ELSEVIER

Available online at www.sciencedirect.com

ScienceDirect

Journal of Functional Analysis 260 (2011) 76–116

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

From U-bounds to isoperimetry with applications to H-type groups [☆]

J. Inglis [a,*], V. Kontis [a], B. Zegarliński [b,1]

[a] *Department of Mathematics, Imperial College London, 180 Queen's Gate, London SW7 2AZ, UK*

[b] *CNRS, Toulouse, France*

Received 6 January 2010; accepted 5 August 2010

Available online 6 September 2010

Communicated by L. Gross

Abstract

In this paper we study U-bounds in relation to L_1-type coercive inequalities and isoperimetric problems for a class of probability measures on a general metric space $(\mathbb{R}^N, d)$. We prove the equivalence of an isoperimetric inequality with several other coercive inequalities in this general framework. The usefulness of our approach is illustrated by an application to the setting of H-type groups, and an extension to infinite dimensions.

Keywords: Logarithmic Sobolev inequalities; Isoperimetric inequalities; H-type groups; Infinite dimensional applications

1. Introduction

An effective technology to study coercive inequalities involving (sub-)gradients and a variety of probability measures on metric measure spaces was recently introduced in [23]. This approach was based on so-called U-bounds, that is estimates of the following form

☆ Supported by EPSRC grant EP/D05379X/1.

* Corresponding author.

E-mail addresses: james.inglis06@imperial.ac.uk (J. Inglis), vasileios.kontis07@imperial.ac.uk (V. Kontis), boguslaw.zegarlinski@math.univ-toulouse.fr (B. Zegarliński).

1 On leave of absence from Imperial College London.

0022-1236/$ – see front matter
doi:10.1016/j.jfa.2010.08.003

$$\int |f|^q U(d)^{\gamma_q}\, d\mu \leqslant C_q \int |\nabla f|^q\, d\mu + D_q \int |f|^q\, d\mu.$$

Here $q \in [1, \infty)$, d is a metric associated to the (sub-)gradient ∇, $\gamma_q, C_q, D_q \in (0, \infty)$ are constants independent of the function f, and $d\mu \equiv e^{-U(d)}\, d\lambda$ is a probability measure, where $U(d)$ is a function that is bounded from below and has suitable growth at infinity, and $d\lambda$ is a natural underlying measure. While the consequences of the bounds corresponding to $q > 1$ were extensively explored there, the limiting case was left open. In this paper we show that there is a natural direct way from U-bounds with $q = 1$ to isoperimetric information. In fact we show an essential equivalence of such a bound with an $L_1\Phi$-entropy inequality

$$\mathbf{Ent}_\mu^\Phi(f) \leqslant c\mu|\nabla f|$$

where

$$\mathbf{Ent}_\mu^\Phi(f) \equiv \mu\Phi(f) - \Phi(\mu f)$$

is defined with a suitable Orlicz function Φ, as well as the equivalence with an isoperimetric inequality with a suitable profile function. We first recall an interesting result of [26] showing that in case of the Gaussian measures on Euclidean spaces, the functions f such that $\mu|f| < \infty$ belong to the Orlicz space defined by a function $\Phi(s) = s(\log(1+s))^{\frac{1}{2}}$. Also, on the level of isoperimetry for probability measures, we would like to recall a comprehensive characterisation of isoperimetric profiles for measures on the real line obtained in [10] (see also [5,12,14,29] and references therein) as well as the isoperimetric functional inequalities studied in [8] (see also [2,3,12,33]). These results provided additional motivation to our work. In particular, in [12] the authors conjecture that for super-Gaussian distributions one should expect an analogue of the isoperimetric functional inequality (IFI_2) introduced in [8], with a suitable non-Gaussian isoperimetric function and a different than Euclidean length of the gradient. In [2] (an alternative to [27]) the authors gave a proof of the $p = 1$ (sub-)gradient bound

$$|\nabla P_t f|^p \leqslant C_p(t) P_t |\nabla f|^p$$

for the heat kernel on the Heisenberg group, and as a consequence obtained an (IFI_2) inequality in this case. We mention that, for $p > 1$, gradient bounds were earlier established in [16], while the logarithmic Sobolev inequality for heat kernels on Heisenberg-type groups was established in [23].

The other interesting question is what are the optimal equivalent conditions, on the one side characterising the properties of the semigroup for which the form associated to the generator is given by the square of a fixed sub-gradient, and on the other side characterising the isoperimetric properties (e.g. in the form of some isoperimetric functional inequality with a given length of the sub-gradient). In the particular situation when $p = 1$ gradient bounds are known, and an equivalence relation (between (IFI_2) and the logarithmic Sobolev inequality) was established in [19]. It seems that we are still away from fully understanding the peculiarity of this situation and in particular answering the question what kind of additional conditions are necessary to establish equivalence between conditions of different orders in the length of the gradient (as well as finding a more direct proof of this equivalence without going through the semigroup route).

From the point of view of applications to an infinite dimensional probabilistic set-up involving an infinite product of non-compact Lie groups, it is important that we are dealing with inequalities satisfying the tensorisation property. Then one can attack the interesting question of for which non-product measures one can prove similar properties. This question, when the underlying space is as we wish, appears to have some new challenging features and so far, besides the results of [24] where logarithmic Sobolev inequalities (LS_q), $q > 1$, are shown for some classes of measures, not much is known. Therefore in the present paper we also contribute to this topic by proving tight $L_1\Phi$-entropy inequalities for suitable infinite dimensional Gibbs measures.

The organisation of our paper is as follows. In Section 2, starting from U-bounds, we prove the $L_1\Phi$-entropy inequality via a route involving "dressing up" the classical Sobolev inequality and a tightening procedure using a generalised Rothaus type lemma of [28], extended relative entropy bounds of [20], and the following Cheeger type inequality

$$\mu|f - \mu f| \leqslant c_0 \mu|\nabla f|.$$

In fact, this type of the Cheeger inequality is shown (in Theorem 2.7) to be a simple consequence of a similar inequality in balls together with U-bounds, provided the function U grows to infinity with the size of the ball.

In Section 3 we discuss some applications to isoperimetric and functional isoperimetric inequalities. Section 4 contains some consequences of the $L_1\Phi$-entropy inequality. In particular this includes the (LS_q) inequality and U-bounds. In Theorem 4.5 we summarise all interrelations between the properties discussed before. Section 5 is devoted to applications of the theory developed in the previous sections to the important class of H-type groups, where one can check the U-bounds for probability measures with density (essentially) dependent on the Carnot–Carathéodory distance. The interesting outcome, which comes out naturally within the presented approach, includes a proof of the $p = 1$ sub-gradient bounds for heat kernels on H-type groups which could potentially be extended to more complicated non-compact groups. Finally in Section 6 we prove the $L_1\Phi$-entropy inequalities for non-product probability measures on an infinite product of H-type groups, which allows us in particular to obtain some new isoperimetric information. Additionally we prove here the (IFI_2) inequality in such a setup; in fact even in the case of the full gradient setup, this provides an interesting extension of results in [33] allowing us to include the important case of unbounded interactions.

2. $L_1\Phi$-entropy inequalities from U-bounds

Throughout this paper we will be working in $\mathbb{R}^N$ equipped with a metric $d : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ and Lebesgue measure $d\lambda$. For $r \geqslant 0$, we will set

$$B(r) := \{x \colon d(x) \leqslant r\},$$

where $d(x) := d(x, 0)$.

We will also let ∇ be a general sub-gradient in $\mathbb{R}^N$ i.e. ∇ is a finite collection $\{X_1, \ldots, X_m\}$ of possibly non-commuting fields. Assume that the divergence of each of these fields with respect to the Lebesgue measure λ on $\mathbb{R}^N$ is zero. Set $\Delta := \sum_{i=1}^m X_i^2$ and $|\nabla f| = (\sum_{i=1}^m (X_i f)^2)^{\frac{1}{2}}$.

Theorem 2.1. *Let U be a locally Lipschitz function on $\mathbb{R}^N$, which is bounded from below and is such that $Z = \int e^{-U}\,d\lambda < \infty$. Let $d\mu = \frac{e^{-U}}{Z}\,d\lambda$, so that μ is a probability measure on $\mathbb{R}^N$. Suppose that the following classical Sobolev inequality is satisfied*

$$\left(\int |f|^{1+\varepsilon}\,d\lambda\right)^{\frac{1}{1+\varepsilon}} \leqslant a\int |\nabla f|\,d\lambda + b\int |f|\,d\lambda \tag{1}$$

for some constants $a, b \in [0,\infty)$, $\varepsilon > 0$ and all locally Lipschitz functions f, and moreover that for some $A, B \in [0,\infty)$ we have

$$\mu\big(|f|\big(|U|^\beta + |\nabla U|\big)\big) \leqslant A\mu|\nabla f| + B\mu|f| \tag{2}$$

for some $\beta \in (0,1]$ and all such f. Then there exist constants $C, D \in [0,\infty)$ such that

$$\mu\left(|f|\left|\log\frac{|f|}{\mu|f|}\right|^\beta\right) \leqslant C\mu|\nabla f| + D\mu|f| \tag{3}$$

for all locally Lipschitz functions f.

Remark 2.2. Inequality (1) should be interpreted as a condition on the gradient ∇.

Proof. Without loss of generality, we may suppose that $f \geqslant 0$ and $U \geqslant 0$. Indeed, otherwise we may apply (3) to the positive and negative parts of f separately. Moreover, if $U \geqslant -K$, with $K \geqslant 0$, we have that $U + K \geqslant 0$ and then we can replace f by fe^{-K} in (3).

First note that

$$\begin{aligned}\mu\left(f\left|\log\frac{f}{\mu f}\right|^\beta\right) &= \mu\left(f\left[\log_+\frac{f}{\mu f}\right]^\beta\right) + \mu\left(f\left[\log\frac{\mu f}{f}\right]^\beta \mathbf{1}_{\{f\leqslant \mu f\}}\right)\\ &\leqslant \mu\left(f\left[\log_+\frac{f}{\mu f}\right]^\beta\right) + e^{-\beta}\beta^\beta\mu(f),\end{aligned}$$

where we have used that $\sup_{x\in(0,1)} x(\log\frac{1}{x})^\beta = e^{-\beta}\beta^\beta$ with $x = \frac{f}{\mu f}$. Thus it suffices to prove that

$$\mu\left(f\left[\log_+\frac{f}{\mu f}\right]^\beta\right) \leqslant C\mu|\nabla f| + D\mu(f), \tag{4}$$

with some constants $C, D \in (0,\infty)$ independent of f. Suppose that $\mu(f) = 1$. With $F \equiv fe^{-U}$ and $\varepsilon \in (0,1)$ sufficiently small, we have

$$\int F\big[\log_+(F)\big]^\beta\,d\lambda = \int_{\{F\geqslant 1\}} F\left[\frac{1}{\varepsilon}\log\big(F^\varepsilon\big)\right]^\beta d\lambda. \tag{5}$$

Now, by Jensen's inequality (since, for $\beta \in (0,1]$, the function $(\log x)^\beta$ is concave on $x \geqslant 1$)

$$\begin{aligned}
\int_{\{F\geqslant 1\}} F\left[\frac{1}{\varepsilon}\log\left(F^{\varepsilon}\right)\right]^{\beta} d\lambda &= \frac{\int_{\{F\geqslant 1\}} F d\lambda}{\varepsilon^{\beta}} \int_{\{F\geqslant 1\}} \frac{F}{\int_{\{F\geqslant 1\}} F\,d\lambda}\left[\log\left(F^{\varepsilon}\right)\right]^{\beta} d\lambda \\
&\leqslant \frac{\int_{\{F\geqslant 1\}} F d\lambda}{\varepsilon^{\beta}}\left[\log \frac{\int_{\{F\geqslant 1\}} F^{1+\varepsilon}\,d\lambda}{\int_{\{F\geqslant 1\}} F\,d\lambda}\right]^{\beta} \\
&= \frac{(1+\varepsilon)^{\beta}\int_{\{F\geqslant 1\}} F\,d\lambda}{\varepsilon^{\beta}}\left[\log \frac{(\int_{\{F\geqslant 1\}} F^{1+\varepsilon}\,d\lambda)^{\frac{1}{1+\varepsilon}}}{(\int_{\{F\geqslant 1\}} F\,d\lambda)^{\frac{1}{1+\varepsilon}}}\right]^{\beta} \\
&\leqslant \frac{(1+\varepsilon)^{\beta}\int_{\{F\geqslant 1\}} F\,d\lambda}{\varepsilon^{\beta}}\left[\log \frac{(\int_{\{F\geqslant 1\}} F^{1+\varepsilon}\,d\lambda)^{\frac{1}{1+\varepsilon}}}{(\int_{\{F\geqslant 1\}} F\,d\lambda)^{\frac{1}{1+\varepsilon}}} + 1\right],
\end{aligned}$$

using the simple fact that $x^{\beta} \leqslant x+1$ for all $x \geqslant 0$. Thus, since $\log x \leqslant x-1$ for all $x \geqslant 0$,

$$\int_{\{F\geqslant 1\}} F\left[\frac{1}{\varepsilon}\log\left(F^{\varepsilon}\right)\right]^{\beta} d\lambda \leqslant \frac{(1+\varepsilon)^{\beta}\int_{\{F\geqslant 1\}} F\,d\lambda}{\varepsilon^{\beta}}\left[\frac{(\int_{\{F\geqslant 1\}} F^{1+\varepsilon}\,d\lambda)^{\frac{1}{1+\varepsilon}}}{(\int_{\{F\geqslant 1\}} F\,d\lambda)^{\frac{1}{1+\varepsilon}}}\right].$$

Since we have assumed that $\mu(f)=1$, we have

$$\int_{\{fe^{-U}\geqslant 1\}} fe^{-U}\,d\lambda/Z \equiv \int_{\{F\geqslant 1\}} F\,d\lambda/Z \leqslant 1,$$

so that

$$\frac{1}{(\int_{\{F\geqslant 1\}} F\,d\lambda)^{\frac{1}{1+\varepsilon}}} \leqslant \frac{Z^{\frac{\varepsilon}{1+\varepsilon}}}{\int_{\{F\geqslant 1\}} F\,d\lambda}.$$

Thus

$$\begin{aligned}
\int_{\{F\geqslant 1\}} F\left[\frac{1}{\varepsilon}\log\left(F^{\varepsilon}\right)\right]^{\beta} d\lambda &\leqslant \frac{(1+\varepsilon)^{\beta} Z^{\frac{\varepsilon}{1+\varepsilon}}}{\varepsilon^{\beta}}\left(\int F^{1+\varepsilon}\,d\lambda\right)^{\frac{1}{1+\varepsilon}} \\
&\leqslant \frac{(1+\varepsilon)^{\beta} Z^{\frac{\varepsilon}{1+\varepsilon}}}{\varepsilon^{\beta}}\left(a\int \left|\nabla(F)\right| d\lambda + bZ\right),
\end{aligned}$$

provided $\varepsilon > 0$ is chosen sufficiently small so that in the last step we can apply the classical Sobolev inequality (1). Dividing both sides by the normalisation factor Z and recalling $F \equiv fe^{-U}$, this implies

$$\int f\left[\log_{+}\left(fe^{-U}\right)\right]^{\beta} d\mu \leqslant c_1\mu|\nabla f| + c_2\mu\left(f|\nabla U|\right) + c_3, \tag{6}$$

with $d\mu \equiv \frac{1}{Z}e^{-U}\,d\lambda$ and $c_1 = c_2 = (1+\varepsilon)^\beta a Z^{\frac{\varepsilon}{1+\varepsilon}}/\varepsilon^\beta$, $c_3 = (1+\varepsilon)^\beta b Z^{\frac{\varepsilon}{1+\varepsilon}}/\varepsilon^\beta$. Now consider the left-hand side of (6). Since $\beta \in (0,1]$, $(x-y)^\beta \geqslant x^\beta - y^\beta$ for $x \geqslant y \geqslant 0$. Applying this with $x = \log f$ and $y = U \geqslant 0$, we have

$$\begin{aligned}
\int f\big[\log_+\big(fe^{-U}\big)\big]^\beta\,d\mu &= \int_{\{f\geqslant e^U\}} f(\log f - U)^\beta\,d\mu \\
&\geqslant \int_{\{f\geqslant e^U\}} f(\log f)^\beta\,d\mu - \int_{\{f\geqslant e^U\}} fU^\beta\,d\mu \\
&= \mu\big(f[\log_+ f]^\beta\big) - \int_{\{1\leqslant f\leqslant e^U\}} f(\log f)^\beta\,d\mu \\
&\quad - \int_{\{f\geqslant e^U\}} fU^\beta\,d\mu \\
&\geqslant \mu\big(f[\log_+ f]^\beta\big) - \int_{\{1\leqslant f\}} fU^\beta\,d\mu.
\end{aligned}$$

Combining this with (6) we see that

$$\begin{aligned}
\mu\big(f[\log_+ f]^\beta\big) &\leqslant c_1\mu|\nabla f| + c_2\mu\big(f|\nabla U|\big) + c_3 + \int_{\{1\leqslant f\}} fU^\beta\,d\mu \\
&\leqslant c_1\mu|\nabla f| + \max\{c_2,1\}\mu\big(f\big(U^\beta + |\nabla U|\big)\big) + c_3 \\
&\leqslant \big(c_1 + \max\{c_2,1\}A\big)\mu|\nabla f| + c_3 + \max\{c_2,1\}B,
\end{aligned}$$

where we have used (2) in the last step. Finally, for general $f \geqslant 0$, we apply the above inequality to $f/\mu(f)$ to arrive at (4). □

As a corollary, we can also state the following perturbation result.

Corollary 2.3. *Let U and μ be as in Theorem* 2.1*, and suppose conditions* (1) *and* (2) *are satisfied. Let W be a locally Lipschitz function such that $\int e^{-W}\,d\mu < \infty$ and*

$$|\nabla W| \leqslant \delta\big(|U|^\beta + |\nabla U|\big) + C(\delta), \qquad |W|^\beta \leqslant a_0\big(|U|^\beta + |\nabla U|\big) + a_1 \tag{7}$$

almost everywhere, with some $0 < \delta < \frac{1}{A}$ and $C(\delta), a_0, a_1 \in (0,\infty)$. Then there exist constants $\tilde{C}$ and $\tilde{D}$ such that, for all locally Lipschitz functions f,

$$\tilde{\mu}\left(|f|\left|\log\frac{|f|}{\tilde{\mu}|f|}\right|^\beta\right) \leqslant \tilde{C}\tilde{\mu}|\nabla f| + \tilde{D}\tilde{\mu}|f|, \tag{8}$$

where $\tilde{\mu}$ is the probability measure on $\mathbb{R}^N$ given by $\tilde{\mu}(d\lambda) := e^{-W}\mu(d\lambda)/Z_{\tilde{\mu}}$, with $Z_{\tilde{\mu}} \equiv \mu(e^{-W})$.

Proof. Take $f \geqslant 0$. Since (2) holds by assumption, we can apply it to the function fe^{-W}. This yields

$$\tilde{\mu}\big(f\big(|U|^\beta + |\nabla U|\big)\big) \leqslant A\tilde{\mu}|\nabla f| + A\tilde{\mu}\big(f|\nabla W|\big) + B\tilde{\mu}(f)$$
$$\leqslant A\tilde{\mu}|\nabla f| + \delta A\tilde{\mu}\big(f\big(|U|^\beta + |\nabla U|\big)\big) + \big(B + AC(\delta)\big)\tilde{\mu}(f)$$

using (7). Thus, since $\delta A < 1$, we have that

$$\tilde{\mu}\big(f\big(|U|^\beta + |\nabla U|\big)\big) \leqslant \tilde{A}\tilde{\mu}|\nabla f| + \tilde{B}\tilde{\mu}(f) \tag{9}$$

for $\tilde{A} = A/(1-\delta A)$, $\tilde{B} = (B + AC(\delta))/(1-\delta A)$. Replacing f by fe^{-W} in (3) of Theorem 2.1, we get

$$\tilde{\mu}\left(f\left|\log\frac{fe^{-W}}{\tilde{\mu}(f)Z_{\tilde{\mu}}}\right|^\beta\right) \leqslant C\tilde{\mu}|\nabla f| + C\tilde{\mu}\big(f|\nabla W|\big) + D\tilde{\mu}(f).$$

Using this together with (9) and the elementary inequality $(x+y)^\beta \leqslant x^\beta + y^\beta$ which holds when $x, y \geqslant 0$ and $\beta \in (0,1]$ yields

$$\begin{aligned}
&\tilde{\mu}\left(f\left|\log\frac{f}{\tilde{\mu}(f)}\right|^\beta\right)\\
&\quad\leqslant C\tilde{\mu}|\nabla f| + \tilde{\mu}\big(f\big(|W|^\beta + C|\nabla W|\big)\big) + \big(D + |\log Z_{\tilde{\mu}}|^\beta\big)\tilde{\mu}(f)\\
&\quad\leqslant C\tilde{\mu}|\nabla f| + a_0\max\{1, C\}\tilde{\mu}\big(f\big(|U|^\beta + |\nabla U|\big)\big) + \big(a_1 + D + |\log Z_{\tilde{\mu}}|^\beta\big)\tilde{\mu}(f)\\
&\quad\leqslant \tilde{C}\tilde{\mu}|\nabla f| + \tilde{D}\tilde{\mu}(f),
\end{aligned}$$

where $\tilde{C} = C + a_0\max\{1, C\}\tilde{A}$ and $\tilde{D} = a_1 + D + |\log Z_{\tilde{\mu}}|^\beta + a_0\max\{1, C\}\tilde{B}$. The inequality for general f follows in similar way by applying the above inequality to the positive and negative parts of f separately. □

The resulting inequality in Theorem 2.1 is a defective inequality, in the sense that it contains a term involving $\mu|f|$ on the right-hand side. For our purposes this type of inequality is not strong enough, and therefore we now aim to prove a tightened inequality of the following form

$$\mathbf{Ent}_\mu^\Phi\big(|f|\big) := \mu\big(\Phi\big(|f|\big)\big) - \Phi\big(\mu|f|\big) \leqslant c\mu|\nabla f|, \tag{10}$$

where $\Phi(x) = x(\log(1+x))^\beta$, $\beta \in (0,1]$, and $c \in (0,\infty)$ is a constant independent of f. We accomplish this in the situation (see Theorem 2.5 below) when we have the following Cheeger type inequality

$$\mu|f - \mu f| \leqslant c_0\mu|\nabla f|$$

with a constant $c_0 \in (0,\infty)$ independent of f.

A bound of the form described in (10) will be called in what follows an *$L_1\Phi$-entropy inequality*. It is an example of a (non-homogeneous) additive Φ-entropy inequality, as studied in [5] and [15]. To arrive at the desired inequality, our strategy will be as follows. We will first use Theorem 2.1 to prove a defective $L_1\Phi$-entropy inequality, that is an inequality of a similar form but containing additionally on its right-hand side a term proportional to $\mu|f|$. Then we will adapt some ideas of Rothaus [30], generalised in [12], to show that such a defective inequality can be tightened. We begin by proving the following lemma.

Lemma 2.4. *Let $\Phi(x) = x(\log(1+x))^\beta$, $\beta \in (0,1]$ and let μ be a given probability measure. Then there exists a constant $\kappa \in [0,\infty)$ such that for any functions f and g satisfying $0 \leqslant g \leqslant f$, $\mu f < \infty$, one has*

$$\mathbf{Ent}_\mu^\Phi(g) \leqslant \mu\left(f\left[\log_+\left(\frac{f}{\mu f}\right)\right]^\beta\right) + \kappa\mu(f).$$

Proof. We have that

$$\begin{aligned}\mathbf{Ent}_\mu^\Phi(g) &= \mu\big(g\big[\big(\log(1+g)\big)^\beta - \big(\log(1+\mu g)\big)^\beta\big]\big)\\ &\leqslant \mu\left(g\left[\log\left(1+\frac{g}{\mu g}\right)\right]^\beta\right)\\ &\leqslant \mu\left(f\left[\log\left(1+\frac{g}{\mu g}\right)\right]^\beta\right),\end{aligned} \tag{11}$$

since $g \leqslant f$. Set $F(x) := (\log(1+x))^\beta$ for $x \in [0,\infty)$. Then F is increasing and concave. Moreover, there exists a constant $\theta \in (0,\infty)$ such that $xF'(x) \leqslant \theta$ for all x. Following [20], we now claim that

$$xF(y) \leqslant xF(x) + \theta y \tag{12}$$

for all $x, y \geqslant 0$. Indeed, if $y \leqslant x$ this is trivial. If $x \leqslant y$, we have

$$\begin{aligned}x\big(F(y) - F(x)\big) &= x\frac{F(y)-F(x)}{y-x}(y-x) \leqslant xF'(x)y\\ &\leqslant \theta y.\end{aligned}$$

Setting $x = \frac{f}{\mu f}$ and $y = \frac{g}{\mu g}$ in (12) and integrating both sides with respect to the measure μ yields

$$\mu\left(f\left[\log\left(1+\frac{g}{\mu g}\right)\right]^\beta\right) \leqslant \mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^\beta\right) + \theta\mu(f).$$

Thus, by (11)

$$\mathbf{Ent}_\mu^\Phi(g) \leqslant \mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^\beta\right) + \theta\mu(f). \tag{13}$$

Now

$$\begin{aligned}\mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^{\beta}\right) &= \mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^{\beta}\mathbf{1}_{\{f\leqslant\mu f\}}\right)\\ &\quad+\mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^{\beta}\mathbf{1}_{\{f\geqslant\mu f\}}\right)\\ &\leqslant (\log 2)^{\beta}\mu(f)+\mu\left(f\left[\log\left(\frac{2f}{\mu f}\right)\right]^{\beta}\mathbf{1}_{\{f\geqslant\mu f\}}\right)\\ &= (\log 2)^{\beta}\mu(f)\\ &\quad+\mu\left(f\left[\log 2+\log\left(\frac{f}{\mu f}\right)\right]^{\beta}\mathbf{1}_{\{f\geqslant\mu f\}}\right)\\ &\leqslant 2(\log 2)^{\beta}\mu(f)+\mu\left(f\left[\log_{+}\left(\frac{f}{\mu f}\right)\right]^{\beta}\right),\end{aligned}$$

using once again the inequality $(x+y)^{\beta}\leqslant x^{\beta}+y^{\beta}$ for $x,y\geqslant 0$ and $\beta\in(0,1]$. Combining this with (13), we arrive at

$$\mathbf{Ent}_{\mu}^{\Phi}(g)\leqslant\mu\left(f\left[\log_{+}\left(\frac{f}{\mu f}\right)\right]^{\beta}\right)+\left(2(\log 2)^{\beta}+\theta\right)\mu(f),$$

which completes the proof. □

Theorem 2.5. *Suppose U, λ and μ are as in Theorem* 2.1. *In addition, suppose that the following Cheeger type inequality holds*

$$\mu|f-\mu f|\leqslant c_0\mu|\nabla f| \tag{14}$$

for some $c_0>0$ and all locally Lipschitz functions f. Then there exists $c\in(0,\infty)$ such that (10) *holds, i.e. for any locally Lipschitz function f, we have*

$$\mathbf{Ent}_{\mu}^{\Phi}\left(|f|\right)\leqslant c\mu|\nabla f|,$$

where $\Phi(x)=x(\log(1+x))^{\beta}$.

Proof. By Lemma A.1 of the appendix of [28], we have that there exist constants $\tilde{a}$ and $\tilde{b}$ such that

$$\mathbf{Ent}_{\mu}^{\Phi}\left(f^2\right)\leqslant\tilde{a}\mathbf{Ent}_{\mu}^{\Phi}\left((f-\mu f)^2\right)+\tilde{b}\mu(f-\mu f)^2.$$

Thus, for any $t\in\mathbb{R}$, we have that

$$\begin{aligned}\mathbf{Ent}_{\mu}^{\Phi}|f+t| &= \mathbf{Ent}_{\mu}^{\Phi}\left[\left(|f+t|^{\frac{1}{2}}\right)^2\right]\\ &\leqslant\tilde{a}\mathbf{Ent}_{\mu}^{\Phi}\left[\left(|f+t|^{\frac{1}{2}}-\mu|f+t|^{\frac{1}{2}}\right)^2\right]+\tilde{b}\mu\left(|f+t|^{\frac{1}{2}}-\mu|f+t|^{\frac{1}{2}}\right)^2.\end{aligned} \tag{15}$$

Let $G = (|f+t|^{\frac{1}{2}} - \mu|f+t|^{\frac{1}{2}})^2$. Note that we can write

$$\begin{aligned} G = \left(|f+t|^{\frac{1}{2}} - \mu|f+t|^{\frac{1}{2}}\right)^2 &= \left(\int \left|f(\omega)+t\right|^{\frac{1}{2}} - \left|f(\tilde{\omega})+t\right|^{\frac{1}{2}} d\mu(\tilde{\omega})\right)^2 \\ &\leqslant \int \left(\left|f(\omega)+t\right|^{\frac{1}{2}} - \left|f(\tilde{\omega})+t\right|^{\frac{1}{2}}\right)^2 d\mu(\tilde{\omega}) \\ &\leqslant \int \left|f(\omega) - f(\tilde{\omega})\right| d\mu(\tilde{\omega}) \\ &\leqslant |f| + \mu|f|, \end{aligned}$$

using the elementary inequality $||x+t|^{\frac{1}{2}} - |y+t|^{\frac{1}{2}}| \leqslant |x-y|^{\frac{1}{2}}$ in the last but one step. Hence, we have by (15) that

$$\mathbf{Ent}_\mu^\Phi |f+t| \leqslant \tilde{a}\mathbf{Ent}_\mu^\Phi(G) + 2\tilde{b}\mu|f|. \tag{16}$$

Since $0 \leqslant G \leqslant |f| + \mu|f|$, by Lemma 2.4 and Theorem 2.1, we have

$$\begin{aligned} \mathbf{Ent}_\mu^\Phi(G) &\leqslant \mu\left(\left(|f|+\mu|f|\right)\left[\log_+ \frac{|f|+\mu|f|}{\mu(|f|+\mu|f|)}\right]^\beta\right) + 2\kappa\mu|f| \\ &\leqslant C\mu|\nabla f| + 2(D+\kappa)\mu|f|. \end{aligned} \tag{17}$$

Combining (16) and (17) yields

$$\sup_{t\in\mathbb{R}} \mathbf{Ent}_\mu^\Phi |f+t| \leqslant \tilde{a}C\mu|\nabla f| + 2\left(\tilde{a}(D+\kappa) + \tilde{b}\right)\mu|f|. \tag{18}$$

This implies the following bound

$$\mathbf{Ent}_\mu^\Phi |f| \leqslant \tilde{a}C\mu|\nabla f| + 2\left(\tilde{a}(D+\kappa) + \tilde{b}\right)\mu|f - \mu f|. \tag{19}$$

Finally we can apply the Cheeger type inequality (14) to the last term on the right-hand side of (19) to arrive at

$$\mathbf{Ent}_\mu^\Phi\left(|f|\right) \leqslant c\mu|\nabla f|,$$

with $c = \tilde{a}C + 2c_0(\tilde{a}(D+\kappa) + \tilde{b})$. □

In the same spirit as Corollary 2.3, this inequality is stable under perturbations of the following type.

Corollary 2.6. *Let* U, λ *and* μ *be as in Theorem* 2.1. *Suppose also that the Cheeger type inequality* (14) *holds. As in Corollary* 2.3, *let* W *be a real function which is locally Lipschitz and such that* $\int e^{-W} d\mu < \infty$ *and*

$$|\nabla W| \leqslant \delta\left(|U|^\beta + |\nabla U|\right) + C(\delta), \qquad |W|^\beta \leqslant a_0\left(|U|^\beta + |\nabla U|\right) + a_1$$

for some $\delta < \frac{1}{A}$, $C(\delta), a_0, a_1 \in (0,\infty)$ *and* $\beta \in (0,1]$. *Moreover, let* V *be a measurable function such that*

$$osc(V) \equiv \sup V - \inf V < \infty.$$

Then there exists a constant $\hat{c}$ *such that, for all locally Lipschitz functions* f,

$$\mathbf{Ent}^{\Phi}_{\hat{\mu}}\big(|f|\big) \leqslant \hat{c}\hat{\mu}|\nabla f|,$$

where $\hat{\mu}$ *is the probability measure on* $\mathbb{R}^N$ *given by*

$$\hat{\mu}(d\lambda) := e^{-W-V}\mu(d\lambda)/\hat{Z},$$

with a normalisation constant $\hat{Z} \in (0,\infty)$ *and* $\Phi(x) = x(\log(1+x))^{\beta}$.

Proof. In the case $V = 0$, the result is obtained by following the proof of Theorem 2.5, using Corollary 2.3 where necessary. In the case $V \neq 0$, by Lemma 3.4.2 of [1], we may write

$$\begin{aligned}\mathbf{Ent}^{\Phi}_{\hat{\mu}}\big(|f|\big) &= \inf_{t\in[0,\infty)} \hat{\mu}\big(\Phi\big(|f|\big) - \Phi'(t)\big(|f|-t\big) - \Phi(t)\big)\\ &\leqslant \frac{e^{osc(V)}Z_0}{\hat{Z}} \inf_{t\in[0,\infty)} \int \big(\Phi\big(|f|\big) - \Phi'(t)\big(|f|-t\big) - \Phi(t)\big)\frac{e^{-W}}{Z_0}\,d\mu\end{aligned}$$

where $Z_0 = \int e^{-W}\,d\mu$. Applying the above case when $V = 0$ to the measure $\frac{e^{-W}}{Z_0}\,d\mu$ yields

$$\begin{aligned}\mathbf{Ent}^{\Phi}_{\hat{\mu}}\big(|f|\big) &\leqslant \frac{e^{osc(V)}Z_0}{\hat{Z}}c' \int |\nabla f|\frac{e^{-W}}{Z_0}\,d\mu\\ &\leqslant c'e^{2\,osc(V)}\hat{\mu}|\nabla f|,\end{aligned}$$

for some constant c', so that the result holds. □

In Theorem 2.5 we assume that the Cheeger type inequality (14) holds, together with inequalities (1) and (2). However, we note below that under some conditions it is possible to deduce the Cheeger type inequality directly from a weaker version of the U-bound (2), using the method in [23].

Theorem 2.7. *Let* $d\mu = \frac{e^{-U}}{Z}\,d\lambda$ *be probability measure on* $\mathbb{R}^N$, *and suppose that the following inequality is satisfied*

$$\mu\big(f|U|^{\beta}\big) \leqslant A\mu|\nabla f| + B\mu|f|, \tag{20}$$

for some $\beta > 0$ *and all locally Lipschitz functions* f. *Suppose also that*:

(a) *for any* $L \geqslant 0$ *there exists* $r = r(L) \in (0, \infty)$ *such that*

$$\{|U|^\beta \leqslant L\} \subset B(r) \tag{21}$$

for some ball $B(r)$ *of radius* r;

(b) *for* $r = r(L)$ *there exists* $m_r \in (0, \infty)$ *such that the following Poincaré inequality in the ball* $B(r)$ *is satisfied*

$$\int_{B(r)} \left| f - \frac{1}{\lambda(B(r))} \int_{B(r)} f \, d\lambda \right| d\lambda \leqslant \frac{1}{m_r} \int_{B(r)} |\nabla f| \, d\lambda \tag{22}$$

for all suitable functions f.

Then there exists a constant c_0 *such that*

$$\mu|f - \mu f| \leqslant c_0 \mu |\nabla f|$$

for all locally Lipschitz functions f.

Proof. We have that

$$\mu|f - \mu f| \leqslant 2\mu|f - m|$$

for all $m \in \mathbb{R}$. Now for $L \geqslant 0$ we have

$$\mu|f - m| \leqslant \mu\big(|f - m| \mathbf{1}_{\{|U|^\beta \leqslant L\}}\big) + \mu\big(|f - m| \mathbf{1}_{\{|U|^\beta \geqslant L\}}\big). \tag{23}$$

We have that $\{|U|^\beta \leqslant L\} \subset B(r)$ for some $r = r(L) \in (0, \infty)$, so that putting $m = \frac{1}{\lambda(B(r))} \times \int_{B(r)} f \, d\lambda$, and noting that on the set $\{|U|^\beta \leqslant R\}$ there exists a constant A_r such that

$$\frac{1}{A_r} \leqslant \frac{d\mu}{d\lambda} \leqslant A_r,$$

we can bound the first term using assumption (a). Indeed,

$$\begin{aligned} \mu\big(|f - m| \mathbf{1}_{\{|U|^\beta \leqslant L\}}\big) &\leqslant A_r \int_{B(r)} \left| f - \frac{1}{\lambda(B(r))} \int_{B(r)} f \, d\lambda \right| d\lambda \\ &\leqslant \frac{A_r}{m_r} \int_{B(r)} |\nabla f| \, d\lambda \leqslant \frac{A_r^2}{m_r} \mu |\nabla f| \end{aligned} \tag{24}$$

using (22). On the other hand, using (20), we have

$$\mu\big(|f-m|\mathbf{1}_{\{|U|^\beta\geqslant L\}}\big)\leqslant\frac{1}{L}\mu\big(|f-m||U|^\beta\big)$$
$$\leqslant\frac{A}{L}\mu|\nabla f|+\frac{B}{L}\mu|f-m|. \tag{25}$$

Using estimates (24) and (25) in (23), and taking L large enough ends the proof. □

We can now combine all the results of this section into the following theorem.

Theorem 2.8. *Let U, λ and μ be as in Theorem* 2.1.

Suppose also that conditions (a) *and* (b) *of Theorem* 2.7 *are satisfied. Then there exists $c\in(0,\infty)$ such that* (10) *holds, i.e.*

$$\mathbf{Ent}_\mu^\Phi\big(|f|\big)\leqslant c\mu|\nabla f|$$

for all locally Lipschitz functions f, where $\Phi(x)=x(\log(1+x))^\beta$.

To conclude this section, we finally note that the $L_1\Phi$-entropy inequality (10) can be tensorised in the following sense.

Lemma 2.9 *(Tensorisation). Let I be a finite index set, and ν_i, $i\in I$ be probability measures. Set $\nu_I:=\bigotimes_{i\in I}\nu_i$. Suppose that for each $i\in I$, ν_i satisfies the $L_1\Phi$-entropy inequality* (10) *with a constant $c(i)\in(0,\infty)$. Then so does ν_I with constant* $\max_{i\in I}\{c(i)\}$.

Proof. The proof follows by induction. The key observation is as follows: for $J\subset I$ and $k\notin J$, one has

$$\begin{aligned}\nu_k\otimes\nu_J\Phi(f)-\Phi(\nu_k\otimes\nu_J f)&=\nu_k\big(\nu_J\Phi(f)-\Phi(\nu_J f)\big)\\&\quad+\big(\nu_k\Phi(\nu_J f)-\Phi\big(\nu_k(\nu_J f)\big)\big)\\&\leqslant\nu_k\bigg(\sum_{j\in J}c_J\nu_J|\nabla_j f|\bigg)+c_k\nu_k|\nabla_k\nu_J f|\\&\leqslant\max(c_J,c_k)\sum_{j\in J\cup k}\nu_k\otimes\nu_J|\nabla_j f|.\end{aligned}$$ □

3. Isoperimetric inequalities

In this section our aim is to derive isoperimetric information for the measure μ starting from $L_1\Phi$-entropy inequalities. We assume that μ is non-atomic and that the distance d on $\mathbb{R}^N$ is related to the modulus of the gradient of a function $f:\mathbb{R}^N\to\mathbb{R}$ by

$$|\nabla f|(x)=\limsup_{d(x,y)\downarrow 0}\frac{|f(x)-f(y)|}{d(x,y)}. \tag{26}$$

As usual, we define the surface measure of a Borel set $A\subset\mathbb{R}^N$ by

$$\mu^+(A) = \liminf_{\varepsilon \downarrow 0} \frac{\mu(A^\varepsilon \setminus A)}{\varepsilon}$$

where $A^\varepsilon = \{x \in \mathbb{R}^n\colon\ d(x, A) < \varepsilon\}$ is the (open) ε-neighbourhood of A (with respect to d). We are concerned with a problem of estimating the isoperimetric profile of the measure μ, that is a function $\mathcal{I}_\mu : [0, 1] \to \mathbb{R}^+$ defined by

$$\mathcal{I}_\mu(t) = \inf\bigl\{\mu^+(A)\colon\ A \text{ Borel such that } \mu(A) = t\bigr\}$$

(with $\mathcal{I}_\mu(0) = \mathcal{I}_\mu(1) = 0$). By definition it is the largest function such that the following isoperimetric inequality holds

$$\mathcal{I}_\mu\bigl(\mu(A)\bigr) \leqslant \mu^+(A). \tag{27}$$

For $q > 1$ and p such that $\frac{1}{q} + \frac{1}{p} = 1$, we define functions $\mathcal{U}_q = f_p \circ F_p^{-1}$ where f_p is the density of the measure $d\nu_p(x) = \frac{e^{-|x|^p}}{Z_p}\,dx$ on $\mathbb{R}$ and $F_p' = f_p$ (here, $|x|$ denotes the Euclidean norm of $x \in \mathbb{R}$). This is motivated by the fact that $\mathcal{U}_q$ is the isoperimetric function of ν_p in dimension 1. It is known (see [12]) that $\mathcal{U}_q(t)$ is symmetric and behaves like $G(t) = t(\log(\frac{1}{t}))^{\frac{1}{q}}$ near the origin so that for some constant $L_q > 0$, we have

$$\frac{1}{L_q} G\bigl(\min(t, 1-t)\bigr) \leqslant \mathcal{U}_q(t) \leqslant L_q G\bigl(\min(t, 1-t)\bigr) \tag{28}$$

for all $t \in [0, 1]$.

Theorem 3.1. *Assume that the $L_1\Phi$-entropy inequality*

$$\mathbf{Ent}_\mu^\Phi\bigl(|f|\bigr) \leqslant c\mu|\nabla f|$$

holds for some constant $c \in (0, \infty)$ and all locally Lipschitz functions f, where $\Phi(x) = x(\log(1+x))^\beta$ and $\beta \in (0, 1]$. Then $\mathcal{I}_\mu \geqslant \frac{1}{\tilde{c}}\mathcal{U}_q$ with some constant $\tilde{c} > 0$, $q = \frac{1}{\beta}$ and the measure μ satisfies an isoperimetric inequality of the form

$$\mathcal{U}_q(t) \leqslant \tilde{c}\mu^+(A) \tag{29}$$

for all a Borel sets A of measure $t = \mu(A)$.

Proof. When applied to a non-negative function f such that $\mu f = 1$, the $L_1\Phi$-entropy inequality becomes

$$\mu\bigl(f\bigl(\bigl(\log(1+f)\bigr)^\beta - (\log 2)^\beta\bigr)\bigr) \leqslant c\mu|\nabla f|,$$

which implies that for all non-negative f (not identically 0) we have

$$\mu\left(f\left(\left(\log\left(1 + \frac{f}{\mu f}\right)\right)^\beta - (\log 2)^\beta\right)\right) \leqslant c\mu|\nabla f|. \tag{30}$$

Let A be a Borel set with measure $t = \mu(A)$. To start with, suppose that $t \in [0, \frac{1}{2}]$. We can approximate the indicator function of A by a sequence of Lipschitz functions $(f_n)_{n\in\mathbb{N}}$ satisfying

$$\limsup_{n\to\infty} \mu|\nabla f_n| \leqslant \mu^+(A)$$

(see [10, Lemma 3.5]). Taking f_n in (30) and passing to the limit as $n \to \infty$ yields

$$t\left(\left(\log\left(1+\frac{1}{t}\right)\right)^\beta - (\log 2)^\beta\right) \leqslant c\mu^+(A). \tag{31}$$

We now observe that for $t \in [0, \frac{1}{2}]$ we have

$$\eta\left(\log\left(\frac{1}{t}\right)\right)^\beta \leqslant \left(\log\left(1+\frac{1}{t}\right)\right)^\beta - (\log 2)^\beta \tag{32}$$

with $\eta = (\frac{\log 3}{\log 2})^\beta - 1 > 0$. This implies

$$t\left(\log\left(\frac{1}{t}\right)\right)^\beta \leqslant \frac{c}{\eta}\mu^+(A), \tag{33}$$

for all $t \in [0, \frac{1}{2}]$. Thus, by the equivalence relation (28), we have that

$$\mathcal{U}_q(t) \leqslant \tilde{c}\mu^+(A) \tag{34}$$

for all $t \in [0, \frac{1}{2}]$, with $\tilde{c} = \frac{c}{\eta}L_q$.

Now suppose that $t = \mu(A) \in (\frac{1}{2}, 1]$. For functions $f \in [0, 1]$, we can apply (30) to $1 - f$, which yields

$$\mu\left((1-f)\left(\left(\log\left(1+\frac{1-f}{1-\mu f}\right)\right)^\beta - (\log 2)^\beta\right)\right) \leqslant c\mu|\nabla f|.$$

If we now take f_n in this inequality (where $(f_n)_{n\in\mathbb{N}}$ is again the Lipschitz approximation of the characteristic function of A) and pass to the limit as $n \to \infty$, we see that

$$(1-t)\left(\left(\log\left(1+\frac{1}{1-t}\right)\right)^\beta - (\log 2)^\beta\right) \leqslant c\mu^+(A).$$

Writing $s = 1 - t \in [0, \frac{1}{2})$ and using (32) now gives

$$s\left(\log\left(\frac{1}{s}\right)\right)^\beta \leqslant \frac{c}{\eta}\mu^+(A). \tag{35}$$

Thus by (28) again, we have $\mathcal{U}_q(1-t) = \mathcal{U}_q(s) \leqslant \tilde{c}\mu^+(A)$ for all $t \in (\frac{1}{2}, 1]$ with $\tilde{c} = \frac{c}{\eta}L_q$. By symmetry of $\mathcal{U}_q$ therefore $\mathcal{U}_q(t) \leqslant \tilde{c}\mu^+(A)$ for $t \in (\frac{1}{2}, 1]$, which combined with (34) yields the result. $\square$

An important corollary of this result is the following:

Corollary 3.2. *Assume that the $L_1\Phi$-entropy inequality*

$$\mathbf{Ent}_\mu^\Phi\big(|f|\big) \leqslant c\mu|\nabla f|$$

holds for some constant $c \in (0,\infty)$ and all locally Lipschitz functions f, where $\Phi(x) = x(\log(1+x))^\beta$ and $\beta \in (0,1]$. Then there exists a constant c_0 such that

$$\mu|f - \mu f| \leqslant c_0\mu|\nabla f|. \tag{36}$$

Proof. We note that if $\beta = 1/q$,

$$\mathcal{U}_q(t) \geqslant \frac{1}{L_q}\min(t,1-t)\log\left(\frac{1}{\min(t,1-t)}\right)^{1/q} \geqslant \frac{(\log 2)^{1/q}}{L_q}\min(t,1-t).$$

Thus by Theorem 3.1, we have that

$$\min(t,1-t) \leqslant \tilde{c}\frac{L_q}{(\log 2)^\beta}\mu^+(A),$$

for $t = \mu(A)$, which is Cheeger's isoperimetric inequality on sets. This is equivalent (up to a constant) to its functional form

$$\mu|f - \mu f| \leqslant c_0\mu|\nabla f|$$

(see for example [11]). □

Following an argument of [26] we can pass from the isoperimetric statement above to inequality (4). We note that in our general setting, the following coarea inequality is available, (for a proof see e.g. [10, Lemma 3.2]),

$$\mu|\nabla f| \geqslant \int_{\mathbb{R}} \mu^+\big(\{f > s\}\big)\,ds \tag{37}$$

for locally Lipschitz functions f.

Proposition 3.3. *If the measure μ satisfies an isoperimetric inequality of the form* (29)*, then there exist constants $K, K' > 0$ such that*

$$\mu\big(f(\log_+ f)^\beta\big) \leqslant K\mu|\nabla f| + K' \tag{38}$$

for all positive locally Lipschitz functions f such that $\mu(f) = 1$, where $\beta = \frac{1}{q}$.

Proof. Let f be non-negative, with $\mu(f)=1$. The coarea inequality (37) together with our assumption imply

$$\mu|\nabla f| \geqslant \int_{\mathbb{R}} \mu^+\big(\{f>s\}\big)\,ds \geqslant \frac{1}{\tilde{c}}\int_{\mathbb{R}} \mathcal{U}_q\big(\mu\big(\{f>s\}\big)\big)\,ds$$

Now, by Markov's inequality $\mu(\{f>s\}) \leqslant \mu(\{f>2\}) \leqslant \frac{1}{2}$ when $s \geqslant 2$. Therefore, since $\mathcal{U}_q(t) \geqslant \frac{1}{L_q} t(\log(\frac{1}{t}))^{\frac{1}{q}}$ for $t \leqslant \frac{1}{2}$ by (28),

$$\begin{aligned}
\int_0^\infty \mathcal{U}_q\big(\mu\big(\{f>s\}\big)\big)\,ds &\geqslant \int_2^\infty \mathcal{U}_q\big(\mu\big(\{f>s\}\big)\big)\,ds \\
&\geqslant \frac{1}{L_q}\int_2^\infty \mu\big(\{f>s\}\big)\left(\log\left(\frac{1}{\mu(\{f>s\})}\right)\right)^{\frac{1}{q}} ds \\
&= \frac{1}{L_q}\int_0^\infty \mu\big(\{f>s\}\big)\left(\log\left(\frac{1}{\mu(\{f>s\})}\right)\right)^{\frac{1}{q}} ds \\
&\quad - \frac{1}{L_q}\int_0^2 \mu\big(\{f>s\}\big)\left(\log\left(\frac{1}{\mu(\{f>s\})}\right)\right)^{\frac{1}{q}} ds \\
&\geqslant \frac{1}{L_q}\int_0^\infty \mu\big(\{f>s\}\big)\left(\log\left(\frac{1}{\mu(\{f>s\})}\right)\right)^{\frac{1}{q}} ds - \frac{2}{L_q}M,
\end{aligned}$$

where $M=\sup_{t\in[0,1]} t(\log\frac{1}{t})^\beta$. Next, again by Markov's inequality, $\mu(\{f>s\}) \leqslant \frac{1}{s}$. Therefore, when $s \geqslant 1$ we have

$$\log\frac{1}{\mu(\{f>s\})} \geqslant \log s$$

and we always have $\log\frac{1}{\mu(\{f>s\})} \geqslant 0$. Therefore, $\log\frac{1}{\mu(\{f>s\})} \geqslant \log_+ s$, which implies

$$\mu|\nabla f| \geqslant \frac{1}{\tilde{c}L_q}\int_{\mathbb{R}} (\log_+ s)^\beta \mu\big(\{f>s\}\big)\,ds - \frac{2M}{\tilde{c}L_q} \geqslant \frac{1}{\tilde{c}L_q}\mu\big(f(\log_+ f)^\beta\big) - K$$

with $K=\frac{2M}{\tilde{c}L_q}+1$. To see the last inequality, let $F(s)=\int_0^s (\log_+ t)^\beta\,dt$ and $H(s)=s(\log_+ s)^\beta - s$. Then $F(s) \geqslant 0 \geqslant H(s)$ on $[0,e]$ and when $s \geqslant e$ $F'(s)=(\log s)^\beta$ and $H'(s)=(\log s)^\beta + \beta(\log s)^{\beta-1}-1$. Therefore, since $\log s \geqslant 1$ and $\beta \in (0,1]$, $F' \geqslant H'$ for $s \geqslant e$ from which it follows that $F \geqslant H$ on $[0,\infty)$. Therefore,

$$\int_0^\infty (\log_+ s)^\beta \mu(\{f > s\})\,ds = \int_0^\infty F'(s)\mu(\{f > s\})\,ds$$
$$= \mu(F(f))$$
$$\geqslant \mu(H(f))$$
$$= \mu(f(\log_+ f)^\beta) - \mu(f)$$
$$= \mu(f(\log_+ f)^\beta) - 1. \qquad \square$$

Remark 3.4. With the above results, we have thus shown the equivalence of the $L_1\Phi$-entropy inequality with the isoperimetric inequality (29) and with inequality (38) together with the Cheeger inequality (36); see Theorem 4.5 below.

Remark 3.5. When $\frac{1}{\beta} = q = 2$, the function $\mathcal{U}_2$ represents the Gaussian isoperimetric function. In this case, the isoperimetric inequality (29) is known to be equivalent to the following inequalities introduced by Bobkov in [8] and [9]:

$$\mathcal{U}_2(\mu(f)) \leqslant \mu(\mathcal{U}_2(f) + \tilde{c}|\nabla f|), \tag{39}$$

$$\mathcal{U}_2(\mu(f)) \leqslant \mu\left(\sqrt{\mathcal{U}_2(f)^2 + \tilde{c}^2|\nabla f|^2}\right) \tag{40}$$

for all locally Lipschitz $f : \mathbb{R} \to [0, 1]$. The equivalence of these inequalities in this case follows by a transportation argument which uses the fact that the standard Gaussian measure γ on $\mathbb{R}$ satisfies (39) and (40) with $\tilde{c} = 1$ (see [6, Proposition 5]).

Remark 3.6. Suppose that the measure μ satisfies an $L_1\Phi$-entropy inequality on a metric space $(\mathcal{M}, d)$. Suppose that on the product space $(\mathcal{M}^n, d_n, \mu^{\otimes n})$ we have $|\nabla f| = \sum_{i=1}^n |\nabla_i f|$, where ∇_i denotes differentiation with respect to the ith coordinate and where the moduli of the gradients are defined via (26) with the supremum distance. The tensorisation property of the $L_1\Phi$-entropy (Lemma 2.9) then allows us to obtain isoperimetric information on the product space (where the surface measure is now defined with respect to supremum distance). This problem was considered in [4].

4. Consequences of $L_1\Phi$-entropy inequalities

In this section we look at some consequences of the $L_1\Phi$-entropy inequality

$$\mathbf{Ent}_\mu^\Phi(|f|) \leqslant c\mu|\nabla f|, \tag{41}$$

with $\Phi(x) = x(\log(1 + x))^\beta$, $\beta \in (0, 1]$, for a general probability measure μ. The first result shows that this inequality implies a q-logarithmic Sobolev inequality, as studied in [12] and [23].

Theorem 4.1. *Let μ be an arbitrary probability measure which satisfies the $L_1\Phi$-entropy inequality* (41) *for some $\beta \in [\frac{1}{2}, 1]$ and set $q = \frac{1}{\beta} \in [1, 2]$. Then there exists a constant c_q such that the following (LS_q) inequality holds*

$$\mu\left(|f|^q \log \frac{|f|^q}{\mu|f|^q}\right) \leqslant c_q \mu|\nabla f|^q \tag{42}$$

for all locally Lipschitz functions f.

Proof. Without loss of generality we assume that $f \geqslant 0$. Applying $L_1\Phi$-entropy inequality (41) to the function $f/\mu f$, we obtain the following homogeneous version

$$\mu\left(f\left[\log\left(1+\frac{f}{\mu f}\right)\right]^\beta\right) \leqslant c\mu|\nabla f| + (\log 2)^\beta \mu(f). \tag{43}$$

We apply this inequality to the function $g = f(1+\log(1+f))^{1-\beta} \geqslant f \geqslant 0$, where f is such that $\mu(f) = 1$. Note that $\mu(g) \geqslant 1$. Then we have

$$\begin{aligned}
\mu\left(g\left[\log\left(1+\frac{g}{\mu g}\right)\right]^\beta\right) &= \mu\left(f\big(1+\log(1+f)\big)^{1-\beta}\left[\log\left(1+\frac{g}{\mu g}\right)\right]^\beta\right) \\
&\geqslant \mu\left(f\big(1+\log(1+f)\big)^{1-\beta}\left[\log\left(1+\frac{f}{\mu g}\right)\right]^\beta\right) \\
&\geqslant \mu\left(f\left(1+\log\left(1+\frac{f}{\mu g}\right)\right)^{1-\beta}\left[\log\left(1+\frac{f}{\mu g}\right)\right]^\beta\right) \\
&\geqslant \mu\left(f\log\left(1+\frac{f}{\mu g}\right)\right) = \mu\big(f\log(\mu g + f)\big) - \log\mu(g) \\
&\geqslant \mu\big(f\log(1+f)\big) - \mu(g).
\end{aligned}$$

Thus for all $f \geqslant 0$ with $\mu(f) = 1$,

$$\begin{aligned}
\mu\big(f\log(1+f)\big) &\leqslant c\mu\big|\nabla\big(f\big(1+\log(1+f)\big)^{1-\beta}\big)\big| + \big((\log 2)^\beta + 1\big)\mu(g) \\
&\leqslant c\mu\big(\big(1+\log(1+f)\big)^{1-\beta}|\nabla f|\big) \\
&\quad + c(1-\beta)\mu\left(\frac{f}{(1+\log(1+f))^\beta}\frac{1}{1+f}|\nabla f|\right) \qquad (44) \\
&\quad + \big((\log 2)^\beta + 1\big)\mu(g) \\
&\leqslant c\mu\big(\big(1+\log(1+f)\big)^{1-\beta}|\nabla f|\big) + c(1-\beta)\mu|\nabla f| \\
&\quad + \big((\log 2)^\beta + 1\big)\mu(g). \qquad (45)
\end{aligned}$$

Since we have assumed $\beta \geqslant \frac{1}{2}$, we have $1-\beta \leqslant \beta$ and hence

$$\begin{aligned}
\mu(g) = \mu\big(f\big(1+\log(1+f)\big)^{1-\beta}\big) &\leqslant 1 + \mu\big(f\big[\log(1+f)\big]^{1-\beta}\big) \\
&\leqslant \mu\big(f\big[\log(1+f)\big]^\beta\big) + 2 \\
&\leqslant c\mu|\nabla f| + (\log 2)^\beta + 2
\end{aligned}$$

by another application of the $L_1\Phi$-entropy inequality (43) in the last step. Using this in (44), we see that for general $f \geqslant 0$,

$$\mu\left(f\log\left(1+\frac{f}{\mu f}\right)\right) \leqslant c\mu\left(\left(1+\log\left(1+\frac{f}{\mu f}\right)\right)^{1-\beta}|\nabla f|\right) + c\big(2-\beta+(\log 2)^{\beta}\big)\mu|\nabla f| + \big((\log 2)^{\beta}+2\big)^2\mu(f). \tag{46}$$

Replacing f by f^q with $q = \frac{1}{\beta}$ in the above yields

$$\begin{aligned}
\mu\left(f^q\log\left(1+\frac{f^q}{\mu f^q}\right)\right) &\leqslant qc\mu\left(\left(1+\log\left(1+\frac{f^q}{\mu f^q}\right)\right)^{1-\beta} f^{q-1}|\nabla f|\right) \\
&\quad + cq\big(2-\beta+(\log 2)^{\beta}\big)\mu\big(f^{q-1}|\nabla f|\big) \\
&\quad + \big((\log 2)^{\beta}+2\big)^2\mu\big(f^q\big) \\
&\leqslant \frac{qc\varepsilon^{p-1}}{p}\mu\left(f^q\left(1+\log\left(1+\frac{f^q}{\mu f^q}\right)\right)\right) \\
&\quad + \left(\frac{c}{\varepsilon}+c\big(2-\beta+(\log 2)^{\beta}\big)\right)\mu|\nabla f|^q \\
&\quad + \left(\frac{cq}{p}\big(2-\beta+(\log 2)^{\beta}\big)+\big((\log 2)^{\beta}+2\big)^2\right)\mu\big(f^q\big)
\end{aligned}$$

where $\varepsilon > 0$ and we have applied Young's inequality with indices $\frac{1}{p}+\frac{1}{q}=1$. Choosing $qc\varepsilon^{p-1}/p < 1$, we can simplify this bound as follows

$$\mu\left(f^q\log\left(1+\frac{f^q}{\mu f^q}\right)\right) \leqslant C'\mu|\nabla f|^q + D'\mu\big(f^q\big)$$

where

$$C' = \frac{\frac{c}{\varepsilon}+c(2-\beta+(\log 2)^{\beta})}{1-\frac{qc\varepsilon^{p-1}}{p}}, \qquad D' = \frac{\frac{cq}{p}(2-\beta+(\log 2)^{\beta})+((\log 2)^{\beta}+2)^2}{1-\frac{qc\varepsilon^{p-1}}{p}}.$$

From this one obtains the defective (LS_q), which for all $f \geqslant 0$ such that $\mu(f^q)=1$ can be equivalently represented as

$$\mu\big(f^q\log f^q\big) \leqslant C'\mu|\nabla f|^q + D'. \tag{47}$$

Let us now recall that by Corollary 3.2, our assumption implies that there exists a constant c_0 such that

$$\mu|f-\mu f| \leqslant c_0\mu|\nabla f|.$$

From this inequality we can use the arguments of [12, Chapter 2] to deduce that there exists a constant c_q such that

$$\mu|f-\mu f|^q \leqslant c_q \mu|\nabla f|^q.$$

Finally, by Rothaus-type arguments ([12, Chapter 3], see also Appendix B), we can then remove the defective term in (47) to arrive at the result. $\square$

Theorem 4.1 has a number of corollaries, which follow from known results about the q-logarithmic Sobolev inequality (LS_q) contained in [12] and [23]. We mention here the following one, which is important for our purposes.

Corollary 4.2. *Let μ be an arbitrary probability measure which satisfies the $L_1\Phi$-entropy inequality* (41) *with $\beta \in [\frac{1}{2}, 1]$. Suppose f is a locally Lipschitz function such that*

$$|\nabla f|^q \leqslant af + b \tag{48}$$

with $q = \frac{1}{\beta}$, for some constants $a, b \in [0, \infty)$. Then for all $t > 0$ sufficiently small

$$\mu\big(e^{tf}\big) < \infty.$$

Proof. Follows from Theorem 4.5 of [23]. $\square$

In Section 2 we proved that, under some conditions, if $d\mu = \frac{e^{-U}}{Z}\,d\lambda$ is a probability measure which satisfies a Cheeger type inequality of the form (14), and a U-bound of the form

$$\mu\big(|f|\big[|U|^\beta + |\nabla U|\big]\big) \leqslant A\mu|\nabla f| + B\mu|f|, \tag{49}$$

then the $L_1\Phi$-entropy inequality (41) holds.

We now aim to show the converse i.e. that under some weak conditions, the $L_1\Phi$-entropy inequality (41) implies a bound of the form (49). We first prove the following useful lemma.

Lemma 4.3. *Let μ be a probability measure. Then*

$$\mu(fh) \leqslant s^{-1}\mathbf{Ent}_\mu^\Phi(f) + s^{-1}\Theta(sh) \tag{50}$$

for all $s > 0$ and suitable functions $f, h \geqslant 0$ such that $\mu(f) = 1$, where $\Phi(x) = x(\log(1+x))^\beta$, $\beta \equiv \frac{1}{q} \in (0, 1]$ and

$$\Theta(h) \equiv \big(\theta + (\log 2)^\beta + \big(\log \mu e^{h^q}\big)^\beta\big)$$

with $\theta = \sup_{x \geqslant 0} \beta x(\log(1+x))^{\beta-1}/(1+x)$.

Moreover, suppose that μ satisfies the $L_1\Phi$-entropy inequality (41) *for some $\beta \in [\frac{1}{2}, 1]$ with constant c, and that $g \geqslant 0$ is a locally Lipschitz function such that*

$$|\nabla g|^q \leqslant ag + b \tag{51}$$

for some constants $a, b \in (0, \infty)$. *Then* $\Theta(s^\beta g^\beta) < \infty$ *for sufficiently small* $s > 0$, *and*

$$\mu\big(fg^\beta\big) \leqslant \frac{c}{s^\beta}\mu|\nabla f| + \frac{c}{s^\beta}\Theta\big(s^\beta g^\beta\big)\mu(f), \tag{52}$$

for all locally Lipschitz functions $f \geqslant 0$.

Proof. We remark first that for functions $f, h \geqslant 0$, $\mu f = 1$, with $s \in (0, \infty)$ and $\beta \equiv \frac{1}{q} \in (0, 1)$, we have

$$\begin{aligned}
\mu(fh) &= s^{-1}\mu\big(f\big(\log e^{s^q h^q}\big)\big)^\beta \\
&\leqslant s^{-1}\mu\left[f\left(\log\left(1 + \frac{e^{s^q h^q}}{\mu e^{s^q h^q}}\right)\right)^\beta \chi\big(e^{s^q h^q} \geqslant \mu e^{s^q h^q}\big)\right] \\
&\quad + s^{-1}\big(\log \mu e^{s^q h^q}\big)^\beta \mu(f).
\end{aligned}$$

By the generalised relative entropy inequality of [20], we have

$$\begin{aligned}
\mu\left[f\left(\log\left(1 + \frac{e^{s^q h^q}}{\mu e^{s^q h^q}}\right)\right)^\beta\right] &\leqslant \mu f\left(\log\left(1 + \frac{f}{\mu f}\right)\right)^\beta + \theta\mu f \\
&\leqslant \mathbf{Ent}_\mu^\Phi(f) + \big(\theta + (\log 2)^\beta\big)\mu f,
\end{aligned}$$

since $\mu f = 1$. We therefore get the following bound

$$\mu(fh) \leqslant s^{-1}\mathbf{Ent}_\mu^\Phi(f) + s^{-1}\big(\theta + (\log 2)^\beta + \big(\log \mu e^{s^q h^q}\big)^\beta\big). \tag{53}$$

This ends the proof of the first part of the lemma.

Replacing h by $g^\beta \equiv g^{\frac{1}{q}}$ and s by s^β in (53), we see that the second part is a consequence of Corollary 4.2. □

Theorem 4.4. *Let* $d\mu = \frac{e^{-U}}{Z}\,d\lambda$ *be a probability measure on* $\mathbb{R}^N$, *with* U *a locally Lipschitz function bounded from below. Suppose* μ *satisfies the* $L_1\Phi$*-entropy inequality* (41) *for some* $\beta \in [\frac{1}{2}, 1]$.

Suppose also that

$$|\nabla U| \leqslant a|U|^\beta + b \tag{54}$$

for some constants $a, b \in (0, \infty)$. *Then there exist constants* $A, B \in [0, \infty)$ *such that*

$$\mu\big(|f|\big(|U|^\beta + |\nabla U|\big)\big) \leqslant A\mu|\nabla f| + B\mu|f|, \tag{55}$$

for all locally Lipschitz f.

Proof. Let $f \geqslant 0$. We may also suppose that $U \geqslant 0$ (otherwise we can shift it by a constant). Note that from (54), it follows that

$$|\nabla U|^q \leqslant \tilde{a} U + \tilde{b}$$

with $q = \frac{1}{\beta}$. Hence we may apply Lemma 4.3, to see that

$$\mu\big(f U^\beta\big) \leqslant \frac{c}{s^\beta} \mu|\nabla f| + \frac{c}{s^\beta} \Theta\big(s^\beta U^\beta\big) \mu(f)$$

with $\Theta(s^\beta U^\beta) < \infty$ for sufficiently small s. □

The following theorem summarises the results of the paper so far.

Theorem 4.5. *Let μ be a non-atomic probability measure on $(\mathbb{R}^N, d)$, $|\nabla f|$ be given by* (26) *and $q \geqslant 1$. Then the following statements are equivalent*:

(i)

$$\mathbf{Ent}_\mu^\Phi\big(|f|\big) \leqslant c\mu|\nabla f|,$$

where $\Phi(x) = x(\log(1+x))^{\frac{1}{q}}$, for some constant $c \in (0,\infty)$ and all locally Lipschitz f;

(ii)

$$\mu\left(f\left(\log_+ \frac{f}{\mu f}\right)^{1/q}\right) \leqslant K\mu|\nabla f| + K'\mu f,$$

for some $K > 0$ and

$$\mu|f - \mu f| \leqslant c_0 \mu|\nabla f|$$

with some $c_0 \in (0,\infty)$ and all locally Lipschitz $f \geqslant 0$;

(iii)

$$\mathcal{U}_q(t) \leqslant \tilde{c}\mu^+(A),$$

for some $\tilde{c} > 0$ and all Borel sets A of measure $t = \mu(A)$.

Moreover, for $q \in (1,2]$ statements (i)–(iii) *imply*

(iv)

$$\mu\left(|f|^q \log \frac{|f|^q}{\mu|f|^q}\right) \leqslant C'\mu|\nabla f|^q \qquad (LS_q)$$

for some $C' \in (0,\infty)$ and all locally Lipschitz functions f, *and*

(v)

$$\mathcal{U}_2(\mu f) \leqslant \mu\sqrt{\mathcal{U}_2^2(f) + C''|\nabla f|^2} \qquad (IFI_2)$$

for some $C'' \in (0, \infty)$ *and all locally Lipschitz functions* $0 \leqslant f \leqslant 1$.

Finally, suppose that the probability measure μ *is given by* $\mu(dx) = \frac{e^{-U}}{Z}\, d\lambda$ *for some locally Lipschitz function* U *on* $\mathbb{R}^N$ *which is bounded from below. Suppose that the measure* $d\lambda$ *satisfies the classical Sobolev inequality* (1) *together with the Poincaré inequality in balls* (22)*, and that* $\forall L \geqslant 0$ *there exists* $r = r(L)$ *such that* $\{U \leqslant L\} \subset B(r)$. *In this situation the following* U*-bound*

$$\mu\big(|f|\big(|U|^{\beta} + |\nabla U|\big)\big) \leqslant A\mu|\nabla f| + B\mu|f| \tag{56}$$

for all locally Lipschitz functions and constants $A, B \in [0, \infty)$, $\beta \in (0, 1]$, *implies that statements* (i)–(iii) *hold with* $q = \frac{1}{\beta}$. *If in addition we have that* (54) *holds i.e. there exist constants* a, b *such that*

$$|\nabla U| \leqslant aU^{\beta} + b$$

then (56) *is actually equivalent to the statements* (i)–(iii).

Proof. (ii) ⇒ (i) was shown in Section 2. (i) ⇒ (iii) is proved in Theorem 3.1. Finally, Proposition 3.3 together with Corollary 3.2 show that (iii) ⇒ (ii). The rest of the theorem, except (v), is a restatement of the results of Section 2 and the current one.

To see (v) we notice that using (28) for small $t > 0$ (as well as small $1 - t > 0$) we have

$$\mathcal{U}_2(t) \leqslant \bar{C}_0 \mathcal{U}_q(t)$$

with some $\bar{C}_0 \in (0, \infty)$, and thus there is a constant $\bar{C} \in (0, \infty)$ such that for all $t \in (0, 1)$

$$\mathcal{U}_2(t) \leqslant \bar{C}\mathcal{U}_q(t).$$

Hence, by (iii), we have the following isoperimetric relation

$$\mathcal{U}_2(t) \leqslant \tilde{C}\mu^{+}(A)$$

for any set A with $\mu(A) = t$. This isoperimetric inequality was shown in [6, Proposition 5] to be equivalent to (IFI_2) in the setting of Riemannian manifolds. The proof remains valid in our setting once we note that the co-area inequality (37) is available. □

Remark 4.6. We remark that generally perturbation of (IFI_2) is a difficult matter if the unbounded log of the density is involved. Our route via U-bounds allows us to achieve that very effectively.

Secondly, as conjectured in [12] for $q \in (1, 2]$ it would be natural to expect the following functional isoperimetric inequality with optimal isoperimetric function

$$\mathcal{U}_q(\mu f) \leqslant \mu \sqrt[q]{\mathcal{U}_q^q(f) + C_q |\nabla f|_q^q} \tag{IFI_q}$$

with some $C_q \in (0, \infty)$ for all locally Lipschitz functions $0 \leqslant f \leqslant 1$. One of the motivations for such a relation is that (as shown in [12]) it implies (LS_q). Using (IFI_2) and the relation of l_q norms, in finite dimension one can see that

$$\mathcal{U}_2(\mu f) \leqslant \mu \sqrt[q]{\mathcal{U}_2^q(f) + C_2' |\nabla f|_q^q}.$$

In the right-hand side, using the asymptotic relation between isoperimetric functions, one could also replace $\mathcal{U}_2$ with $\mathcal{U}_q$. The question remains if adjusting the left-hand side in a similar way would still preserve the inequality in the desired sharp form.

5. Application of results

In order to see where these results can be applied, suppose we are still working in the general situation described at the start of this paper, and define a probability measure

$$d\mu_p := \frac{e^{-\alpha d^p}}{Z} d\lambda \tag{57}$$

on $\mathbb{R}^N$, with $\alpha > 0$, $p \in (1, \infty)$ and normalisation constant Z. Recall that here $d : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ is a metric on $\mathbb{R}^N$. We have the following result which can be found in [23].

Proposition 5.1. *Let μ_p be given by* (57)*. Suppose that we have*

(i) $\frac{1}{\sigma} \leqslant |\nabla d| \leqslant 1$ *almost everywhere for some* $\sigma \in [1, \infty)$;
(ii) $\Delta d \leqslant K + \alpha p \varepsilon d^{p-1}$ *on* $\{x\colon d(x) \geqslant 1\}$, *for some* $K \in [0, \infty)$, $\varepsilon \in [0, \frac{1}{\sigma^2})$.

Then there exist constants $A, B \in [0, \infty)$ *such that*

$$\mu_p\big(|f| d^{p-1}\big) \leqslant A \mu_p |\nabla f| + B \mu_p |f|$$

for all locally Lipschitz functions f.

This proposition gives conditions under which the bound (56) in Theorem 4.5 holds for a particular choice of U and β. Indeed, we thus have the following corollary:

Corollary 5.2. *Let μ_p be given by* (57)*. Suppose that conditions* (i) *and* (ii) *of Proposition* 5.1 *are satisfied. Suppose also that the measure $d\lambda$ satisfies the classical Sobolev inequality* (1) *together with the Poincaré inequality in balls* (22)*. Then inequalities* (i)–(iii) *of Theorem* 4.5 *are satisfied, with q such that* $\frac{1}{p} + \frac{1}{q} = 1$.

Moreover, if $p \geqslant 2$ (iv)–(v) *are also true.*

Proof. For $U = \alpha d^p$ and $\beta = \frac{1}{q}$ we have

$$\mu_p\big(|f|\big(U^\beta + |\nabla U|\big)\big) \leqslant \mu_p\big(|f|\big(\alpha^\beta d^{\beta p} + \alpha p d^{p-1}\big)\big)$$
$$\leqslant \big(\alpha^\beta + \alpha p\big)\mu_p\big(|f| d^{p-1}\big).$$

Therefore by Proposition 5.1, we have

$$\mu\big(|f|\big(U^\beta + |\nabla U|\big)\big) \leqslant \tilde{A}\mu|\nabla f| + \tilde{B}\mu|f|$$

where $\tilde{A} = (\alpha^\beta + \alpha p)A$ and $\tilde{B} = (\alpha^\beta + \alpha p)B$. Thus we can apply Theorem 4.5. □

We can perturb the measure in this result and all the inequalities will hold for the perturbed measure, as follows.

Corollary 5.3. *Let $d\hat{\mu} = e^{-W-V}/\hat{Z}\, d\mu_p$ be the probability measure described in Corollary* 2.6 *with unbounded locally Lipschitz W and bounded measurable V. Then $\hat{\mu}$ enjoys all properties as μ_p in Corollary* 5.2.

Remark 5.4. The conditions of Corollary 5.2 are easily seen to be satisfied in the Euclidean case, when we are dealing with the standard gradient and Laplacian in $\mathbb{R}^N$, and $d(x) = |x|$. In this situation, with $p = 2$, the inequalities we prove are already known (see [26]), though the proof we give here is new.

The value of our results is that they can be used in more general situations than the Euclidean one. In particular it can be applied in the following setting.

Example 5.5. [H-type groups] Let $\mathfrak{g}$ be a (finite dimensional real) Lie algebra and let $\mathfrak{z}$ denote its centre (i.e. $[\mathfrak{g}, \mathfrak{z}] = 0$). We say that $\mathfrak{g}$ is of H-type if it admits a vector space decomposition

$$\mathfrak{g} = \mathfrak{v} \oplus \mathfrak{z}$$

where $[\mathfrak{v}, \mathfrak{v}] \subseteq \mathfrak{z}$, such that there exists an inner product $\langle\cdot,\cdot\rangle$ on $\mathfrak{g}$ such that $\mathfrak{z}$ is an orthogonal complement to $\mathfrak{v}$, and the map $J_Z : \mathfrak{v} \mapsto \mathfrak{v}$ given by

$$\langle J_Z X, Y\rangle = \big\langle [X, Y], Z\big\rangle$$

for $X, Y \in \mathfrak{v}$ and $Z \in \mathfrak{z}$ satisfies $J_Z^2 = -|Z|^2 I$ for each $Z \in \mathfrak{z}$. An H-type group is a simply connected Lie group $\mathbb{G}$ whose Lie algebra is of H-type.

Such a group is a Carnot group of step 2 (see [13] for details). In particular the Heisenberg group is an H-type group with a one-dimensional centre. However, there also exist H-type groups with centre of any dimension.

On an H-type group $\mathbb{G}$ we consider vector fields $X_1, \ldots, X_m$ which form an orthonormal basis of $\mathfrak{v}$. The sub-Laplacian (or Kohn operator) is given by $\Delta_{\mathbb{G}} := \sum_{i=1}^m X_i^2$ and sub-gradient by $\nabla_{\mathbb{G}} := (X_1, \ldots, X_m)$. The associated Carnot–Carathéodory distance is defined by

$$d(x, y) := \sup\big\{f(x) - f(y):\ f \text{ such that } |\nabla_{\mathbb{G}} f| \leqslant 1\big\}.$$

It is shown in [23] that conditions (i) and (ii) of Proposition 5.1 are satisfied in this setting. Moreover, the Lebesgue measure $d\lambda$ satisfies the classical Sobolev inequality (1) and Poincaré inequality in balls (22) with the sub-gradient $\nabla_{\mathbb{G}}$ (see [32]). Thus, by Corollary 5.2 we arrive at the following:

Theorem 5.6. *Let $\mathbb{G}$ be an H-type group, equipped with Carnot–Carathéodory distance d and canonical sub-gradient $\nabla_{\mathbb{G}}$ as described above. Let*

$$d\mu_p := \frac{e^{-\alpha d^p}}{Z}\, d\lambda$$

with $p > 1$ and $\alpha > 0$ be a probability measure on $\mathbb{G}$ and

$$d\hat{\mu} = e^{-W-V}/\hat{Z}\, d\mu_p$$

with $W \equiv W(d)$ satisfying conditions as in Corollary 5.2 with horizontal gradient and V a bounded measurable function. Then inequalities (i)–(iii) of Theorem 4.5 are satisfied with q such that $\frac{1}{p} + \frac{1}{q} = 1$. Moreover for $p \geqslant 2$, the measure $\hat{\mu}$ satisfies (LS_q) and (IFI_2).

See [23, Theorem 2.2] for details of the perturbation technique necessary to achieve the relevant U-bounds.

5.1. U-bounds versus gradient bounds for heat kernel

As a conclusion to this section we mention that our setup is naturally inclusive for the following gradient bounds for the heat kernel on H-type groups which has recently attracted considerable attention (see e.g. [2,16–18,27] and references therein).

Indeed, in the following let $\mathbb{G}$ be an H-type group.

Corollary 5.7. *The semigroup $P_t \equiv e^{t\Delta_{\mathbb{G}}}$ satisfies the following*

$$|\nabla_{\mathbb{G}} P_t f| \leqslant C_1(t) P_t |\nabla_{\mathbb{G}} f|$$

for all suitable functions, with $C_1(t) \in (0, \infty)$ independent of f.

Proof. Due to the group covariance, it is sufficient to show the bound at the identity element and thanks to the action of the dilations, one only needs to establish it at $t = 1$. Denoting the corresponding heat kernel by h, we see that

$$\left| \int f \nabla_{\mathbb{G}} h\, d\lambda \right| = \left| \int \big(f - \langle f \rangle\big) \nabla_{\mathbb{G}} h d\lambda \right| \leqslant \int \big|f - \langle f \rangle\big| |\nabla_{\mathbb{G}} \log h| h\, d\lambda \tag{58}$$

with $\langle f \rangle \equiv \int f h\, d\lambda$. To bound the right-hand side of this expression, suppose that we have a function V growing to infinity such that

$$|\nabla_{\mathbb{G}} \log h| \leqslant k_1 V(d) + k_2 \tag{59}$$

for some $k_1, k_2 > 0$, and for which the following U-bound is satisfied

$$\int \left|f - \langle f\rangle\right| V(d) h\, d\lambda \leqslant C \int |\nabla_{\mathbb{G}} f| h\, d\lambda + D \int \left|f - \langle f\rangle\right| h\, d\lambda \tag{60}$$

with some $C, D \in [0, \infty)$ independent of f. Combining this with (58), we would then be able to conclude that

$$\left|\int f \nabla_{\mathbb{G}} h\, d\lambda\right| \leqslant Ck_1 \int |\nabla_{\mathbb{G}} f| h\, d\lambda + (Dk_1 + k_2) \int \left|f - \langle f\rangle\right| h\, d\lambda.$$

Moreover, under the assumptions of Theorem 2.7, (60) implies that

$$\int \left|f - \langle f\rangle\right| h\, d\lambda \leqslant \alpha \int |\nabla_{\mathbb{G}} f| h\, d\lambda,$$

for some $\alpha > 0$, allowing us to conclude that

$$\left|\int f \nabla_{\mathbb{G}} h\, d\lambda\right| \leqslant \big(Ck_1 + (Dk_1 + k_2)\alpha\big) \int |\nabla_{\mathbb{G}} f| h\, d\lambda.$$

Therefore, to complete the proof we must check that, in the setting of H-type groups, there exists a V such that (59) and (60) hold, and that the assumptions of Theorem 2.7 are satisfied, namely that for $r > 0$, $\{V < r\}$ is contained in some ball and inequality (22) holds.

In an H-type group with $\dim \mathfrak{z} = m$ and $\dim \mathfrak{z}^{\perp} = 2n$ we have the following heat kernel bounds of [17] (see also [27] and [7]): there exists $R > 0$ such that for all points $(x, z) = (x_1, \ldots, x_{2n}, z_1, \ldots, z_m)$ with $d(0, (x, z)) > R$,

$$h(x, z) \asymp \frac{d(0, (x, z))^{2n-m-1}}{1 + (|x| d(0, (x, z)))^{n-1/2}} e^{-\frac{1}{4} d(0,(x,z))^2}, \tag{61}$$

$$\left|\nabla_{\mathbb{G}} \log h(x, z)\right| \leqslant K\big(1 + d\big(0, (x, z)\big)\big). \tag{62}$$

Here $f \asymp g$ means that $c_1 f \leqslant g \leqslant c_2 f$ for some constants $c_1, c_2 > 0$. In this case we may write $f = e^{\psi} g$ with a function ψ of bounded oscillation. We can therefore take $V(d) = d$, so that (59) holds, with $k_1 = K$ and $k_2 = \sup\{|\nabla_{\mathbb{G}} \log h|(x, z)\colon d(0, (x, z)) \leqslant R\} + K$, as do the assumptions of Theorem 2.7 (see for example [25,31,32]).

Finally, to show that (60) holds we use similar perturbative arguments as in Theorem 7.1 of [23] (where the corresponding U-bound with $V(d) = d$ was established for the first Heisenberg group, where $n = m = 1$).

More precisely, letting $W = \log(\frac{d(0,(x,z))^{2n-m-1}}{[1+\varepsilon|x| d(0,(x,z))]^{n-1/2}})$ with some $\varepsilon \in (0, 1)$ to be determined later, we show that we may write

$$h(x, z) = e^{-\psi - W} e^{-\frac{1}{4} d^2}$$

where $osc(\psi) < \infty$. This follows from the fact that

$$C_1\big(1+\big[|x|d\big(0,(x,z)\big)\big]^{n-1/2}\big) \leqslant \big[1+|x|d\big(0,(x,z)\big)\big]^{n-1/2} \leqslant C_2\big(1+\big[|x|d\big(0,(x,z)\big)\big]^{n-1/2}\big),$$

for some constants $C_1, C_2 > 0$, together with the estimate

$$\varepsilon^{n-1/2}\big[1+|x|d\big(0,(x,z)\big)\big]^{n-1/2} \leqslant \big[1+\varepsilon|x|d\big(0,(x,z)\big)\big]^{n-\frac{1}{2}} \leqslant \big[1+|x|d\big(0,(x,z)\big)\big]^{n-1/2}$$

and (61).

Moreover, using the triangle inequality we compute

$$\begin{aligned}|\nabla_{\mathbb{G}} W| &\leqslant (2n-m-1)\frac{|\nabla_{\mathbb{G}} d(0,(x,z))|}{d(0,(x,z))} \\ &\quad + \varepsilon\left(n-\frac{1}{2}\right)\frac{|\nabla_{\mathbb{G}}|x||d(0,(x,z)) + |x||\nabla_{\mathbb{G}} d(0,(x,z))|}{1+\varepsilon|x|d(0,(x,z))} \\ &\leqslant \frac{2n-m-1}{R} + \varepsilon(2n-1)d\big(0,(x,z)\big)\end{aligned}$$

where we have used that $|\nabla_{\mathbb{G}}|x|| = |\nabla_{\mathbb{G}} d| = 1$ and $|x| \leqslant d(0,(x,z))$.

Choosing ε small enough, Theorem 2.2 of [23] gives the following U-bound for positive functions f

$$\int f(x)d(x)h(x)\,dx \leqslant C\int |\nabla_{\mathbb{G}} f|(x)h(x)\,dx + D\int f(x)h(x)\,dx.$$

Applying this to $|f-\langle f\rangle|$ we arrive at (60) with $V(d)=d$. □

While the gradient bounds still remain a challenge for more complicated groups, it may be useful to keep this observation in mind, as in principle it allows for a heat kernel bound (61) with far less precise description of the slowly varying factor (provided the corresponding control distance d satisfies a sufficiently good Laplacian bound outside some compact set).

5.2. U-bounds versus integrated Gaussian bounds for heat kernel

Assuming a bound of the following form

$$\mu(fd) \leqslant C\mu|\nabla f| + D\mu(f), \tag{63}$$

for a function $f = e^{\lambda \min(d,L)^2}$ with L a positive number, we get

$$\begin{aligned}\mu\big(e^{\lambda\min(d,L)^2}\min(d,L)\big) &\leqslant 2\lambda C\mu\big(e^{\lambda\min(d,L)^2}\min(d,L)\big|\nabla\min(d,L)\big|\big) \\ &\quad + D\mu\big(e^{\lambda\min(d,L)^2}\big) \\ &\leqslant 2\lambda C\mu\big(e^{\lambda\min(d,L)^2}\min(d,L)\big) \\ &\quad + D\mu\big(e^{\lambda\min(d,L)^2}\big).\end{aligned}$$

If $2\lambda C < 1$, this implies

$$\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)\big) \leqslant D'\mu\big(e^{\lambda \min(d,L)^2}\big) \tag{64}$$

with $D' \equiv D(1-2\lambda C)^{-1}$. Next, choosing $f = e^{\lambda \min(d,L)^2} \min(d,L)$ instead in (63), we obtain

$$\begin{aligned}
\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)^2\big) &\leqslant C\mu\big|\nabla\big(e^{\lambda \min(d,L)^2} \min(d,L)\big)\big| + D\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)\big) \\
&\leqslant 2\lambda C\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)^2\big) + D\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)\big) \\
&\quad + C\mu\big(e^{\lambda \min(d,L)^2}\big).
\end{aligned}$$

Thus using (64), we obtain

$$\begin{aligned}
\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)^2\big) &\leqslant 2\lambda C\mu\big(e^{\lambda \min(d,L)^2} \min(d,L)^2\big) \\
&\quad + \big(D'+C\big)\mu\big(e^{\lambda \min(d,L)^2}\big).
\end{aligned}$$

Rearranging this, for $2\lambda C \leqslant 2\lambda_0 C < 1$,

$$\frac{d}{d\lambda}\mu\big(e^{\lambda \min(d,L)^2}\big) = \mu\big(e^{\lambda \min(d,L)^2} \min(d,L)^2\big) \leqslant D''\mu\big(e^{\lambda \min(d,L)^2}\big)$$

with $D'' \equiv (D'+C)(1-2\lambda_0 C)^{-1}$. Solving this differential inequality and passing with $L \to \infty$, we arrive at the following:

Theorem 5.8 *(Integrated Gaussian bound). Suppose the following is true*

$$\mu(fd) \leqslant C\mu|\nabla f| + D\mu(f)$$

for all locally Lipschitz functions f, with some constants $C, D \in (0,\infty)$. Then

$$\mu\big(e^{\lambda d^2}\big) \leqslant e^{\lambda D''}$$

for $2\lambda C \leqslant 2\lambda_0 C < 1$ with some constant $D'' \in (0,\infty)$.

See Appendix A for a generalisation of this idea.

Remark 5.9. We point out that using this result and idea of [21] one can obtain Gaussian-type upper bounds on the heat kernel.

Remark 5.10. From the point of view of the computations of [23] we start with

$$h\nabla f = \nabla(fh) - f\nabla h$$

and, with a unitary linear functional $\boldsymbol{\alpha}$, we get

$$\int \boldsymbol{\alpha}(\nabla f)h\,d\lambda = \int \boldsymbol{\alpha}\big(\nabla(fh)\big)\,d\lambda + \int f\boldsymbol{\alpha}\left(\nabla \log\frac{1}{h}\right)h\,d\lambda.$$

Hence, one gets

$$\int f\left(\boldsymbol{\alpha}\left(\nabla \log\frac{1}{h}\right) - div\,\boldsymbol{\alpha}\right)h\,d\lambda \leqslant \int |\nabla f|\cdot|\boldsymbol{\alpha}|h\,d\lambda.$$

If the expression in the bracket on the left-hand side can be shown to have a treatable bound from below, such a bound can be a useful source of analysis (though the implementation of this idea in case of other than H-type groups remains open).

6. Extension to infinite dimensions

In this section we aim to extend the $L_1\Phi$-entropy inequality to the infinite dimensional setting, where we include some bounded interactions. The setup will be as follows.

The spin space: Let $\mathcal{M} = (\mathbb{R}^N, d)$ be a metric space equipped with Lebesgue measure $d\lambda$, general sub-gradient $\nabla = (X_1, \ldots, X_m)$ consisting of divergence free (possibly non-commuting) vector fields and sub-Laplacian $\Delta := \sum_{i=1}^m X_i^2$, as above.

The lattice: Let $\mathbb{Z}^D$ be the D-dimensional lattice for some fixed $D \in \mathbb{N}$, equipped with the lattice metric $dist(\cdot,\cdot)$ defined by

$$dist(\mathbf{i},\mathbf{j}) := \sum_{l=1}^{D} |i_l - j_l|$$

for $\mathbf{i} = (i_1, \ldots, i_D), \mathbf{j} = (j_1, \ldots, j_D) \in \mathbb{Z}^D$. For $\mathbf{i},\mathbf{j} \in \mathbb{Z}^D$ we will also write

$$\mathbf{i} \sim \mathbf{j} \quad \Leftrightarrow \quad dist(\mathbf{i},\mathbf{j}) = 1$$

i.e. $\mathbf{i} \sim \mathbf{j}$ when $\mathbf{i}$ and $\mathbf{j}$ are nearest neighbours in the lattice. For $\Lambda \subset \mathbb{Z}^D$, we will write $\Lambda^c \equiv \mathbb{Z}^D \setminus \Lambda$, $|\Lambda|$ for the cardinality of Λ, and $\Lambda \subset\subset \mathbb{Z}^D$ when $|\Lambda| < \infty$.

The configuration space: Let $\Omega := (\mathcal{M})^{\mathbb{Z}^D}$ be the *configuration space*. Given $\Lambda \subset \mathbb{Z}^D$ and $\omega = (\omega_\mathbf{i})_{\mathbf{i}\in\mathbb{Z}^D} \in \Omega$, let $\omega_\Lambda := (\omega_\mathbf{i})_{\mathbf{i}\in\Lambda} \in (\mathcal{M})^\Lambda$ (so that $\omega \mapsto \omega_\Lambda$ is the natural projection of Ω onto $\mathcal{M}^\Lambda$).

Given $\omega \in \Omega$ we introduce the injection: $\mathcal{M}^\Lambda \to \Omega$, defined by $\eta \in \mathcal{M}^\Lambda \mapsto \eta \bullet_\Lambda \omega$ where $(\eta \bullet_\Lambda \omega)_\mathbf{i} = \eta_\mathbf{i}$ when $\mathbf{i} \in \Lambda$ and $(\eta \bullet_\Lambda \omega)_\mathbf{i} = \omega_\mathbf{i}$ when $\mathbf{i} \in \Lambda^c$.

Let $f : \Omega \to \mathbb{R}$. Then for $\mathbf{i} \in \mathbb{Z}^D$ and $\omega \in \Omega$ define $f_\mathbf{i}(\cdot|\omega) : \mathcal{M} \to \mathbb{R}$ by

$$f_\mathbf{i}(x|\omega) := f(x \bullet_{\{\mathbf{i}\}} \omega).$$

Let $C^{(n)}(\Omega)$, $n \in \mathbb{N}$ denote the set of all functions f for which we have $f_\mathbf{i}(\cdot|\omega) \in C^{(n)}(\mathcal{M})$ for all $\mathbf{i} \in \mathbb{Z}^D$. For $\mathbf{i} \in \mathbb{Z}^D$, $k \in \{1, \ldots, m\}$ and $f \in C^{(1)}(\Omega)$, define

$$X_{\mathbf{i},k} f(\omega) := X_k f_\mathbf{i}(x|\omega)|_{x=\omega_\mathbf{i}},$$

where $X_1, \ldots, X_m$ are the vector fields on $\mathcal{M}$.

Define similarly $\nabla_{\mathbf{i}} f(\omega) := \nabla f_{\mathbf{i}}(x|\omega)|_{x=\omega_{\mathbf{i}}}$ and $\Delta_{\mathbf{i}} f(\omega) := \Delta f_{\mathbf{i}}(x|\omega)|_{x=\omega_{\mathbf{i}}}$ for suitable f, where ∇ and Δ are the sub-gradient and the sub-Laplacian on $\mathcal{M}$ respectively. For $\Lambda \subset \mathbb{Z}^D$, set $\nabla_\Lambda f = (\nabla_{\mathbf{i}} f)_{\mathbf{i}\in\Lambda}$ and

$$|\nabla_\Lambda f| := \sum_{\mathbf{i}\in\Lambda} |\nabla_{\mathbf{i}} f|.$$

Finally, a function f on Ω is said to be *localised* in a set $\Lambda \subset \mathbb{Z}^D$ if f is only a function of those coordinates in Λ.

Local specification and Gibbs measure: Let $\Psi = (\psi_X)_{X\subset\subset\mathbb{Z}^D}$ be a family of C^2 functions such that ψ_X is localised in $X \subset\subset \mathbb{Z}^D$. Assume that $\psi_X \equiv 0$ whenever the diameter of X is greater than positive constant R. We will also assume that there exists a constant $M \in (0,\infty)$ such that $\|\psi_X\|_\infty \leqslant M$ and $\|\nabla_{\mathbf{i}}\psi_X\|_\infty \leqslant M$ for all $\mathbf{i} \in \mathbb{Z}^D$. We say Ψ is a bounded potential of range R. For $\omega \in \Omega$, define

$$H_\Lambda^\omega(x_\Lambda) = \sum_{\Lambda\cap X\neq\emptyset} \psi_X(x_\Lambda \bullet_\Lambda \omega),$$

for $x_\Lambda = (x_{\mathbf{i}})_{\mathbf{i}\in\Lambda} \in \mathcal{M}^\Lambda$.

Let U be a locally Lipschitz function on $\mathcal{M}$ which is bounded from below and such that $\int_{\mathcal{M}} e^{-U}\,d\lambda < \infty$. Suppose also that $\forall L \geqslant 0$ there exists $r = r(L)$ such that

$$\{U \leqslant L\} \subset B(r).$$

Let $d\mu = \frac{e^{-U}}{Z}\,d\lambda$, so that μ is a probability measure on $\mathcal{M}$, and let

$$\mu_\Lambda(dx_\Lambda) := \bigotimes_{\mathbf{i}\in\Lambda} \mu(dx_{\mathbf{i}})$$

be the product measure on $\mathcal{M}^\Lambda$. Now define

$$\mathbb{E}_\Lambda^\omega(dx_\Lambda) = \frac{e^{JH_\Lambda^\omega(x_\Lambda)}}{\int e^{JH_\Lambda^\omega(x_\Lambda)}\mu_\Lambda(dx_\Lambda)}\mu_\Lambda(dx_\Lambda) \equiv \frac{e^{JH_\Lambda^\omega(x_\Lambda)}}{Z_\Lambda^\omega}\mu_\Lambda(dx_\Lambda) \tag{65}$$

for $J \in \mathbb{R}$. We will write $\mu_{\{\mathbf{i}\}} = \mu_{\mathbf{i}}$ and $\mathbb{E}_{\{\mathbf{i}\}}^\omega = \mathbb{E}_{\mathbf{i}}^\omega$ for $\mathbf{i} \in \mathbb{Z}^D$. We finally define an infinite volume *Gibbs measure* ν on Ω to be a solution of the (DLR) equation:

$$\nu\mathbb{E}_\Lambda^{\cdot} f = \nu f \tag{66}$$

for all bounded measurable functions f on Ω. ν is a measure on Ω which has $\mathbb{E}_\Lambda^\omega$ as its finite volume conditional measures.

Following for example [22,24], the extension of Theorem 2.8 to this infinite dimensional setting will take the following form.

Theorem 6.1. *Suppose that the classical Sobolev inequality* (1) *and that the Poincaré inequality in balls* (22) *are both satisfied. Suppose also that inequality* (2) *is satisfied, i.e. there exist constants* $A, B \in (0, \infty)$ *such that*

$$\mu\big(|f|\big(|U|^{\beta} + |\nabla U|\big)\big) \leqslant A\mu|\nabla f| + B\mu|f|$$

for some $\beta \in (0, 1]$ *and locally Lipschitz functions* $f : \mathcal{M} \to \mathbb{R}$. *Then there exists* $J_0 > 0$ *such that for* $|J| < J_0$, *the Gibbs measure* ν *is unique and there exists a constant* C *such that*

$$\mathbf{Ent}_{\nu}^{\Phi}\big(|f|\big) \leqslant C\nu\bigg(\sum_{\mathbf{i}\in\mathbb{Z}^D} |\nabla_{\mathbf{i}} f|\bigg), \tag{67}$$

where $\Phi(x) = x(\log(1+x))^{\beta}$, *for all* f *for which the right-hand side is well defined.*

For notational simplicity, we will only prove Theorem 6.1 in the case $R = 1$ and $D = 2$, but the method can easily be extended to general R and D, (see e.g. [22] for the idea of the general scheme).

Define the sets

$$\begin{aligned} \Gamma_0 &= (0, 0) \cup \big\{\mathbf{j} \in \mathbb{Z}^2\colon\ dist\big(\mathbf{j}, (0, 0)\big) = 2n \text{ for some } n \in \mathbb{N}\big\}, \\ \Gamma_1 &= \mathbb{Z}^2 \setminus \Gamma_0. \end{aligned}$$

Note that $dist(\mathbf{i}, \mathbf{j}) > 1$ for all $\mathbf{i}, \mathbf{j} \in \Gamma_k$, $k = 0, 1$ and $\Gamma_0 \cap \Gamma_1 = \emptyset$. Moreover $\mathbb{Z}^2 = \Gamma_0 \cup \Gamma_1$. For the sake of notation, we will write $\mathbb{E}_{\Gamma_k} = \mathbb{E}_{\Gamma_k}^{\omega}$ for $k = 0, 1$. We will also define

$$\mathcal{P} := \mathbb{E}_{\Gamma_1}\mathbb{E}_{\Gamma_0}.$$

The proof will rely on the following few lemmata.

Lemma 6.2. *Under the conditions of Theorem* 6.1, *there exist constants* $\hat{c}_0$ *and* $\hat{c}$ *independent of* $\mathbf{i} \in \mathbb{Z}^D$ *and* $\omega \in \Omega$ *such that*

$$\mathbb{E}_{\mathbf{i}}^{\omega}\big|f - \mathbb{E}_{\mathbf{i}}^{\omega} f\big| \leqslant \hat{c}_0\mathbb{E}_{\mathbf{i}}^{\omega}|\nabla_{\mathbf{i}} f| \tag{68}$$

and

$$\mathbf{Ent}_{\mathbb{E}_{\mathbf{i}}^{\omega}}^{\Phi}\big(|f|\big) \leqslant \hat{c}\mathbb{E}_{\mathbf{i}}^{\omega}|\nabla_{\mathbf{i}} f| \tag{69}$$

for all suitable functions f, $\mathbf{i} \in \mathbb{Z}^D$ *and* $\omega \in \Omega$.

Proof. Firstly, by Theorem 2.7, we have that there exists a constant c_0 independent of $\mathbf{i}$ such that

$$\mu_{\mathbf{i}}|f - \mu_{\mathbf{i}} f| \leqslant c_0\mu_{\mathbf{i}}|\nabla_{\mathbf{i}} f|.$$

Since

$$osc\big(H_{\mathbf{i}}^{\omega}\big) \leqslant 2\big\|H_{\mathbf{i}}^{\omega}\big\|_{\infty} \leqslant 2\sum_{\{\mathbf{i}\}\cap X\neq\emptyset}\|\psi_X\|_{\infty} \leqslant 8M,$$

by a standard result about bounded perturbations of Poincaré type inequalities (see [12]), inequality (68) holds.

Moreover, by the assumptions and Theorem 2.8, we have

$$\mathbf{Ent}_{\mu_{\mathbf{i}}}^{\Phi}\big(|f|\big) = \mu_{\mathbf{i}}\big(\Phi\big(|f|\big)\big) - \Phi\big(\mu_{\mathbf{i}}|f|\big) \leqslant c\mu_{\mathbf{i}}|\nabla_{\mathbf{i}} f|$$

for all $\mathbf{i} \in \mathbb{Z}^D$. Thus by the bounded perturbation Corollary 2.6, (69) holds. □

Lemma 6.3. *Under the conditions of Theorem* 6.1*, there exists* $J_0 > 0$ *such that for* $|J| < J_0$*, there exists a constant* $\varepsilon \in (0,1)$ *such that*

$$\nu\big|\nabla_{\Gamma_k}(\mathbb{E}_{\Gamma_l} f)\big| \leqslant \nu|\nabla_{\Gamma_k} f| + \varepsilon\nu|\nabla_{\Gamma_1} f|$$

for all suitable f *and* $k, l \in \{0,1\}$ *such that* $k \neq l$.

Proof. We suppose $k = 1$ and $l = 0$. The case $k = 0, l = 1$ follows similarly. We can write

$$\begin{aligned}\nu\big|\nabla_{\Gamma_1}(\mathbb{E}_{\Gamma_0} f)\big| &= \nu\bigg(\sum_{\mathbf{i}\in\Gamma_1}\big|\nabla_{\mathbf{i}}(\mathbb{E}_{\Gamma_0} f)\big|\bigg) \leqslant \nu\bigg(\sum_{\mathbf{i}\in\Gamma_1}\big|\nabla_{\mathbf{i}}(\mathbb{E}_{\{\sim\mathbf{i}\}} f)\big|\bigg)\\ &\leqslant \nu\sum_{\mathbf{i}\in\Gamma_1}|\nabla_{\mathbf{i}} f| + |J|\nu\bigg(\sum_{\mathbf{i}\in\Gamma_1}\big|\mathbb{E}_{\{\sim\mathbf{i}\}}\big(f[\nabla_{\mathbf{i}} H_{\{\sim\mathbf{i}\}} - \mathbb{E}_{\{\sim\mathbf{i}\}}\nabla_{\mathbf{i}} H_{\{\sim\mathbf{i}\}}]\big)\big|\bigg)\end{aligned}$$

where we have used (66) and denoted $\{\sim \mathbf{i}\} = \{\mathbf{j}\colon\ \mathbf{j} \sim \mathbf{i}\}$. Now set $\mathcal{W}_{\mathbf{i}} = W_{\mathbf{i}} - \mathbb{E}_{\{\sim\mathbf{i}\}} W_{\mathbf{i}}$, where $W_{\mathbf{i}} = \nabla_{\mathbf{i}} H_{\{\sim\mathbf{i}\}}^{\omega}$. Then since $\mathbb{E}_{\{\sim\mathbf{i}\}}\mathcal{W}_{\mathbf{i}} = 0$, we have that

$$\nu\big|\nabla_{\Gamma_1}(\mathbb{E}_{\Gamma_0} f)\big| \leqslant \nu\sum_{\mathbf{i}\in\Gamma_1}|\nabla_{\mathbf{i}} f| + |J|\nu\bigg(\sum_{\mathbf{i}\in\Gamma_1}\big|\mathbb{E}_{\{\sim\mathbf{i}\}}(f - \mathbb{E}_{\{\sim\mathbf{i}\}} f)\mathcal{W}_{\mathbf{i}}\big|\bigg). \tag{70}$$

Now, by our assumptions on the potential, we have $\|\mathcal{W}_{\mathbf{i}}\|_{\infty} \leqslant 8M$ for all $\mathbf{i} \in \mathbb{Z}^D$, so that

$$\big|\mathbb{E}_{\{\sim\mathbf{i}\}}(f - \mathbb{E}_{\{\sim\mathbf{i}\}} f)\mathcal{W}_{\mathbf{i}}\big| \leqslant 8M\mathbb{E}_{\{\sim\mathbf{i}\}}|f - \mathbb{E}_{\{\sim\mathbf{i}\}} f|. \tag{71}$$

Note that by construction, $\mathbb{E}_{\{\sim\mathbf{i}\}}$ is a product measure. Now by Lemma 6.2 together with Lemma 2.9 there exists a constant $\hat{c}_0$ such that

$$\mathbb{E}_{\{\sim\mathbf{i}\}}|f - \mathbb{E}_{\{\sim\mathbf{i}\}} f| \leqslant \hat{c}_0\mathbb{E}_{\{\sim\mathbf{i}\}}|\nabla_{\{\sim\mathbf{i}\}} f|. \tag{72}$$

Using (71) and (72) in (70), we then arrive at

$$\begin{aligned}\nu\big|\nabla_{\Gamma_1}(\mathbb{E}_{\Gamma_0}f)\big| &\leqslant \nu\sum_{\mathbf{i}\in\Gamma_1}|\nabla_{\mathbf{i}}f| + 8M\hat{c}_0|J|\nu\bigg(\sum_{\mathbf{i}\in\Gamma_1}|\nabla_{\{\sim\mathbf{i}\}}f|\bigg)\\ &= \nu\sum_{\mathbf{i}\in\Gamma_1}|\nabla_{\mathbf{i}}f| + 32M\hat{c}_0|J|\nu\bigg(\sum_{\mathbf{i}\in\Gamma_0}|\nabla_{\mathbf{i}}f|\bigg).\end{aligned}$$

Thus taking $J_0 = \frac{1}{32M\hat{c}_0}$ proves the lemma. □

Lemma 6.4. *Under the conditions of Theorem* 6.1*, there exists $J_0 > 0$* (*given by Lemma* 6.3) *such that for $|J| < J_0$, $\mathcal{P}^r f$ converges almost everywhere to νf, where we recall that $\mathcal{P} = \mathbb{E}_{\Gamma_1}\mathbb{E}_{\Gamma_0}$. In particular ν is unique.*

Proof. The proof is standard: see for example Lemma 5.6 of [24]. □

Proof of Theorem 6.1. We may suppose $f \geqslant 0$. Using (66), write

$$\begin{aligned}\nu\big(\Phi(f)\big) - \Phi(\nu f) &= \nu\mathbb{E}_{\Gamma_0}\big(\Phi(f)\big) - \nu\big(\Phi(\mathbb{E}_{\Gamma_0}f)\big)\\ &\quad + \nu\big(\Phi(\mathbb{E}_{\Gamma_0}f)\big) - \Phi(\nu f)\\ &= \nu\big(\mathbf{Ent}^{\Phi}_{\mathbb{E}_{\Gamma_0}}(f)\big) + \nu\big(\mathbf{Ent}^{\Phi}_{\mathbb{E}_{\Gamma_1}}(\mathbb{E}_{\Gamma_0}f)\big)\\ &\quad + \nu\big(\Phi(\mathbb{E}_{\Gamma_1}\mathbb{E}_{\Gamma_0}f)\big) - \Phi(\nu f).\end{aligned}$$

Since probability measures $\mathbb{E}_{\Gamma_0}$ and $\mathbb{E}_{\Gamma_1}$ are product measures by construction, we have by Lemmas 2.9 and 6.2 that they both satisfy $L_1\Phi$-entropy inequalities with constant $\hat{c}$. Therefore, the above yields

$$\begin{aligned}\nu\big(\Phi(f)\big) - \Phi(\nu f) &\leqslant \hat{c}\nu|\nabla_{\Gamma_0}f| + \hat{c}\nu\big|\nabla_{\Gamma_1}(\mathbb{E}_{\Gamma_0}f)\big|\\ &\quad + \nu\big(\Phi(\mathcal{P}f)\big) - \Phi(\nu f).\end{aligned}$$

We can similarly write

$$\begin{aligned}\mu\big(\Phi(\mathcal{P}f)\big) &= \nu\big(\mathbf{Ent}^{\Phi}_{\mathbb{E}_{\Gamma_0}}(\mathcal{P}f)\big) + \nu\big(\mathbf{Ent}^{\Phi}_{\mathbb{E}_{\Gamma_1}}(\mathbb{E}_{\Gamma_0}\mathcal{P}f)\big) + \nu\big(\Phi\big(\mathcal{P}^2f\big)\big)\\ &\leqslant \hat{c}\nu|\nabla_{\Gamma_0}\mathcal{P}f| + \hat{c}\nu\big|\nabla_{\Gamma_1}(\mathbb{E}_{\Gamma_0}f)\big| + \nu\big(\Phi\big(\mathcal{P}^2f\big)\big).\end{aligned}$$

Repeating this process, after r steps we see that

$$\nu\big(\Phi(f)\big) - \Phi(\nu f) \leqslant \hat{c}\sum_{k=0}^{r-1}\nu\big|\nabla_{\Gamma_0}\mathcal{P}^kf\big| + \hat{c}\sum_{k=0}^{r-1}\nu\big|\nabla_{\Gamma_1}\big(\mathbb{E}_{\Gamma_0}\mathcal{P}^kf\big)\big| \tag{73}$$

$$+ \nu\big(\Phi\big(\mathcal{P}^rf\big)\big) - \Phi(\nu f). \tag{74}$$

We may control the first and second terms using Lemma 6.3. Indeed

$$\begin{aligned}\nu\big|\nabla_{\Gamma_0}\mathcal{P}^k f\big| &\leqslant \varepsilon^2\nu\big|\nabla_{\Gamma_0}\mathcal{P}^{k-1} f\big| \\ &\leqslant \varepsilon^{2k-1}\nu|\nabla_{\Gamma_1}\mathbb{E}_{\Gamma_0} f| \\ &\leqslant \varepsilon^{2k-1}\nu|\nabla_{\Gamma_1} f| + \varepsilon^{2k}\nu|\nabla_{\Gamma_0} f|. \end{aligned} \tag{75}$$

Similarly

$$\nu\big|\nabla_{\Gamma_1}\big(\mathbb{E}_{\Gamma_0}\mathcal{P}^k f\big)\big| \leqslant \varepsilon^{2k}\nu|\nabla_{\Gamma_1} f| + \varepsilon^{2k+1}\nu|\nabla_{\Gamma_0} f|. \tag{76}$$

Using (75) and (76) in (73) yields

$$\begin{aligned}\nu\big(\Phi(f)\big) - \Phi(\nu f) \leqslant \hat{c}\big(1+\varepsilon^{-1}\big)\left[\left(\sum_{k=0}^{r-1}\varepsilon^{2k}\right)\nu|\nabla_{\Gamma_1} f| + \left(\sum_{k=0}^{r-1}\varepsilon^{2k+1}\right)\nu|\nabla_{\Gamma_0} f|\right] \\ + \nu\big(\Phi\big(\mathcal{P}^r f\big)\big) - \Phi(\nu f).\end{aligned}$$

By Lemma 6.4 we have that $\lim_{r\to\infty}\mathcal{P}^r f = \nu f$, ν-almost surely. Therefore taking the limit as $r\to\infty$ in the above (which exists since $\varepsilon\in(0,1)$) yields

$$\nu\big(\Phi(f)\big) - \Phi(\nu f) \leqslant C\nu|\nabla_{\mathbb{Z}^D} f|$$

where $C = \hat{c}\frac{1+\varepsilon^{-1}}{1-\varepsilon^2}$. □

Next, we consider (IFI_2) for a family of examples. In particular we restrict ourselves to a situation when $\mathcal{M}$ is an H-type group and assume that for $\mathbf{i}\in\mathbb{Z}^D$

$$U_{\mathbf{i}} \equiv \sum_{k=0,\dots,p-1}\alpha_k d_{\mathbf{i}}^{p-k} \equiv \sum_{k=0,\dots,p-1}\alpha_k d^{p-k}(\omega_{\mathbf{i}}) \tag{77}$$

with $d(\cdot)$ denoting the Carnot–Carathéodory distance from the unit element, $p\geqslant 2$, where $\alpha_0\in(0,\infty)$ and $\alpha_k\in\mathbb{R}$. As above we consider an interaction

$$H_\Lambda^\omega(x_\Lambda) = \sum_{\Lambda\cap X\neq\emptyset}\psi_X(x_\Lambda\bullet_\Lambda\omega), \tag{78}$$

which is assumed to be bounded with bounded (sub-)gradient and for simplicity is of finite range, as specified at the beginning of the current section. Moreover we are given a family of regular conditional expectations defined by (6.1). Combining the previous results with those of this section the previous we arrive at the following theorem.

Theorem 6.5. *Suppose $p\geqslant 2$. Then there exists $J_0>0$ such that for $|J|<J_0$ the unique Gibbs measure ν corresponding to the interaction* (77)–(78) *satisfies the following inequalities*:

(i)

$$\mathbf{Ent}_\nu^\Phi\big(|f|\big) \leqslant C_1\nu\left(\sum_{\mathbf{i}\in\mathbb{Z}^D}|\nabla_{\mathbf{i}} f|\right), \tag{79}$$

where $\Phi(x) = x(\log(1+x))^{\frac{1}{q}}$, $\frac{1}{q}+\frac{1}{p}=1$, *with some constant* $C_1 \in (0,\infty)$, *for any* f *for which the right-hand side is well defined*;

(ii)

$$\mathcal{U}_2(\nu f) \leqslant \nu\Big(\mathcal{U}_2(f)^2 + C_2 \sum_{\mathbf{i}\in\mathbb{Z}^D} |\nabla_{\mathbf{i}} f|^2\Big)^{\frac{1}{2}} \tag{80}$$

where $\mathcal{U}_2$ *is the Gaussian isoperimetric profile function* (*as defined in Section* 3), *with some constant* $C_2 \in (0,\infty)$ *for any function* $0 \leqslant f \leqslant 1$ *for which the right-hand side is well defined.*

Proof. To begin we notice that the reference measure $d\mu$ satisfies a U-bound, and therefore the conditional expectation (as a perturbation of the reference measure by strictly bounded and strictly positive density), also satisfies the following inequality

$$\int_{\mathbb{H}} f|U|^{\frac{1}{q}}\, d\mathbb{E}_{\mathbf{i}} \leqslant A \int_{\mathbb{H}} |\nabla_{\mathbf{i}} f|\, d\mathbb{E}_{\mathbf{i}} + B \int_{\mathbb{H}} f\, d\mathbb{E}_{\mathbf{i}} \tag{81}$$

with some constants $A, B \in (0,\infty)$ independent of $\mathbf{i}$ and $\omega_{\mathbf{j}}$, where $\mathbb{E}_{\mathbf{i}}$ denotes the corresponding conditional expectations. Thus we can apply Theorem 4.5 to conclude that the $\mathbb{E}_{\mathbf{i}}$'s satisfy Cheeger's inequality, as well as $L_1\Phi$-entropy and (IFI_2) bounds with constants independent of $\mathbf{i}$ and $\omega_{\mathbf{j}}$'s. With this bound the proof of (ii) follows via the strategy developed in [33]. □

Remark 6.6. We remark that once the conditional measures satisfy $L_1\Phi$-entropy or (IFI_2) inequalities with constants independent of external conditions, one can show that the Gibbs measure also satisfies (IFI_2) even when the interactions $H_{\mathbf{i}}$ contain an unbounded component, provided we have Cheeger's inequality and appropriate U-bounds. In particular one obtains the following generalisation of the results of [33] where only the bounded interaction case was studied.

Theorem 6.7. *Suppose* $\mathcal{M} \equiv \mathbb{R}$ *and that* U *is a semibounded polynomial of degree at least* 2. *Let*

$$H_{\mathbf{i}}^{\omega}(x_{\mathbf{i}}) \equiv \varepsilon \sum_{\{\mathbf{i}\}\cap X \neq \emptyset} \psi_X(x_{\mathbf{i}} \bullet_{\mathbf{i}} \omega) + \varepsilon \sum_{\mathbf{j}} G_{\mathbf{ij}} x_{\mathbf{i}} \omega_{\mathbf{j}}$$

with ψ_X *satisfying conditions of Theorem* 6.5, $\sum_{\mathbf{j}} |G_{\mathbf{ij}}| < \infty$ *and* $\varepsilon \in (0,\infty)$. *Then, if* $\varepsilon \in (0,\infty)$ *is sufficiently small, the corresponding Gibbs measure satisfies* (IFI_2).

Remark 6.8. For cylinder functions dependent on N coordinates, adapting the length of the gradient in part (i) of Theorem 6.5, we get

$$\mathbf{Ent}_{\nu}^{\Phi}\big(|f|\big) \leqslant C_1 \sqrt{N} \nu \Big(\sum_{\mathbf{i}_l \in \mathbb{Z}^D, l=1,\dots,N} |\nabla_{\mathbf{i}_l} f|^2\Big)^{\frac{1}{2}} \tag{82}$$

for all functions f for which the right-hand side is well defined. Now, choosing a Lipschitz approximation of a cylinder set A_N (specified by conditions on coordinates $\omega_{\mathbf{i}_l}, l = 1, \dots, N$), by Theorem 3.1 we arrive at

$$\mathcal{U}_q\big(\nu(A_N)\big) \leqslant \tilde{c}\sqrt{N}\nu_2^+(A_N) \tag{83}$$

with suitable constant $\tilde{c} \in (0, \infty)$ independent of N, and with use of the subscript 2 on the right-hand side to emphasise that we have here the surface measure with respect to the quadratic distance. On the other hand using part (ii) of Theorem 6.5 yields

$$\mathcal{U}_2\big(\nu(A_N)\big) \leqslant \sqrt{C_2}\nu_2^+(A_N). \tag{84}$$

Thus we obtain a potentially useful tool for optimisation of isoperimetric relations for finite dimensional marginals of the measure ν.

Acknowledgment

We are very grateful to the referee for their helpful suggestions and careful reading of this manuscript.

Appendix A. Exponential bounds

Suppose for $d\mu \equiv e^{-U}\, d\lambda/Z$, with $U \geqslant \varepsilon$, for some $\varepsilon > 0$, and Z a normalisation constant, we have

$$\mu\big(fU^\beta\big) \leqslant C\mu|\nabla f| + D\mu f$$

for all locally Lipschitz functions f. In particular, for a Lipschitz cut-off function $0 < \varepsilon \leqslant U_L \leqslant U$, for $f \equiv e^{\lambda U_L} U_L^\alpha$, with $\alpha, \beta > 0$, $\alpha + \beta = 1$, we have

$$\begin{aligned}\mu\big(e^{\lambda U_L} U_L\big) &= \mu\big(e^{\lambda U_L} U_L^\alpha \cdot U_L^\beta\big) \leqslant C\mu\big|\nabla\big(e^{\lambda U_L} U_L^\alpha\big)\big| + D\mu\big(e^{\lambda U_L} U_L^\alpha\big) \\ &\leqslant \lambda C\mu\big(e^{\lambda U_L} U_L^\alpha \cdot |\nabla U_L|\big) + \alpha C\mu\big(e^{\lambda U_L} U_L^{\alpha-1} \cdot |\nabla U_L|\big) \\ &\quad + D\mu\big(e^{\lambda U_L} U_L^\alpha\big).\end{aligned}$$

If we assume that

$$|\nabla U_L| \leqslant a U_L^\beta$$

with $a \in (0, \infty)$ independent of L, then we get

$$\begin{aligned}\mu\big(e^{\lambda U_L} U_L\big) &\leqslant \lambda C\mu\big(e^{\lambda U_L} U_L^\alpha \cdot \big(aU_L^\beta\big)\big) \\ &\quad + \alpha C\mu\big(e^{\lambda U_L} U_L^{\alpha-1} \cdot \big(aU_L^\beta\big)\big) + D\mu\big(e^{\lambda U_L} U_L^\alpha\big) \\ &\leqslant \lambda a C\mu\big(e^{\lambda U_L} U_L\big) + \alpha a C\mu\big(e^{\lambda U_L}\big) + D\mu\big(e^{\lambda U_L} U_L^\alpha\big).\end{aligned}$$

Using our assumption that $U_L \geqslant \varepsilon > 0$ and a bound

$$U_L^\alpha \leqslant \lambda\delta U_L + A(\lambda\delta)$$

with some $\delta, A(\lambda\delta) \in (0,\infty)$ independent of L, we get

$$\mu\big(e^{\lambda U_L} U_L\big) \leqslant \lambda(aC + D\delta)\mu\big(e^{\lambda U_L} U_L\big) + \big(\alpha a C + D \cdot A(\lambda\delta)\big)\mu\big(e^{\lambda U_L}\big).$$

Hence for $\lambda \in (0,\lambda_0)$, with $\lambda_0 \equiv (aC + D\delta)^{-1}$, we have

$$\frac{d}{d\lambda}\mu\big(e^{\lambda U_L}\big) = \mu\big(e^{\lambda U_L} U_L\big) \leqslant B\mu\big(e^{\lambda U_L}\big)$$

with

$$B \equiv B(\lambda_0, \delta) \equiv \big(\alpha a C + D \cdot A(\lambda\delta)\big)\big(1 - \lambda_0(aC + D\delta)\big)^{-1}.$$

Solving this differential inequality for $\lambda \in (0,\lambda_0)$, we obtain

$$\mu\big(e^{\lambda U_L}\big) \leqslant e^{\lambda B}.$$

Since the constant B is independent of L, by the dominated convergence theorem we obtain the following bound

$$\mu\big(e^{\lambda U}\big) \leqslant e^{\lambda B},$$

which holds true for $\lambda \in (0,\lambda_0)$.

Appendix B. Rothaus argument

Here we give a brief outline of how to tighten a defective q-logarithmic Sobolev inequality under the additional assumption of a q-Poincaré inequality. To do this we need the following results from [12] which generalise the so-called Rothaus argument.

Suppose that $q \in [1,2]$ and define the Orlicz space $L_{N_q}(\mu)$ generated by the function $N_q(x) = x^q \log(1 + x^q)$ to be the space of measurable functions f such that

$$\|f\|_{N_q}^q := \inf\left\{\lambda > 0: \int N_q\left(\frac{f}{\lambda}\right) d\mu \leqslant 1\right\} < \infty.$$

Lemma B.1. *(See [12].) If $f \geqslant 0 \in L_{N_q}(\mu)$ then*

$$\|f\|_{N_q}^q \leqslant \mu\left(f^q \log\frac{f^q}{\mu(f^q)}\right) + \mu\big(f^q\big).$$

Lemma B.2. *(See [12].) If $f \in L_{N_q}(\mu)$ and $\mu(f) = 0$ then*

$$\sup_{a\in\mathbb{R}} \int |f + a|^q \log\frac{|f + a|^q}{\mu|f + a|^q}\, d\mu \leqslant 16\|f\|_{N_q}^q.$$

With these results in hand, suppose that we have a defective (LS_q) inequality

$$\mu\left(|f|^q \log \frac{|f|^q}{\mu|f|^q}\right) \leqslant C\mu|\nabla f|^q + D\mu|f|^q, \tag{85}$$

as well as a q-Poincaré inequality

$$\mu|f-\mu f|^q \leqslant K\mu|\nabla f|^q. \tag{86}$$

By Lemma B.1 applied to the function $|f-\mu f|$ we have

$$\begin{aligned}\|f-\mu f\|_{N_q}^q &\leqslant \mu\left(|f-\mu f|^q \log \frac{|f-\mu f|^q}{\mu|f-\mu f|^q}\right) + \mu|f-\mu f|^q \\ &\leqslant C\mu|\nabla f|^q + (D+1)\mu|f-\mu f|^q \\ &\leqslant \big(C+K(D+1)\big)\mu|\nabla f|^q,\end{aligned}$$

where we have first used (85) and then (86). By applying Lemma B.2 to $f-\mu f$ we obtain

$$\begin{aligned}\mu\left(|f|^q \log \frac{|f|^q}{\mu|f|^q}\right) &= \mu\left(|f-\mu f+\mu f|^q \log \frac{|f-\mu f+\mu f|^q}{\mu|f-\mu f+\mu f|^q}\right) \\ &\leqslant 16\|f-\mu f\|_{N_q}^q \\ &\leqslant 16\big(C+K(D+1)\big)\mu|\nabla f|^q,\end{aligned}$$

so that we arrive at the desired tight form of the logarithmic Sobolev inequality.

References

[1] C. Ané, S. Blachère, D. Chafaï, P. Fougères, I. Gentil, F. Malrieu, C. Roberto, G. Scheffer, Sur les inégalités de Sobolev logarithmiques, Panor. Synthèses, vol. 10, Soc. Math. France, Paris, 2000.

[2] D. Bakry, F. Baudoin, M. Bonnefont, D. Chafaï, On gradient bounds for the heat kernel on the Heisenberg group, J. Funct. Anal. 255 (2008) 1905–1938.

[3] D. Bakry, M. Ledoux, Lévy–Gromov's isoperimetric inequality for an infinite dimensional diffusion generator, Invent. Math. 123 (1996) 259–281.

[4] F. Barthe, Infinite dimensional isoperimetric inequalities in product spaces with the supremum distance, J. Theoret. Probab. 17 (2) (2004) 293–308.

[5] F. Barthe, P. Cattiaux, C. Roberto, Interpolated inequalities between exponential and Gaussian, Orlicz hypercontractivity and isoperimetry, Rev. Mat. Iberoamericana 22 (3) (2006) 993–1067.

[6] F. Barthe, B. Maurey, Some remarks on isoperimetry of Gaussian type, Ann. Inst. H. Poincaré Probab. Statist. 36 (4) (2000) 419–434.

[7] R. Beals, B. Gaveau, P. Greiner, Hamilton–Jacobi theory and the heat kernel on Heisenberg groups, J. Math. Pures Appl. 79 (7) (2000) 633–689.

[8] S. Bobkov, A functional form of the isoperimetric inequality for the Gaussian measure, J. Funct. Anal. 135 (1) (1996) 39–49.

[9] S. Bobkov, An isoperimetric inequality on the discrete cube, and an elementary proof of the isoperimetric inequality in Gauss space, Ann. Probab. 25 (1) (1997) 206–214.

[10] S. Bobkov, C. Houdré, Isoperimetric constants for product probability measures, Ann. Probab. 25 (1) (1997) 184–205.

[11] S. Bobkov, C. Houdré, Some Connections Between Isoperimetric and Sobolev-Type Inequalities, Mem. Amer. Math. Soc., vol. 616, 1997.

[12] S. Bobkov, B. Zegarliński, Entropy Bounds and Isoperimetry, Mem. Amer. Math. Soc., vol. 829, 2005.
[13] A. Bonfiglioli, E. Lanconelli, F. Uguzzoni, Stratified Lie Groups and Potential Theory for Their Sub-Laplacians, Springer Monogr. Math., Springer-Verlag, 2007.
[14] P. Cattiaux, N. Gozlan, A. Guillin, C. Roberto, Functional inequalities for heavy tails distributions and application to isoperimetry, preprint.
[15] D. Chafaï, Entropies, convexity and functional inequalities: on ϕ-entropies and ϕ-Sobolev inequalities, J. Math. Kyoto Univ. 44 (2) (2004) 325–363.
[16] B. Driver, T. Melcher, Hypoelliptic heat kernel inequalities on the Heisenberg group, J. Funct. Anal. 221 (2) (2005) 340–365.
[17] N. Eldredge, Precise estimates for the subelliptic heat kernel on H-type groups, J. Math. Pures Appl. 92 (2009) 52–85.
[18] N. Eldredge, Gradient estimates for the subelliptic heat kernel on H-type groups, J. Funct. Anal. 258 (2) (2010) 504–533.
[19] P. Fougères, Hypercontractivité et isopérimétrie gaussienne. Applications aux systémes de spins, Ann. Inst. H. Poincaré Probab. Statist. 36 (5) (2000) 647–689.
[20] P. Fougères, C. Roberto, B. Zegarliński, Sub-Gaussian measures and associated semilinear problems, preprint.
[21] A. Grigor'yan, Gaussian upper bounds for the heat kernel on arbitrary manifolds, J. Differential Geom. 45 (1997) 33–52.
[22] A. Guionnet, B. Zegarliński, Lectures on logarithmic Sobolev inequalities, in: Séminaire de Probabilités, XXXVI, in: Lecture Notes in Math., vol. 1801, Springer-Verlag, 2003, pp. 1–134.
[23] W. Hebisch, B. Zegarliński, Coercive inequalities on metric measure spaces, J. Funct. Anal. 258 (2010) 814–851.
[24] J. Inglis, I. Papageorgiou, Logarithmic Sobolev inequalities for infinite dimensional Hörmander type generators on the Heisenberg group, J. Pot. Anal. 31 (1) (2009) 79–102.
[25] D. Jerison, The Poincaré inequality for vector fields satisfying Hörmander's condition, Duke Math. J. 53 (2) (1986) 503–523.
[26] M. Ledoux, Isopérimétrie et inégalités de Sobolev logarithmiques Gaussiennes, C. R. Acad. Sci. Paris Sér. I Math. 306 (2) (1988) 79–82.
[27] H.-Q. Li, Estimation optimale du gradient du semi-groupe de la chaleur sur le groupe de Heisenberg, J. Funct. Anal. 236 (2) (2006) 369–394.
[28] P. Ługiewicz, B. Zegarliński, Coercive inequalities for Hörmander type generators in infinite dimensions, J. Funct. Anal. 247 (2) (2007) 438–476.
[29] E. Milman, On the role of convexity in functional and isoperimetric inequalities, Proc. Lond. Math. Soc. 99 (3) (2009) 32–66.
[30] O. Rothaus, Analytic inequalities, isoperimetric inequalities and logarithmic Sobolev inequalities, J. Funct. Anal. 64 (1985) 296–313.
[31] L. Saloff-Coste, Aspects of Sobolev-Type Inequalities, London Math. Soc. Lecture Note Ser., vol. 289, Cambridge University Press, 2002.
[32] N.T. Varopoulos, L. Saloff-Coste, T. Coulhon, Analysis and Geometry on Groups, Cambridge Tracts in Math., vol. 100, Cambridge University Press, 1992.
[33] B. Zegarliński, Isoperimetry for Gibbs measures, Ann. Probab. 29 (2001) 802–819.