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Internal Direct Products of Groupoids

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When generalizing the internal direct product from groups to all groupoids (binary systems), it has been customary to imitate the group case by making restrictions, especially the existence of an identity element. This article develops what seems a natural basic definition of internal direct product, then uses it as a background against which to compare more popular restricted versions. © 1999 Academic Press

1. PROLOGUE

The fruitfulness of Jónsson and Tarski's work [5] on direct product decompositions is very clear from Willard's summary [7, pp. 130–131]. Yet [5] includes a curious oversight concerning the definition of internal direct products of groupoids. That definition generalizes the familiar group case, which can be described as follows without mentioning normal subgroups.

A group G may be the internal direct product of subgroups G_1 and G_2. In that case each element of G is uniquely expressible as x_1x_2 with $x_1 \in G_1$, $x_2 \in G_2$; and the product of any such x_1x_2 and y_1y_2 can be written $(x_1y_1)(x_2y_2)$. The connection with the ordinary external direct product $G_1 \times G_2$ is that the mapping θ: $(x_1, x_2) \mapsto x_1x_2$ is an isomorphism from $G_1 \times G_2$ onto G.

If G is an external direct product $H_1 \times H_2$ of groups, then it is also an internal direct product as follows. Let e_1 be the identity element of H_1, and e_2 of H_2, then let $G_1 = H_1 \times \{e_2\}$ and $G_2 = \{e_1\} \times H_2$. It is easy to

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check that G is the internal direct product of G_1 and G_2, although the isomorphism θ is the notationally clumsy

$$((x_1, e_2), (e_1, x_2)) \mapsto (x_1, e_2)(e_1, x_2) = (x_1, x_2).$$

Such arguments depend so vitally upon e_1 and e_2 that past generalizations of the internal direct product from groups to groupoids have treated identity elements as indispensable. The classic statement of that belief occurs within the book [5] of Jónsson and Tarski. Their algebras are groupoids with multiple operators, but the essential problem with internal direct products lies in the groupoid structure. On p. 7 they make the following claim.

"On the other hand, the definition of a [internal] direct product implicitly involves the notion of a zero element, and it cannot be applied to systems without a zero unless we agree to use the term "subalgebra" in a more general sense. What is even more important, a detailed examination of the proof of 1.5 [characterizing internal direct products] reveals that condition 1.1(ii) [existence and properties of zero] plays an essential role in this proof. It can easily be shown by means of examples that, in general, Theorem 1.5 does not apply to systems $\underline{A}$ in which either part (ii′) or part (ii″) of this condition fails; for it may then happen that $\underline{A}$ is isomorphic to the cardinal [i.e., external direct] product of two systems $\underline{B}$ and $\underline{C}$, without containing any subsystem isomorphic to $\underline{B}$ or $\underline{C}$. We can thus say that only by restricting ourselves to systems which satisfy 1.1(ii) are we in a position to introduce an adequate notion of an inner direct product."

Example 7.1 illustrates the difficulty they described and firmly blamed on the lack of an identity element. The strong but unproved statement of Jónsson and Tarski seems to have started a tradition, and not only in later papers involving the same authors [2–4]. Łoś [6] uses more general algebras, but imposes conditions from which he can derive essentially the existence of the groupoid identity element as a theorem.

That tradition is renounced by Definition 2.2 below, whose consequences do not include the existence of an identity element. It seems the most natural and economical definition of internal direct product. (Evidently Cohn thinks so, as his [1, p. 56, Exercise 11] gives the same definition; but he then reverts to tradition by citing [5].)

2. FUNDAMENTALS

The notation is reversed Polish. Various groupoids use the same operator symbol β, so the result of the appropriate binary operation on x and y is written $xy\beta$. Also the notation introduced in each definition is assumed from then on.

DEFINITION 2.1. The *external direct product* of groupoids (G_1, β) and (G_2, β) is $(G_1 \times G_2, \beta)$, where

$$\forall x_1, y_1 \in G_1, \qquad \forall x_2, y_2 \in G_2,$$
$$(x_1, x_2)(y_1, y_2)\beta = (x_1 y_1 \beta, x_2 y_2 \beta).$$

DEFINITION 2.2. A groupoid (G, β) is the *internal direct product* of its non-empty subgroupoids G_1 and G_2 if and only if the mapping θ: $(x_1, x_2) \mapsto x_1 x_2 \beta$ is an isomorphism from $G_1 \times G_2$ onto G.

Prescribing that $G \neq \varnothing$ is a minor convenience. Of course θ is just the mapping which interprets the binary operator symbol β in G.

Next, the isomorphism θ^{-1} can be used to define the projections α_1 and α_2 from G onto G_1 and G_2.

DEFINITION 2.3. With the notation of Definition 2.2,

$$\forall x \in G, \, x\theta^{-1} = (x\alpha_1, x\alpha_2).$$

From Definitions 2.2 and 2.3 easily follow $G\alpha_1 = G_1$ and $G\alpha_2 = G_2$. Hence an element of G_1 may be written either x_1 or $x\alpha_1$ as convenient, and a similar remark applies to G_2. This is assumed in, for example, the proof of Theorem 2.5(b). Also the consequence that $G_1\alpha_1 \subseteq G_1$ and $G_2\alpha_2 \subseteq G_2$ is useful from Theorem 3.2 onward.

THEOREM 2.4. $\forall x_1, y_1 \in G_1$, $\forall x_2, y_2 \in G_2$, $x_1 x_2 \beta y_1 y_2 \beta\beta = x_1 y_1 \beta x_2 y_2 \beta\beta$; *i.e., elements can be combined componentwise.*

Proof.

$$
\begin{aligned}
& x_1 x_2 \beta y_1 y_2 \beta\beta \\
&\quad = (x_1, x_2)\theta(y_1, y_2)\theta\beta = (x_1, x_2)(y_1, y_2)\beta\theta && 2.2 \\
&\quad = (x_1 y_1 \beta, x_2 y_2 \beta)\theta && 2.1 \\
&\quad = x_1 y_1 \beta x_2 y_2 \beta\beta. && 2.2
\end{aligned}
$$

∎

The second paragraph of Section 1 has now been formalized for general groupoids. Such pedestrian results are easily obtained by going back and forth to the external direct product, but for faster progress an internal direct product can be characterized purely by intrinsic properties.

THEOREM 2.5. *Suppose a set G is closed to a binary operation β and unary operations α_1 and α_2. The groupoid (G, β) is the internal direct product of $G\alpha_1$ and $G\alpha_2$ if and only if the algebra $(G, \beta, \alpha_1, \alpha_2)$ has the* *following properties*:

(a) $\forall x \in G,\ x\alpha_1 x\alpha_2\,\beta = x$

(b) $\forall x, y \in G,\ x\alpha_1 y\alpha_2\,\beta\alpha_1 = x\alpha_1$ *and* $x\alpha_1 y\alpha_2\,\beta\alpha_2 = y\alpha_2$

(c) $\forall x, y \in G,\ x\alpha_1 y\alpha_1\,\beta = xy\beta\alpha_1$ *and* $x\alpha_2 y\alpha_2\,\beta = xy\beta\alpha_2$.

Proof. Write $G\alpha_1 = G_1$ and $G\alpha_2 = G_2$. First, assume (G, β) is the internal direct product of G_1 and G_2. Then:

(a) $\forall x \in G$,
$$x = x\theta^{-1}\theta = (x\alpha_1, x\alpha_2)\theta \qquad 2.3$$
$$= x\alpha_1 x\alpha_2\,\beta \qquad 2.2$$

(b) $\forall x_1 \in G_1,\quad \forall y_2 \in G_2,$
$$(x_1, y_2) = (x_1, y_2)\theta\theta^{-1} = x_1 y_2\,\beta\theta^{-1} \qquad 2.2$$
$$= (x_1 y_2\,\beta\alpha_1, x_1 y_2\,\beta\alpha_2) \qquad 2.3$$

so $x_1 = x_1 y_2\,\beta\alpha_1$ and $y_2 = x_1 y_2\,\beta\alpha_2$

(c) $\forall x, y \in G$,

$(xy\beta\alpha_1, xy\beta\alpha_2)$
$$= xy\beta\theta^{-1} \qquad 2.3$$
$$= x\theta^{-1} y\theta^{-1}\beta \qquad \because \theta^{-1} \text{ is an isomorphism by 2.2}$$
$$= (x\alpha_1, x\alpha_2)(y\alpha_1, y\alpha_2)\beta \qquad 2.3$$
$$= (x\alpha_1 y\alpha_1\,\beta, x\alpha_2 y\alpha_2\,\beta) \qquad 2.1$$

so $xy\beta\alpha_1 = x\alpha_1 y\alpha_1\,\beta$ and $xy\beta\alpha_2 = x\alpha_2 y\alpha_2\,\beta$.

Conversely, assume (a)–(c) are satisfied. Define η: $G \to G_1 \times G_2$ by $x\eta = (x\alpha_1, x\alpha_2)$. Then

$$x\eta = y\eta \;\Rightarrow\; x\alpha_1 = y\alpha_1 \text{ and } x\alpha_2 = y\alpha_2$$
$$\Rightarrow\; x\alpha_1 x\alpha_2\,\beta = y\alpha_1 y\alpha_2\,\beta$$
$$\Rightarrow\; x = y \quad \text{by (a)}. \quad \therefore \eta \text{ is injective}.$$

Also

$$\forall x_1 \in G_1, \quad \forall y_2 \in G_2, \quad x_1 y_2\,\beta\eta = (x_1 y_2\,\beta\alpha_1, x_1 y_2\,\beta\alpha_2)$$
$$= (x_1, y_2) \quad \text{by (b)}. \quad \therefore \eta \text{ is surjective}.$$

Also

$$\begin{aligned}\forall x, y \in G, \quad x\eta y\eta\beta &= (x\alpha_1, x\alpha_2)(y\alpha_1, y\alpha_2)\beta \\ &= (x\alpha_1 y\alpha_1 \beta, x\alpha_2 y\alpha_2 \beta) && 2.1 \\ &= (xy\beta\alpha_1, xy\beta\alpha_2) && \text{by (c)} \\ &= xy\beta\eta. && \therefore \eta \text{ is a homomorphism.}\end{aligned}$$

Finally let $\theta = \eta^{-1}$. Then G is an internal direct product as in Definition 2.2. ∎

Theorem 2.5 offers a useful alternative point of view. The class of all groupoid internal direct products may now be regarded as a variety of $(\beta, \alpha_1, \alpha_2)$-algebras, with the formulae 2.5(a)–(c) as axioms. (These axioms are in fact independent.) Subsequent proofs refer back to 2.5(a)–(c) rather than directly to Definition 2.2.

Between those general proofs there are also references to special properties commonly expected of internal direct products. The following list seems to cover most such properties.

CONDITIONS 2.6. Custom prefers an internal direct product to satisfy:

(a) G has a neutral (identity) element.

(b) Every element of G_1 commutes with every element of G_2.

(c) $|G_2\alpha_1| = |G_1\alpha_2| = 1$.

(d) $G_2\alpha_1 = G_1\alpha_2 =$ a singleton invariant under α_1 and α_2.

(e) $|G_1 \cap G_2| = 1$.

(f) $\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ are identity mappings. Equivalently, $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$.

The statement of (d) may appear awkward, especially since the singleton in (d) and (e) traditionally contains just the identity element mentioned in (a). However the six statements (a)–(f) as they stand are useful for reference later. Rather obviously, they are not independent. Much less obviously, they can all be false. (Cf. the last of Examples 7.8).

3. BASIC THEOREMS

Examples 7.3–7.9 show that 2.6(f) cannot be proven in general. However 2.5(c) means that α_1 and α_2 must be endomorphisms, and Theorem 3.2 shows that their restrictions to G_1 and G_2, respectively, must be injective endomorphisms.

First it is convenient to have notation for right and left translations.

DEFINITION 3.1. For any $a \in G$, $\forall x \in G$, $x\rho_a = xa\beta$ and

$$x\lambda_a = ax\beta.$$

THEOREM 3.2. *In any internal direct product* $(G, \beta, \alpha_1, \alpha_2)$ *with* $G_1 = G\alpha_1$ *and* $G_2 = G\alpha_2$, $\alpha_1|_{G_1}$ *and* $\alpha_2|_{G_2}$ *are injections.*

Proof. Choose any $a_2 \in G_2$ (which is non-empty by 2.2). Then

$$\begin{aligned} \forall x_1 \in G_1, \quad x_1\alpha_1\rho_{a_2\alpha_1} &= x_1\alpha_1 a_2\alpha_1\beta && 3.1 \\ &= x_1 a_2 \beta\alpha_1 && 2.5(c) \\ &= x_1. && 2.5(b) \end{aligned}$$

Thus $\alpha_1|_{G_1}$ has a right inverse, so it is injective. Similarly $\alpha_2|_{G_2}$ is injective. ∎

COROLLARY. *For any* $a_2 \in G_2$, $\rho_{a_2\alpha_1}|_{G_1\alpha_1}$ *is a right inverse of* $\alpha_1|_{G_1}$. *For any* $a_1 \in G_1$, $\lambda_{a_1\alpha_2}|_{G_2\alpha_2}$ *is a right inverse of* $\alpha_2|_{G_2}$.

In light of Theorems 2.5(c) and 3.2, the question whether $\alpha_1|_{G_1}$ is an actual automorphism is simply the question whether $G_1\alpha_1 = G_1$. Examples 7.5–7.9 show that this is not always so. But the condition $G_1\alpha_1 = G_1$ is a significant restriction (as is $G_2\alpha_2 = G_2$), which the next theorem expresses in various forms.

THEOREM 3.3. *In any internal direct product* $(G, \beta, \alpha_1, \alpha_2)$ *with* $G_1 = G\alpha_1$ *and* $G_2 = G\alpha_2$, *the following conditions are equivalent.*

(a) $G_1\alpha_1 = G_1$.

(b) $G_2\alpha_1$ *is a singleton* $\{d_1\}$ *with* $d_1\alpha_1 = d_1$.

(c) $G_1\alpha_1 \cap G_2\alpha_1$ *is a singleton* $\{d_1\}$.

(d) $G_1\alpha_1 \cap G_2\alpha_1 \neq \varnothing$.

Four other mutually equivalent conditions are obtained by transposing the subscripts 1 *and* 2 *above.*

Proof. (a) $\Rightarrow$ (b). Choose any $c_1 \in G_1$ (which is non-empty). Then $c_1\alpha_2\alpha_1 \in G_1 = G_1\alpha_1$ from (a), so $\exists d_1 \in G_1$: $c_1\alpha_2\alpha_1 = d_1\alpha_1$.

$$\begin{aligned} \therefore \forall x_2 \in G_2, \quad x_2\alpha_1 &= c_1 x_2 \beta\alpha_2\alpha_1 && 2.5(b) \\ &= c_1\alpha_2\alpha_1 x_2\alpha_2\alpha_1\beta && 2.5(c) \\ &= d_1\alpha_1 x_2\alpha_2\alpha_1\beta && \\ &= d_1 x_2\alpha_2\beta\alpha_1 && 2.5(c) \\ &= d_1, && 2.5(b) \end{aligned}$$

$\therefore G_2\alpha_1 = \{d_1\}$, so d_1 is unique. Also $d_1\alpha_1 = c_1\alpha_2\alpha_1 \in G_2\alpha_1 = \{d_1\}$. Hence (b).

(b) $\Rightarrow$ (c). From (b), $G_2\alpha_1 = \{d_1\} = \{d_1\alpha_1\} \subseteq G_1\alpha_1$. Hence (c).

(c) $\Rightarrow$ (d) obviously.

(d) $\Rightarrow$ (a). From (d), $\exists a_1 \in G_1, \exists a_2 \in G_2$: $a_1\alpha_1 = a_2\alpha_1$.

$$\begin{aligned} \therefore \forall x_1 \in G_1, \qquad x_1 &= x_1 a_2 \beta\alpha_1 && 2.5(\text{b}) \\ &= x_1\alpha_1 a_2\alpha_1\beta && 2.5(\text{c}) \\ &= x_1\alpha_1 a_1\alpha_1\beta && \\ &= x_1 a_1 \beta\alpha_1 && 2.5(\text{c}) \\ &\in G_1\alpha_1. && \end{aligned}$$

$\therefore G_1 \subseteq G_1\alpha_1$. Also $G_1\alpha_1 \subseteq G\alpha_1 = G_1$. Hence (a). Similar proofs apply throughout when subscripts 1 and 2 are transposed. ∎

COROLLARY. *Under the conditions* 3.3(a)–(d), $\rho_{d_1}|_{G_1}$ *is the (two-sided) inverse automorphism of* $\alpha_1|_{G_1}$. *Under the corresponding conditions with transposed subscripts,* $\lambda_{d_2}|_{G_2}$ *is the (two-sided) inverse automorphism of* $\alpha_2|_{G_2}$.

Proof. Rewrite the corollary to Theorem 3.2 using 3.3(a) and (b), and also 2.5(c). ∎

Of the two statements in 3.3(b), the invariance under α_1 is the vital one. Condition 3.3(b) can in fact be weakened to $G_2\alpha_1{}^2 = G_2\alpha_1$ with impunity, but Examples 7.5–7.7 show that it cannot be weakened merely to $|G_2\alpha_1| = 1$.

Conditions 3.3(a)–(d) are not equivalent to the four conditions obtained from them by transposing subscripts. Examples 7.6 (with $k \geq 1$) satisfy the former but not the latter, and Examples 7.5 and 7.8 satisfy the latter but not the former. However the next theorem deals with the situation where both are satisfied simultaneously.

THEOREM 3.4. *In any internal direct product* $(G, \beta, \alpha_1, \alpha_2)$ *with* $G_1 = G\alpha_1$ *and* $G_2 = G\alpha_2$, *the following conditions are equivalent.*

(a) $G_1\alpha_1 = G_1$ *and* $G_2\alpha_2 = G_2$.

(b) $G_2\alpha_1 = G_1\alpha_2 =$ *a singleton* $\{d\}$ *with* $d\alpha_1 = d\alpha_2 = d$.

(c) $G_1 \cap G_2 =$ *a singleton* $\{d\}$.

(d) $G_1 \cap G_2 \neq \varnothing$.

Proof. (a) $\Rightarrow$ (b). From (a), Theorem 3.3 gives $G_2\alpha_1 = \{d_1\}$ with $d_1\alpha_1 = d_1$, and $G_1\alpha_2 = \{d_2\}$ with $d_2\alpha_2 = d_2$. Also $d_2\alpha_1 \in G_2\alpha_1 =$

$\{d_1\}$, so $d_2\alpha_1 = d_1$. Similarly $d_1\alpha_2 = d_2$.

$$\begin{aligned} \therefore\ d_1 &= d_1\alpha_1 d_1\alpha_2\beta && 2.5(\mathrm{a}) \\ &= d_1 d_2\beta = d_2\alpha_1 d_2\alpha_2\beta = d_2 && 2.5(\mathrm{a}) \\ &= d, \end{aligned}$$

say. Hence (b).

(b) $\Rightarrow$ (c). From (b), $d \in G_2\alpha_1 \cap G_1\alpha_2 \subseteq G_1 \cap G_2$. Also $\forall x \in G_1 \cap G_2$, $x\alpha_1 \in (G_1 \cap G_2)\alpha_1 \subseteq G_2\alpha_1 = \{d\}$ and

$$x\alpha_2 \in (G_1 \cap G_2)\alpha_2 \subseteq G_1\alpha_2 = \{d\},$$

so

$$\begin{aligned} x &= x\alpha_1 x\alpha_2\beta && 2.5(\mathrm{a}) \\ &= dd\beta = d && \because \{d\} \text{ is a subgroupoid.} \end{aligned}$$

Hence (c).

(c) $\Rightarrow$ (d) obviously.

(d) $\Rightarrow$ (a). $(G_1 \cap G_2)\alpha_1 \subseteq G_1\alpha_1 \cap G_2\alpha_1$. Hence

$$\begin{aligned} G_1 \cap G_2 \neq \varnothing \quad &\Rightarrow \quad G_1\alpha_1 \cap G_2\alpha_1 \neq \varnothing \\ &\Rightarrow \quad G_1\alpha_1 = G_1 && 3.3 \end{aligned}$$

and similarly

$$G_2\alpha_2 = G_2.$$

▮

COROLLARY 1. *Under the conditions* 3.4(a)–(d), $\rho_d|_{G_1}$ *and* $\alpha_1|_{G_1}$ *are inverse automorphisms,* $\lambda_d|_{G_2}$ *and* $\alpha_2|_{G_2}$ *are inverse automorphisms, and* $dd\beta = d$.

Proof. In the corollary to Theorem 3.3, put $d_1 = d_2 = d$. Also observe that $\{d\}$ is a subgroupoid. ▮

COROLLARY 2. $|G_1 \cap G_2|$ *can only be* 1 *or* 0.

Proof. 3.4(c) $\Leftrightarrow$ 3.4(d). ▮

Cases with $|G_1 \cap G_2| = 1$ satisfy all the Theorem 3.4 conditions and clearly generalize traditional internal direct products, whereas cases with $|G_1 \cap G_2| = 0$ satisfy none of the Theorem 3.4 conditions and are much more novel. Examples 7.5–7.9 (and Section 8) exhibit some of the tangled consequences of $G_1 \cap G_2 = \varnothing$. In the meantime Theorems 3.5–6.6 explore the tidier situation where $G_1 \cap G_2 = \{d\}$.

First, Theorem 3.4 Corollary 1 has something like a converse.

THEOREM 3.5. *If a groupoid (G_1, β) has an idempotent element d_1 such that ρ_{d_1} is an automorphism, then G_1 is isomorphic to the left factor of an internal direct product.*

If a groupoid (G_2, β) has an idempotent element d_2 such that λ_{d_2} is an automorphism, then G_2 is isomorphic to the right factor of an internal direct product.

Any such G_1 can be combined with any such G_2, and the resulting internal direct product satisfies the Theorem 3.4 conditions.

Proof. Both G_1 and G_2 will be needed in the construction. So, if given only G_1, let G_2 be the trivial group $\{d_2\}$; if given only G_2, let G_1 be the trivial group $\{d_1\}$. Then in all cases both G_1 and G_2 are available.

ρ_{d_1} is an automorphism of G_1, so $\rho_{d_1}{}^{-1}$ is also. Likewise $\lambda_{d_2}{}^{-1}$ is an automorphism of G_2. Hence we can define within $G_1 \times G_2$

$$(x_1, x_2)\alpha_1 = \left(x_1 \rho_{d_1}{}^{-1}, d_2\right),$$

and

$$(x_1, x_2)\alpha_2 = \left(d_1, x_2 \lambda_{d_2}{}^{-1}\right).$$

The verification of 2.5(a)–(c) is then quite routine, and is omitted here. Also

$$\begin{aligned}(G_1 \times G_2)\alpha_1 &= G_1 \times \{d_2\} &\quad &\because \rho_{d_1}{}^{-1} \text{ is an automorphism of } G_1\\ &\cong G_1 &\quad &\because d_2 \text{ is idempotent,}\end{aligned}$$

and similarly $(G_1 \times G_2)\alpha_2 = \{d_1\} \times G_2 \cong G_2$. Thus $(G_1 \times G_2, \beta)$ is an internal direct product of subgroupoids isomorphic to G_1 and G_2. These subgroupoids have intersection $(G_1 \times \{d_2\}) \cap (\{d_1\} \times G_2) = \{(d_1, d_2)\}$ as in Theorem 3.4(c). ▮

The main difficulty in applying Theorem 3.5 is notational. In practice the construction is easier if the embeddings of G_1 and G_2 are treated as identifications. For this write $d_1 = d_2 = d$, choose notation to make $G_1 \cap G_2 = \{d\}$ only, and write the elements of G as $x_1 x_2 \beta$ rather than (x_1, x_2). Then G_1 and G_2 are themselves subgroupoids of G, as in Examples 7.2–7.4.

The idempotence condition $dd\beta = d$ is vital to the Theorem 3.5 construction. In Example 7.1, both ρ_a and ρ_b are automorphisms of G_1, and both λ_f and λ_g are automorphisms of G_2; but the direct product cannot be made internal as above, because none of a, b, f, and g is idempotent.

4. ANALYZING THE TRADITION

It is now possible to look in more detail at the traditional properties 2.6(a)–(f) and relations between them. Theorem 3.4 already includes 2.6(d) and (e), and the following theorems treat internal direct products which are closer to tradition in various other ways. The first of them involves 2.6(e) and (b).

THEOREM 4.1. *If $G_1 \cap G_2 = \{d\}$, and d commutes with every element of G, then every element of G_1 commutes with every element of G_2.*

Proof. $\forall x_1 \in G_1, \forall x_2 \in G_2,$

$$
\begin{aligned}
x_2 x_1 \beta &= d x_2 \alpha_2 \beta x_1 \alpha_1 d \beta\beta && \text{2.5(a), 3.4 Corollary 1}\\
&= d x_1 \alpha_1 \beta x_2 \alpha_2 d \beta\beta && \text{2.4} \quad (\because d \in G_1 \text{ and } d \in G_2)\\
&= x_1 \alpha_1\, d\beta\, d x_2 \alpha_2\, \beta\beta && \text{if } d \text{ commutes with every element}\\
&= x_1 x_2 \beta. && \text{2.5(a), 3.4 Corollary 1}
\end{aligned}
$$

▮

This generalizes the traditional proof which assumes an identity element and simply substitutes it for x_1 and y_2 in Theorem 2.4.

The converse of Theorem 4.1 is false. Example 7.6 (with $k = 1$) has every element of G_1 commuting with every element of G_2, but $G_1 \cap G_2$ empty.

Now, 2.6(f) includes the claim that $\alpha_1|_{G_1}$ is an identity mapping. Theorem 3.3 and its corollary give conditions merely for $\alpha_1|_{G_1}$ to be an automorphism, but by an easy deduction it is idempotent and hence the identity mapping of G_1 if and only if $G_2\alpha_1 = \{d_1\}$ where d_1 is a right identity element for G_1. Rather than spend time on that and on the corresponding result for α_2, the next theorem strengthens Theorem 3.4 (involving both α_1 and α_2) to the whole of 2.6(f).

THEOREM 4.2. *$\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ are both identity mappings (i.e., $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$) if and only if some element d in $G_1 \cap G_2$ is both a right identity element for G_1 and a left identity element for G_2. In such a case $\{d\} = G_1 \cap G_2 = G_2\alpha_1 = G_1\alpha_2$.*

Proof. Assume both $\alpha_1|_{G_1} = 1_{G_1}$ and $\alpha_2|_{G_2} = 1_{G_2}$. This is a special case of 3.4(a). Hence 3.4(c) and (b) give $\{d\} = G_1 \cap G_2 = G_2\alpha_1 = G_1\alpha_2$, and 3.4 Corollary 1 gives $\rho_d|_{G_1} = 1_{G_1}$ and $\lambda_d|_{G_2} = 1_{G_2}$ so d has the required identity properties.

Conversely, assume an element d in $G_1 \cap G_2$ is both a right identity element for G_1 and a left identity element for G_2. Then 3.4(d) holds, so 3.4(c) and (b) give $\{d\} = G_1 \cap G_2 = G_2\alpha_1 = G_1\alpha_2$, and 3.4 Corollary 1 gives $\alpha_1|_{G_1} = 1_{G_1}$ and $\alpha_2|_{G_2} = 1_{G_2}$. ∎

Łoś included 2.6(f) in his definition of internal direct product [6, p. 34, (1)], but he missed the consequence that $G_1 \cap G_2 \neq \emptyset$. (Cf. [6, p. 35, Lemma 3].)

An effect of combining 2.6(b) and (f) can be inferred from 4.2 as

THEOREM 4.3. *If every element of G_1 commutes with every element of G_2, and $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$, then G has an identity element.*

Proof. Assuming $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$, apply Theorem 4.2 but write e instead of d. Then in particular $e \in G_1$, and e is a left identity element for G_2. Thus, assuming every element of G_1 commutes with every element of G_2, e is also a right identity element for G_2.

$\therefore \forall x \in G,$	$xe\beta = x\alpha_1 x\alpha_2\,\beta ee\beta\beta$	2.5(a)
	$= x\alpha_1 e\beta x\alpha_2 e\beta\beta$	2.4, $\because e \in G_1$ and $e \in G_2$
	$= x\alpha_1 x\alpha_2\,\beta$	$\because e$ is a right identity for G_1 and G_2
	$= x$	2.5(a)
	$= ex\beta$	similarly.

∎

The last of these miscellaneous theorems combines 2.6(a) and (e).

THEOREM 4.4. *If G has identity element e, and $G_1 \cap G_2 = \{d\}$, then $d = e$.*

Proof. Assume $G_1 \cap G_2 = \{d\}$ as in 3.4(c). Then 3.4(b) gives $d = d\alpha_1 = d\alpha_2$, so

$d\alpha_1 = d\alpha_1^2 = ed\beta\alpha_1^2$	assuming identity element e
$= e\alpha_1 d\alpha_1\,\beta\alpha_1$	2.5(c)
$= e\alpha_1\,d\alpha_2\,\beta\alpha_1$	
$= e\alpha_1,$	2.5(b)

and similarly $d\alpha_2 = e\alpha_2$. Combine these two equations using 2.5(a). ∎

Theorem 4.4 does *not* assert that if G has identity element e, then $G_1 \cap G_2 = \{e\}$: the hypothesis $G_1 \cap G_2 = \{d\}$ is vital. Some of Examples 7.8 show that when $G_1 \cap G_2$ is empty, G may yet have an identity element somewhere else.

Implications connecting 2.6(a)–(f) can now be assembled. Theorem 4.2 shows that (f) $\Rightarrow$ (e). Theorem 3.4 shows that (e) $\Leftrightarrow$ (d). Trivially (d) $\Rightarrow$ (c). Theorems 4.4 and 4.1 show that (a) $\wedge$ (e) $\Rightarrow$ (b). Theorems 4.4 and 4.2 show that (a) $\wedge$ (e) $\Rightarrow$ (f). Theorem 4.3 shows that (b) $\wedge$ (f) $\Rightarrow$ (a).

From all that, the appropriate Boolean disentangling sorts internal direct products into 12 classes partially ordered by inclusion as in Fig. 4.5. (For simplicity, one-sided conditions such as in Theorem 3.3 are not included here.) The figure's notation (c, d, e), for example, means the class of all internal direct products satisfying 2.6(c), (d), and (e).

The 12 classes in the figure are all distinct. Example 7.2 satisfies 2.6(c–f) but neither (a) nor (b), so every class containing (c, d, e, f) is distinct from every class contained in (a) or (b). Example 7.3 shows likewise that every class containing (b, c, d, e) is distinct from every class contained in (a) or (c, d, e, f). Various cases of Example 7.8 show that every class containing (a, b, c) is distinct from every class contained in (c, d, e), every class containing (a, b) is distinct from every class contained in (c), and every class containing (a, c) is distinct from every class contained in (b).

Perhaps Theorem 3.4 Corollary 2 locates the major watershed, between internal direct products having $|G_1 \cap G_2| = 1$ and those having $|G_1 \cap G_2| = 0$, i.e., between members and non-members of (c, d, e). Within that class, Fig. 4.5 shows that (b) and (f) (essentially as in Theorems 4.1 and 4.2) may be regarded as two independent components which together strengthen the condition $|G_1 \cap G_2| = 1$ to the fully classical (a, b, c, d, e, f).

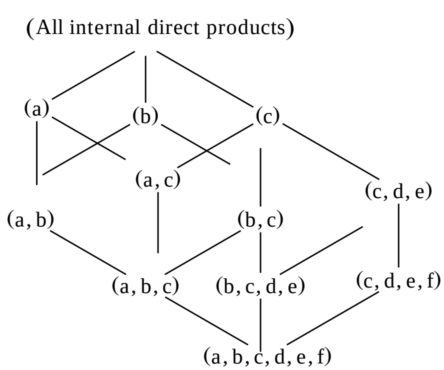

FIGURE 4.5

5. STANDARD SPECIAL PROPERTIES

Some familiar groupoid properties force internal direct products to satisfy 2.6(c, d, e) or even (c, d, e, f). The next three theorems show why. As usual $G \neq \varnothing$ from Definition 2.2.

THEOREM 5.1. *If G is finite, then* $|G_1 \cap G_2| = 1$.

Proof. By Theorem 3.2, $\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ are injective. Hence in the finite case $G_1\alpha_1 = G_1$ and $G_2\alpha_2 = G_2$. The result now follows from Theorem 3.4. ∎

THEOREM 5.2. *If G satisfies either the right or left cancellation law, then* $|G_1 \cap G_2| = 1$.

Proof. Assume G satisfies the left cancellation law. Choose any $a_1 \in G_1$ and $a_2 \in G_2$. Then

$$\begin{aligned} a_1\alpha_1 a_1\alpha_2\beta &= a_1 && 2.5(\mathrm{a}) \\ &= a_1 a_2 \beta\alpha_1 && 2.5(\mathrm{b}) \\ &= a_1\alpha_1 a_2\alpha_1\beta. && 2.5(\mathrm{c}) \end{aligned}$$

Cancel $a_1\alpha_1$ from the left. $\therefore\ a_1\alpha_2 = a_2\alpha_1$. $\therefore\ G_1 \cap G_2 \neq \varnothing$. The result now follows from Theorem 3.4.

The proof is similar if G has right cancellation. ∎

It may be observed that the foregoing proof really requires only one left cancellable element in $G_1\alpha_1$ or one right cancellable element in $G_2\alpha_2$.

The last two theorems show that either finiteness or a cancellation law puts an internal direct product into the class (c, d, e) of Fig. 4.5. However, it need not be in either of the subclasses (b, c, d, e) or (c, d, e, f): Example 7.4 is a finite cancellation groupoid in which d is neither central (as in 4.1) nor a one-sided identity element for G_1 or G_2 (as in 4.2).

The next result is stronger: the associative law puts an internal direct product into the class (c, d, e, f). In fact 2.6(f) is equivalent to a restricted associative condition, as follows.

THEOREM 5.3. $\alpha_1^{\,2} = \alpha_1$ *and* $\alpha_2^{\,2} = \alpha_2$ *if and only if*

$$\forall x_1 \in G_1, \qquad \forall y \in G, \qquad \forall z_2 \in G_2, \qquad x_1 y\beta z_2\beta = x_1 y z_2\beta\beta.$$

Proof. Assume every $x_1 y\beta z_2\,\beta = x_1 y z_2\,\beta\beta$. Choose any $a_2, b_2 \in G_2$. Then

$$\begin{aligned}
\forall x_1 \in G_1, \quad x_1\alpha_1 &= x_1\alpha_1 a_2 b_2\,\beta\beta\alpha_1 && \text{2.5(b)}\\
&= x_1\alpha_1 a_2\,\beta b_2\,\beta\alpha_1 && \text{from the above assumption}\\
&= x_1\alpha_1 a_2\,\beta\alpha_1 b_2\alpha_1\,\beta && \text{2.5(c)}\\
&= x_1\alpha_1 b_2\alpha_1\,\beta && \text{2.5(b)}\\
&= x_1 b_2\,\beta\alpha_1 && \text{2.5(c)}\\
&= x_1. && \text{2.5(b)}
\end{aligned}$$

$\therefore\ \alpha_1|_{G_1} = 1_{G_1}$. Similarly $\alpha_2|_{G_2} = 1_{G_2}$.

Conversely, assume $\alpha_1^2 = \alpha_1$ and $\alpha_2^2 = \alpha_2$. Then an element d has properties as in Theorem 4.2. Now, for every $x_1 \in G_1$ and $y \in G$,

$$\begin{aligned}
x_1 y\beta\alpha_1 &= x_1\alpha_1 y\alpha_1\,\beta && \text{2.5(c)}\\
\therefore\ x_1 y\beta\alpha_1 d\beta &= x_1\alpha_1 y\alpha_1 d\,\beta\beta && \because d \text{ is a right identity for } G_1, \text{ by 4.2.}
\end{aligned}$$

Hence, for every $z_2 \in G_2$,

$$\begin{aligned}
x_1 y\beta\alpha_1 z_2\alpha_1\,\beta &= x_1\alpha_1 y\alpha_1 z_2\alpha_1\,\beta\beta && \because G_2\alpha_1 = \{d\} \text{ by 4.2}\\
\therefore\ x_1 y\beta z_2\,\beta\alpha_1 &= x_1\alpha_1 y z_2\,\beta\alpha_1\,\beta && \text{2.5(c)}\\
&= x_1 y z_2\,\beta\beta\alpha_1. && \text{2.5(c)}
\end{aligned}$$

Similarly $x_1 y\beta z_2\,\beta\alpha_2 = x_1 y z_2\,\beta\beta\alpha_2$. Combine the last two equations using 2.5(a). $\therefore\ x_1 y\beta z_2\,\beta = x_1 y z_2\,\beta\beta$. ∎

COROLLARY. *If G is a semigroup, then $\alpha_1^2 = \alpha_1$, $\alpha_2^2 = \alpha_2$, and $G_1 \cap G_2 = \{d\}$ where d is a right identity element for G_1 and a left identity element for G_2.*

Proof. The equation in Theorem 5.3 is a restricted case of the associative law. The result follows from Theorems 5.3 and 4.2. ∎

The corollary shows that the associative law puts an internal direct product into the class (c, d, e, f) of Fig. 4.5. However it need not be in the subclass (a, b, c, d, e, f): Example 7.2 is a semigroup in which d is neither an identity element (as in 4.4) nor even central (as in 4.1).

Theorems 5.1–5.3 ensure that most well-known groupoids can form internal direct products only with $|G_1 \cap G_2| = 1$. Any example with $|G_1 \cap G_2| = 0$ must be infinite and without the associative law or either cancellation law. (Perhaps this helps to explain why Examples 7.5–7.9 may seem far-fetched.) However, the commutative law does not force $|G_1 \cap G_2|$ to be 1: Example 7.6 (with $k = 1$) is commutative but has $G_1 \cap G_2$ empty.

Also the condition $|G_1 \cap G_2| = 1$ does not enforce any of the special properties of Theorems 5.1–5.3: Example 7.6 (with $k = 0$) has $|G_1 \cap G_2| = 1$ but enjoys neither finiteness nor the associative nor cancellation law.

6. CLOSER TO TRADITION

Consider any internal direct product G having $|G_1 \cap G_2| = 1$, and hence belonging to the class (c, d, e) of Fig. 4.5. Although G need not satisfy 2.6(f) and the corresponding properties in 4.2, it is always *isotopic* to a groupoid which has those properties and so belongs to the class (c, d, e, f). If G is in (b, c, d, e), then the isotopic groupoid is in (a, b, c, d, e, f).

Several construction steps lead to the main result in Theorem 6.6.

DEFINITIONS 6.1 (of γ_{11}, δ, ε_1, ε_2).

$\forall x \in G$, $x\gamma_{11} = x\alpha_1{}^2 x\alpha_2{}^2\beta$.

The binary operation $\delta = \beta\gamma_{11}$.

When $G_1 \cap G_2 = \{d\}$, then $\forall x \in G$, $x\varepsilon_1 = x\alpha_1 d\beta$, and $x\varepsilon_2 = dx\alpha_2\,\beta$.

The notation γ_{11} is part of a systematic notation γ_{mn} mentioned in Section 8.

Theorem 6.6 shows that $(G, \delta, \varepsilon_1, \varepsilon_2)$ is an internal direct product of G_1 and G_2, satisfying 2.6(c–f), and having (G, δ) isotopic to (G, β). The proof uses several lemmas.

LEMMA 6.2. *γ_{11} is an endomorphism of (G, β).*

Proof.

$$\forall x, y \in G, \qquad xy\beta\gamma_{11} = xy\beta\alpha_1{}^2 xy\beta\alpha_2{}^2\beta \qquad 6.1$$
$$= x\alpha_1{}^2 y\alpha_1{}^2\beta x\alpha_2{}^2 y\alpha_2{}^2\beta\beta \qquad 2.5(c)$$
$$= x\alpha_1{}^2 x\alpha_2{}^2\beta y\alpha_1{}^2 y\alpha_2{}^2\beta\beta \qquad 2.4$$
$$= x\gamma_{11} y\gamma_{11}\,\beta. \qquad 6.1$$

∎

LEMMA 6.3. *If* $G_1 \cap G_2 = \{d\}$, *then*

$$\gamma_{11}\varepsilon_1 = \varepsilon_1\gamma_{11} = \alpha_1, \qquad \gamma_{11}\varepsilon_2 = \varepsilon_2\gamma_{11} = \alpha_2,$$

$$G\varepsilon_1 = G_1, \qquad \textit{and} \qquad G\varepsilon_2 = G_2.$$

Proof.

$$\begin{aligned} \forall x \in G, \quad x\gamma_{11}\varepsilon_1 &= x\alpha_1{}^2 x\alpha_2{}^2\beta\alpha_1 d\beta && 6.1\\ &= x\alpha_1{}^2 d\beta && 2.5(\mathrm{b})\\ &= x\alpha_1 && 3.4 \text{ Corollary } 1\end{aligned}$$

$$\begin{aligned} \forall x \in G, \quad x\varepsilon_1\gamma_{11} &= x\alpha_1 d\beta\alpha_1{}^2 x\alpha_1 d\beta\alpha_2{}^2\beta && 6.1\\ &= x\alpha_1{}^2 d\alpha_2\beta && 2.5(\mathrm{b}), \quad \because d \in G_2\\ &= x\alpha_1{}^2 d\beta && 3.4(\mathrm{b})\\ &= x\alpha_1. && 3.4 \text{ Corollary } 1\end{aligned}$$

Similarly $\gamma_{11}\varepsilon_2 = \varepsilon_2\gamma_{11} = \alpha_2$.

$$\begin{aligned} \forall x \in G, \quad x\alpha_1 &= x\gamma_{11}\varepsilon_1 && \text{from above}\\ &\in G\varepsilon_1. \end{aligned}$$

$\therefore G_1 \subseteq G\varepsilon_1$.

$$\begin{aligned} \forall x \in G, \quad x\varepsilon_1 &= x\alpha_1 d\beta && 6.1\\ &\in G_1. && \because d \in G_1\end{aligned}$$

$\therefore G\varepsilon_1 \subseteq G_1$. $\therefore G\varepsilon_1 = G_1$. Similarly $G\varepsilon_2 = G_2$. ∎

LEMMA 6.4. *If* $G_1 \cap G_2 = \{d\}$, *then* $(G, \delta, \varepsilon_1, \varepsilon_2)$ *is an internal direct product*.

Proof. The appropriate versions of 2.5(a)–(c) are verified as follows.

(a)
$$\begin{aligned} \forall x \in G, \quad x\varepsilon_1 x\varepsilon_2\delta &= x\varepsilon_1 x\varepsilon_2\beta\gamma_{11} && 6.1\\ &= x\varepsilon_1\gamma_{11}x\varepsilon_2\gamma_{11}\beta && 6.2\\ &= x\alpha_1 x\alpha_2\beta && 6.3\\ &= x. && 2.5(\mathrm{a})\end{aligned}$$

(b)
$$\begin{aligned} \forall x, y \in G, \quad x\varepsilon_1 y\varepsilon_2\delta\varepsilon_1 &= x\varepsilon_1 y\varepsilon_2\beta\gamma_{11}\varepsilon_1 && 6.1\\ &= x\varepsilon_1 y\varepsilon_2\beta\alpha_1 && 6.3\\ &= x\varepsilon_1; && 2.5(\mathrm{b})\end{aligned}$$

since 6.3 shows $x\varepsilon_1 \in G_1$ and $y\varepsilon_2 \in G_2$.

Similarly $x\varepsilon_1 y\varepsilon_2 \delta\varepsilon_2 = y\varepsilon_2$.

(c) For every x, y ∈ G,

$$xy\beta\alpha_1 = x\alpha_1 y\alpha_1 \beta \qquad 2.5(c)$$

$$\therefore\ xy\beta\gamma_{11}\varepsilon_1 = x\varepsilon_1\gamma_{11} y\varepsilon_1\gamma_{11}\beta \qquad 6.3$$

$$= x\varepsilon_1 y\varepsilon_1 \beta\gamma_{11} \qquad 6.2$$

$$\therefore\ xy\delta\varepsilon_1 = x\varepsilon_1 y\varepsilon_1 \delta. \qquad 6.1$$

Similarly $xy\delta\varepsilon_2 = x\varepsilon_2 y\varepsilon_2 \delta$. ∎

LEMMA 6.5. *If $G_1 \cap G_2 = \{d\}$, then γ_{11} is an automorphism of (G, β), having inverse $x \mapsto x\varepsilon_1 x\varepsilon_2 \beta$.*

Proof.

$$\forall x \in G, \qquad x\varepsilon_1 x\varepsilon_2 \beta\gamma_{11} = x\varepsilon_1 x\varepsilon_2 \delta \qquad 6.1$$

$$= x, \qquad 6.4(a)$$

and

$$x\gamma_{11}\varepsilon_1 x\gamma_{11}\varepsilon_2 \beta = x\alpha_1 x\alpha_2 \beta \qquad 6.3$$

$$= x. \qquad 2.5(a)$$

Thus $x\gamma_{11}^{-1} = x\varepsilon_1 x\varepsilon_2 \beta$, so the endomorphism γ_{11} of Lemma 6.2 is in fact an automorphism. ∎

Lemma 6.2 does not require that $G_1 \cap G_2 \neq \emptyset$, but Lemma 6.5 does: cf. Examples 7.5–7.9 and Section 8. (In fact the converse of Lemma 6.5 can be proven.)

THEOREM 6.6. *Any internal direct product $(G, \beta, \alpha_1, \alpha_2)$ with $G_1 \cap G_2 = \{d\}$ has the following properties.*

(a) *(G, β) is isotopic to (G, δ).*

(b) *$(G, \delta, \varepsilon_1, \varepsilon_2)$ is an internal direct product of G_1 and G_2.*

(c) *$\varepsilon_1^2 = \varepsilon_1$ and $\varepsilon_2^2 = \varepsilon_2$.*

(d) *d is a right identity element for (G_1, δ) and a left identity element for (G_2, δ).*

Proof.

(a) γ_{11} is a permutation of G 6.5

and

$$\forall x, y \in G, \qquad xy\beta\gamma_{11} = xy\delta. \qquad 6.1$$

$\therefore$ $(1_G, 1_G, \gamma_{11})$ is an isotopism from (G, β) onto (G, δ). (Lemma 6.5 shows that $(\gamma_{11}, \gamma_{11}, 1_G)$ is another.)

(b) Lemma 6.3 shows $G\varepsilon_1 = G_1$ and $G\varepsilon_2 = G_2$. The result then follows from Lemma 6.4.

$$
\begin{aligned}
\text{(c)}\quad \forall x \in G, \qquad x\varepsilon_1^{\,2} &= x\alpha_1 d\beta\alpha_1 d\beta && 6.1\\
&= x\alpha_1 d\beta && 2.5(\text{b}),\quad \because \mathrm{d} \in \mathrm{G}_2\\
&= \mathrm{x}\varepsilon_1. && 6.1
\end{aligned}
$$

Similarly $\varepsilon_2^{\,2} = \varepsilon_2$.

(d) now follows from Theorem 4.2. ∎

7. EXAMPLES

In each of the finite examples 7.1–7.4, G_2 is the opposite of G_1. The infinite examples begin with 7.5.

EXAMPLE 7.1. An external direct product which cannot be made internal.

G_1	a	b
a	b	b
b	a	a

G_2	f	g
f	g	f
g	g	f

$G_1 \times G_2$	(a, f)	(a, g)	(b, f)	(b, g)
(a, f)	(b, g)	(b, f)	(b, g)	(b, f)
(a, g)	(b, g)	(b, f)	(b, g)	(b, f)
(b, f)	(a, g)	(a, f)	(a, g)	(a, f)
(b, g)	(a, g)	(a, f)	(a, g)	(a, f)

$G_1 \times G_2$ has no non-trivial subgroupoids at all, so it is certainly not an internal direct product of two subgroupoids. Section 1 referred to this difficulty.

As mentioned at the end of Section 3,

$$\rho_a = \rho_b = \text{the transposition } (a \quad b) = \text{an automorphism of } G_1$$

and

$$\lambda_f = \lambda_g = \text{the transposition } (f \quad g) = \text{an automorphism of } G_2.$$

However none of a, b, f, g is idempotent, so Theorem 3.5 does not apply.

EXAMPLE 7.2. A semigroup which is an internal direct product satisfying 2.6 (c, d, e, f) but neither (a) nor (b).

G_1	d	a
d	d	d
a	a	a

G_2	d	f
d	d	f
f	d	f

Here $dd\beta = d$, ρ_d is the identity automorphism of G_1, and λ_d is the identity automorphism of G_2, so the 3.5 conditions are satisfied. Hence combine G_1 and G_2 as described after Theorem 3.5, writing t for $af\beta$. That gives:

x	d	a	f	t
$x\alpha_1$	d	a	d	a
$x\alpha_2$	d	d	f	f

Thus, for example,

$$\begin{aligned} tf\beta\alpha_1 &= t\alpha_1 f\alpha_1\,\beta && 2.5(\mathrm{c}) \\ &= ad\beta = a && \text{from tables above} \end{aligned}$$

and

$$tf\beta\alpha_2 = t\alpha_2 f\alpha_2\,\beta = ff\beta = f$$

so

$$\begin{aligned} tf\beta &= tf\beta\alpha_1 tf\beta\alpha_2\,\beta && 2.5(\mathrm{a}) \\ &= af\beta = t. \end{aligned}$$

Calculating by components in that way leads to this table:

G	d	a	f	t
d	d	d	f	f
a	a	a	t	t
f	d	d	f	f
t	a	a	t	t

Of course the last two tables could have been given in the first place, but then they would need to be checked for conditions 2.5(a)–(c). The G_1 and G_2 tables occur within the G table.

The table for α_1 and α_2 attests 2.6(f): $\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ are identity mappings. Theorem 4.2 restates this in terms of d. Indeed, G_1 has d (and also a) as a right but not left identity element, and G_2 has d (and also f) as a left but not right identity element. But neither d nor anything else is even a one-sided identity element for G itself, so 2.6(a) is false. Also $a \in G_1$ and $d \in G_2$ but a does not commute with d, so 2.6(b) is false.

Since G_1 and G_2 are obviously associative, G is also.

Finally, G_1 alone (or G_2 alone) illustrates the introductory paragraph of the proof of Theorem 3.5. Since $dd\beta = d$ and ρ_d is an automorphism of G_1, the groupoid G_1 can be seen as the internal direct product of itself and $\{d\}$.

EXAMPLE 7.3. A commutative quasigroup which is an internal direct product satisfying 2.6(b, c, d, e) but neither (a) nor (f).

(G_1, β)	d	a	b
d	d	b	a
a	b	a	d
b	a	d	b

(G_2, β)	d	f	g
d	d	g	f
f	g	f	d
g	f	d	g

Here $\rho_d|_{G_1}$ is the transposition $(a\ b)$ which is easily seen to be an automorphism of G_1. Likewise $\lambda_d|_{G_2}$ is an automorphism of G_2, and also $dd\beta = d$. Hence Theorem 3.5 applies, so G_1 and G_2 can be combined as in the previous example to give a groupoid G of order 9, not tabulated here. By Theorem 3.4 Corollary 1, $\alpha_1|_{G_1}$ is the inverse of $\rho_d|_{G_1}$ so it also is $(a\ b)$, and similarly $\alpha_2|_{G_2} = (f\ g)$.

Now, d commutes with every element, so Theorem 4.1 ensures 2.6(b): every element of G_1 commutes with every element of G_2. As an example of the Theorem 4.1 proof, $ga\beta = df\beta bd\beta\beta = db\beta fd\beta\beta = bd\beta df\beta\beta = ag\beta$. But neither d nor anything else is even a one-sided identity element, so 2.6(a) is false. Also $\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ as given above are not identity mappings, so 2.6(f) is false.

Incidentally, G is a finite commutative quasigroup with $G_1 \cong G_2$, but nevertheless it satisfies neither 2.6(a) nor (f).

This is a convenient example to illustrate Section 6. Using Definition 6.1, γ_{11} turns out to be the permutation $(a\ b)(f\ g)(af\beta\ \ bg\beta)(ag\beta\ \ bf\beta)$ leaving d invariant. Hence $\delta = \beta\gamma_{11}$ gives, in particular, new tables for G_1 and G_2:

(G_1, δ)	d	a	b
d	d	a	b
a	a	b	d
b	b	d	a

(G_2, δ)	d	f	g
d	d	f	g
f	f	g	d
g	g	d	f

These are both groups, so their direct product (G, δ) is just the elementary Abelian group of order 9. Other details from Theorem 6.6 are easy to check.

EXAMPLE 7.4. A finite quasigroup which is an internal direct product satisfying 2.6(c, d, e) but neither (a), (b), nor (f).

G_1	d	a	b	c
d	d	c	a	b
a	b	a	c	d
b	c	d	b	a
c	a	b	d	c

G_2	d	f	g	h
d	d	g	h	f
f	h	f	d	g
g	f	h	g	d
h	g	d	f	h

Here $\rho_d|_{G_1}$ is the cycle $(a\ b\ c)$ which is an automorphism of G_1. Likewise $\lambda_d|_{G_2}$ is an automorphism of G_2, and also $dd\beta = d$. Hence Theorem 3.5 applies, so G_1 and G_2 can be combined as in Example 7.2 to give a groupoid G of order 16, not tabulated here. By Theorem 3.4 Corollary 1, $\alpha_1|_{G_1}$ is the inverse of $\rho_d|_{G_1}$ so it is $(a\ c\ b)$, and similarly $\alpha_2|_{G_2} = (f\ h\ g)$.

G is a finite cancellation groupoid (because both G_1 and G_2 are), and indeed satisfies 2.6(c–e) as Theorems 5.1 and 5.2 require. But it has not even a one-sided identity element, so 2.6(a) is false. Also d commutes with no other element, so 2.6(b) is false. Also $\alpha_1|_{G_1}$ and $\alpha_2|_{G_2}$ as given above are not identity mappings, so 2.6(f) is false.

Like the previous example, this one can be used to illustrate Theorem 6.6, although in this case (G, δ) is not a group.

EXAMPLES 7.5. The simplest internal direct products having $G_1 \cap G_2 = \varnothing$, i.e., not satisfying 2.6(c, d, e).

Theorem 5.1 shows that any such G must be infinite, so at least one of G_1 and G_2 is infinite. Hence the smallest possible orders $|G_1|$ and $|G_2|$ are $\aleph_0$ and 1 (or vice versa), as in these examples.

$$G = \mathbb{N} \cup \{0\} = \{0, 1, 2, \ldots\}.$$

$$\forall m, n \in G, \quad mn\beta = \begin{cases} m - 1 & \text{if } m > n \\ m & \text{if } m = n, \\ m + (n - m)\phi & \text{if } m < n \end{cases}$$

where ϕ is any mapping $\mathbb{N} \to \mathbb{N} \cup \{0\}$. (Different choices of ϕ give different examples.)

$$\forall n \in G, \qquad n\alpha_1 = n + 1 \quad \text{and} \quad n\alpha_2 = 0.$$

Since cases with $G_1 \cap G_2 = \varnothing$ are unfamiliar, 2.5(a)–(c) are here verified in full.

(a)

$$\forall n \in G, \qquad n\alpha_1 n\alpha_2\,\beta = (n+1)0\beta = (n+1) - 1 \qquad \because n+1 > 0$$
$$= n.$$

(b)

$$\forall m, n \in G, \qquad m\alpha_1 n\alpha_2\,\beta\alpha_1 = (m+1)0\beta\alpha_1 = m\alpha_1$$

and

$$m\alpha_1 n\alpha_2\,\beta\alpha_2 = 0 = n\alpha_2.$$

(c)

$$m > n \quad \Rightarrow \quad m\alpha_1 n\alpha_1\,\beta = (m+1)(n+1)\beta = m$$
$$= (m-1)\alpha_1 = mn\beta\alpha_1$$
$$m = n \quad \Rightarrow \quad m\alpha_1 n\alpha_1\,\beta = m\alpha_1 = mn\beta\alpha_1$$
$$m < n \quad \Rightarrow \quad m\alpha_1 n\alpha_1\,\beta = (m+1)(n+1)\beta$$
$$= m + 1 + (n-m)\phi = mn\beta + 1 = mn\beta\alpha_1$$

Also $\forall m, n \in G$, $m\alpha_2 n\alpha_2\,\beta = 00\beta = 0 = mn\beta\alpha_2$.

Hence G is indeed an internal direct product with $G_1 = G\alpha_1 = \mathbb{N}$ and $G_2 = G\alpha_2 = \{0\}$, so $|G_1| = \aleph_0$ and $|G_2| = 1$. Incidentally also $G_1 \cup G_2 = G$, showing another sense in which these examples are small.

Clearly $G_1 \cap G_2 = \varnothing$, so 2.6(e) is false. In fact G is easily seen to satisfy 2.6(c) but neither (a), (d), (e), nor (f). It satisfies 2.6(b) if and only if ϕ is the mapping $n \mapsto n - 1$ (as in Example 7.6 with $k = 1$).

It follows from Theorems 5.1–5.3 that G is neither finite nor a cancellation groupoid nor a semigroup. Indeed, $00\beta = 0 = 10\beta$ and $20\beta = 1 = 21\beta$ are counter-examples to the two cancellation laws, and $21\beta0\beta = 10\beta = 0 \neq 1 = 20\beta = 210\beta\beta$ is a counter-example to the associative law.

Since $G_1\alpha_1 = \{2, 3, 4, \ldots\} \neq G_1$, the conditions 3.3(a)–(d) are not satisfied. However, they are satisfied when the subscripts 1 and 2 are transposed, since $G_2\alpha_2 = \{0\} = G_2$.

Section 6 mentioned that although Lemma 6.2 showed γ_{11} to be an endomorphism in all cases, Lemma 6.5 only showed it to be an automorphism in cases where $G_1 \cap G_2 \neq \varnothing$. In the present examples, $n\gamma_{11} = n\alpha_1^{\,2} n\alpha_2^{\,2}\beta = (n+2)0\beta = n + 1$, so indeed γ_{11} is not surjective. (Similar remarks apply in Examples 7.6–7.9.)

Theorem 2.5(c) shows that α_1 is an endomorphism, and in these examples α_1: $n \mapsto n + 1$ is injective. Hence $G\alpha_1$, $G\alpha_1^{\,2}$, etc. are subgroupoids isomorphic to G itself, and thus internal direct products in their own right. In particular, $G\alpha_1 = \{1, 2, 3, 4, \ldots\}$ is an internal direct product of $G\alpha_1{}^2 = \{2, 3, 4, \ldots\}$ and $\{1\}$. Thus

$$\begin{aligned} G &= \text{internal direct product of } G\alpha_1 \text{ and } G\alpha_2 \\ &= \text{internal direct product of } \big(\text{internal direct product of } G\alpha_1{}^2 \text{ and } \{1\}\big) \\ &\quad \text{and } \{0\}. \end{aligned}$$

But this *internal* direct multiplication is not associative. There is no internal direct product of $\{1\}$ and $\{0\}$; since their external direct product has order one, whereas there are two elements to accommodate. Hence any extension of internal direct products to more than two factors requires special care.

EXAMPLES 7.6. Groupoids from 7.5 which factorize as internal direct products in more than one way.

With the same (G, β) as in Examples 7.5, can the projections α_1 and α_2 be different from before? This can happen if (and in fact only if) there is some integer $k \geq 0$ such that $\forall n \geq \mathrm{Max}(k, 1)$, $n\phi = n - k$. (If $k \geq 2$, the values $1\phi, 2\phi, \ldots, (k-1)\phi$ are still arbitrary non-negative integers.) Then (G, β) is an internal direct product not only as in 7.5 but also in a new way where $\forall n \in G$, $n\alpha_1 = 0$, and $n\alpha_2 = n + k$. The verification of 2.5(a)–(c) is omitted this time. $G_1 = G\alpha_1 = \{0\}$ and $G_2 = G\alpha_2 = \{k, k+1, k+2, \ldots\}$.

Case $k = 0$.

$$\forall n \in \mathbb{N}, \qquad n\phi = n, \quad \text{so } mn\beta = \begin{cases} m - 1 & \text{if } m > n \\ n & \text{if } m \leq n \end{cases}$$

$$\forall n \in G, \qquad n\alpha_1 = 0 \quad \text{and} \quad n\alpha_2 = n.$$

G is the internal direct product of $G_1 = \{0\}$ and $G_2 = G = \mathbb{N} \cup \{0\}$.

Hence $G_1 \cap G_2 = \{0\} \neq \varnothing$, and indeed G satisfies all of 2.6(c–f) as in Theorem 4.2. Observe then that a groupoid may factorize as an internal direct product in one way which satisfies 2.6(e) as well as in another way (cf. 7.5) which does not.

Case $k = 1$.

$$\forall n \in \mathbb{N}, \qquad n\phi = n - 1, \quad \text{so } mn\beta = \begin{cases} m - 1 & \text{if } m > n \\ m & \text{if } m = n \\ n - 1 & \text{if } m < n \end{cases}$$

$$\forall n \in G, \qquad n\alpha_1 = 0 \quad \text{and} \quad n\alpha_2 = n + 1.$$

G is the internal direct product of $G_1 = \{0\}$ and $G_2 = \mathbb{N}$.

The formula for $mn\beta$ shows that G is commutative, so the commutative law does not force $G_1 \cap G_2$ to be non-empty (as mentioned near the end of Section 5). In particular, every element of G_1 commutes with every element of G_2, and indeed G satisfies 2.6(b, c) but neither (a), (d), (e), nor (f), so the converse of Theorem 4.1 is false.

The present factorization of G as an internal direct product is just the 7.5 factorization with the subscripts 1 and 2 transposed. The commutative law makes this possible.

Cases $k \geq 1$ have $G_1\alpha_1 = \{0\} = G_1$ and $G_2\alpha_2 = \{2k, 2k+1, 2k+2, \ldots\} \neq G_2$, so they satisfy the conditions 3.3(a)–(d) but not the same conditions with the subscripts 1 and 2 transposed.

Cases $k \geq 2$ have $G \neq G_1 \cup G_2$.

EXAMPLES 7.7. Groupoids resembling Examples 7.5 but with G_1 and G_2 both infinite.

$$G = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\}).$$

$\forall m, n, p, q \in \mathbb{N} \cup \{0\}$, $(m, n)(p, q)\beta$ is the ordered pair whose first component

$$= \begin{cases} m-1 & \text{if } m > p \\ m & \text{if } m = p \\ m + (p-m)\phi_1 & \text{if } m < p \end{cases}$$

and second component

$$= \begin{cases} q-1 & \text{if } q > n \\ q & \text{if } q = n \\ q + (n-q)\phi_2 & \text{if } q < n, \end{cases}$$

where ϕ_1 and ϕ_2 are any mappings $\mathbb{N} \to \mathbb{N} \cup \{0\}$. (Different choices of ϕ_1 and ϕ_2 give different examples.)

$\forall m, n \in \mathbb{N} \cup \{0\}$,

$$(m, n)\alpha_1 = (m+1, 0) \quad \text{and} \quad (m, n)\alpha_2 = (0, n+1).$$

Verification of 2.5(a)–(c) is no harder than in Examples 7.5. $G_1 = G\alpha_1 = \mathbb{N} \times \{0\}$ and $G_2 = G\alpha_2 = \{0\} \times \mathbb{N}$, so $G_1 \cap G_2 = \varnothing$. Indeed G is easily seen to satisfy 2.6(c) but neither (a), (d), (e), nor (f). It satisfies 2.6(b) if and only if $\forall n \in \mathbb{N}$, $n\phi_1 = n\phi_2 = n - 1$, in which case G is commutative. It violates both parts of 3.4(a), since $G_1\alpha_1 \neq G_1$ and $G_2\alpha_2 \neq G_2$.

EXAMPLES 7.8. Internal direct products in which $|G_1\alpha_2| = 1$ but $|G_2\alpha_1|$ is any non-zero cardinal number at all, and the other traditions also can be violated.

Let (I, ω) be any groupoid whose subgroupoid J has identity element j. (This j need not be an identity for the whole of I.) Let χ and ψ be mappings $\mathbb{N} \times I \to \mathbb{N} \cup \{0\}$, arbitrary except that $\forall n \in \mathbb{N}$, $\forall y \in J$, $ny\chi = n - 1$.

Then G is the set $(\mathbb{N} \cup \{0\}) \times I \times J$.

$\forall m, n \in \mathbb{N} \cup \{0\}$, $\forall u, v \in I$, $\forall x, y \in J$,

$$(m,u,x)(n,v,y)\beta = \begin{cases} ((m-n)v\chi + n, u, xy\omega) & \text{if } m > n \\ (n, uv\omega, xy\omega) & \text{if } m = n \\ ((n-m)u\psi + m, v, xy\omega) & \text{if } m < n \end{cases}$$

$$(m,u,x)\alpha_1 = (m+1, u, j)$$

$$(m,u,x)\alpha_2 = (0, x, x).$$

After all that, it may be heartening to observe that Examples 7.5 are the special cases having $I = J = \{j\}$ and $nj\chi = n - 1$, with (n, j, j) abbreviated to n, and $nj\psi$ abbreviated to $n\phi$. Even so, perhaps the check of 2.5(a)–(c) had better not be left to the hapless reader.

First,

$$\begin{aligned}(m,u,x)\alpha_1(n,v,y)\alpha_2\,\beta &= (m+1,u,j)(0,y,y)\beta \\ &= ((m+1)y\chi, u, jy\omega) && \because m+1 > 0 \\ &= (m,u,y) && \because y \in J.\end{aligned}$$

Thus

(a) $$(m,u,x)\alpha_1(m,u,x)\alpha_2\,\beta = (m,u,x)$$

(b) $$\begin{aligned}(m,u,x)\alpha_1(n,v,y)\alpha_2\,\beta\alpha_1 &= (m,u,y)\alpha_1 \\ &= (m+1,u,j) \\ &= (m,u,x)\alpha_1\end{aligned}$$

$$\begin{aligned}(m,u,x)\alpha_1(n,v,y)\alpha_2\,\beta\alpha_2 &= (m,u,y)\alpha_2 \\ &= (0,y,y) \\ &= (n,v,y)\alpha_2\end{aligned}$$

(c) From the definitions of β and α_1,

$$\begin{aligned}(m,u,x)(n,v,y)\beta\alpha_1 &= \begin{cases} ((m-n)v\chi + n + 1, u, j) & \text{if } m > n \\ (n+1, uv\omega, j) & \text{if } m = n \\ ((n-m)u\psi + m + 1, v, j) & \text{if } m < n \end{cases} \\ &= (m+1,u,j)(n+1,v,j)\beta \qquad \because jj\omega = j \\ &= (m,u,x)\alpha_1(n,v,y)\alpha_1\,\beta.\end{aligned}$$

$$\begin{aligned}(m,u,x)(n,v,y)\beta\alpha_2 &= (\text{something}, \text{something}, xy\omega)\alpha_2\\ &= (0, xy\omega, xy\omega)\\ &= (0,x,x)(0,y,y)\beta\\ &= (m,u,x)\alpha_2(n,v,y)\alpha_2\beta.\end{aligned}$$

The definitions of α_1 and α_2 show that $G_1 = \{(p,u,j) : p \in \mathbb{N}, u \in I\}$, so $G_1\alpha_2 = \{(0,j,j)\}$; and also that $G_2 = \{(0,x,x) : x \in J\}$, so $G_2\alpha_1 = \{(1,x,j) : x \in J\}$. It follows easily that 2.6(d)–(f) are all false. However 2.6(a)–(c) depend upon the choice of I, J, χ, and ψ.

Conditions for 2.6(c) are almost immediate. From the previous paragraph $|G_1\alpha_2| = 1$, but the definition of β shows that $(G_2\alpha_1, \beta) \cong (J, \omega)$. Thus G satisfies 2.6(c) if and only if $|J| = 1$, i.e., $J = \{j\}$. However $G_2\alpha_1$ can be isomorphic to any groupoid with identity, and hence of any non-zero order. This makes Examples 7.8 very different from orthodox internal direct products as in 2.6 or even in 3.4 where every member of G_2 has the same first component d.

Next consider a condition which is sufficient (and in fact necessary) for G to satisfy 2.6(b). Suppose $\forall n \in \mathbb{N}$, $\forall y \in J$, $ny\psi = n - 1$. Then

$$(m+1,u,j)(0,y,y)\beta = ((m+1)y\chi, u, jy\omega) = (m,u,y),$$

and

$$(0,y,y)(m+1,u,j)\beta = ((m+1)y\psi, u, yj\omega) = (m,u,y).$$

Thus every element $(m+1,u,j)$ of G_1 commutes with every element $(0,y,y)$ of G_2, so 2.6(b) is satisfied.

Now consider some conditions which are sufficient (and in fact necessary) for G to satisfy 2.6(a). Suppose (I, ω) has an identity element $i \notin J$, and $\forall n \in \mathbb{N}$, $ni\chi = ni\psi = n$.

Then

$$\begin{aligned}(m,u,x)(0,i,j)\beta &= \begin{cases}(mi\chi, u, xj\omega) & \text{if } m > 0\\ (0, ui\omega, xj\omega) & \text{if } m = 0\end{cases}\\ &= (m,u,x),\end{aligned}$$

and

$$\begin{aligned}(0,i,j)(n,v,y)\beta &= \begin{cases}(0, iv\omega, jy\omega) & \text{if } 0 = n\\ (ni\psi, v, jy\omega) & \text{if } 0 < n\end{cases}\\ &= (n,v,y).\end{aligned}$$

Thus G has the identity element $(0, i, j)$, so 2.6(a) is satisfied. Notice that $G_1 \cap G_2$ is empty, yet G has an identity element somewhere else.

All those conditions suggest examples to separate some of the classes in Fig. 4.5. First:

(I, ω)	i	j
i	i	j
j	j	j

with $J = \{j\}$

and

$$\forall n \in \mathbb{N}, \qquad ni\chi = ni\psi = n, \qquad nj\chi = n - 1.$$

The conditions used in defining G are fulfilled, and the foregoing discussion shows that G satisfies 2.6(a) and (c). It also satisfies 2.6(b) if $\forall n \in \mathbb{N}$, $nj\psi = n - 1$, which provides an example satisfying 2.6(a–c) but neither (d), (e), nor (f). But otherwise $\exists n \in \mathbb{N}$: $nj\psi \neq n - 1$, say, for example, $5j\psi = 8$. In that case $(5, i, j)(0, j, j)\beta = (4, i, j) \neq (8, i, j) = (0, j, j)(5, i, j)\beta$ contrary to 2.6(b); which provides an example satisfying 2.6(a, c) but neither (b), (d), (e), nor (f).

Next:

(I, ω)	i	j	k
i	i	j	k
j	j	j	k
k	k	k	k

with $J = \{j, k\}$,

and

$$\forall n \in \mathbb{N}, \qquad ni\chi = ni\psi = n, \qquad nj\chi = nk\chi = nj\psi = nk\psi = n - 1.$$

The conditions used in defining G are fulfilled, and the earlier discussion shows that G satisfies 2.6(a, b) but neither (c), (d), (e), nor (f).

Finally:

(I, ω)	j	k
j	j	k
k	k	k

with $J = I$,

and

$$\forall n \in \mathbb{N}, \qquad nj\chi = nk\chi = n - 1,$$

but $5j\psi = 8$. The reader may check that G is then an internal direct product satisfying none of the six conditions 2.6(a)–(f). This highly unorthodox example vindicates the claim at the end of Section 2.

EXAMPLES 7.9. Internal direct products in which $|G_2\alpha_1|$ and $|G_1\alpha_2|$ are any non-zero cardinal numbers.

The construction details will not be given. But just as each Example 7.7 is compounded from two versions of 7.5, so each Example 7.9 can be constructed analogously from two versions of 7.8. The elements are now ordered sextuples so the notation looks intricate, but the generalization from Examples 7.8 raises no other difficulties.

8. EPILOGUE

Examples 7.5–7.9 perhaps look like conjuring tricks. It seems only fair to disclose briefly how the rabbits got into the hats.

Consider any internal direct product $(G, \beta, \alpha_1, \alpha_2)$. For all non-negative integers m and n, define $\gamma_{mn}: G \to G$ by $x\gamma_{mn} = x\alpha_1{}^{m+1}x\alpha_2{}^{n+1}\beta$. Every such γ_{mn} is an endomorphism of G (as shown in Theorem 2.5(a) for γ_{00} and in Lemma 6.2 for γ_{11}). If G_{mn} is defined as $G\gamma_{mn}$, then $G_{mn} \cap G_{pq} = G_{\mathrm{Max}(m,p),\,\mathrm{Max}(n,q)}$. This can be extended by defining $G_{\infty 0} = \bigcap\{G_{m0}: m \in \mathbb{N}\}$ and $G_{0\infty} = \bigcap\{G_{0n}: n \in \mathbb{N}\}$ and then $\forall r, s \in \{0\} \cup \mathbb{N} \cup \{\infty\}$, $G_{rs} = G_{r0} \cap G_{0s}$. Incidentally the Theorem 3.3 conditions are also equivalent to $G_{10} = G$, the same conditions with transposed subscripts are equivalent to $G_{01} = G$, and the Theorem 3.4 conditions are equivalent to $G_{10} = G = G_{01}$. G can now be partitioned into four subsets as follows. Define $K = G - (G_{10} \cup G_{01})$, $L_1 = G_{\infty 0} - G_{01}$, and $L_2 = G_{0\infty} - G_{10}$. Also write $\Gamma = \{\gamma_{mn}: m, n \in \mathbb{N} \cup \{0\}\}$, so, for example, $K\Gamma = \{x\gamma_{mn}: x \in K,\ m \geq 0,\ n \geq 0\}$. Then G is the disjoint union $K\Gamma \mathbin{\dot\cup} L_1\Gamma \mathbin{\dot\cup} L_2\Gamma \mathbin{\dot\cup} G_{\infty\infty}$. The Theorem 3.4 conditions such as $|G_1 \cap G_2| = 1$ are equivalent to $K = L_1 = L_2 = \varnothing$, i.e., $G = G_{\infty\infty}$. (Lemma 6.5 partly proves this.) Examples 7.5 and 7.8 have $K = L_1 = G_{\infty\infty} = \varnothing$, so $G = L_2\Gamma$. Examples 7.5 were constructed by writing $L_2 = \{0\}$ and each $0\gamma_{mn} = m$ (and in fact include isomorphic copies of all groupoids $L_2\Gamma$ having $|L_2| = 1$). After a long struggle, Examples 7.8 were constructed by writing $L_2 = \{0\} \times I \times J$ and each $(0, u, x)\gamma_{mn} = (m, u, x)$. Examples 7.7 and 7.9 have $L_1 = L_2 = G_{\infty\infty} = \varnothing$, so $G = K\Gamma$. Examples 7.7 were constructed by writing $K = \{(0,0)\}$ and each $(0,0)\gamma_{mn} = (m, n)$. Examples 7.9 have $K = \{0\} \times I_1 \times J_1 \times \{0\} \times I_2 \times J_2$.

An obvious final question (vital for the refinement property of [5] etc.) is how to generalize this whole theory of internal direct products to more than the two factors G_1 and G_2. The ideal would be to find minimum restrictions permitting such a generalization. Some restrictions would probably be needed because Examples 7.5 display a non-associative aspect of internal direct multiplication, and perhaps also because the related

Example 7.6 (with $k = 0$) has one factorization which satisfies the conditions 2.6(c–e) and another which does not.

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