all $h \in L_p(X; \rho)$ where $p_1$ is dual to $p_0$. Since the Hardy–Littlewood maximal function $M$ is bounded on $L_p$ it then follows that

$$\|\sum_i S_{\lambda_i} f_i\|_p \leq c \lambda \|X_i\|_p \leq c \lambda \sum_i |B_i| \leq c \lambda^{-1} \|f\|_1.$$ 

Finally with $\|\xi\|_\infty \leq 1$ one has

$$\rho(\{x \in X : |(\hat{\xi}(L) h_1)(x)| > \lambda\}) \leq c \lambda^{-p} \|\sum_i S_{\lambda_i} f_i\|_p \leq c \lambda^{-1} \|f\|_1.$$ 

Thus the $h_1$-term has a bound $c \lambda^{-1} \|f\|_1$.

Combination of these estimates allows the conclusion that $L$ has a bounded holomorphic functional calculus on $L_p(X; \rho)$, $p \in (1, \infty)$, for all $\xi \in H_\omega(S^\mu)$ with $\mu_2 = \mu_1 \vee \mu$. To reduce the angle to $\mu$ we then use an $L_2$ argument.

The assumed holomorphy implies that $L$ is of type $\omega = \pi/2 - \theta$ on $L_2(\mu)$. Since this is a Hilbert space the existence of a bounded $H_\omega(S^\mu)$-functional calculus implies the existence of a bounded $H_\omega(S^\mu)$-functional
calculus for all \( \mu > \omega \). Therefore we can apply Theorem 3.1 to obtain a bounded \( H_{\omega}(S^0_\mu) \)-functional calculus on \( L_p(X; \rho) \), \( p \in \langle 1, \infty \rangle \), for all \( \mu > \pi/2 - \varphi \).

One disadvantage of the foregoing discussion is that it only applies to generators of holomorphic semigroups, i.e., operators \( L \) of type \( \omega \) with \( \omega \in [0, \pi/2) \). If \( L \) is of type \( \omega \) with \( \omega \in [\pi/2, \pi) \) one could consider the fractional powers \( L^\alpha \), \( 0 < \alpha < 1 \), which are of type \( \omega \). Therefore by choosing \( \alpha \) sufficiently small one could arrange that \( \omega \alpha \in [0, \pi/2) \). But then it is unclear what conditions ensure that the semigroup generated by \( L^\alpha \) has a kernel satisfying Poisson bounds. Alternatively, one can handle the case \( \omega \in [\pi/2, \pi) \) by considering the resolvents of \( L \) in place of the semigroup.

In order to motivate our subsequent hypotheses first consider the case that \( L \) generates a semigroup with a kernel \( K \) satisfying Poisson bounds of order \( m \). Then \((L - I)^{-1}\) has a kernel \( G_{L,I} \) satisfying bounds

\[
|G_{L,I}(x; y)| \leq R_{L,I}(x; y) \wedge R_{L,I}(y; x)
\]

for all \( x, y \in X \) and \( \lambda \in \mathbb{C} \) with \( |\arg \lambda| > \mu > \omega \), where

\[
R_{L,I}(x; y) = c \int_0^\infty dt t^{1-1-|\Re \lambda|} |B(t; t^{1/m})|^{-1} g(d(x; y)^m t^{-1}).
\]

But using the strong homogeneity property (5) one immediately finds bounds

\[
R_{L,I}(x; y) \leq |\lambda|^{-1-|\Re \lambda|} |B(x; |\lambda|^{-1/m})|^{-1} \tilde{g}(d(x; y)^m |\lambda|)
\]

for all \( \lambda \in \mathbb{C} \) with \( |\arg \lambda| > \mu > \omega \) and \( |\lambda| \) sufficiently large where \( \tilde{g} \) is given by

\[
\tilde{g}(r^m) = c \int_0^\infty ds s^{1-1-m} e^{-s^{1-n_m} (1 + r^m s^{-1})^{-(n+\beta)/m}}.
\]

Thus if \( l > n_m \) then \( \tilde{g} \) is again a bounded decreasing function satisfying the asymptotic property (8). Therefore we adopt bounds of this general form in the next statement.

**Theorem 4.3.** Let \( L \) be an operator of type \( \omega \in [0, \pi) \) on \( L_p(X; \rho) \) where \( p \in \langle 1, \infty \rangle \) and \((X, \rho, d)\) is a space of homogenous type with the doubling property.

Assume that

(i) \( L \) has a bounded \( H_{\omega}(S^0_\mu) \)-functional calculus on \( L_p(X; \rho) \) for some \( \mu > \omega \).
(ii) there is an $l \geq 1$ such that the resolvent $\lambda \mapsto (\lambda I - L)^{-1}$ has a kernel $G_{t, \lambda}$ satisfying the bounds

$$|G_{t, \lambda}(x; y)| \leq R_{t, \lambda}(x; y) \wedge R_{t, \lambda}(y; x)$$

for all $x, y \in X$ and $\lambda \in \mathbb{C}$ with $|\arg \lambda| \geq \mu > \omega$ and $|\lambda|$ sufficiently large where

$$R_{t, \lambda}(x; y) = |\lambda|^{-1} |B(x; |\lambda|^{-1/m})|^{-1} g(d(x; y)^m |\lambda|)$$

and $g$ is a bounded function satisfying (8).

It follows that $L$ has a bounded $H_{\infty}(S^\mu_0)$-functional calculus on $L_p(X; \rho)$ for each $p \in (1, \infty)$ and $\mu > \omega$.

**Proof.** The Cauchy integral representation gives

$$\zeta^{(k)}(z) = (2\pi i)^{-1} k! \int_{C} d\lambda \zeta(\lambda)(\lambda - z)^{-(k+1)}$$

and hence

$$\zeta(L) = (2\pi i)^{-1} (l - 1)! \int_{C} d\lambda \zeta(\lambda)(L - \lambda I)^{-l},$$

where $\zeta^{(l-1)} = \zeta$. In particular

$$|\zeta(\lambda)| \leq c |\lambda|^{l-1} ||\zeta||_\infty.$$  

But then the kernel $K_\zeta$ of $\zeta(L)$ is related to the kernel $G_{t, \lambda}$ of the $l$th power of the resolvent by

$$K_\zeta = (2\pi i)^{-1} (l - 1)! \int_{C} d\lambda \zeta(\lambda) G_{t, \lambda}.$$  

Consequently the resolvent bounds in the second assumption of the theorem give

$$|K_\zeta(x; y)| \leq c ||\zeta||_\infty \int_{0}^{\infty} dt |\lambda|^{-1} |B(x; t^{-1/m})|^{-1} g(d(x; y)^m t)$$

$$\leq c ||\zeta||_\infty \int_{0}^{\infty} dt t^{-1} |B(x; t^{1/m})|^{-1} g(d(x; y)^m t^{-1}).$$

But the latter bound is of exactly the same form as the estimates obtained earlier from the heat kernel bounds. Therefore the proof of the
Theorem is now a repetition of the previous arguments. There is one cosmetic change. In the semigroup argument we split the function $h$ in the Calderón–Zygmund decomposition into a sum of functions $S_t f_i$ and $(I - S_t) f_i$. But in the present case these are replaced by $(t, L - I)^{-1} f_i$ and $(I - (t, L - I)^{-1}) f_i$, respectively.

5. EXPONENTIAL GROWTH

The previous results on the bounded $H_\infty$-calculus were restricted to spaces with the doubling property and spaces for which the volume grows exponentially are automatically excluded. In particular none of the results apply to non-unimodular Lie groups equipped with Haar measure. Nevertheless, much of the material on the $H_\infty$-calculus can be extended to spaces of exponential growth.

The space $(X, \rho)$ is defined to have exponential growth if there is an integer $D'$ and $\lambda, \mu \geq 0$ such that

$$e^{-1} r^{D'} \leq |B(x; r)| \leq e^{\lambda r}$$

for all $r \leq 1$ and

$$C e^{\mu r} \leq |B(x; r)| \leq C e^{\lambda r}$$

for all $r \geq 1$ uniformly for all $x \in X$.

Since spaces of this type are locally polynomial one automatically has a local version of the doubling property: there is a $c_2 \geq 1$ such that

$$|B(x; 2r)| \leq c_2 |B(x; r)|$$

for all $r \in (0, 1]$ uniformly for $x \in X$.

Moreover, one has a finite covering property: there exists a sequence $x_i \in X$, $i = 1, 2, \ldots$ such that

$$X = \bigcup_{i=1}^\infty B(x_i; 1)$$

and each $x \in X$ lies in at most $N_1$ balls $B(x_i; 1)$ and at most $N_2$ balls $B(x_i; 2)$.

The finite covering property is a consequence of the local doubling property by the following argument. Let $C \subset X$ be a countable set such that

$$X = \bigcup_{x \in C} B(x; 1)$$
and the balls $B(x;1/2)$ are disjoint for distinct points $x \in C$. Then for $r \in [1,2]$ and $y \in X$ introduce the finite set $I_r(y) = \{ x \in C : y \in B(x; r) \}$. It follows that

$$|B(y;2r)| \geq \sum_{x \in I_r(y)} |B(y;1/2)| \geq c_r \sum_{x \in I_r(y)} |B(y;4r)| \geq c_r |I_r(y)| \frac{|B(y;2r)|}{2r},$$

where the second estimate uses the local doubling property and $|I_r(y)|$ denotes the number of points in $I_r(y)$. Thus $|I_r(y)|$ is bounded uniformly in $y$ for all $r \in [1,2]$ and the finite covering property follows immediately.

One requires stronger kernel bounds to deal with exponential growth. The kernel $K$ is defined to satisfy Gaussian bounds of order $m > 0$ if

$$|K_t(x;y)| \leq G_t(x;y) \wedge G_t(y;x)$$

for all $x,y \in X$ and $t > 0$ where

$$G_t(x;y) = a |B(x;t^{\mu})|^{-1} e^{-b(d(x,y)^{\nu}r^{1-m}t^{-1})^{1-m}}, \quad (18)$$

Now one has analogues of many of the earlier results.

**Proposition 5.1.** Let $(X, \rho, d)$ be a space with exponential growth and $K$ a semigroup kernel satisfying Gaussian bounds of order $m$. Then there exist $a^*, b^* > 0$ and $\omega > 0$ such that

$$\text{ess sup}_{x \in X} \int_{d(x,y) \geq r} dp(y) |K_t(x;y)| \leq a^* e^{-b^*(\rho)^{1-m}t^{-1}} e^{\omega t}$$

for all $r \geq 0$ and $t > 0$.

The proof is very similar to the proof of Corollary 2.2. It then follows from Proposition 2.3 that the semigroup $S$ extends to a continuous semigroup on each of the $L_p$-spaces.

**Theorem 5.2.** Let $L$ be the generator of a strongly continuous semigroup $S$ on $L_2(X;\rho)$ where $(X, \rho, d)$ is a space with exponential growth and $S$ is bounded and holomorphic in a sector $\Delta(\theta) = \{ z : |\arg z| < \theta \leq \pi/2 \}$.

Assume that

(i) $L$ has a bounded $H_{s,\mu}(S^\mu)$-functional calculus on $L_2(X;\rho)$ for some $\mu \in (\pi/2 - \theta, \pi)$,

(ii) $S_z, z \in \Delta(\varphi)$, has a kernel $K_z$ satisfying the Gaussian bounds

$$|K_z(x; y)| \leq G_{\Re z}(x; y) \wedge G_{\Re z}(y; x)$$

for all $z \in \Delta(\varphi)$ where $\varphi \in (0, \theta)$ and $G$ is given by (18).
It follows that for all sufficiently large values of $\tau$ the operator $L + \tau I$ has a bounded $H_{\infty}(S_0^0)$-functional calculus on $\mathcal{L}_p(X; \rho)$ for each $p \in (1, \infty)$ and $\mu > \pi/2 - \varphi$. Moreover, $\hat{\zeta}(L + \tau I)$ is of weak type $(1, 1)$ for each $\hat{\zeta} \in H_{\infty}(S_0^0)$.

**Proof.** First, choose a covering with the finite covering property, set $B_i = B(x_i, 1)$ and $B_{2,i} = B(x_i, 2)$. Secondly, given $f \in L_1(X; \rho)$ define $f_i$ by $f_i(x) = f(x)/k$ if $x \in B_i$ and $x$ is in $k$ ball $B_{2,i}$ and $f_i(x) = 0$ if $x \notin B_i$. Then supp $f_i \subseteq B_i$, $f = \sum_{i=1}^{\infty} f_i$, $\sum_{i=1}^{\infty} \|f_i\|_1 \leq N_1 \|f\|_1$ and

$$\hat{\zeta}(L) f = \sum_{i=1}^{\infty} \hat{\zeta}(L) f_i.$$  

Note that although supp $f_i \subseteq B_i$, the support of $\hat{\zeta}(L) f_i$ can be much larger. Next let $\epsilon \geq 0$ and decompose $\hat{\zeta}(L + \tau I) f_i$ into two parts

$$\hat{\zeta}(L + \tau I) f_i = T_i f_i + R_i f_i,$$

where $T_i = \chi_{2,i} \hat{\zeta}(L + \tau I)$, $R_i = (I - \chi_{2,i}) \hat{\zeta}(L + \tau I)$ and $\chi_{2,i}$ denotes multiplication by the characteristic function of the ball $B_{2,i}$. Adaptation of the reasoning of [BER94] combined with the estimation techniques of Section 3 now allows one to prove that $\hat{\zeta}(L + \tau I)$ is of weak type $(1, 1)$.

The starting point of the proof is the inequality

$$\rho(\{x \in X: |(T_i f_i)(x)| > \lambda/2\}) \leq \rho(\{x \in X: \sum_{i=1}^{\infty} |(T_i f_i)(x)| > \lambda/2\})$$

$$+ \rho(\{x \in X: \sum_{i=1}^{\infty} |(R_i f_i)(x)| > \lambda/2\}).$$

Consider the first term on the right hand side. The finite covering property ensures that for each $x \in X$ the sum $\sum_{i=1}^{\infty} |(T_i f_i)(x)|$ has at most $N_2$ non-zero terms. Therefore

$$\rho(\{x \in X: \sum_{i=1}^{\infty} |(T_i f_i)(x)| > \lambda/2\}) \leq \sum_{i=1}^{\infty} \rho(\{x \in X: |(T_i f_i)(x)| > \lambda/2N_2\}).$$

But since supp $f_i \subseteq B_i$ and supp $T_i f_i \subseteq B_{2,i}$ the estimate is now local. As the space is locally of homogenous type, with the local doubling property, we may now argue as in the proof of Theorem 3.1 and deduce that

$$\rho(\{x \in X: |(T_i f)(x)| > \lambda/2N_2\}) \leq c\lambda^{-1} \|f\|_1$$

for each $f \in L_1(B_{2,i}; \rho)$. The replacement of $\hat{\zeta}(L + \tau I)$ by its localization $T_i = \chi_{2,i} \hat{\zeta}(L + \tau I)$ does not affect the previous arguments and the value of $c$.
can be chosen independent of the value of \( i \). The uniformity of the estimate in \( i \) follows from the uniformity of the local doubling estimate. Then using the covering property and the last two estimates one concludes that

\[
\rho(\{x \in X: \sum_{j=1}^{\infty} |(T_j f_j)(x)| > \lambda/2\}) \leq c N_2 \lambda^{-1} \| f \|_1.
\]

Next we estimate the terms involving \( R f_j \). One has

\[
\rho(\{x \in X: \sum_{j=1}^{\infty} |(R_j f_j)(x)| > \lambda/2\}) \leq 2\lambda^{-1} \sum_{j=1}^{\infty} \int_X dp(x) \sum_{j=1}^{\infty} |(R_j f_j)(x)|
\]

\[
\leq 2\lambda^{-1} \sum_{j=1}^{\infty} \int_X dp(x) |(\xi(L) f_j)(x)|.
\]

But \( \xi = \xi_+ + \xi_- \) with

\[
\xi_{\pm} \sim 1 + \xi(1 + \xi) = \int_{F_{\pm}} dz \, S_x e^{-\eta}(z)
\]

and

\[
\eta(x) = -(2\pi i)^{-1} \int_{\mathbb{R}} dv \, e^{i\xi(y)}
\]

where we have again used the notation of the proof of Theorem 3.1. Hence, setting \( \| \xi \|_\infty = 1 \) for simplicity, one finds

\[
\int_{X \setminus B_i} dp(x) |(\xi(L + \tau I) f_j)(x)| \leq \int_0^{\infty} dz \int_0^{\infty} |v| \int_0^{\infty} dp(y) |f(y)|
\]

\[
\times \int_{d(x; y) \geq 1} dp(x) |K(x; y)|
\]

\[
\leq a \sigma^{-1} \| f \|_1 \int_0^{\infty} dz |z| |z|^{-1}
\]

\[
\times e^{-\delta |z|^{-1/2}} e^{-\delta |z - \tau \cos \phi|}
\]

where we have used the brownian estimates of Proposition 5.1. Therefore choosing \( \tau \) such that \( \omega - \tau \cos \phi > 0 \) one obtains bounds

\[
\int_{X \setminus B_i} dp(x) |(\xi(L + \tau I) f_j)(x)| \leq c \| f \|_1.
\]
Since a similar estimate is valid for $\zeta_-$ it follows that

$$
\rho \left( \{ x \in X : \sum_{i=1}^{\infty} \| (R_i f)(x) \| > \lambda^2/2 \} \right) \leq c \lambda^{-1} \| f \|_1. 
$$

Combination of these various estimates establishes that

$$
\rho(\{ x \in X : \| (\zeta(L+\tau I) f)(x) \| > \lambda \} ) \leq c \lambda^{-1} \| f \|_1
$$

and $\zeta(L+\tau I)$ is of weak type $(1, 1)$. The statements concerning the bounded $H_\sigma$-functional calculus on the $L_\mu$-spaces now follow by standard interpolation and duality argument.

6. ALTERNATIVE CALCULI

The $H_\sigma$-functional calculus is valid for a large class of operators but a relatively small family of functions. In this section we examine two other types of calculus which are only defined for operators of type 0 but have the advantage of containing the $C^\infty$-functions. First we consider the classical Hörmander functional calculus and then the more recent calculus of Davies and Helffer and Sjöstrand [Dav93a] [Dav93b] [HS89].

The Hörmander functional calculus is defined in terms of spaces $A_{\alpha,1}(\mathbb{R}^n), \alpha > 0$, of Lipschitz, or Besov, functions. (We adopt the definitions and notation of [CDMY96].) It was proved in [CDMY96, Theorem 4.10, that if $L$ is a one-to-one operator of type 0 then it admits a bounded $A_{\alpha,1}(\mathbb{R}^n)$-functional calculus if and only if it admits a bounded $H_\sigma(S_0^1)$-functional calculus for all small positive $\mu$ and there exist positive constants $C$ and $\alpha$ such that

$$
\| \zeta(L) \| \leq C \mu^{-\alpha} \| \zeta \|_\infty 
$$

(19)

for all $\zeta \in H_\sigma(S_0^1)$. Therefore we examine criteria in terms of semigroup kernels which allow the verification of (19) for suitable values of $\alpha$.

The second type of functional calculus, developed by Davies [Dav93a] [Dav93b] based on an idea of Helffer and Sjöstrand [HS89], involves the space $\alpha_\mu$ of $C^\infty$-functions over $\mathbb{R}$ for which $\| \zeta \|_{\alpha_\mu} < \infty$ where

$$
\| \zeta \|_{\alpha_\mu} = \| \zeta \|_\infty + \sum_{r=1}^{n} \int_{\mathbb{R}} dx (1+x^2)^{(n-1)/2} \left| \frac{d^r \zeta}{dx^r}(x) \right|.
$$
Let \( \tau \) be a non-negative \( C^\infty \)-function such that \( \tau(x) = 1 \), if \( |x| \leq 1 \), and \( \tau(x) = 0 \), if \( |x| \geq 2 \), and introduce the “almost analytic extension” of \( \xi \) by

\[
\overline{\xi}(x, y) = \left( \xi(x) + \sum_{r=1}^{n} \frac{\partial^r \xi}{\partial x^r} (x)(iy)^r/r! \right) \tau(y)(1 + x^2)^{1/2}.
\]

The operator \( \overline{\xi}(L) \) is then defined by the algorithm

\[
\overline{\xi}(L) = \pi^{-1} \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \frac{\delta^r \xi}{\delta z^r} (z)(L - zI)^{-1},
\]

where \( z = x + iy \). (If \( L \) is a self-adjoint operator this definition of \( \overline{\xi}(L) \) agrees with the definition by spectral theory.) Now if \( \overline{\xi}(L) \) is bounded for all \( \xi \in \mathcal{D}' \) and \( \|\overline{\xi}(L)\| \leq c\|\xi\|_{\mathcal{D}'} \) then \( L \) is defined to have a bounded \( \mathcal{D}' \)-functional calculus. We give conditions for which this occurs which extend results of Davies [Dav93a] [Dav93b]. Again the degree of singularity of the semigroup norm \( \|e^{-tL}\|_{p \rightarrow p} \) as \( \text{Re} z \to 0 \) is critical.

Let \( r > 0 \) denote a positive function such that \( |B_r| = O(r^n) \) as \( r \to 0 \), \( |B_r| = O(r^n) \) as \( r \to \infty \) and

\[
e^{-1} |B_r| \leq |B(x; r)| \leq c |B_r|
\]

for all \( x \in X, r > 0 \) and some \( c > 0 \). This measure of volume exists for spaces of polynomial type by definition and the second-order Gaussian bounds for the semigroup kernel \( K \) can be expressed in the form

\[
|K_{t}(x; y)| \leq c |B_t|^2 e^{-bt(x; y)^2 t^{-1}}
\]

for all \( t > 0 \). But these Gaussian bounds extend to the right half-plane by Davies’ use of the Phragmén–Lindelöf theorem [Dav94] (see also [Dav89], Section 3.4, and [Ouh93], [Dav93b]).

**Proposition 6.1.** Let \( (X, \rho, d) \) be a space of polynomial type with dimensions \((D', D)\) and \( S \) a continuous semigroup on \( L_2(X; \rho) \). Assume that \( S \) is holomorphic in the open right half-plane and \( S_t, t > 0, \) has a kernel \( K_t, t > 0, \) satisfying second-order Gaussian bounds.

It follows that \( S \) extends to a uniformly bounded continuous semigroup on each of the spaces \( L_p(X; \rho), \rho \in [1, \infty] \), which is holomorphic in the open right half-plane and \( z \in D(\pi/2) \mapsto S_z \) has a kernel which is analytic and satisfies bounds

\[
|K_{t}(x; y)| \leq d_t |B_{(\text{Re} z)^2} t^{-1} e^{-b(x; y)^2 (\text{Re} z)^{-1}}
\]

for each \( \epsilon \in (0, 1] \) and all \( z \) with \( \text{Re} z > 0 \).
Moreover, there is a $c_\epsilon > 0$ such that

$$\text{ess sup}_{x,y : |d(x,y)| \geq r} dp(y) \ |K_\epsilon(x; y)| \leq c_\epsilon \beta^{-\theta \epsilon \nu - D} e^{-\mu(1 - \epsilon) \beta^2 |z|^{-1}}$$

(22)

for all $r > 0$ and $z$ with $\text{Re\ z} > 0$ where $\beta = \cos (\text{arg\ z})$.

**Proof.** Since the semigroup kernel $K$ satisfies Gaussian bounds $S$ extends to a uniformly bounded continuous semigroup on each of the spaces $L_p(X; \rho)$, $p \in [1, \infty)$, by the discussion in Section 2.

Next, it follows by an elaboration of the arguments in Section 3.4 of [Dav89] that $t > 0 \mapsto K_t$ extends to a function which is analytic, with respect to the uniform norm on $L_\infty(X \times X; \rho \times \rho)$, in the open right half-plane and which satisfies the bounds (21). Care has to be taken in the argument establishing the bounds for two reasons. First, since we do not assume any continuity properties of the kernel as a function over $X \times X$ one cannot simply consider the function $K_t$ with its arguments fixed. Secondly, since the volume $r \mapsto |B_r|$ behaves differently for small and large values of $r$ the arguments of [Dav89] have to be split into two parts. Since these problems are handled straightforwardly we omit the details.

Finally, it follows from the Gaussian bounds and the proof of Proposition 2.1 that

$$\text{ess sup}_{x \in X} \int_{|d(x,y)| \geq r} dp(y) \ |K_\epsilon(x; y)| \leq c_\epsilon |B_{(\text{Re\ z})^{-1}}|^{-1} |B_{(\text{Re\ z})^{-1}}|^{-1} e^{-\mu(1 - \epsilon) \beta^2 |z|^{-1}}.$$  

But

$$|B_{(\text{Re\ z})^{-1}}| = |B_{(\text{Re\ z})^{-1}}|^{-1} c \beta^{-\theta \epsilon \nu - D} |B_{(\text{Re\ z})^{-1}}|$$

by the strong homogeneity property. Thus the brownian estimates in the last statement of the proposition follow immediately. Note that the same argument also gives a similar bound with the roles of the variables interchanged.

The brownian estimates on the kernel immediately give bounds on the $L_p$-norms of the semigroup $S$ and then, by Laplace transformation, on the resolvent of the semigroup generator.

**Proposition 6.2.** Adopt the assumptions of Proposition 6.1. Then there is a $c > 0$ such that

$$\|S\|_{p \rightarrow p} \leq c (\text{cos\ (arg\ z)})^{-\theta \epsilon \nu - D}$$

(23)
uniformly for $p \in [1, \infty]$ and $z \in A(\pi/2)$. Moreover, if $L$ denotes the generator of $S$ then there is a $c' > 0$ such that

$$\|(\lambda I - L)^{-1}\|_{p \to p} \leq c' |\lambda|^{-1} \left( \sin(\arg \lambda) \right)^{-(D' + D + 1)}$$

uniformly for $p \in [1, \infty]$ and $\lambda \in \mathbb{C} \setminus [0, \infty)$. Hence $L$ admits a bounded $\omega_n$-functional calculus on $L_p(X; p)$, $p \in [1, \infty]$, for all $n \geq D' + D + 1$.

**Proof.** The brownian estimates (22) with $r = 0$ give the $L_\infty$-bound

$$\|S_p\|_{\infty} = \text{ess sup}_{x \in X} \int_X dp(y) |K_p(x; y)| \leq c(\cos(\arg z))^{-(D' + D)}$$

for all $z \in A(\pi/2)$. But a similar estimate on the adjoint kernel gives an identical bound on $\|S_p\|_{1, \infty}$. Therefore the $L_p$-bounds follow by interpolation. If, however, $T$ is a continuous semigroup on a Banach space, holomorphic in the open right half-plane, then bounds

$$\|T_r\| \leq c(\cos(\arg z))^{-\sigma} e^{\sigma \Re z} \quad (24)$$

with $\sigma \geq 0$ lead by Laplace transformation to bounds

$$\|(\lambda I - L)^{-1}\| \leq c |\lambda|^{-1} (|\sigma + \lambda|/|\lambda|)^\sigma (\sin(\arg \lambda))^{-\sigma - 1} \quad (25)$$

on the resolvent of the generator $L$ of $T$ for all $\lambda \in \mathbb{C} \setminus [\sigma, \infty)$. This is explicitly established by the calculation in the proof of Lemma 3 in [Dav93b]. Now if $\sigma > 0$ the bounds (25) are Davies’ Condition H3, and if $\sigma = 0$ Davies’ Condition H2. But these conditions suffice for the development of a functional calculus from the algorithm of Helffer and Sjöstrand by Davies’ arguments [Dav93a] [Dav93b]. Thus Davies’ results apply with the index $\sigma = D' + D$. Therefore the second statement of the proposition follows from the first statement. If $p \in [1, \infty]$ the last statement follows from [Dav93b], Theorem 2. But if $p = \infty$ it follows from duality and the $p = 1$ result applied to the adjoint semigroup.

Next we relate the behaviour of the resolvent bounds with the behaviour of the estimates in the holomorphic functional calculus by refining a result of McIntosh [McI86] for operators on a Hilbert space.

**Lemma 6.3.** Let $L$ be an operator of type $\omega$ on a Hilbert space $\mathcal{H}$ with

$$\|(\lambda I - L)^{-1}\| \leq c_{\mu} |\lambda|^{-1} \quad (26)$$

for all $\mu = |\arg \lambda| \in (\omega, \pi]$. Further assume that $L$ has a bounded $H_{\mu}(S^0_{\mu})$-functional calculus for some $\mu_1 \in (\omega, \pi]$. 
It follows that \( L \) has a bounded \( H_{\omega}(S_{0}^{0}) \)-functional calculus, and there is a \( c > 0 \) such that

\[
\| \xi(L) \| \leq c \| \xi \|_{\infty}
\]  

(27)

for all \( \xi \in H_{\omega}(S_{0}^{0}) \), for all \( \mu \in \langle \omega, \pi \rangle \).

**Proof.** If \( \mu \in [\mu_{1}, \pi] \) then \( H_{\omega}(S_{0}^{0}) \subseteq H_{\omega}(S_{\mu_{2}}^{0}) \) and one automatically has a bounded \( H_{\omega}(S_{\mu_{2}}^{0}) \)-functional calculus for all \( \mu \in \langle \mu_{1}, \pi \rangle \). The statement concerning \( \mu \in \langle \omega, \mu_{1} \rangle \) is a consequence of the results of McIntosh [McI86], Sections 7 and 8. The existence of a bounded \( H_{\omega}(S_{0}^{0}) \)-functional calculus for \( \mu \in \langle \omega, \mu_{1} \rangle \) is contained in the theorem in Section 8. The second statement, which identifies the \( \mu \)-dependence in the estimate (27) with the \( \mu \)-dependence in the resolvent bound (26), is a consequence of the proofs of the theorems in Sections 7 and 8. The proof of (a) implies (g) in Section 8 establishes that \( L \) satisfies quadratic estimates for a particular function \( \Psi_{(\alpha)} \in H_{\omega}(S_{\mu_{2}}^{0}), \mu > \omega \), and the constant in these estimates is independent of \( \mu \). Therefore using these functions in the proof of the theorem in Section 7 one obtains bounds, analogous to those at the top of page 223 in [McI86],

\[
\| \xi(\theta_{\gamma})(L) \| \leq (2\pi)^{-1} \| \xi \|_{\infty} \int_{\gamma} |d\lambda| \| (\lambda I - L)^{-1} \| |\theta(\lambda)|
\]

\[
\leq c_{\omega}(\pi)^{-1} \| \xi \|_{\infty} \int_{0}^{\infty} d\lambda \lambda^{-1}C\lambda^{t}(1 + t^{2}\lambda^{2t})^{-1},
\]

where the value of the constant \( C \) depends upon the special choice of function \( \Psi_{(\alpha)} \) but is independent of \( \mu \). Therefore the desired bounds are a direct consequence of the approximation procedure used in Section 7 of [McI86].

Combination of Proposition 6.2 and Lemma 6.3 immediately gives the following statement about functional calculus on \( L_{2}(X; \mu) \).

**Corollary 6.4.** Adopt the assumptions of Propositions 6.1 and 6.2. Further assume that \( L \) has a bounded \( H_{\omega}(S_{\mu_{2}}^{0}) \)-functional calculus on \( L_{2}(X; \mu) \) for some \( \mu_{2} \in \langle 0, \pi \rangle \).

It follows that \( L \) has a bounded \( H_{\omega}(S_{\mu}^{0}) \)-functional calculus on \( L_{2}(X; \mu) \) for all \( \mu \in \langle 0, \pi \rangle \) and there is a \( c > 0 \) such that

\[
\| \xi(L) \|_{2 \rightarrow 2} \leq c \mu^{-D'} \| \xi \|_{\infty}
\]

for all \( \xi \in H_{\omega}(S_{\mu}^{0}) \) and \( \mu \in \langle 0, \pi \rangle \).
One can deduce from this corollary and Theorem 4.10 of [CDMY96] that $L$ admits a bounded $A^*_a(R_+)$-functional calculus on $L^2(X; \rho)$ for all $\alpha \geq D' + D + 1$. The next result gives a slightly weaker conclusion for the action of the generator on the $L_p$-spaces. An earlier result of the following type was given in [Duo94].

**Theorem 6.5.** Let $L$ be the generator of a strongly continuous semigroup $S$ on $L^2(X; \rho)$ where $(X, \rho, d)$ is a space of polynomial type with dimensions $(D', D)$. Assume that $S$ is bounded and holomorphic in the open right half-plane and that

1. $L$ has a bounded $H_\mu(S^\mu_0)$-functional calculus on $L^2(X; \rho)$ for one $\mu \in (0, \pi]$,
2. $S_t$ has a kernel $K_t$ satisfying the Gaussian bounds

$$|K_t(x; y)| \leq a |B_t|^\mu e^{-h(x, y)^\mu}$$

for all $t > 0$.

It follows that $L$ admits a bounded $A^*_a(R_+)$-functional calculus on $L^p(X; \rho)$, $p \in (1, \infty)$, for all $\alpha > D' + D + 1$.

**Proof.** Since $S$ is holomorphic in the open right half-plane the generator $L$ is of type 0 and our aim is to verify (19) on each of the $L_p$-spaces.

It follows from Proposition 6.1 that the analytically continued kernel satisfies Gaussian bounds in the open right half-plane. Then Theorem 3.1 implies that $L$ has a bounded $H_\mu(S^\mu_0)$-functional calculus, for all positive $\mu$, on each $L^p$-space, $p \in (1, \infty)$. The rest of the proof consists of an elaboration of the arguments used to prove Theorem 3.1.

Fix the Calderón–Zygmund decomposition of the integrable function $f$ at height $\lambda$ as in the proof of Theorem 3.1. Then the weak type $(1, 1)$ estimates on the "good" part $g$ and on the $h_1$-term follow as before. Now to estimate the $h_2$-term we again use the decomposition of the $\xi_j$ into the two parts $\xi_{j+}$. Consider the estimation of $\xi_+$. Let $\gamma_+$ be the ray $te^{\theta t}$, $t > 0$, and $\Gamma_+$ the ray $te^{i(\pi - \theta)/2}$, $t > 0$. With this choice we reconsider the estimate

$$J = \int_{X \setminus B_j} dp(x) |(\xi_+ + L) f_j(x)|$$

$$\leq \int_0^\infty d|z| \int_0^\infty d|v| |e^{\gamma z} (1 - e^{\gamma t})| \int_{B_j} dp(y) |f_j(y)|$$

$$\times \int_{d(x, y) \geq \epsilon_j} dp(z) |K_j(z; y)|$$
The integral with respect to $|z|$ in the expression for $J$ will be estimated in two parts, $I_1$ and $I_2$, according as to whether $|z| \leq \beta t_j$ or $|z| \geq \beta t_j$ where $t_j = r_j^2$ and $\beta = \sin (\theta/2) = \cos (\arg z)$.

First, since $\text{Re} z = -\beta |z| |v|$ for $z \in \Gamma_+$ and $v \in \mathbb{C}$ one can use the brownian estimates of Proposition 6.1 to obtain

$$I_1 \leq c \beta^{-d} \int_0^{\beta t_j} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v| e^{-br_j^2 |v|}}$$

$$\leq c \beta^{-d} \int_0^{\beta t_j} dt \int_0^{|z|} ds \ e^{-\beta r_j^2 |v|}$$

$$= c \beta^{-(d+1)} \int_0^{\beta t_j} dt \int_0^{|z|} ds \ e^{-\beta r_j^2 |v|}$$

$$= c \beta^{-(d+1)} \int_0^{\beta t_j} dt \int_0^{|z|} ds \ e^{-\beta r_j^2 |v|}$$

where $d = D' \vee D$. Therefore the contribution $J_1$ of $I_1$ to the bound on $J$ is estimated by

$$J_1 \leq c \beta^{-(D' \vee D + 1)} \| f_j \|_1.$$

Secondly, the estimation of $I_2$ only requires the brownian estimate for $r = 0$. Using this estimate one immediately has

$$I_2 \leq c \beta^{-d} \int_0^{\beta t_j} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v| (1 - e^{-\beta |v|})}$$

$$\leq c \beta^{-d} \int_0^{\beta t_j} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v| (1 \wedge (|v| t_j))}.$$

Now the integral on the right hand side is separated into two parts $I_{21}$ and $I_{22}$ according as $t$ whether $|v| t_j > 1$ or $|v| t_j \leq 1$. It follows that

$$I_{21} \leq c \beta^{-(d+1)} \int_0^{\beta t_j} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v|}$$

$$\leq c \beta^{-(d+1)} \int_0^{\beta t_j} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v|} = c \beta^{-(d+1)} \int_0^{|z|} d |z| \int_0^{|z|} d |v| \ e^{-\beta |z| |v|}.$$
But since $x \mapsto x^\varepsilon e^{-x}$ is uniformly bounded for all $\varepsilon > 0$ one then has

$$I_21 = c\beta^{-(d + 1)} \int_0^\infty ds s^{-(\beta s)^-} = c\beta^{-(d + 1 + \varepsilon)} \int_0^\infty ds s^{-(1 - \varepsilon)} = c\beta^{-(d + 1 + 2\varepsilon)}.$$  

Alternatively,

$$I_{22} \leq c\beta^{-(d + 1)} \int_0^\infty ds \int_0^{1/\beta} d|z| |v| \int_0^{|v|} t_j e^{-\beta |v|} |v|$$

$$\leq c\beta^{-(d + 1)} \int_0^\infty ds \int_0^{1/\beta} d|z| |v| \int_0^{|v|} t_j e^{-\beta |v|} |v|$$

$$= c\beta^{-(d + 1 + \varepsilon)} \int_0^\infty ds |z| |v|^{-1 + \varepsilon} t_j \leq c\beta^{-(d + 1 + 2\varepsilon)}$$

and this integral gives a contribution to $J$ similar to $I_{21}$.

Combining these estimates one concludes that

$$\int_{\mathcal{X} \setminus \mathcal{B}_2} dp(x) \|\langle \xi \rangle (\mathcal{L}) f_j(x)\| \leq c\beta^{-(d + 1 + 2\varepsilon)} \|f\|_1,$$

for all $\varepsilon \in (0, 1]$, with a similar estimate for $\xi (\mathcal{L}) f_j$. Therefore, piecing together the bounds for the components of the Calderón–Zygmund decomposition of the function $f$ and the operator $\xi (\mathcal{L})$, one deduces the weak type $(1, 1)$ estimate

$$p\{ x \in \mathcal{X} : |\langle \xi \rangle (\mathcal{L}) f_j(x)\| > \lambda \} \leq c\lambda^{-1} \beta^{-(d + 1 + 2\varepsilon)} \|f\|_1$$  

(28)

for all $\xi \in H_{\infty}(S^n_0)$ with $\|\xi\|_{\infty} = 1$ where $\beta = \sin \mu$.

Since $\mathcal{L}$ has a bounded $H_{\infty}$-functional calculus on $L_2$, by assumption, it follows from Corollary 6.4 that one has bounds $\|\xi (\mathcal{L})\|_{2 \to 2} \leq c\beta^{-(d + 1)} \times \|\xi\|_{\infty}$ for all $\xi \in H_{\infty}(S^n_0)$. Therefore the Marcinkiewicz interpolation theorem gives bounds $\|\xi (\mathcal{L})\|_{p \to p} \leq c\beta^{-(d + 1 + \varepsilon)} \|\xi\|_{\infty}$ for $\varepsilon > 0$ and all $p \in (1, 2]$, and the statement of the theorem follows from (19).

The only information about the semigroup kernel for complex values of $t$ which was used in the above proof came from the brownian estimates (22). Therefore any improvement in the index $D' \lor D$ in the brownian bounds would improve the conclusion of the theorem. Such improvements are possible if $\mathcal{L}$ is self-adjoint on $L_2$.

**Theorem 6.6.** Let $\mathcal{L}$ be the generator of a strongly continuous self-adjoint semigroup $S$ on $L_2(\mathcal{X}; \rho)$ where $(\mathcal{X}, \rho, d)$ is a space of polynomial
type with dimensions \((D', D)\). Assume that \(S_t\) has a kernel \(K_t\), satisfying the Gaussian bounds
\[
|K_t(x; y)| \leq a |B_{r^2}|^{-1} e^{-b|a(x, y)|^2/2} \tag{29}
\]
for all \(t > 0\).

It follows that \(L\) admits a bounded \(A_{\mu,n}(\mathbb{R}_+)\)-functional calculus on the spaces \(L_p(X; \rho), p \in (1, \infty)\), for all \(n \geq (D' \lor D)/2 + 1\) and a bounded \(\mathcal{A}_n\)-functional calculus for all \(n \geq (D' \lor D)/2 + 1\).

**Proof.** First, it follows from the Gaussian bounds, by the argument given at the end of the proof of Theorem 6.5, that \(t \mapsto |S_t|_{p \to p}\) is uniformly bounded for each \(p \in [1, \infty)\). Hence \(L\) must be non-negative, by spectral theory, and \(S\) contractive on \(L_2\).

Secondly, it follows from spectral theory that \(L\) has a bounded \(H_n(S_0')\)-functional calculus on \(L_2(X; \rho)\), for all \(\mu \in (0, \pi]\), and one has bounds
\[
\| \xi(L) \|_{2 \to 2} \leq \| \xi \|_{\infty} \tag{29}
\]
for all \(\xi \in H_n(S_0')\) and all \(\mu \in (0, \pi]\). Hence \(L\) admits a bounded \(A_{\mu,n}(\mathbb{R}_+)\)-functional calculus on \(L_2(X; \rho)\), for all sufficiently large \(\alpha\), by Theorem 6.5. The lower bound on \(p\) can be improved by use of the uniform \(L_2\)-bounds (29) but better bounds can be obtained by observing that the self-adjointness of \(L\) allows one to improve the previous brownian estimates.

**Proposition 6.7.** Adopt the assumptions of Theorem 6.6. It follows that for each \(\varepsilon \in (0, 1]\) there are \(c, c' > 0\) such that
\[
\text{ess sup}_{x \in X} \int_{|d(x, y)| > r} dp(y) |K_t(x; y)| \leq c \rho^{-(1+\varepsilon)|D' \lor D|/2} e^{-c \beta r^2 |z|^2} \tag{30}
\]
for all \(r > 0\) and \(z\) with \(\text{Re } z > 0\) where \(\beta = \cos (\text{arg } z)\).

**Proof.** First, introduce the norm
\[
\|K_t\|_2 = \text{ess sup}_{x \in X} \left( \int_X dp(y) |K_t(x; y)|^2 \right)^{1/2}.
\]
Then, setting \(z = s + it\) with \(s, t \in \mathbb{R}\) one has
\[
\|K_t\|_2 = \|S_{s+it}\|_{2 \to \infty} \leq \|S_{s}\|_{2 \to \infty} \|S_{t}\|_{2 \to \infty} = \|S_{s}\|_{2 \to \infty} = \|K_{\text{Re } z}\|_2.
\]
Secondly, estimating the norm $\|K_{\Re z}\|_{L^2}$ as in the proof of Proposition 2.1 one has a bound

$$\|K_{\Re z}\|_{L^2} \leq c |B_{(\Re z)^{1/2}}|^{-1/2}.$$ 

Thirdly,

$$\text{ess sup}_{x \in X} \int_{d(x, y) \geq r} dp(y) |K_x(x; y)|$$

$$\leq \|K_x\|_{L^2}^{-e} \text{ess sup}_{x \in X} \left( \int_{d(x, y) \geq r} dp(y) |K_x(x; y)|^{2c(1+e)} \right)^{(1+e)/2}$$

by the Hölder inequality. Now it follows from Proposition 6.1 that the kernel satisfies the Gaussian bounds (21) for complex $z$ and hence

$$\int_{d(x, y) \geq r} dp(y) |K_x(x; y)|^{2c(1+e)}$$

$$\leq c |B_{(\Re z)^{1/2}}|^{-2c(1+e)} \int_{d(x, y) \geq r} dp(y) e^{-\epsilon d(x, y)^2 |\Re z|^{-1}}$$

$$\leq c |B_{(\Re z)^{1/2}}|^{-2c(1+e)} |B_{(\Re z^{-1})^{-1/2}}| e^{-\epsilon d(x, y)^2 |\Re z|^{-1}},$$

where the last estimate again follows from the proof of Proposition 6.1.

Combination of these estimates then gives

$$\text{ess sup}_{x \in X} \int_{d(x, y) \geq r} dp |K_x(x; y)|$$

$$\leq c |B_{(\Re z)^{1/2}}|^{-1-(1+e)/2} |B_{(\Re z)^{1/2}}|^{-e} |B_{(\Re z^{-1})^{-1/2}}|^{(1+e)/2} e^{-\epsilon d(x, y)^2 |\Re z|^{-1}}$$

$$= (|B_{(\Re z)^{1/2}}|^{-1} |B_{(\Re z^{-1})^{-1/2}}|)^{1+e/2} e^{-\epsilon d(x, y)^2 |\Re z|^{-1}}.$$

But one again has

$$|B_{(\Re z^{-1})^{-1/2}}| = |B_{\beta^{-1}(\Re z)^{1/2}}| \leq c \beta^{-d} |B_{(\Re z)^{1/2}}|$$

with $d = (D' \vee D)$ by the strong homogeneity property.

The desired bound then is an immediate consequence of the last two inequalities. 

Now consider the proof of Theorem 6.6. If one repeats the argument used to prove Theorem 6.5 but with the brownian bounds of Proposition 6.1
replaced by those of Proposition 6.7 one establishes the weak type \((1, 1)\) estimate

\[
\rho(\{ x \in X : |(L f)(x)| > \lambda \}) \lesssim c \lambda^{-1} \beta^{-1(d+1)/2} \| f \|_1
\]

instead of (28). Therefore the Marcinkiewicz interpolation theorem applied to (29) and (31) gives bounds

\[
\| \zeta(L) \|_{p \to p} \lesssim c \beta^{-(d+1)/2 + 2}\| \xi \|_{\infty}
\]

for all \( p \in (1, 2) \). But if \( p \in (1, 2) \) one may choose \( \varepsilon > 0 \) such that \( (d+1)/2 + 2\varepsilon)(2p^{-1} - 1) \leq (d+1)/2 \) and hence

\[
\| \zeta(L) \|_{p \to p} \lesssim c \beta^{-(d+1)/2} \| \xi \|_{\infty}
\]

for all \( p \in (1, 2) \). A similar result follows by duality for \( p \in [2, \infty) \) and since \( \beta = \sin \mu \) one concludes that

\[
\| \zeta(L) \|_{p \to p} \lesssim c \mu^{-(d+1)/2} \| \xi \|_{\infty}
\]

for all \( p \in (1, \infty) \). Therefore the statement of the theorem concerning the Hörmander functional calculus follows from (19).

The statement concerning the bounded \( \mathcal{A}_\nu \)-functional calculus is somewhat easier to deduce. It follows as in the previous section from the brownian estimates and interpolation. The improved brownian bounds and the uniform \( L_2 \)-bounds give the improved estimate. 

**ACKNOWLEDGMENTS**

This work originated with a remark to one of us (X. D.) by Michael Cowling that Hebisch’s approach to singular integration theory might be used to remove Hölder continuity requirements in earlier results on holomorphic functional calculus. The final form of the paper was affected by Michael Christ’s suggestion that Gaussian bounds on the semigroup kernels could be replaced by weaker bounds of the Poisson type. The authors are also grateful to Detlef Müller for pointing out a serious omission in an earlier version of the paper.

During the course of our collaboration we both profited from generous help and constructive suggestions from Alan McIntosh. Discussions with Michael Cowling were also stimulating. Our understanding of spaces of exponential growth was aided by discussions with Andrea Nahmod and Thierry Coulhon and we are grateful to Tom ter Elst for numerous useful comments.

In the early stages of this work X. T. Duong was supported, at the University of New South Wales, by a grant from the Australian Research Council and during the later stages by a Macquarie University Research Fellowship.
REFERENCES


