Uniform attractor for non-autonomous plate equation with a localized damping and a critical nonlinearity ✪

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Abstract

In this paper, we consider dynamical behavior of non-autonomous plate-type evolutionary equations with critical nonlinearity. We prove the existence of a uniform attractor in the space $H_0^2(\Omega) \times L^2(\Omega)$.
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1. Introduction

In this paper, we consider the following non-autonomous plate equation with critical nonlinearity:

$$
\begin{cases}
  u_{tt} + a(x)u_t + \Delta^2 u + \lambda u + f(u) = g(x,t), & x \in \Omega, \\
  u|_{\partial \Omega} = \frac{\partial}{\partial \nu} u|_{\partial \Omega} = 0, \\
  u(x, \tau) = u_0(x), \quad u_t(x, \tau) = u_1(x).
\end{cases}
$$

Here $\Omega \subset \mathbb{R}^n$ is a bounded domain with a sufficiently smooth boundary, $\lambda > 0$, and the functions $a(x)$ and $f$ satisfy the following conditions:

1. $a(x) \in L^\infty(\Omega)$, $a(x) \geq a_0 > 0$ in $\Omega$,
2. $f \in C^1(\mathbb{R})$, $|f'(s)| \leq C(1 + |s|^p)$, $0 < p \leq \frac{4}{n-4}$,
3. $\lim \inf_{|s| \to \infty} \frac{f(s)}{s} > -\lambda_1^2$.

The external force $g$ satisfies

$$
g(x,t) \in L^\infty(\mathbb{R}; L^2(\Omega))
$$

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and
\[ \partial_t g \in L^r_b(\mathbb{R}; L^r(\Omega)) \quad \text{with } r > \frac{2n}{n+4}. \] (1.6)

The parameter \( \alpha_0 \) in (1.2) is a constant and \( \lambda_1 \) in (1.4) is the first eigenvalue of \( -\Delta \) with zero Dirichlet data. The number \( \frac{4}{n-4} \) is called the critical exponent, since the nonlinearity \( f \) is not compact in this case, which is an essential difficulty in studying the asymptotic behavior even for the autonomous case (e.g., [6,7,10,17,23] etc.). It is clear that the external force \( g \) satisfying (1.5) and (1.6) is translation bounded but not translation compact (the definition can be found in the beginning of the next section).

In present paper, we consider the non-autonomous system (1.1) via the uniform attractors of the corresponding family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \). The feature of the model (1.1) is that: (i) the nonlinearity \( f(u) \) has critical exponent, and (ii) the external forcing \( g(x, t) \) is not translation compact in \( L^2(\mathbb{R}; L^2(\Omega)) \).

Let us recall some relevant research in this area. For the autonomous case, when \( a(x) \equiv \alpha_0 > 0 \), the existence of a global attractor for plate equation with subcritical exponent was studied in [1] and the critical exponent case was studied in [15]. Khanmamedov [14] showed the global attractor for Eq. (1.1) with critical nonlinearity. In the case of semilinear wave equations, if \( a(x) \equiv \text{constant} \), the long-time behavior of weak solutions was investigated by many authors, see, for example, [6,10] for the linear damping and [7,17,23] for the nonlinear damping. In the locally distributed damping form (i.e. \( a(x) \not\equiv \text{constant} \)), the exponential decay property of solutions was successively treated in [4,5,22] and references therein. Using finite speed of propagation and a unique continuation result for the wave equations, the existence of a global attractor was established in [3]. Other results of global attractor can be found in [2,12,18] and references therein. For the special case of wave equations with localized damping, i.e., \( f(u) = g(x, t) \equiv 0 \), the problem was also considered extensively, for instance, [20] for the localized nonlinear damping and [19,21] for the localized linear dissipation. The study of wave equation with localized damping in autonomous case has attracted much attention in recent years. However, non-autonomous hyperbolic equation with localized damping is less discussed, especially for the plate equation. When \( a(x) \equiv \text{constant} \), in Chepyzhov and Vishik [8], the authors obtained the existence of a uniformly attractor when \( g \) is translation compact and nonlinearity \( f \) is subcritical. In Zelik [9], under some assumptions (i.e., \( g, \partial_t g \in L^\infty(\mathbb{R}; L^2(\Omega)), f \in C^2(\mathbb{R}), f(0) = 0, \) and \( f \) is critical exponent) and by using a bootstrap argument, the author obtained some regularity estimates for the solutions, which naturally implies the existence of a uniform attractor. Belleri and Pata [24] considered the strongly damped semilinear wave equation with localized dissipation and proved the existence of uniform absorbing set.

Our main goal in this paper is to prove the existence of an uniform attractor for the problem (1.1)–(1.6) in \( H^2(\Omega) \times L^2(\Omega) \), by using a “contractive function” method, which was proposed by Sun et al. in [16]. They used the method to testify the uniform asymptotic compactness. The initial idea of the method occurred in the reference [13] for autonomous case. The key idea of “contractive function” is to verify asymptotic compactness of the corresponding semigroup (or process). Comparing to the autonomous case, the main complexity of non-autonomous system is placed on the external force. Under appropriate assumptions, we need to prove that the external force belongs to the contractive function. To our best knowledge, the present paper is the first one of dealing with the non-autonomous plate equation with critical nonlinearity.

This paper is organized as follows: In Section 2, we give some preparations for our consideration; in Section 3, we prove the existence of an uniformly absorbing set in \( H^2(\Omega) \times L^2(\Omega) \); in the last section, we derive uniform asymptotic compactness of the corresponding family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), generated by problem (1.1).

2. Preliminaries

In this section, we first recall some basic concepts about non-autonomous systems, we refer to [8] for more details. Space of translation bounded functions in \( L^r(\mathbb{R}; L^k(\Omega)) \), with \( r, k \geq 1 \):

\[ L^r_b(\mathbb{R}; L^k(\Omega)) = \left\{ g \in L^r(\mathbb{R}; L^k(\Omega)) : \sup_{t \in \mathbb{R}} \frac{1}{t+1} \left( \int_\Omega \left| \int_0^t g(x, s) \right|^k dx \right)^{\frac{r}{k}} ds < \infty \right\}. \]
Space of translation compact functions in $L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega))$:

$$L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) = \{ g \in L^2_{\text{loc}}(\mathbb{R}; L^2(\Omega)) : \text{for any interval } [t_1, t_2] \subset \mathbb{R}, \}
\{ g(x, h + s) : h \in \mathbb{R} \}_{[t_1, t_2]} \text{ is precomputed in } L^2(t_1, t_2; L^2(\Omega)) \}. $$

Let $X$ be a Banach space, and $\Sigma$ a parameter set.

The operators $\{ U_\sigma(t, \tau), \sigma \in \Sigma \}$, are said to be a family of processes in $X$ with symbol space $\Sigma$ if for any $\sigma \in \Sigma$

$$U_\sigma(t, s) \circ U_\sigma(s, \tau) = U_\sigma(t, \tau), \quad \forall t \geq s \geq \tau, \quad \tau \in \mathbb{R},$$

$$U_\sigma(\tau, \tau) = \text{Id} \text{(identity)}, \quad \forall \tau \in \mathbb{R}. \quad \text{(2.1)}$$

Let $(T(s))_{s \geq 0}$ be the translation semigroup on $\Sigma$. We say that a family of processes $\{ U_\sigma(t, \tau), \sigma \in \Sigma \}$, satisfies the translation identity if

$$U_\sigma(t + s, \tau + s) = U_{T(s)\sigma}(t, \tau), \quad \forall \sigma \in \Sigma, \quad t \geq \tau, \quad \tau \in \mathbb{R}, \quad s \geq 0,$$

$$T(s) \Sigma = \Sigma, \quad \forall s \geq 0. \quad \text{(2.3)}$$

We denote by $B(X)$ the collection of the bounded sets of $X$, and $\mathbb{R}_t = \{ t \in \mathbb{R}, \ t \geq \tau \}$.

**Definition 2.1.** (See [8].) A bounded set $B_0 \in B(X)$ is said to be a bounded uniformly (w.r.t. $\sigma \in \Sigma$) absorbing set for $\{ U_\sigma(t, \tau), \sigma \in \Sigma \}$, if for any $\tau \in \mathbb{R}$ and $B \in B(X)$ there exists $T_0 = T_0(B, \tau)$ such that $\bigcup_{\sigma \in \Sigma} U_\sigma(t; \tau)B \subset B_0$ for all $t \geq T_0$.

**Definition 2.2.** (See [8].) A set $\mathcal{A} \subset X$ is said to be uniformly (w.r.t. $\sigma \in \Sigma$) attracting for the family of processes $\{ U_\sigma(t, \tau), \sigma \in \Sigma \}$, if for any fixed $\tau \in \mathbb{R}$ and any $B \in B(X)$

$$\lim_{t \to +\infty} \left( \sup_{\sigma \in \Sigma} \text{dist}(U_\sigma(t; \tau)B; \mathcal{A}) \right) = 0,$$

where $\text{dist}(\cdot, \cdot)$ is the usual Hausdorff semidistance in $X$ between two sets.

In particular, a closed uniformly attracting set $\mathcal{A}_\Sigma$ is said to be the uniform (w.r.t. $\sigma \in \Sigma$) attractor of the family of processes $\{ U_\sigma(t, \tau), \sigma \in \Sigma \}$, if it is contained in any closed uniformly attracting set (minimality property).

Similar to the autonomous cases (e.g., see [14]), we can easily obtain the following existence and uniqueness results. The proof is based on the Galerkin approximation method (e.g., see [25]) and the time-dependent terms make no essential complications.

**Lemma 2.3.** Let $\Omega$ be a bounded domain of $\mathbb{R}^n$ with smooth boundary, $a(x)$ and $f$ satisfy (1.2)–(1.4), $g \in L^\infty(\mathbb{R}; L^2(\Omega))$. Then for any initial data $(u_0^\tau, u_1^\tau) \in H^2_\Omega(\Omega) \times L^2(\Omega)$, the problem (1.1) has an unique solution $u(t)$ which satisfies $(u(t), u_\tau(t)) \in C(\mathbb{R}_\tau; H^2_\Omega(\Omega) \times L^2(\Omega))$ and $\partial_t u_\tau \in L^2_{\text{loc}}(\mathbb{R}_\tau; H^{-2}(\Omega))$.

For convenience, hereafter, the norm and scalar product in $L^2(\Omega)$ are denoted by $\| \cdot \|$ and $(\cdot, \cdot)$, respectively. We use the notations as in Chepyzhov and Vishik [8]. Let $y(t) = (u(t), u_\tau(t))$, $y_\tau = (u_0^\tau, u_1^\tau), E_0 = H^2_\Omega(\Omega) \times L^2(\Omega)$ with finite energy norm

$$\| y \|_{E_0} = \| A u \|^2 + \| u_\tau \|^2. $$

Then system (1.1) is equivalent to

$$\partial_t u_\tau = -\Delta^2 u - a(x)u_\tau - \lambda u - f(u) + g(x, t), \quad \text{for any } t \geq \tau,$$

$$u|_{\partial \Omega} = \frac{\partial}{\partial \nu} u|_{\partial \Omega} = 0, \quad u(x, \tau) = u_0^\tau(x), \quad u_\tau(x, \tau) = u_1^\tau(x). \quad \text{(2.5)}$$

We can also write (2.5) in the operator form

$$\partial_t y = A_\sigma(t)(y), \quad y|_{t=\tau} = y_\tau. \quad \text{(2.6)}$$

where $\sigma(t) = g(x, t)$ is symbol of Eq. (2.6).
We now define the symbol space for (2.6). Take a fixed symbol \( \sigma_0(x, s) = g_0(x, s) \), \( g_0 \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega)) \) for some \( r > \frac{2n}{n+4} \). Set
\[
\Sigma_0 = \{ (x, t) \mapsto g_0(x, t + h) : h \in \mathbb{R} \}
\] (2.7)
and
\[
\Sigma \text{ the } ^* -\text{weakly closure of } \Sigma_0 \text{ in } L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega)).
\] (2.8)
Then we have the following simple properties.

**Proposition 2.4.** \( \Sigma \) is bounded in \( L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega)) \), and for any \( \sigma \in \Sigma \), the following estimate holds:
\[
\| \sigma \|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega))} \leq \| g_0 \|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega))}.
\]

Thus, from Lemma 2.3, we know that (1.1) is well posed for all \( \sigma(s) \in \Sigma \) and generates a family of processes \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), given by the formula \( U_\sigma(t, \tau)y = y(t) \). The \( y(t) \) is the solution of (1.1)–(1.6) and \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), satisfies (2.1)–(2.2). At the same time, due to the unique solvability, we know \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), satisfies the translation identity (2.3)–(2.4).

In what follows, we denote by \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), the family of processes which is generated by (2.6)–(2.8).

### 3. Uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing set in \( E_0 \)

**Theorem 3.1.** Under assumptions (1.2)–(1.6), the family of processes \( \{ U_\sigma(t, \tau) \} \), \( \sigma \in \Sigma \), corresponding to (1.1) has a bounded uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing set \( B_0 \) in \( E_0 \).

**Proof.** From the definition of \( \Sigma \) we know that for all \( \sigma \in \Sigma \),
\[
\| \sigma \|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega))}^2 \leq \| g_0 \|_{L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_b^{1, r}(\mathbb{R}; L^r(\Omega))}^2.
\] (3.1)
Let \( F(s) = \int_0^s f(\tau) d\tau \), taking the scalar product with \( v = u + \delta u \) in \( L^2 \), where \( 0 < \delta \leq \delta_0 \) (the parameter \( \delta_0 \) will be determined later), we get
\[
\frac{1}{2} \frac{d}{dt} \left( \| v \|^2 + \| \Delta u \|^2 + \lambda \| u \|^2 + 2 \int_\Omega F(u) dx \right) + (a(x) v, v) - \delta \| v \|^2 + \delta \| \Delta u \|^2
\]
\[
+ \delta \lambda \| u \|^2 - \delta (a(x) u, v) + \delta^2 (u, v) + \delta \int_\Omega f(u) u dx = (g, v).
\] (3.2)
From (1.3) and (1.4) we know that there are \( \lambda^2_1 > \lambda' > 0 \) and \( C_0 > 0 \) such that
\[
(f(u), u) \geq \lambda' \int_\Omega F(u) dx - C_0, \quad F(u) \geq -\frac{\lambda'}{2} u^2 - C_0.
\]
Due to (1.2), there exists \( L > 0 \), such that
\[
-\delta (a(x) u, v) \geq -\delta L (u, v), \quad (a(x) v, v) \geq \alpha_0 \| v \|^2.
\]
From now on, \( \delta_0 \) is chosen as
\[
\delta_0 = \min \left\{ \alpha_0, \frac{\lambda_1^2 \alpha_0}{4} \right\} \frac{\lambda_1^2 \alpha_0}{2L^2}.
\]
Consequently, we obtain the estimate by using Hölder and Young inequalities.
(a(x)v, v) − δ∥v∥^2 + δ∥Du∥^2 − δ(a(x)u, v) + δ²(u, v)
\geq (α_0 − δ)∥v∥^2 + δ∥Du∥^2 − δ(L − δ)(u, v)
\geq δ∥Du∥^2 + \frac{3}{4}α_0∥v∥^2 − \left(\frac{δ}{2}∥Du∥^2 + \frac{δL^2}{2λ_1}∥v∥^2\right)
\geq \frac{δ}{2}∥Du∥^2 + \frac{α_0}{2}∥v∥^2.

(3.3)

Inserting the inequality (3.3) in (3.2), we have
\frac{1}{2} d\frac{d}{dt}(∥v∥^2 + ∥Du∥^2 + λ∥u∥^2 + 2 \int_Ω F(u) dx) + \frac{δ}{2}∥Du∥^2 + \frac{α_0}{2}∥v∥^2
+ δλ∥u∥^2 + δλ' \int_Ω F(u) dx ≤ δC_0 + \frac{α_0}{4}∥v∥^2 + \frac{1}{α_0}∥g∥^2.

(3.4)

Choosing γ = \min(δ, δλ') and setting z(t) = ∥v∥^2 + ∥Du∥^2 + λ∥u∥^2 + 2 \int_Ω F(u) dx, we get
\gamma(t) \leq z(t) ≤ z(τ) \exp(-γ(t − τ)) + (1 + γ^{-1})\left(2δC_0 + \frac{2}{α_0}∥g∥_{L_b^2}^2\right).

(3.5)

where we have also used the Gronwall’s inequality. From (3.1), we know
∥g∥_{L_b^2}^2 \leq ∥g_0∥_{L_b^2}^2 \quad \text{for all } g ∈ Σ.

Therefore, we obtain the uniformly (w.r.t. σ ∈ Σ) absorbing set B₀ in E₀,
B₀ = \{y = (u, u_t) | ∥y∥^2 ≤ ρ₀^2\},
where ρ₀ = 2(1 + γ^{-1})(2δC_0 + \frac{2}{α_0}∥g_0∥_{L_b^2}^2). That is to say, for any bounded subset B in E₀, there exists a t₀ = t₀(τ, B) ≥ τ such that
\bigcup_{g ∈ Σ} U_g(t, τ)B ⊂ B₀, \quad ∀t ≥ t₀.

4. Uniform (w.r.t. σ ∈ Σ) asymptotic compactness in E₀

In this section, we first give some useful preliminaries and then obtain some a priori estimates about the energy inequalities based on the idea presented in [7,13,16,23]. Finally, we use Theorem 4.5 to establish the uniform (w.r.t. σ ∈ Σ) asymptotic compactness in E₀.

Hereafter, we always denote by B₀ the bounded uniformly (w.r.t. σ ∈ Σ) absorbing set obtained in Theorem 3.1.

4.1. Preliminaries

We now recall the simple criterion developed in [16].

Definition 4.1. (See [16].) Let X be a Banach space, B a bounded subset of X and Σ a symbol (or parameter) space.
We call a function φ(·, ·, ·, ·), defined on (X × X) × (Σ × Σ), to be a contractive function on B × B if for any sequence \{x_n\}_{n=1}^∞ \subset B and any \{σ_n\} ⊂ Σ, there is a subsequence \{x_{nk}\}_{k=1}^∞ \subset \{x_n\}_{n=1}^∞ and \{σ_{nk}\}_{k=1}^∞ \subset \{σ_n\}_{n=1}^∞ such that
\lim_{k → ∞} \lim_{l → ∞} φ(x_{nk}, x_{nl}; σ_{nk}, σ_{nl}) = 0.

We denote the set of all contractive functions on B × B by Contr(B, Σ).
Proposition 4.3. Let \( \{u_\sigma(t, \tau)\}, \sigma \in \Sigma \) be a family of processes which satisfies the translation identity (2.3)–(2.4) on Banach space \( X \) and has a bounded uniformly (w.r.t. \( \sigma \in \Sigma \)) absorbing set \( B_0 \subset X \). Moreover, assume that for any \( \varepsilon > 0 \) there exist \( T = T(B_0, \varepsilon) \) and \( \phi_T \in \text{Con}(B_0, \Sigma) \) such that

\[
\| U_{\sigma_1}(T, 0)x - U_{\sigma_2}(T, 0)y \| \leq \varepsilon + \phi_T(x, y; \sigma_1, \sigma_2), \quad \forall x, y \in B_0, \forall \sigma_1, \sigma_2 \in \Sigma.
\]

Then \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), is uniformly (w.r.t. \( \sigma \in \Sigma \)) asymptotically compact in \( X \).

Applying Proposition 7.1 of Robinson [11], we immediately get the following results. The proof is based on the technique in [16], for reader’s convenience, we replicate it here and only make a few minor changes for our problem.

Proposition 4.3. Let \( g \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R}; L^r(\Omega))(r > \frac{2n}{n+4}) \). Then there is an \( M > 0 \) such that

\[
\sup_{t \in \mathbb{R}} \| g(x, t+s) \|_{L^2(\Omega)} \leq M \quad \text{for all } s \in \mathbb{R}.
\]

Proposition 4.4. Let \( s_i \in \mathbb{R} \) (\( i = 1, 2, \ldots \)), \( g \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R}; L^r(\Omega))(r > \frac{2n}{n+4}) \), \( \{u_n(t) \mid t \geq 0, \ n = 1, 2, \ldots \} \) be bounded in \( H_0^2(\Omega) \), and for any \( T_1 > 0 \), \( \{u_{n_i}(t) \mid n = 1, 2, \ldots \} \) bounded in \( L^\infty(0, T_1; L^2(\Omega)) \). Then for any \( T > 0 \), there exist subsequences \( \{u_{n_k}\}_{k=1}^\infty \) of \( \{u_n\}_{n=1}^\infty \) and \( \{s_{n_k}\}_{k=1}^\infty \) of \( \{s_n\}_{n=1}^\infty \) such that

\[
\lim_{k \to \infty} \lim_{l \to \infty} \int_0^T \int_\Omega (g(x, \tau + s_{n_k}) - g(x, \tau + s_{n_l}))(u_{n_k} - u_{n_l})(\tau) \, dx \, d\tau \, ds = 0.
\]

Proof. Since \( \{u_n(t) \mid t \geq 0, \ n = 1, 2, \ldots \} \) is bounded in \( H_0^2(\Omega) \) and for any \( T_1 > 0 \), \( \{u_{n_i}(t) \mid n = 1, 2, \ldots \} \) is bounded in \( L^\infty(0, T_1; L^2(\Omega)) \), we assume for any \( T > 0 \), without loss of generality (at most by passing subsequence), that

\[ u_n(T) \to u_0 \quad \text{in } L^2(\Omega) \]

and

\[ u_n \to v \quad \text{in } L^k(0, T; L^k(\Omega)) \quad \text{(this require } r > \frac{2n}{n+4} \text{)} \].

where \( k < \frac{2n}{n-4} \).

Note that for any \( w \in W^{1,2}_b(\mathbb{R}; L^2(\Omega)) \),

\[
(g(x, t+s_i) - g(x, t+s_j))w_l(t) = \frac{d}{dt}((g(x, t+s_i) - g(x, t+s_j))w(t)) - (g_t(x, t+s_i) - g_t(x, t+s_j))w_l(t).
\]

Then by using Hölder inequality, we obtain

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_\Omega (g(x, \tau + s_n) - g(x, \tau + s_m))(u_n - u_m)(\tau) \, dx \, d\tau \, ds
\]

\[
\leq \lim_{n \to \infty} \lim_{m \to \infty} 2MT \left( \int_\Omega |u_n(T) - u_m(T)|^2 \, dx \right)^{\frac{1}{2}} + \lim_{n \to \infty} \lim_{m \to \infty} 2MT^2 \left( \int_0^T \int_\Omega |u_{nx}(s) - u_m(s)|^2 \, dx \, ds \right)^{\frac{1}{2}}
\]

\[
+ \lim_{n \to \infty} \lim_{m \to \infty} T \int_0^T \int_\Omega ((g_t(x, s + s_n) - g_t(x, s + s_m))(u_n(s) - u_m(s)) \, dx \, ds
\]

\[
= \lim_{n \to \infty} \lim_{m \to \infty} T \int_0^T \int_\Omega ((g_t(x, s + s_n) - g_t(x, s + s_m))(u_n(s) - u_m(s)) \, dx \, ds
\]
\[
\leq \lim_{n \to \infty} \lim_{m \to \infty} T \left( \int_0^T \int_\Omega \left| g_t(x, s + s_n) - g_t(x, s + s_m) \right|^r \right)^{\frac{1}{r}} \left( \int_0^T \int_\Omega \left| u_n(s) - u_m(s) \right|^k \right)^{\frac{1}{k}} = 0. \]

4.2. A priori estimates

The main purpose of this subsection is to establish (4.10)–(4.12), which will be used to obtain the uniform (w.r.t. \( \sigma \in \Sigma \)) asymptotic compactness.

For any \((u^i_0, v^i_0) \in B_0\), let \((u^i(t), u^i(t))\) be the corresponding solution to \(\sigma^i\) with respect to initial data \((u^i_0, v^i_0)\), \(i = 1, 2\), that is, \((u^i(t), u^i(t))\) is the solution of the following equation:

\[
\begin{cases}
  u_{tt} + a(x)u_t + \Delta^2 u + \lambda u + f(u(t)) = \sigma^i(x, t), \\
  u|_{\partial \Omega} = \partial_{\nu} u \big|_{\partial \Omega} = 0, \\
  (u(0), u_t(0)) = (u^i_0, v^i_0).
\end{cases}
\]

(4.1)

**Lemma 4.5.** Assume the condition (1.2) is satisfied. Then for any fixed \(T > 0\), there exist a constant \(C_T\) and a function \(\phi_T((u^1_0, v^1_0), (u^2_0, v^2_0); \sigma_1, \sigma_2)\) such that

\[
\| u_1(T) - u_2(T) \|_{E_0} \leq C_T + \phi_T((u^1_0, v^1_0), (u^2_0, v^2_0); \sigma_1, \sigma_2),
\]

where \(C_T\) and \(\phi_T\) depend on \(T\).

**Proof.** For convenience, we denote

\[ g_i(t) = \sigma^i(x, t), \quad t \geq 0, \quad i = 1, 2, \]

and

\[ w(t) = u_1(t) - u_2(t). \]

Then \(w(t)\) satisfies

\[
\begin{cases}
  w_{tt} + a(x)w_t + \Delta^2 w + \lambda w + f(u_1(t)) - f(u_2(t)) = g_1(t) - g_2(t), \\
  w|_{\partial \Omega} = \partial_{\nu} w \big|_{\partial \Omega} = 0, \\
  (w(0), w_t(0)) = (u^1_0, v^1_0) - (u^2_0, v^2_0).
\end{cases}
\]

(4.2)

Set

\[
E_w(t) = \frac{1}{2} \int_\Omega |w_t(t)|^2 + \frac{1}{2} \int_\Omega |\Delta w(t)|^2.
\]

At first, multiplying (4.2) by \(w\) and integrating over \([0, T] \times \Omega\), we get

\[
\int_0^T \int_\Omega |\Delta w(s)|^2 \, dx \, ds = \int_\Omega w_t(0)w(0) \, dx - \int_\Omega w_t(T)w(T) \, dx + \frac{1}{2} \int_\Omega a(x)|w(0)|^2 \, dx - \frac{1}{2} \int_\Omega a(x)|w(T)|^2 \, dx - \lambda \int_0^T \int_\Omega |w(s)|^2 \, dx \, ds + \int_0^T \int_\Omega |w_t(s)|^2 \, dx \, ds - \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s) \, dx \, ds + \int_0^T (g_1 - g_2)w \, dx \, ds.
\]

(4.3)
Then we have

\[
\int_0^T E_w(s) \, ds = \frac{1}{2} \int_\Omega w_1(0)w(0) \, dx - \frac{1}{2} \int_\Omega w_1(T)w(T) \, dx + \frac{1}{4} \int_\Omega a(x) |w(0)|^2 \, dx \\
- \frac{1}{4} \int_\Omega a(x) |w(T)|^2 \, dx - \frac{1}{2} \int_0^T \int_\Omega |w(s)|^2 \, dx \, ds + \int_0^T \int_\Omega |w_1(s)|^2 \, dx \, ds \\
- \frac{1}{2} \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s) \, dx \, ds + \frac{1}{2} \int_0^T \int_\Omega (g_1 - g_2)w \, dx \, ds.
\] (4.4)

From condition (1.2), we get

\[
\int_0^s \int_\Omega |w_1(\tau)|^2 \, dx \, d\tau \leq \tilde{c} \int_s^T \int_\Omega a(x) |w_1(s)|^2 \, dx \, ds,
\]

where \(\tilde{c}\) is a constant. So we have

\[
\int_0^T E_w(s) \, ds \leq \frac{1}{2} \int_\Omega w_1(0)w(0) \, dx - \frac{1}{2} \int_\Omega w_1(T)w(T) \, dx + \frac{1}{4} \int_\Omega a(x) |w(0)|^2 \, dx \\
- \frac{1}{4} \int_\Omega a(x) |w(T)|^2 \, dx - \frac{1}{2} \int_0^T \int_\Omega |w(s)|^2 \, dx \, ds + \tilde{c} \int_0^T \int_\Omega a(x) |w_1(s)|^2 \, dx \, ds \\
- \frac{1}{2} \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s)))w(s) \, dx \, ds + \frac{1}{2} \int_0^T \int_\Omega (g_1 - g_2)w \, dx \, ds.
\] (4.5)

Secondly, multiplying (4.2) by \(w_1\) and integrating over \([s, T] \times \Omega\), we get

\[
E_w(T) + \int_s^T \int_\Omega a(x) |w_1(\tau)|^2 \, dx \, d\tau + \frac{1}{2} \lambda \|w(T)\|^2 \\
= E_w(s) - \int_s^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau)))w_1(\tau) \, dx \, d\tau \\
+ \frac{1}{2} \lambda \|w(s)\|^2 + \int_s^T \int_\Omega (g_1 - g_2)w_1 \, dx \, d\tau,
\] (4.6)

where \(0 \leq s \leq T\). Integrating (4.6) over \([0, T]\) with respect to \(s\), we obtain that

\[
TE_w(T) \leq \int_0^T E_w(s) \, ds - \int_0^T \int_s^T (f(u_1(\tau)) - f(u_2(\tau)))w_1(\tau) \, dx \, d\tau \, ds \\
+ \frac{1}{2} \lambda \int_0^T \int_\Omega |w(s)|^2 \, dx \, ds + \int_0^T \int_s^T (g_1 - g_2)w_1 \, dx \, d\tau \, ds.
\] (4.7)

From (4.6) we also have
\[
\int_0^T \int_\Omega a(x) w_t(\tau)^2 \, dx \, d\tau \leq E_w(0) - \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \, dx \, d\tau
\]
\[
+ \frac{1}{2} \lambda \| w(0) \|^2 + \int_0^T \int_\Omega (g_1 - g_2) w_t \, dx \, d\tau.
\]  
(4.8)

Therefore, from (4.5), (4.7) and (4.8), we get
\[
TE_w(T) \leq \tilde{c} E_w(0) + \frac{1}{2} \tilde{c} \lambda \| w(0) \|^2 + \frac{1}{4} \int_\Omega a(x) |w(0)|^2 \, dx
\]
\[
- \frac{1}{4} \int_\Omega a(x) |w(T)|^2 \, dx + \frac{1}{2} \int_\Omega w_t(0) w(0) \, dx - \frac{1}{2} \int_\Omega w_t(T) w(T) \, dx
\]
\[
+ \tilde{c} \int_0^T \int_\Omega (g_1 - g_2) w_t \, dx \, ds - \tilde{c} \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s))) w_t(s) \, dx \, ds
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega (g_1 - g_2) w \, dx \, ds - \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \, dx \, d\tau \, ds
\]
\[
+ \int_0^T \int_\Omega (g_1 - g_2) w_t \, dx \, ds - \frac{1}{2} \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \, dx \, d\tau \, ds.
\]  
(4.9)

Set
\[
C_M = \tilde{c} E_w(0) + \frac{1}{2} \tilde{c} \lambda \| w(0) \|^2 + \frac{1}{4} \int_\Omega a(x) |w(0)|^2 \, dx - \frac{1}{2} \int_\Omega w_t(0) w(0) \, dx - \frac{1}{2} \int_\Omega w_t(T) w(T) \, dx,
\]  
(4.10)

\[
\phi_T((u_0^1, v_0^1), (u_0^2, v_0^2); \sigma_1, \sigma_2)
\]
\[
= \frac{1}{4} \int_\Omega a(x) |w(0)|^2 \, dx - \frac{1}{4} \int_\Omega a(x) |w(T)|^2 \, dx + \int_0^T \int_\Omega (g_1 - g_2) w_t \, dx \, d\tau \, ds
\]
\[
+ \tilde{c} \int_0^T \int_\Omega (g_1 - g_2) w_t \, dx \, ds - \tilde{c} \int_0^T \int_\Omega (f(u_1(s)) - f(u_2(s))) w_t(s) \, dx \, ds
\]
\[
+ \frac{1}{2} \int_0^T \int_\Omega (g_1 - g_2) w \, dx \, ds - \frac{1}{2} \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w(s) \, dx \, ds
\]
\[
- \int_0^T \int_\Omega (f(u_1(\tau)) - f(u_2(\tau))) w_t(\tau) \, dx \, d\tau \, ds.
\]  
(4.11)

Then we have
\[
E_w(T) \leq \frac{C_M}{T} + \frac{1}{T} \phi_T((u_0^1, v_0^1), (u_0^2, v_0^2); \sigma_1, \sigma_2).
\]  
(4.12)

The proof is complete.  \(\square\)
4.3. Uniform asymptotic compactness

In this subsection, we shall prove the uniform (w.r.t. \( \sigma \in \Sigma \)) asymptotic compactness in \( H_0^2(\Omega) \times L^2(\Omega) \), which is given in the following theorem.

**Theorem 4.6.** Assume that \( a(x) \) and \( f \) satisfy (1.2)-(1.4). If \( g_0 \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W_0^{1,r}(\mathbb{R}; L^r(\Omega)) \) for some \( r > \frac{2n}{n+4} \) and \( \Sigma \) is defined by (2.8), then the family of processes \( \{ U_{\sigma}(t, \tau) \} \), \( \sigma \in \Sigma \), corresponding to (1.1), is uniformly (w.r.t. \( \sigma \in \Sigma \)) asymptotically compact in \( H_0^2(\Omega) \times L^2(\Omega) \).

**Proof.** Since the family of processes \( \{ U_{\sigma}(t, \tau) \} \), \( \sigma \in \Sigma \), has a bounded uniformly absorbing set and from Lemma 4.5, for any fixed \( \varepsilon > 0 \), we can choose \( T \) large enough, such that \( \frac{C_M}{T} \leq \varepsilon \).

Hence, thanks to Theorem 4.2, it is sufficient to prove \( \phi_T(\cdot, \cdot; \cdot, \cdot) \in \text{Contr}(B_0, \Sigma) \) for each fixed \( T \).

From the proof procedure of Theorem 3.1, we can deduce that for any fixed \( T \),

\[
\bigcup_{\sigma \in \Sigma} \bigcup_{t \in [0, T]} U_{\sigma}(t, 0)B_0 \text{ is bounded in } E_0,
\]

and the bound depends on \( T \).

Let \( (u_n, u_{nt}) \) be the solutions corresponding to initial data \( (u^n_0, v^n_0) \in B_0 \) with respect to symbol \( \sigma_n \in \Sigma \), \( n = 1, 2, \ldots \). Then, from (4.13), without loss of generality (at most by passing subsequence), we assume that

\[
u_n \to u \text{ in } L^\infty(0, T; H_0^2(\Omega)),
\]

\[
u_n \to u_t \text{ in } L^\infty(0, T; L^2(\Omega)),
\]

\[
u_n \to u \text{ in } L^2(0, T; L^2(\Omega)),
\]

\[
u_n \to u \text{ in } L^k(0, T; L^k(\Omega)),
\]

\[
u_n(0) \to u(0) \text{ and } \nu_n(T) \to u(T) \text{ in } L^k(\Omega),
\]

for \( k < \frac{2n}{n+4} \), where we have used the compact embedding \( H_0^2 \hookrightarrow L^k \).

We now deal with each term on the right-hand side of (4.11). At first, from Proposition 4.3 and (4.17), by the similar method used in the proof of Proposition 4.4, we can obtain

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_{\Omega} \left( g_n(x, s) - g_m(x, s) \right) \left( u_n(s) - u_m(s) \right) dx ds = 0,
\]

and from Proposition 4.4 we can get

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_{\Omega} \left( f(u_n(s)) - f(u_m(s)) \right) \left( u_n(s) - u_m(s) \right) dx ds = 0.
\]

Secondly, from (4.16), we have

\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_{\Omega} \left( f(u_n(s)) - f(u_m(s)) \right) \left( u_n(s) - u_m(s) \right) dx ds = 0.
\]

From (4.18), noticing the condition (1.2), we obtain
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_{\Omega} a(x) |u_n(T) - u_m(T)|^2 \, dx = 0, \tag{4.23}
\]

and
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_{\Omega} a(x) |u_n(0) - u_m(0)|^2 \, dx = 0. \tag{4.24}
\]

Finally, using the similar method in the proof of Lemma 2.2 in [23], we get
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_{\Omega} \left( f(u_n(s)) - f(u_m(s)) \right) \left( u_{n_1}(s) - u_{m_1}(s) \right) \, dx \, ds = 0, \tag{4.25}
\]
\[
\lim_{n \to \infty} \lim_{m \to \infty} \int_0^T \int_s^T \int_{\Omega} \left( f(u_n(\tau)) - f(u_m(\tau)) \right) \left( u_{n_1}(\tau) - u_{m_1}(\tau) \right) \, dx \, d\tau \, ds = 0. \tag{4.26}
\]

Hence, combining (4.19)–(4.26), we get \( \phi_T (\cdot, \cdot; \cdot, \cdot) \in \text{Contr}(B_0, \Sigma) \) immediately. \( \Box \)

### 4.4. Existence of uniform attractor

**Theorem 4.7.** Assume that \( a(x) \) and \( f \) satisfy (1.2)–(1.4). If \( g_0 \in L^\infty(\mathbb{R}; L^2(\Omega)) \cap W^{1,r}_b(\mathbb{R}; L^r(\Omega)) \) for some \( r > \frac{2n}{n + 4} \) and \( \Sigma \) is defined by (2.8), then the family of processes \( \{U_\sigma(t, \tau)\}, \sigma \in \Sigma \), corresponding to (1.1) has a compact uniform (w.r.t. \( \sigma \in \Sigma \)) attractor \( A_\Sigma \).

**Proof.** Theorems 3.1 and 4.6 imply the existence of an uniform attractor immediately. \( \Box \)

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### References

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