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Global well-posedness for the Euler–Boussinesq system with axisymmetric data

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Abstract

In this paper we prove the global well-posedness for the three-dimensional Euler–Boussinesq system with axisymmetric initial data without swirl. This system couples the Euler equation with a transport-diffusion equation governing the temperature.

Keywords: Axisymmetric flows; Global well-posedness; Harmonic analysis

Contents

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1. Introduction

Boussinesq systems are widely used to model the dynamics of the ocean or the atmosphere. They arise from the density dependent fluid equations by using the so-called Boussinesq approximation which consists in neglecting the density dependence in all the terms but the one involving the gravity. This approximation can be justified from compressible fluid equations by a simultaneous low Mach number/Froude number limit, we refer to [15] for a rigorous justification. In this paper we shall assume that the fluid is inviscid but heat-conducting and hence the system reads

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = \rho e_z, & (t,x) \in \mathbb{R}_+ \times \mathbb{R}^3, \\ \partial_t \rho + v \cdot \nabla \rho - \Delta \rho = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0, \quad \rho_{|t=0} = \rho_0. \end{cases} \tag{1}$$

Here, the velocity $v = (v^1, v^2, v^3)$ is a three-component vector field with zero divergence, the scalar function ρ denotes the density or the temperature and p the pressure of the fluid. Note that we have assumed that the heat conductivity coefficient is one, one can always reduce the problem to this situation by a change of scale (as soon as the fluid is assumed to be heat conducting) which is not important for global well-posedness issues with data of arbitrary size that we shall consider here. The term ρe_z where $e_z = (0, 0, 1)^t$ takes into account the influence of the gravity and the stratification on the motion of the fluid. Note that when the initial density ρ_0 is identically zero (or constant) then the above system reduces to the classical incompressible Euler equation:

$$\begin{cases} \partial_t v + v \cdot \nabla v + \nabla p = 0, \\ \operatorname{div} v = 0, \\ v_{|t=0} = v_0. \end{cases} \tag{2}$$

From this observation, one cannot expect to have a better theory for the Boussinesq system than for the Euler equation. For the Euler equation, a well-known criterion for the existence of global smooth solution is the Beale–Kato–Majda criterion [3]. It states that the control of the vorticity of the fluid $\omega = \operatorname{curl} v$ in $L^1_{loc}(\mathbb{R}_+, L^\infty)$ is sufficient to get global well-posedness. In space dimension two, the vorticity ω can be identified to a scalar function which solves the transport equation

$$\partial_t \omega + v \cdot \nabla \omega = 0.$$

From this transport equation, one immediately gets that

$$\|\omega(t)\|_{L^p} \leqslant \|\omega_0\|_{L^p}$$

for every $p \geqslant 1$ and hence the global well-posedness follows from the Beale–Kato–Majda criterion.

In a similar way, the global well-posedness for two-dimensional Boussinesq systems which has recently drawn a lot of attention seems to be in a satisfactory state. More precisely global well-posedness has been shown in various function spaces and for different viscosities, we refer for example to [1,5,7,12,13,17–20,22]. In particular, for the model (1) in 2D, the main idea is that by studying carefully the coupling between the two equations and by using the smoothing effect of the second equation, it is still possible to get an a priori estimate in L^∞ (or in $B^0_{\infty,1}$) for ω and hence the global well-posedness.

In the three-dimensional case, very few is known: even for the Euler equation, the vorticity ω solves the equation

$$\partial_t \omega + v \cdot \nabla \omega = \omega \cdot \nabla v \tag{3}$$

and the way to control the vortex stretching term $\omega \cdot \nabla v$ in the right-hand side is a widely open problem. Nevertheless, a classical situation where one can get global existence is the case that v is axisymmetric without swirl [30,23]. Our aim here is to study how this classical global existence result for axisymmetric data for the Euler equation can be extended to the Boussinesq system (1).

Before stating our main result, let us recall the main ingredient in the global existence proof for the Euler equation with axisymmetric data. The assumption that the vector field v is axisymmetric without swirl means that it has the form:

$$v(t,x) = v^r(t,r,z)e_r + v^z(t,r,z)e_z, \quad x = (x_1, x_2, z), \quad r = \left(x_1^2 + x_2^2\right)^{\frac{1}{2}}, \tag{4}$$

where (e_r, e_θ, e_z) is the local basis of $\mathbb{R}^3$ corresponding to cylindrical coordinates. Note that we assume that the velocity is invariant by rotation around the vertical axis (axisymmetric flow) and that the angular component v^θ of v is identically zero (without swirl). For these flows, the vorticity is under the form

$$\omega = \left(\partial_3 v^r - \partial_r v^z\right)e_\theta := \omega_\theta e_\theta$$

and the vortex stretching term reads

$$\omega \cdot \nabla v = \frac{v^r}{r}\omega.$$

In particular ω_θ satisfies the equation

$$\partial_t \omega_\theta + v \cdot \nabla \omega_\theta = \frac{v^r}{r}\omega_\theta. \tag{5}$$

The crucial fact is then that the quantity $\zeta := \frac{\omega_\theta}{r}$ solves the transport equation

$$\partial_t \zeta + v \cdot \nabla \zeta = 0$$

from which we get that for every $p \in [1, \infty]$

$$\left\|\zeta(t)\right\|_{L^p} \leqslant \|\zeta_0\|_{L^p}. \tag{6}$$

It was shown by Ukhoviskii and Yudovich [30] and independently by Ladyzhenskaya [23] that these new a priori estimates are strong enough to prevent the formation of singularities in finite time for axisymmetric flows without swirl. More precisely global existence and uniqueness was established for axisymmetric initial data with finite energy and satisfying in addition $\omega_0 \in L^2 \cap L^\infty$ and $\frac{\omega_0}{r} \in L^2 \cap L^\infty$. In terms of Sobolev regularity these assumptions are satisfied if the velocity v_0 belongs to H^s with $s > \frac{7}{2}$. This condition was improved more recently. In [29], it was proven that global well-posedness still holds if v_0 is in H^s with $s > \frac{5}{2}$ (note that this is the natural regularity requirement for the initial velocity in the Sobolev scale in view of the standard local existence result) and in the recent work [11], Danchin has obtained global existence and uniqueness for initial data such that $\omega_0 \in L^{3,1} \cap L^\infty$ and $\zeta_0 \in L^{3,1}$ (here, $L^{3,1}$ denotes the Lorentz space, the definition of the Lorentz spaces $L^{p,q}$ as interpolation spaces is recalled below). Their proof is based on the observation that one can deduce from the Biot–Savart law, the pointwise estimate

$$\left|\frac{v^r}{r}\right| \lesssim \frac{1}{|\cdot|^2} * |\zeta|. \tag{7}$$

By convolution laws in Lorentz spaces (again recalled below) and (6), this yields the estimate

$$\left\|\frac{v^r}{r}(t)\right\|_{L^\infty} \lesssim \left\|\zeta(t)\right\|_{L^{3,1}} \lesssim \|\zeta_0\|_{L^{3,1}}.$$

Since one gets from a crude estimate on (5) that

$$\left\|\omega_\theta(t)\right\|_{L^\infty} \leqslant \left\|\omega_0(t)\right\|_{L^\infty} e^{\int_0^t \|v^r/r\|_{L^\infty}},$$

the global well-posedness in H^s $s > 5/2$ (the assumption $\zeta_0 \in L^{3,1}$ is automatically satisfied) then follows from the Beale–Kato–Majda criterion. It is actually possible, as shown in [2], to get global well-posedness in the critical Besov regularity, that is, $v_0 \in B^{\frac{3}{p}+1}_{p,1}$, $\forall p \in [1, \infty]$, in the sense that it is possible to propagate globally the critical Besov regularity if $\zeta_0 \in L^{3,1}$.

Our aim here is to extend these global well-posedness results to the Boussinesq system (1). Our main result reads:

Theorem 1.1. *Consider the Boussinesq system* (1). *Let $s > \frac{5}{2}$, $v_0 \in H^s$ be an axisymmetric divergence free vector field without swirl and let ρ_0 be an axisymmetric function belonging to $H^{s-2} \cap L^m$ with $m > 6$ and such that $r^2 \rho_0 \in L^2$. Then there is a unique global solution (v, ρ) such that*

$$(v, \rho) \in \mathcal{C}\big(\mathbb{R}_+; H^s\big) \times \big(\mathcal{C}\big(\mathbb{R}_+; H^{s-2} \cap L^m\big) \cap L^1_{\mathrm{loc}}\big(\mathbb{R}_+; W^{1,\infty}\big)\big) \quad \textit{and} \quad r^2\rho \in \mathcal{C}\big(\mathbb{R}_+; L^2\big).$$

Let us give a few comments about our result.

Remark 1.2. By axisymmetric scalar function we mean again a function that depends only on the variables (r, z) but not on the angle θ in cylindrical coordinates. One can easily check that for smooth local solutions, if (ρ_0, v_0) is axisymmetric (and v_0 without swirl), this property is preserved by the evolution.

Remark 1.3. The assumption on the moment of ρ is probably technical. The control of the moments of ρ are needed in our proof in some commutator estimates (see (9) for example).

Note that in view of the proof for the Euler equation, the crucial part is to get an a priori estimate for ζ in $L^{3,1}$. The equation for $\zeta = \omega_\theta / r$ becomes

$$\partial_t \zeta + v \cdot \nabla \zeta = -\frac{\partial_r \rho}{r} \tag{8}$$

and consequently, the main difficulty is to find some strong a priori estimates on ρ to control the term in the right-hand side of (8). The rough idea is that on the axis $r = 0$ the singularity $\frac{1}{r}$ scales as a derivative and hence that the forcing term $\partial_r \rho / r$ can be thought as a Laplacian of ρ and thus one may try to use smoothing effects to control it. We observe that if we neglect for the moment the advection term $v \cdot \nabla \rho$ in the equation of the density then using the maximal smoothing effects of the heat semigroup we can gain two derivatives by integrating in time which is exactly what we need. From this point of view we see that our model is in some sense critical for the global well-posedness analysis. The main difficulty if one wants to use this argument is to deal with the advection term. Indeed, the only control on v that we have at our disposal is a $L^\infty_{\mathrm{loc}} L^2$ estimate (which comes from the basic energy estimate) and this is not sufficient to obtain an estimate for $D^2 \rho$ in $L^1_{\mathrm{loc}}(L^p)$ by considering the convection term as a source term and by using the maximal smoothing effect of the heat equation. Even more refined maximal regularity estimates on convection–diffusion equations ([9,16] for example) do not seem to provide useful information when the control of the velocity field is so poor. Consequently, our strategy for the proof will be to use more carefully the structure of the coupling between the two equations of (1) in order to find suitable a priori estimates for (ζ, ρ). Since the coupling between the two equations does not make the original Boussinesq system well suited for a priori estimates, our main idea is to use an approach that was successfully used for the study of two-dimensional systems with a critical dissipation, see [19,20] and the Navier–Stokes–Boussinesq system with axisymmetric data [21]. It consists in diagonalizing the linear part of the system satisfied by ζ and ρ. We introduce a new unknown Γ which here formally reads

$$\Gamma = \zeta + \frac{\partial_r}{r} \Delta^{-1} \rho$$

and we study the system satisfied by (Γ, ρ) which is given by:

$$\partial_t \Gamma + v \cdot \nabla \Gamma = -\left[\frac{\partial_r}{r} \Delta^{-1}, v \cdot \nabla\right] \rho, \quad \partial_t \rho + v \cdot \nabla \rho = \Delta \rho$$

where $[\frac{\partial_r}{r} \Delta^{-1}, v \cdot \nabla]$ is the commutator defined by

$$\left[\frac{\partial_r}{r} \Delta^{-1}, v \cdot \nabla\right] \rho = \frac{\partial_r}{r} \Delta^{-1} (v \cdot \nabla \rho) - v \cdot \nabla \left(\frac{\partial_r}{r} \Delta^{-1} \rho\right).$$

Note that if we forget the commutator for a while, we immediately get an a priori L^p estimate for Γ for every p from which we can hope to get an L^p estimate for ζ, if the operator $\frac{\partial_r}{r} \Delta^{-1}$ behaves well.

To make this argument rigorous, we need first to study the action of the operator $\frac{\partial_r}{r}\Delta^{-1}$ over axisymmetric functions. This is done in Proposition 2.9 where we prove that this operator takes the form

$$\frac{\partial_r}{r}\Delta^{-1} = \sum_{i,j} a_{ij}(x)\mathcal{R}_{ij}$$

where $\mathcal{R}_{ij} = \partial_i\partial_j\Delta^{-1}$ are Riesz operators and the functions a_{ij} are bounded. This yields that $\frac{\partial_r}{r}\Delta^{-1}$ acts continuously on $L^{3,1}$ and hence that

$$\left\|\frac{\partial_r}{r}\Delta^{-1}\rho(t)\right\|_{L^{3,1}} \lesssim \|\rho(t)\|_{L^{3,1}} \lesssim \|\rho_0\|_{L^{3,1}}.$$

It follows that the control of Γ is equivalent to the control of ζ in $L^{3,1}$. Now it remains to estimate in a suitable way the commutator term $[\frac{\partial_r}{r}\Delta^{-1}, v\cdot\nabla]\rho$ which is the main technical part. It seems that there is no hope to bound the commutator without using unknown quantities because there is no other known a priori estimates of the velocity except that given by energy estimate which is not strong enough. We shall prove (Theorem 3.1) that

$$\big\|\big[(\partial_r/r)\Delta^{-1}, v\cdot\nabla\big]\rho\big\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|\rho x_h\|_{B^0_{\infty,1}\cap L^2} + \|\rho\|_{B^{\frac{1}{2}}_{2,1}}\big). \tag{9}$$

This estimate is the heart of our argument, its proof combines the use of paradifferential calculus and some harmonic analysis results and also requires a careful use of the property that velocity v is axisymmetric without swirl in the Biot–Savart law.

The main reason for which we need some moments of ρ in the right-hand side of (9) is that we want an estimate of the commutator involving ω_θ/r and not ω.

In the right-hand side of (9), $\|\rho\|_{B^{\frac{1}{2}}_{2,1}}$ and $\|\rho x_h\|_{L^2}$ can be controlled in terms of the initial data only by using the smoothing effect of the convection–diffusion equation for ρ and standard energy estimates. Consequently, from this commutator estimate, we obtain that

$$\|\zeta(t)\|_{L^{3,1}} \leqslant C(t)e^{C\|\rho x_h\|_{L^1_t B^0_{\infty,1}}} \tag{10}$$

and the next difficult step is to control $\|\rho x_h\|_{L^1_t B^0_{\infty,1}}$. This is done in two steps. The first step is to get a global L^∞ estimate of ρx_h in terms of the initial data only and then in a second step, we shall prove a logarithmic estimate for the $B^0_{\infty,1}$ norm of $x_h\rho$ in terms of the $L^{3,1}$ norm of ζ.

For the first step, let us observe that $f = \rho x_h$ solves the equation

$$\partial_t f + v\cdot\nabla f - \Delta f = v^h\rho - 2\nabla_h\rho. \tag{11}$$

Note that for the moment, we only have at our disposal the standard energy estimate for v (thus we control $\|v\|_{L^\infty_t L^2}$ only), consequently to obtain an L^∞ estimate for f we need to use an $L^2 \to L^\infty$ estimate for the convection–diffusion equation since the source term in the right-hand side can be estimated only in L^2. Note that the convection term cannot be neglected (again because of the weak control on v that we have at this stage) and hence this estimate cannot be obtained from heat kernel estimates. We shall obtain this estimate by using the Nash–Moser–De Giorgi iterations [14,26,25]. Indeed, the main interest of this approach is that since it is based on energy type estimates, the convection term does not contribute. A general result is recalled in Appendix A. For technical reasons, some higher order moment estimates which are easier to obtain are also needed, they are stated in Proposition 4.2.

Once the estimate of $\|\rho x_h\|_{L^\infty}$ is known in terms of the initial data, one can establish logarithmic Besov space estimates for the convection–diffusion equation (11) by using a special time dependent frequency cut-off of $x_h\rho$ where we combine the L^∞ estimate with some smoothing effects for $x_h\rho$. This yields (see (56))

$$\|\rho x_h\|_{L^1_t B^0_{\infty,1}} \leqslant C_0(t)\left(1+\int_0^t h(\tau)\log\big(2+\|\zeta\|_{L^\infty_\tau L^{3,1}}\big)\,d\tau\right) \tag{12}$$

where $C_0(t)$ is a given continuous function and h is some $L^1_{\mathrm{loc}}(\mathbb{R}_+)$ function. We point out that the use of the moment of order two $|x_h|^2\rho$ is due to the treatment of the commutator $[\Delta_q, v\cdot\nabla](x_h\rho)$ which appears when we deal with the smoothing effects.

The combination of the estimates (12) and (10) with Gronwall inequality allows to control $\|\zeta(t)\|_{L^{3,1}}$ globally in time.

The final step is to deduce, as for the incompressible Euler equation, from the control of $\|\zeta\|_{L^\infty_t L^{3,1}}$ an estimate of $\|\omega\|_{L^\infty_t L^\infty}$ and of $\|\nabla v\|_{L^\infty_t L^\infty}$. This is the aim of Propositions 4.5 and 4.6. Estimates in Sobolev spaces then follow in a rather classical way.

Once a priori estimates for sufficiently smooth functions are known, the result of Theorem 1.1 follows from an approximation argument.

The paper is organized as follows. In Section 2 we fix the notations, give the definitions of the functional spaces, in particular Besov and Lorentz spaces, that we shall use and state some of their useful properties. We also study the operator $\frac{\partial_r}{r}\Delta^{-1}$ in Proposition 2.9. Next, in Section 3, we study the commutator $[\frac{\partial_r}{r}\Delta^{-1}, v\cdot\nabla]$. In Section 4, we turn to the proof of a priori estimates for sufficiently smooth solutions of (1). We first prove in Proposition 4.1 some basic energy estimates, next, we study the moments of ρ in Proposition 4.2 and then we control $\|\zeta\|_{L^{3,1}}$ in Proposition 4.4. Lipschitz and Sobolev estimates are finally obtained in Proposition 4.5 and Proposition 4.6. In Section 5, we give the proof of Theorem 1.1: we obtain the existence part by using the a priori estimates and an approximation argument and then we prove the uniqueness part. Finally, Appendix A is devoted to the proof of a priori estimates for convection–diffusion equations by the Nash–De Giorgi iterations which are needed in the estimate of the moments of ρ. In Appendix B we give the proof of Lemma 2.7 which is a technical commutator lemma used in several places.

2. Preliminaries

2.1. Dyadic decomposition and functional spaces

Throughout this paper, C stands for some real positive constant which may be different in each occurrence and C_0 denotes a positive number depending on the initial data only. We shall sometimes alternatively use the notation $X \lesssim Y$ for the inequality $X \leqslant CY$.

When B is a Banach space, we shall use the shorthand $L^p_T(B)$ for $L^p(0,T,B)$.

Now to introduce Besov spaces which are a generalization of Sobolev spaces we need to recall the dyadic decomposition of the unity in the whole space (see [8]).

Proposition 2.1. *There exist two positive radial functions* $\chi \in \mathcal{D}(\mathbb{R}^3)$ *and* $\varphi \in \mathcal{D}(\mathbb{R}^3\backslash\{0\})$ *such that*

(1) $\chi(\xi) + \sum_{q\in\mathbb{N}} \varphi(2^{-q}\xi) = 1, \qquad \frac{1}{3} \leqslant \chi^2(\xi) + \sum_{q\in\mathbb{N}} \varphi^2(2^{-q}\xi) \leqslant 1 \quad \forall \xi \in \mathbb{R}^3,$

(2) $\operatorname{supp}\varphi(2^{-p}\cdot) \cap \operatorname{supp}\varphi(2^{-q}\cdot) = \emptyset$, *if* $|p-q| \geqslant 2$,

(3) $q \geqslant 1 \Rightarrow \operatorname{supp}\chi \cap \operatorname{supp}\varphi(2^{-q}) = \emptyset$.

For every $u \in \mathcal{S}'(\mathbb{R}^3)$ we define the nonhomogeneous Littlewood–Paley operators by,

$$\Delta_{-1}u = \chi(\mathrm{D})u; \quad \forall q \in \mathbb{N}, \qquad \Delta_q u = \varphi\big(2^{-q}\mathrm{D}\big)u \quad \text{and} \quad S_q u = \sum_{-1\leqslant j\leqslant q-1} \Delta_j u.$$

One can easily prove that for every tempered distribution u,

$$u = \sum_{q\geqslant -1} \Delta_q u. \tag{13}$$

In the sequel we will frequently use Bernstein inequalities (see for example [8]).

Lemma 2.2. *There exists a constant* C *such that for* $k \in \mathbb{N}$, $1 \leqslant a \leqslant b$ *and* $u \in L^a$, *we have*

$$\sup_{|\alpha|=k} \big\|\partial^\alpha S_q u\big\|_{L^b} \leqslant C^k 2^{q(k+3(\frac{1}{a}-\frac{1}{b}))} \|S_q u\|_{L^a},$$

and for $q \in \mathbb{N}$

$$C^{-k} 2^{qk} \|\Delta_q u\|_{L^a} \leqslant \sup_{|\alpha|=k} \big\|\partial^\alpha \Delta_q u\big\|_{L^a} \leqslant C^k 2^{qk} \|\Delta_q u\|_{L^a}.$$

The basic tool of the paradifferential calculus is Bony's decomposition [4]. It distinguishes in a product uv three parts as follows:

$$uv = T_u v + T_v u + \mathcal{R}(u,v),$$

where

$$T_u v = \sum_q S_{q-1} u \Delta_q v, \quad \text{and} \quad \mathcal{R}(u,v) = \sum_q \Delta_q u \widetilde{\Delta}_q v,$$

with

$$\widetilde{\Delta}_q = \sum_{i=-1}^{1} \Delta_{q+i}.$$

The term $T_u v$ is called the paraproduct of v by u and $\mathcal{R}(u,v)$ the remainder term. The main interest of the paraproduct term is that each term $S_{q-1}u\Delta_q v$ has the support of its Fourier transform still localized in an annulus of size 2^q and thus $T_u v$ is a sum of almost orthogonal functions.

Let $(p,r)\in[1,+\infty]^2$ and $s\in\mathbb{R}$, then the nonhomogeneous Besov space $B^s_{p,r}$ is the set of tempered distributions u such that

$$\|u\|_{B^s_{p,r}} := \big(2^{qs}\|\Delta_q u\|_{L^p}\big)_{\ell^r} < +\infty.$$

We remark that the Sobolev space H^s coincides with the Besov space $B^s_{2,2}$. Also, by using the Bernstein inequalities we get easily the embeddings

$$B^s_{p_1,r_1} \hookrightarrow B^{s+3(\frac{1}{p_2}-\frac{1}{p_1})}_{p_2,r_2}, \quad p_1 \leqslant p_2 \text{ and } r_1 \leqslant r_2.$$

Finally, let us notice that we can also characterize L^p spaces in terms of the dyadic decomposition, see [28]. For $p\in]1,+\infty[$, there exists $C>0$ such that: f belongs to L^p if and only if $(\Delta_q f)_{q\geqslant -1}\in L^p l^2$ and

$$C^{-1}\bigg\|\bigg(\sum_{q\geqslant -1}|\Delta_q f|^2\bigg)^{\frac12}\bigg\|_{L^p} \leqslant \|f\|_{L^p} \leqslant C\bigg\|\bigg(\sum_{q\geqslant -1}|\Delta_q f|^2\bigg)^{\frac12}\bigg\|_{L^p}. \tag{14}$$

2.2. *Lorentz spaces and interpolation*

For $p\in]1,\infty[$, $q\in[1,+\infty]$, the Lorentz space $L^{p,q}$ can be defined by real interpolation from Lebesgue spaces:

$$\big(L^{p_0},L^{p_1}\big)_{(\theta,q)} = L^{p,q},$$

where $1\leqslant p_0<p<p_1\leqslant\infty$, θ satisfies $\frac1p=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}$ and $1\leqslant q\leqslant\infty$.

From this definition, we get:

$$L^{p,q}\hookrightarrow L^{p,q'}, \quad L^{p,p}=L^p \tag{15}$$

for every $1<p<\infty$, $1\leqslant q\leqslant q'\leqslant\infty$.

Lorentz spaces will arise in a natural way in our problem because of the following classical convolution results, for the proof see for instance [24,27].

Theorem 2.3. *For every α, $0<\alpha<d$, $p_i\in]1,+\infty[$, $q_i\in[1,+\infty]$, such that $1+\frac{1}{p_1}=\frac{1}{p_2}+\frac{1}{p_3}$ and $\frac{1}{q_1}=\frac{1}{q_2}+\frac{1}{q_3}$, there exists $C>0$ such that*

$$\|f*g\|_{L^{p_1,q_1}} \leqslant C\|f\|_{L^{p_2,q_2}}\|g\|_{L^{p_3,q_3}}. \tag{16}$$

Moveover, in the case that $p_1=\infty$, we have

$$\|f*g\|_{L^\infty(\mathbb{R}^d)} \leqslant C\|f\|_{L^{\frac{d}{\alpha},\infty}(\mathbb{R}^d)}\|g\|_{L^{\frac{d}{d-\alpha},1}(\mathbb{R}^d)}. \tag{17}$$

In particular, by using this result and the fact that $1/|x|^2$ belongs to $L^{\frac{3}{2},\infty}(\mathbb{R}^3)$, we have that

$$\|\nabla\Delta^{-1}f\|_{L^\infty(\mathbb{R}^3)} \lesssim \|f\|_{L^{3,1}(\mathbb{R}^3)} \tag{18}$$

and thanks to the pointwise estimate (7) that

$$\left\|\frac{v_r}{r}\right\|_{L^\infty} \lesssim \|\zeta\|_{L^{3,1}}. \tag{19}$$

To establish some functional inequalities involving Lorentz spaces the following classical interpolation result (see [24] for example) will be very useful.

Theorem 2.4. *Let $1 \leqslant p_1 < p_2 \leqslant \infty$, $1 \leqslant r_1 < r_2 \leqslant \infty$, $q \in [1,\infty]$ and T be a linear bounded operator from L^{p_i} to L^{r_i}. Let $\theta \in]0,1[$ and p, r such that $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ and $\frac{1}{r} = \frac{\theta}{r_1} + \frac{1-\theta}{r_2}$. Then T is also bounded from $L^{p,q}$ to $L^{r,q}$ with*

$$\|T\|_{\mathcal{L}(L^{p,q};L^{r,q})} \leqslant C\|T\|^{\theta}_{\mathcal{L}(L^{p_1};L^{r_1})}\|T\|^{1-\theta}_{\mathcal{L}(L^{p_2};L^{r_2})}.$$

As a consequence, we obtain the following results.

Proposition 2.5. *For $1 < p < +\infty$, $q \in [1,+\infty]$, then exists a constant $C > 0$ such that the following estimates hold true*

(1) $\|uv\|_{L^{p,q}} \leqslant C\|u\|_{L^\infty}\|v\|_{L^{p,q}}$.
(2) $\|T_u v\|_{L^{p,q}} \leqslant C\|u\|_{L^\infty}\|v\|_{L^{p,q}}$.
(3) *Let us define the Riesz transform $\mathcal{R}_{ij} = \partial_i\partial_j\Delta^{-1}$, $i, j \in \{1,2\}$, then*

$$\|\mathcal{R}_{ij}u\|_{L^{p,q}} \leqslant C\|u\|_{L^{p,q}}.$$

(4) *For $s > \frac{1}{2}$ we have $H^s \hookrightarrow L^{3,1}$. For $1 \leqslant p < 3$ we have $B^{\frac{3}{p}-1}_{p,1} \hookrightarrow L^{3,1}$.*

Proof. (1) For a fixed function $u \in L^\infty$, the linear operator $T : v \mapsto uv$ belongs to $\mathcal{L}(L^p, L^p)$ with norm smaller that $\|u\|_{L^\infty}$ and hence the result follows by interpolation from Theorem 2.4.

(3) In a similar way, for every $p \in]1,+\infty[$, $\mathcal{R}_{ij} \in \mathcal{L}(L^p, L^p)$ thanks to the Calderón–Zygmund theorem and hence (3) follows again by using Theorem 2.4.

(2) To establish the inequality, it is again sufficient thanks to Theorem 2.4 to prove that for $u \in L^\infty$, $v \in L^p$ we have $\|T_u v\|_{L^p} \leqslant C\|u\|_{L^\infty}\|v\|_{L^p}$. For this last purpose we will make use of the maximal functions tool. We will start with some classical results in this subject. For a locally integrable function $f : \mathbb{R}^3 \to \mathbb{R}$, we shall define its maximal function $\mathcal{M}f$ by

$$\mathcal{M}f(x) = \sup_{r>0}\frac{1}{r^3}\int_{B(x,r)}\big|f(y)\big|\,dy.$$

From the definition we get

$$0 \leqslant \mathcal{M}(fg)(x) \leqslant \|g\|_{L^\infty}\mathcal{M}f(x). \tag{20}$$

It is well known that $\mathcal{M}$ maps continuously L^p to itself for $p \in]1, \infty]$. Moreover, we have the following lemma. We refer to [28] for a proof.

Lemma 2.6.

(1) *Let $\psi \in \mathcal{S}(\mathbb{R}^3)$ and define $\psi_\varepsilon(x) = \varepsilon^{-3}\psi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. Then there exists $C > 0$ such that for every $p \in [1, \infty]$, $\varepsilon \in (0, 1]$, we have*

$$\sup_{\varepsilon>0}\bigl|\psi_\varepsilon \star f(x)\bigr| \leqslant C\mathcal{M}f(x).$$

In particular, we have

$$\sup_{q\geqslant -1}\bigl|\Delta_q f(x)\bigr| \leqslant C\mathcal{M}f(x).$$

(2) *Let $p \in]1, \infty]$ and $\{f_q, q \geqslant -1\}$ be a sequence belonging to $L^p\ell^2$. Then we have*

$$\biggl\|\Bigl(\sum_q \bigl|\mathcal{M}f_q(x)\bigr|^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} \leqslant C\biggl\|\Bigl(\sum_q \bigl|f_q(x)\bigr|^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p}.$$

Let us now come back to the proof of (2). By using (14), we have

$$\|T_u v\|_{L^p} \lesssim \biggl\|\Bigl(\sum_{j\geqslant -1}\bigl|\Delta_j(T_u v)\bigr|^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} = \biggl\|\Bigl(\sum_{j\geqslant -1}\Bigl|\Delta_j\Bigl(\sum_{|j-q|\leqslant 4} S_{q-1}u\Delta_q v\Bigr)\Bigr|^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p}.$$

This yields according to Lemma 2.6 and (20),

$$\begin{aligned}
\|T_u v\|_{L^p} &\lesssim \biggl\|\Bigl(\sum_{j\geqslant -1}\Bigl(\sum_{|j-q|\leqslant 4}\mathcal{M}(S_{q-1}u\Delta_q v)\Bigr)^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} \\
&\lesssim \|u\|_{L^\infty}\biggl\|\Bigl(\sum_{j\geqslant -1}\Bigl(\sum_{|j-q|\leqslant 4}\mathcal{M}\,\Delta_q v\Bigr)^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} \\
&\lesssim \|u\|_{L^\infty}\biggl\|\Bigl(\sum_{q\geqslant -1}(\mathcal{M}\Delta_q v)^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} \lesssim \|u\|_{L^\infty}\biggl\|\Bigl(\sum_{q\geqslant -1}(\Delta_q v)^2\Bigr)^{\frac{1}{2}}\biggr\|_{L^p} \\
&\lesssim \|u\|_{L^\infty}\|v\|_{L^p}
\end{aligned}$$

where the last estimate follows from a new use of (14). This ends the proof of (2).

(4) The first embedding follows from Sobolev embeddings combined with Theorem 2.4. This is left to the reader. For the second one we refer for example to the proof of Proposition 2.2 of [2]. □

2.3. Some useful commutator estimates

This section is devoted to the study of some basic commutators which will be needed in our main commutator estimates, especially in Theorem 3.1 and Proposition 3.2 . Our first result reads as follows. The proof is postponed to Appendix B.

Lemma 2.7. *Given* $(p, r, \rho, m) \in [1, +\infty]^4$ *such that*

$$1 + \frac{1}{p} = \frac{1}{m} + \frac{1}{\rho} + \frac{1}{r}, \quad p \geqslant r \text{ and } \rho > 3\left(1 - \frac{1}{r}\right).$$

Let f, g *and* h *be three functions such that* $\nabla f \in L^\rho$, $g \in L^m$ *and* $x\mathcal{F}^{-1}h \in L^r$. *Then*

$$\left\|\left[h(\mathrm{D}), f\right]g\right\|_{L^p} \leqslant C\left\|x\mathcal{F}^{-1}h\right\|_{L^r}\|\nabla f\|_{L^\rho}\|g\|_{L^m},$$

where C *is a constant.*

As an application of Lemma 2.7 we get the following commutator estimates.

Lemma 2.8. *Let* $p, m, \rho \in [1, +\infty]$ *such that* $\frac{1}{p} = \frac{1}{m} + \frac{1}{\rho}$. *Then, there exists* $C > 0$ *such that for* $\nabla f \in L^\rho$, $g \in L^m$ *and for every* $q \in \mathbb{N} \cup \{-1\}$

$$\left\|[\Delta_q, f]g\right\|_{\dot{W}^{1,p}} \leqslant C\|\nabla f\|_{L^\rho}\|g\|_{L^m},$$

with the following definition $\|\varphi\|_{\dot{W}^{1,p}} = \|\nabla\varphi\|_{L^p}$.

Proof. We write for $i = 1, 2, 3$,

$$\partial_i\left([\Delta_q, f]g\right) = [\partial_i\Delta_q, f]g - \partial_i f\Delta_q g = \left[h_q(\mathrm{D}), f\right]g - \partial_i f\Delta_q g,$$

with $h_q(\xi) = 2^q\phi(2^{-q}\xi)$, and $\phi \in \mathcal{S}(\mathbb{R}^3)$. Using Lemma 2.7 we get

$$\left\|\left[h_q(\mathrm{D}), f\right]g\right\|_{L^p} \leqslant C\left\|x\mathcal{F}^{-1}h_q\right\|_{L^1}\|\nabla f\|_{L^\rho}\|g\|_{L^m} \leqslant C\|\nabla f\|_{L^\rho}\|g\|_{L^m}.$$

For the other term, the Hölder inequality yields

$$\begin{aligned}\|\partial_i f\Delta_q g\|_{L^p} &\leqslant C\|\nabla f\|_{L^\rho}\|\Delta_q g\|_{L^m}\\ &\leqslant C\|\nabla f\|_{L^\rho}\|g\|_{L^m}. \qquad \square\end{aligned}$$

2.4. Some algebraic identities

We intend in this paragraph to describe first the action of the operator $\frac{\partial_r}{r}\Delta^{-1}u$ over axisymmetric functions. We will show that it behaves like Riesz transforms. The second part is concerned with the study of some algebraic identities involving some multipliers which will appear in a natural way when try to study our main commutator $[\partial_r/r)\Delta^{-1}, v\cdot\nabla]\rho$.

Proposition 2.9. *We have for every axisymmetric smooth scalar function u*

$$(\partial_r/r)\Delta^{-1}u(x) = \frac{x_2^2}{r^2}\mathcal{R}_{11}u(x) + \frac{x_1^2}{r^2}\mathcal{R}_{22}u(x) - 2\frac{x_1x_2}{r^2}\mathcal{R}_{12}u(x), \tag{21}$$

with $\mathcal{R}_{ij} = \partial_{ij}\Delta^{-1}$. *Moreover, for* $p \in]1,\infty[$, $q \in [1,\infty]$ *there exists* $C > 0$ *such that*

$$\left\|(\partial_r/r)\Delta^{-1}u\right\|_{L^{p,q}} \leqslant C\|u\|_{L^{p,q}}. \tag{22}$$

Proof. We set $f = \Delta^{-1}u$, then we can show from Biot–Savart law that f is also axisymmetric. Hence we get by using polar coordinates that

$$\partial_{11}f + \partial_{22}f = (\partial_r/r)f + \partial_{rr}f \tag{23}$$

where

$$\partial_r = \frac{x_1}{r}\partial_1 + \frac{x_2}{r}\partial_2.$$

By using this expression of ∂_r, we obtain

$$\begin{aligned}\partial_{rr} &= \left(\frac{x_1}{r}\partial_1 + \frac{x_2}{r}\partial_2\right)^2 = \partial_r\left(\frac{x_1}{r}\right)\partial_1 + \partial_r\left(\frac{x_2}{r}\right)\partial_2 + \frac{x_1^2}{r^2}\partial_{11} + \frac{x_2^2}{r^2}\partial_{22} + \frac{2x_1x_2}{r^2}\partial_{12}.\\ &= \frac{x_1^2}{r^2}\partial_{11} + \frac{x_2^2}{r^2}\partial_{22} + \frac{2x_1x_2}{r^2}\partial_{12}\end{aligned}$$

since

$$\partial_r\left(\frac{x_i}{r}\right) = 0, \quad \forall i \in \{1,2\}.$$

This yields by using (23) that

$$\begin{aligned}\frac{\partial_r}{r}f &= \left(1 - \frac{x_1^2}{r^2}\right)\partial_{11}f + \left(1 - \frac{x_2^2}{r^2}\right)\partial_{22}f - \frac{2x_1x_2}{r^2}\partial_{12}f\\ &= \frac{x_2^2}{r^2}\partial_{11}f + \frac{x_1^2}{r^2}\partial_{22}f - \frac{2x_1x_2}{r^2}\partial_{12}f.\end{aligned}$$

To get (21), it suffices replace f by $\Delta^{-1}u$.

The estimate (22) is a consequence of (21) and the estimates (1) and (3) of Proposition 2.5 since for every $i, j \in \{1,2\}$, $\frac{x_ix_j}{r^2} \in L^\infty$. □

We shall also need the following identities and estimates.

Lemma 2.10. *For every* $f \in \mathcal{S}(\mathbb{R}^3, \mathbb{R})$*, we have*

(1) *For* $i, j \in \{1, 2, 3\}$

$$\Delta^{-1}(x_i \partial_j f) = x_i \partial_j \Delta^{-1} f + \mathcal{L}_{ij} f$$

where $\mathcal{L}_{ij} f = -2\mathcal{R}_{ij}\Delta^{-1} f$*. Moreover, we have the estimates*:

$$\|\nabla \mathcal{L}_{ij} f\|_{L^\infty} \leqslant C\|f\|_{L^{3,1}}, \tag{24}$$

$$\left\|\nabla^2 \mathcal{L}_{ij} f\right\|_{L^{p,q}} \leqslant C\|f\|_{L^{p,q}}, \quad p \in]1, +\infty[,\ q \in [1, +\infty]. \tag{25}$$

(2) *For* $i, j, k \in \{1, 2, 3\}$

$$\mathcal{R}_{ij}(x_k f) = x_k \mathcal{R}_{ij} f + \mathcal{L}_{ij}^k f,$$

with

$$\mathcal{L}_{ij}^k := -2\partial_k \Delta^{-1}\mathcal{R}_{ij} + \delta_{ik}\partial_j \Delta^{-1} + \delta_{jk}\partial_i \Delta^{-1}$$

where δ_{ij} *denotes the Kronecker symbol. Moreover we have the estimates*

$$\left\|\mathcal{L}_{ij}^k f\right\|_{L^\infty} \leqslant C\|f\|_{L^{3,1}}, \tag{26}$$

$$\left\|\nabla \mathcal{L}_{ij}^k f\right\|_{L^{p,q}} \leqslant C\|f\|_{L^{p,q}}, \quad p \in]1, +\infty[,\ q \in [1, +\infty]. \tag{27}$$

Proof. (1) We first expand

$$\Delta\left(x_i \partial_j \Delta^{-1} f - 2\mathcal{R}_{ij}\Delta^{-1} f\right) = 2\partial_{ij}\Delta^{-1} f + x_i \partial_j f - 2\mathcal{R}_{ij} f = x_i \partial_j f.$$

This yields

$$\Delta^{-1}(x_i \partial_j f) = x_i \partial_j \Delta^{-1} f - 2\mathcal{R}_{ij}\Delta^{-1} f + P(x),$$

with P a harmonic polynomial. We can easily see that the r.h.s of this identity and $\mathcal{R}_{ij}\Delta^{-1} f$ are decreasing at infinity. Thus to prove that P is zero it suffices to prove that $x_i \partial_j \Delta^{-1} f$ goes to zero at infinity. Since

$$\left|x_i \partial_j \Delta^{-1} f\right| \lesssim |x_j| \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|^2}\, dy \lesssim \int_{\mathbb{R}^3} \frac{|f(y)|}{|x-y|}\, dy + \int_{\mathbb{R}^3} \frac{|y_j f(y)|}{|x-y|^2}\, dy.$$

Using Theorem 2.3, we get that $x_i \partial_j \Delta^{-1} f \in L^p$, for every $p > 3$. Hence we get $P = 0$.

The estimates (24), (25) are a direct consequence of the above expression and (18) and the estimate (3) of Proposition 2.5.

(2) We use the same idea as previously. We first get the identity

$$\begin{aligned}\Delta \mathcal{L}_{ij}^k f &= \Delta\left(\mathcal{R}_{ij}(x_k f) - x_k \mathcal{R}_{ij} f\right) = \partial_{ij}(x_k f) - 2\partial_k \mathcal{R}_{ij} f - x_k \partial_{ij} f \\ &= \delta_{ik}\partial_j f + \delta_{jk}\partial_i f - 2\partial_k \mathcal{R}_{ij} f\end{aligned}$$

and by the same argument as above, we finally obtain that

$$\mathcal{R}_{ij}(x_k f) - x_k \mathcal{R}_{ij} f = \delta_{ik}\partial_j \Delta^{-1} f + \delta_{jk}\partial_i \Delta^{-1} f - 2\partial_k \mathcal{R}_{ij}\Delta^{-1} f.$$

The estimates (26), (27) are a direct consequence of the above expression and (18) and the estimate (3) of Proposition 2.5. □

3. Commutator estimates

3.1. The commutator between the advection operator and $\frac{\partial_r}{r}\Delta^{-1}$

In this part we discuss the commutation between the operators $\frac{\partial_r}{r}\Delta^{-1}$ and $v \cdot \nabla$. This is a crucial estimate in order to get better a priori estimates for the solution of (1) by using our transformation. Our result reads as follows.

Theorem 3.1. *Let v be an axisymmetric smooth and divergence free without swirl vector field and ρ an axisymmetric smooth scalar function. Then we have, with the notation $x_h = (x_1, x_2)$, that*

$$\big\| \big[(\partial_r/r)\Delta^{-1}, v\cdot\nabla\big]\rho \big\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}} \big(\|\rho x_h\|_{B^0_{\infty,1}\cap L^2} + \|\rho\|_{B^{\frac{1}{2}}_{2,1}} \big).$$

Proof. Since the functions ρ and $v\cdot\nabla\rho$ are axisymmetric then using the identity of Proposition 2.9 we have

$$(\partial_r/r)\Delta^{-1}\rho(x) = \frac{x_2^2}{r^2}\mathcal{R}_{11}\rho(x) + \frac{x_1^2}{r^2}\mathcal{R}_{22}\rho(x) - 2\frac{x_1x_2}{r^2}\mathcal{R}_{12}\rho(x) := \sum_{i,j=1}^{2} a_{ij}(x)\mathcal{R}_{i,j}\rho(x)$$

and also

$$(\partial_r/r)\Delta^{-1}(v\cdot\nabla\rho)(x) = \sum_{i,j=1}^{2} a_{ij}(x)\mathcal{R}_{i,j}(v\cdot\nabla\rho)(x).$$

Since v has no swirl and the functions $a_{i,j}$ do not depend on r and z, we have for every $1 \leqslant i,j \leqslant 2$

$$v\cdot\nabla a_{i,j}(x) = v^r\partial_r a_{i,j} + v^z\partial_3 a_{i,j} = 0.$$

Consequently our commutator can be rewritten as

$$\big[(\partial_r/r)\Delta^{-1}, v\cdot\nabla\big]\rho(x) = \sum_{i,j=1}^{2} a_{i,j}(x)[\mathcal{R}_{ij}, v\cdot\nabla]\rho = \sum_{i,j=1}^{2} a_{i,j}(x)\,\mathrm{div}\big\{[\mathcal{R}_{ij}, v]\rho\big\}$$

where we have used the fact that v is divergence free to get the last equality. By using that $a_{ij} \in L^\infty$ and the estimate (1) of Proposition 2.5, we first obtain that

$$\big\|\big[(\partial_r/r)\Delta^{-1}, v\cdot\nabla\big]\rho\big\|_{L^{3,1}} \leqslant \sum_{i,j=1}^{2}\big\|\mathrm{div}\big([\mathcal{R}_{ij}, v]\rho\big)\big\|_{L^{3,1}}. \tag{28}$$

The terms $\partial_1([\mathcal{R}_{ij}, v^1]\rho)$ and $\partial_2([\mathcal{R}_{ij}, v^2]\rho)$ can be treated in same way and hence, we shall prove the estimate of the first one only. The estimate of $\partial_3([\mathcal{R}_{ij}, v^3]\rho)$ which is easier will be done in a second step.

• *Estimate of* $\partial_1([\mathcal{R}_{ij}, v^1]\rho)$. Since v is divergence free, we have that $\Delta v = -\nabla\wedge\omega$. Hence for axisymmetric flows (where in particular $\omega = \omega_\theta e_\theta$), we obtain that

$$v^1(x) = \Delta^{-1}\partial_3\omega^2 = \Delta^{-1}\partial_3\big(x_1(\omega_\theta/r)\big).$$

Applying Lemma 2.10(1) we get

$$v^1(x) = x_1\Delta^{-1}\partial_3(\omega_\theta/r) + \mathcal{L}(\omega_\theta/r), \quad \text{with } \mathcal{L} = -2\partial_{13}\Delta^{-2} \tag{29}$$

(we omit the subscript ij for notational convenience). Consequently the commutator can be rewritten under the form

$$\begin{aligned}
\partial_1\big\{\big[\mathcal{R}_{ij}, v^1\big]\rho\big\} &= \partial_1\big(\big[\mathcal{R}_{i,j}, \mathcal{L}(\omega_\theta/r)\big]\rho\big) + \partial_1\big(\big[\mathcal{R}_{i,j}, \big(\Delta^{-1}\partial_3(\omega_\theta/r)\big)x_1\big]\rho\big)\\
&= \partial_1\big(\big[\mathcal{R}_{i,j}, \mathcal{L}(\omega_\theta/r)\big]\rho\big) + \partial_1\big(\big[\mathcal{R}_{i,j}, \big(\Delta^{-1}\partial_3(\omega_\theta/r)\big)\big]x_1\rho\big)\\
&\quad + \partial_1\big(\big(\Delta^{-1}\partial_3(\omega_\theta/r)\big)[\mathcal{R}_{ij}, x_1]\rho\big)\\
&= \partial_1\big(\big(\Delta^{-1}\partial_3(\omega_\theta/r)\big)\mathcal{L}^1_{ij}\rho\big) + \partial_1\big(\big[\mathcal{R}_{i,j}, \mathcal{L}(\omega_\theta/r)\big]\rho\big)\\
&\quad + \partial_1\big(\big[\mathcal{R}_{i,j}, \big(\Delta^{-1}\partial_3(\omega_\theta/r)\big)\big]x_1\rho\big)\\
&= \mathrm{I} + \mathrm{II} + \mathrm{III}
\end{aligned} \tag{30}$$

where we have used the identity (2) of Lemma 2.10.

Estimate of I. We write

$$\partial_1\big(\partial_3\Delta^{-1}(\omega_\theta/r)\mathcal{L}^1_{ij}\rho\big) = \mathcal{R}_{13}(\omega_\theta/r)\mathcal{L}^1_{ij}\rho + \partial_3\Delta^{-1}(\omega_\theta/r)\partial_1\mathcal{L}^1_{ij}\rho. \tag{31}$$

By using (1) and (3) of Proposition 2.5 and (26) we have

$$\big\|\mathcal{R}_{13}(\omega_\theta/r)\mathcal{L}^1_{ij}\rho\big\|_{L^{3,1}} \leqslant \big\|\mathcal{R}_{13}(\omega_\theta/r)\big\|_{L^{3,1}}\big\|\mathcal{L}^1_{ij}\rho\big\|_{L^\infty} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^{3,1}}$$

and by using Proposition 2.5(1) and (18), (27), we also obtain

$$\big\|\partial_3\Delta^{-1}(\omega_\theta/r)\partial_1\mathcal{L}^1_{ij}\rho\big\|_{L^{3,1}} \leqslant \big\|\partial_3\Delta^{-1}(\omega_\theta/r)\big\|_{L^\infty}\big\|\partial_1\mathcal{L}^1_{ij}\rho\big\|_{L^{3,1}} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^{3,1}}.$$

Combining these estimates we find

$$\|\mathrm{I}\|_{L^{3,1}} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^{3,1}}. \tag{32}$$

Estimate of II. We will use Bony decomposition

$$\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3,$$

with

$$\mathrm{II}_1 = \partial_1 \sum_{q\geqslant 0} \big[\mathcal{R}_{ij}, S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho,$$
$$\mathrm{II}_2 = \partial_1 \sum_{q\geqslant 0} \big[\mathcal{R}_{ij}, \Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]S_{q-1}\rho,$$
$$\mathrm{II}_3 = \partial_1 \sum_{q\geqslant -1} \big[\mathcal{R}_{ij}, \Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q\rho.$$

For the first term we easily get that there exists a function $\psi \in \mathcal{S}(\mathbb{R}^3)$ such that

$$\mathrm{II}_1 = \sum_{q\geqslant 0} \partial_1\big\{\big[\psi_q(\mathrm{D}), S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho\big\},$$

with $\psi_q = 2^{3q}\psi(2^q\cdot)$. By using the Bernstein inequality, this yields

$$\big\|\partial_1\big\{\big[\psi_q(\mathrm{D}), S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho\big\}\big\|_{L^2} \leqslant C2^q\big\|\big[\psi_q(\mathrm{D}), S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho\big\|_{L^2}.$$

Thanks to Lemma 2.7 and (24), we find

$$\begin{aligned}\big\|\partial_1\big\{\big[\psi_q(\mathrm{D}), S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho\big\}\big\|_{L^2} &\leqslant C2^q\|x\psi_q\|_{L^1}\big\|\nabla\mathcal{L}(\omega_\theta/r)\big\|_{L^\infty}\|\Delta_q\rho\|_{L^2}\\ &\leqslant C\|x\psi\|_{L^1}\|\omega_\theta/r\|_{L^{3,1}}\|\Delta_q\rho\|_{L^2}.\end{aligned}$$

It follows that

$$\|\mathrm{II}_1\|_{B^{\frac12}_{2,1}} \leqslant C\sum_{q\in\mathbb{N}} 2^{q\frac12}\big\|\partial_1\big\{\big[\psi_q(\mathrm{D}), S_{q-1}\big(\mathcal{L}(\omega_\theta/r)\big)\big]\Delta_q\rho\big\}\big\|_{L^2} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{B^{\frac12}_{2,1}}$$

and hence by using the embedding $B^{\frac12}_{2,1} \hookrightarrow L^{3,1}$ (see Proposition 2.5(4)), we obtain

$$\|\mathrm{II}_1\|_{L^{3,1}} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{B^{\frac12}_{2,1}}. \tag{33}$$

To estimate the term II_2 we do not need to detect cancellation in the structure of the commutator, we just write

$$\mathrm{II}_2 = \sum_{q\geqslant 0} \partial_1\mathcal{R}_{ij}\big(\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)S_{q-1}\rho\big) - \sum_{q\geqslant 0} \partial_1\big\{\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\mathcal{R}_{ij}S_{q-1}\rho\big\}.$$

A useful remark is that thanks to the Bernstein inequalities and (25), we have

$$\|\Delta_q\mathcal{L}f\|_{L^p} \lesssim 2^{-2q}\big\|\nabla^2\mathcal{L}\Delta_q f\big\|_{L^p} \lesssim 2^{-2q}\|f\|_{L^p}, \quad \forall q\geqslant 0,\ p\in\,]1,+\infty[. \tag{34}$$

This yields by using the Hölder inequality and Proposition 2.5(3) that

$$\begin{aligned}\|\mathrm{II}_2\|_{B^{\frac12}_{2,1}} &\lesssim \sum_{q\geqslant 0} 2^{\frac32 q}\big\|\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)S_{q-1}\rho\big\|_{L^2} + \sum_{q\geqslant 0} 2^{q\frac32}\big\|\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\mathcal{R}_{ij}S_{q-1}\rho\big\|_{L^2}\\ &\lesssim \sum_{q\geqslant 0} 2^{q\frac32}\big\|\Delta_q\mathcal{L}(\omega_\theta/r)\big\|_{L^3}\big(\|S_{q-1}\rho\|_{L^6}+\|\mathcal{R}_{ij}S_{q-1}\rho\|_{L^6}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^3}\sum_{q\geqslant 0}2^{-q\frac12}\|S_{q-1}\rho\|_{L^6}\\ &\lesssim \|\omega_\theta/r\|_{L^3}\sum_{q\geqslant 0}\sum_{k\leqslant q-2}2^{\frac12(k-q)}\big(2^{\frac{k}{2}}\|\Delta_k\rho\|_{L^2}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^3}\|\rho\|_{B^{\frac12}_{2,1}}.\end{aligned}$$

Hence we get from Proposition 2.5(4) that

$$\|\mathrm{II}_2\|_{L^{3,1}} \leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{B^{\frac12}_{2,1}}. \tag{35}$$

For the term II_3 we write

$$\begin{aligned}\mathrm{II}_3 &= \partial_1\sum_{q\geqslant 1}\big[\mathcal{R}_{ij},\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde\Delta_q\rho + \partial_1\sum_{-1\leqslant q\leqslant 0}\big[\mathcal{R}_{ij},\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde\Delta_q\rho\\ &:= \mathrm{II}_{31}+\mathrm{II}_{32}.\end{aligned}$$

To estimate the first term we first use the Bernstein inequality to get

$$\|\Delta_k\mathrm{II}_{31}\|_{L^2}\lesssim 2^k\sum_{q\geqslant k-4}\big\|\big[\mathcal{R}_{ij},\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde\Delta_q\rho\big\|_{L^2}.$$

Next, to estimate the terms inside the sum we do not need to use the structure of the commutator. By using again the Hölder inequality, (34) and the Bernstein inequality, we obtain

$$\begin{aligned}&\big\|\big[\mathcal{R}_{ij},\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde\Delta_q\rho\big\|_{L^2}\\ &\quad\lesssim \big\|\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big\|_{L^3}\|\tilde\Delta_q\rho\|_{L^6} + \big\|\Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big\|_{L^3}\|\mathcal{R}_{ij}\tilde\Delta_q\rho\|_{L^6}\\ &\quad\lesssim 2^{-q}\|\omega_\theta/r\|_{L^3}\|\tilde\Delta_q\rho\|_{L^2}.\end{aligned}$$

It follows by using again Proposition 2.5(4) that

$$\|\mathrm{II}_{31}\|_{L^{3,1}}\lesssim\|\mathrm{II}_{31}\|_{B^{\frac12}_{2,1}}\lesssim\|\omega_\theta/r\|_{L^3}\sum_{k\geqslant -1}\sum_{q\geqslant k-4}2^{\frac32(k-q)}2^{q\frac12}\|\tilde\Delta_q\rho\|_{L^3}\lesssim\|\omega_\theta/r\|_{L^3}\|\rho\|_{B^{\frac12}_{2,1}}.$$

For the estimate of the low frequencies term II_{32} we need to use more deeply the structure of the commutator. We first write

$$\mathrm{II}_{32} = \sum_{-1\leqslant q\leqslant 0} \big[\partial_1 \mathcal{R}_{ij}, \Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q\rho - \sum_{-1\leqslant q\leqslant 0} \partial_1 \mathcal{L}\Delta_q(\omega_\theta/r)\mathcal{R}_{ij}\tilde{\Delta}_q\rho.$$

The last term of the above identity is estimated as follows by using again Proposition 2.5 (1) and (3) and (24)

$$\begin{aligned}\Big\| \sum_{-1\leqslant q\leqslant 0} \partial_1 \mathcal{L}\Delta_q(\omega_\theta/r)\mathcal{R}_{ij}\tilde{\Delta}_q\rho\Big\|_{B^{\frac12}_{2,1}} &\lesssim \sum_{-1\leqslant q\leqslant 0} \big\|\partial_1 \mathcal{L}\Delta_q(\omega_\theta/r)\mathcal{R}_{ij}\tilde{\Delta}_q\rho\big\|_{L^2}\\ &\lesssim \big\|\partial_1\mathcal{L}(\omega_\theta/r)\big\|_{L^\infty}\|\rho\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{B^{\frac12}_{2,1}}.\end{aligned}$$

To estimate the first term of II_{32} we write for every $-1\leqslant q\leqslant 0$ thanks to Lemma 2.7 that

$$\big\|\big[\partial_1\mathcal{R}_{ij}, \Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q\rho\big\|_{L^{\frac52}} \lesssim \|xh\|_{L^{\frac{10}{9}}}\big\|\nabla\mathcal{L}(\omega_\theta/r)\big\|_{L^\infty}\|\tilde{\Delta}_q\rho\|_{L^2},$$

where $\widehat{h}(\xi) = \xi_1\frac{\xi_i\xi_j}{|\xi|^2}\tilde{\chi}(\xi)$ and $\tilde{\chi}\in\mathcal{D}(\mathbb{R}^3)$. Using Mikhlin–Hörmander Theorem we have

$$\big|h(x)\big|\leqslant C\big(1+|x|\big)^{-4},\quad \forall x\in\mathbb{R}^3.$$

This gives in particular $xh\in L^{\frac{10}{9}}$. Therefore we get by using again (24) that

$$\Big\|\sum_{-1\leqslant q\leqslant 0}\big[\partial_1\mathcal{R}_{ij}, \Delta_q\big(\mathcal{L}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q\rho\Big\|_{B^{\frac15}_{\frac52,1}} \lesssim \big\|\nabla\mathcal{L}(\omega_\theta/r)\big\|_{L^\infty}\|\rho\|_{L^2}\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.$$

By using the embedding $B^{\frac15}_{\frac52,1}\hookrightarrow L^{3,1}$, (see Proposition 2.5(4)), we find that

$$\|\mathrm{II}_{32}\|_{L^{3,1}}\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.$$

We have thus obtained that the term II_3 enjoys the estimate

$$\|\mathrm{II}_3\|_{L^{3,1}}\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.$$

Consequently, by gathering this last estimate and the estimates (33), (35), we finally get that

$$\|\mathrm{II}\|_{L^{3,1}}\leqslant C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{B^{\frac12}_{2,1}}. \tag{36}$$

Estimate of III. We also decompose the term III by using Bony's formula as follows:

$$\mathrm{III} = \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3,$$

with

$$\mathrm{III}_1=\partial_1\sum_{q\geqslant 0}\big[\mathcal{R}_{ij},S_{q-1}\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\Delta_q(x_1\rho),$$
$$\mathrm{III}_2=\partial_1\sum_{q\geqslant 0}\big[\mathcal{R}_{ij},\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]S_{q-1}(x_1\rho),$$
$$\mathrm{III}_3=\partial_1\sum_{q\geqslant -1}\big[\mathcal{R}_{ij},\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho).$$

As we have done to handle the term II_1, we can use that there exists a function $\psi\in\mathcal{S}(\mathbb{R}^3)$ such that

$$\mathrm{III}_1=\sum_{q\geqslant 0}\partial_1\big(\big[\psi_q(\mathrm{D}),S_{q-1}\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\Delta_q(x_1\rho)\big),$$

with $\psi_q=2^{3q}\psi(2^q\cdot)$. For every $p\in]1,\infty[$, we first write thanks to the Bernstein inequality that

$$\begin{aligned}&\big\|\partial_1\big\{\big[\psi_q(\mathrm{D}),S_{q-1}\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\Delta_q(x_1\rho)\big\}\big\|_{L^p}\\&\quad\leqslant C2^q\big\|\big[\psi_q(\mathrm{D}),S_{q-1}\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\Delta_q(x_1\rho)\big\|_{L^p}.\end{aligned}$$

Then by using successively Lemma 2.7 and the continuity of the Riesz transform (i.e. Proposition 2.5(3)), we get

$$\begin{aligned}&\big\|\big[\psi_q(\mathrm{D}),S_{q-1}\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\Delta_q(x_1\rho)\big\|_{L^p}\\&\quad\lesssim\|x\psi_q\|_{L^1}\big\|S_{q-1}\big(\nabla\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big\|_{L^p}\big\|\Delta_q(x_1\rho)\big\|_{L^\infty}\\&\quad\lesssim 2^{-q}\|x\psi\|_{L^1}\big\|\nabla\partial_3\Delta^{-1}(\omega_\theta/r)\big\|_{L^p}\big\|\Delta_q(x_1\rho)\big\|_{L^\infty}\\&\quad\lesssim 2^{-q}\|\omega_\theta/r\|_{L^p}\big\|\Delta_q(x_1\rho)\big\|_{L^\infty}.\end{aligned}$$

It follows that

$$\|\mathrm{III}_1\|_{L^p}\lesssim\sum_{q\geqslant 0}\|\omega_\theta/r\|_{L^p}\big\|\Delta_q(x_1\rho)\big\|_{L^\infty}\lesssim\|\omega_\theta/r\|_{L^p}\|x_1\rho\|_{B^0_{\infty,1}}.$$

This proves that the linear operator T

$$f\mapsto\sum_{q\geqslant 0}\partial_1\big\{\big[\psi_q(\mathrm{D}),S_{q-1}\big(\partial_3\Delta^{-1}f\big)\big]\Delta_q(x_1\rho)\big\}$$

is continuous from L^p into itself for every $p\in]1,\infty[$ and that

$$\|T\|_{\mathcal{L}(L^p)}\leqslant C_p\|x_1\rho\|_{B^0_{\infty,1}}.$$

Consequently, by using the interpolation result of Theorem 2.4, we get that T is continuous on $L^{p,q}$ for every $1<p<\infty$ and $q\in[1,\infty]$. In particular, this yields

$$\|\mathrm{III}_1\|_{L^{3,1}}\lesssim\|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}}.\tag{37}$$

For the term III_3, we use split it into

$$\begin{aligned}\mathrm{III}_3 &= \partial_1 \sum_{q\geqslant 1}\big[\mathcal{R}_{ij}, \Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho)\\ &\quad + \partial_1 \sum_{-1\leqslant q\leqslant 0}\big[\mathcal{R}_{ij}, \Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho)\\ &:= \mathrm{III}_{31} + \mathrm{III}_{32}.\end{aligned}$$

Let $p \in]1, \infty[$ from the Bernstein inequality, we have that

$$\|\Delta_k \mathrm{III}_{31}\|_{L^p} \lesssim 2^k \sum_{q\geqslant k-4} \big\|\big[\mathcal{R}_{ij}, \Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho)\big\|_{L^p}$$

and the terms inside the sum can be controlled without using the structure of the commutator. We just write

$$\begin{aligned}\big\|\big[\mathcal{R}_{ij}, \Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho)\big\|_{L^p} &\lesssim \big\|\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big\|_{L^p}\big\|\tilde{\Delta}_q(x_1\rho)\big\|_{L^\infty}\\ &\quad + \big\|\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big\|_{L^p}\big\|\mathcal{R}_{ij}\tilde{\Delta}_q(x_1\rho)\big\|_{L^\infty}\\ &\lesssim 2^{-q}\|\omega_\theta/r\|_{L^p}\big\|\tilde{\Delta}_q(x_1\rho)\big\|_{L^\infty}.\end{aligned}$$

Note that we have used the Bernstein inequality, the continuity of the Riesz transform on L^p and the fact that the support of the Fourier transform of $\tilde{\Delta}_q$ does not contain zero which gives that the operator $\mathcal{R}_{ij}\Delta_q$ also acts continuously on L^∞ (since it can be written as the convolution with an L^1 function). It follows that for every $p \in]1, +\infty[$, we have

$$\|\mathrm{III}_{31}\|_{L^p} \lesssim \|\omega_\theta/r\|_{L^p} \sum_{k\geqslant -1}\sum_{q\geqslant k-4} 2^{k-q}\big\|\tilde{\Delta}_q(x_1\rho)\big\|_{L^\infty} \lesssim \|\omega_\theta/r\|_{L^p}\|x_1\rho\|_{B^0_{\infty,1}}.$$

By using again the interpolation result of Theorem 2.4, this yields

$$\|\mathrm{III}_{31}\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}}.$$

We can also estimate the term III_{32} without using the structure of the commutator. By using the continuity of the Riesz transform on L^2 and (18), we obtain

$$\begin{aligned}\big\|\big[\mathcal{R}_{ij}, \Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big]\tilde{\Delta}_q(x_1\rho)\big\|_{L^2} &\lesssim \big\|\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big\|_{L^\infty}\big\|\tilde{\Delta}_q(x_1\rho)\big\|_{L^2}\\ &\quad + \big\|\Delta_q\big(\partial_3\Delta^{-1}(\omega_\theta/r)\big)\big\|_{L^\infty}\big\|\mathcal{R}_{ij}\tilde{\Delta}_q(x_1\rho)\big\|_{L^2}\\ &\lesssim \big\|\partial_3\Delta^{-1}(\omega_\theta/r)\big\|_{L^\infty}\big\|\tilde{\Delta}_q(x_1\rho)\big\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{L^2}.\end{aligned}$$

Therefore we get

$$\|\mathrm{III}_{32}\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{L^2}.$$

Consequently we obtain

$$\|\mathrm{III}_3\|_{B^0_{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}} \|x_1\rho\|_{B^0_{\infty,1}\cap L^2}. \tag{38}$$

Let us now turn to the estimate of the term III_2. We write

$$\begin{aligned}\mathrm{III}_2 &= \sum_{q\geqslant 0} \big[\mathcal{R}_{ij}, \Delta_q(\partial_{13}\Delta^{-1}(\omega_\theta/r)\big] S_{q-1}(x_1\rho) + \big[\mathcal{R}_{ij}, \Delta_q(\partial_3\Delta^{-1}(\omega_\theta/r)\big]\partial_1 S_{q-1}(x_1\rho)\\ &= \mathrm{III}_{21} + \mathrm{III}_{22}.\end{aligned}$$

We have by definition of the paraproducts that

$$\mathrm{III}_{21} = \mathcal{R}_{ij}\big(T_{x_1\rho}\mathcal{R}_{13}(\omega_\theta/r)\big) - T_{\mathcal{R}_{ij}(x_1\rho)}\mathcal{R}_{13}(\omega_\theta/r).$$

Thanks to Proposition 2.5, we get that

$$\begin{aligned}\|\mathrm{III}_{21}\|_{L^{3,1}} &\lesssim \big\|\mathcal{R}_{13}(\omega_\theta/r)\big\|_{L^{3,1}}\big(\|x_1\rho\|_{L^\infty} + \big\|\mathcal{R}_{13}(x_1\rho)\big\|_{L^\infty}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|x_1\rho\|_{L^\infty} + \big\|\mathcal{R}_{13}(x_1\rho)\big\|_{L^\infty}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}\cap L^2}.\end{aligned}$$

Note that the L^2 norm in the right-hand side comes from the low frequency term in the Littlewood–Paley decomposition: we have

$$\big\|\mathcal{R}_{13}\Delta_{-1}(x_1\rho)\big\|_{L^\infty} \lesssim \big\|\mathcal{R}_{13}\Delta_{-1}(x_1\rho)\big\|_{L^2} \lesssim \|x_1\rho\|_{L^2} \tag{39}$$

thanks to the Bernstein inequality and the L^2 continuity of the Riesz transform.

For the estimate of III_{22}, we shall use that thanks to the Bernstein inequality, we have for every f that,

$$\big\|\Delta_q\partial_3\Delta^{-1} f\big\|_{L^p} \lesssim 2^{-q}\|f\|_{L^p}, \quad \forall q\geqslant 0,\ p\in\,]1,+\infty[.$$

This yields

$$\begin{aligned}\|\mathrm{III}_{22}\|_{L^p} &\lesssim \|\omega_\theta/r\|_{L^p}\sum_{q\geqslant 0} 2^{-q}\big(\big\|\partial_1 S_{q-1}(x_1\rho)\big\|_{L^\infty} + \big\|\partial_1 S_{q-1}\mathcal{R}_{ij}(x_1\rho)\big\|_{L^\infty}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^p}\sum_{q\geqslant 0}\sum_{q-2\geqslant p\geqslant -1} 2^{p-q}\big(\big\|\Delta_p(x_1\rho)\big\|_{L^\infty} + \big\|\Delta_p\mathcal{R}_{ij}(x_1\rho)\big\|_{L^\infty}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^p}\big(\|x_1\rho\|_{B^0_{\infty,1}} + \big\|\mathcal{R}_{ij}(x_1\rho)\big\|_{B^0_{\infty,1}}\big)\\ &\lesssim \|\omega_\theta/r\|_{L^p}\|x_1\rho\|_{B^0_{\infty,1}\cap L^2}\end{aligned}$$

by using again (39). Consequently, by interpolation, we also find

$$\|\mathrm{III}_{22}\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}\cap L^2}.$$

We have thus shown that

$$\|\mathrm{III}_2\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}\cap L^2}. \tag{40}$$

Gathering (37), (38) and (40), we obtain

$$\|\mathrm{III}\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{B^0_{\infty,1}\cap L^2}. \tag{41}$$

Finally we obtain

$$\big\|\partial_1\big\{\big[\mathcal{R}_{ij}, v^1\big]\rho\big\}\big\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|x_1\rho\|_{B^0_{\infty,1}\cap L^2} + \|\rho\|_{B^{\frac{1}{2}}_{2,1}}\big)$$

thanks to (41), (36), (32) and (30). In the same way, we also obtain the estimate

$$\big\|\partial_2\big\{\big[\mathcal{R}_{ij}, v^2\big]\rho\big\}\big\|_{L^{3,1}} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|x_2\rho\|_{B^0_{\infty,1}\cap L^2} + \|\rho\|_{B^{\frac{1}{2}}_{2,1}}\big).$$

In view of (28), it remains to estimate the term $\partial_3([\mathcal{R}_{ij}, v^3]\rho)$ which has a different structure.

• *Estimate of* $\partial_3([\mathcal{R}_{ij}, v^3]\rho)$. Since we can write that

$$\Delta v^3 = -(\operatorname{curl}\omega)_3 = -\left(\partial_r\omega_\theta + \frac{\omega_\theta}{r}\right) = -\left(r\partial_r\left(\frac{\omega_\theta}{r}\right) + 2\frac{\omega_\theta}{r}\right) = -x_h\cdot\nabla_h\left(\frac{\omega_\theta}{r}\right) - 2\frac{\omega_\theta}{r},$$

we obtain that

$$v^3(x) = -\Delta^{-1}\left(x_h\cdot\nabla_h\left(\frac{\omega_\theta}{r}\right)\right) - 2\Delta^{-1}\left(\frac{\omega_\theta}{r}\right)$$

and hence by using Lemma 2.10 that

$$\begin{aligned} -v^3(x) &= x_h\cdot\nabla_h\Delta^{-1}(\omega_\theta/r) - 2\sum_{i=1}^{2}\Delta^{-1}\mathcal{R}_{ii}(\omega_\theta/r) + 2\Delta^{-1}(\omega_\theta/r) \\ &= x_h\cdot\nabla_h\Delta^{-1}(\omega_\theta/r) + 2\Delta^{-1}\mathcal{R}_{33}(\omega_\theta/r). \end{aligned} \tag{42}$$

Thus, we have a decomposition of the commutator under the form

$$\begin{aligned} -\partial_3\big(\big[\mathcal{R}_{ij}, v^3\big]\rho\big) &= \sum_{k=1}^{2}\partial_3\big(\partial_k\Delta^{-1}(\omega_\theta/r)[\mathcal{R}_{ij}, x_k]\rho\big) + 2\partial_3\big(\big[\mathcal{R}_{ij}, \Delta^{-1}\mathcal{R}_{33}(\omega_\theta/r)\big]\rho\big) \\ &\quad + \sum_{k=1}^{2}\partial_3\big(\big[\mathcal{R}_{ij}, \partial_k\Delta^{-1}(\omega_\theta/r)\big](x_k\rho)\big) \\ &= \bar{\mathrm{I}} + \bar{\mathrm{II}} + \bar{\mathrm{III}}. \end{aligned}$$

To estimate the first term $\bar{\mathrm{I}}$, we use Lemma 2.10(2) to obtain that

$$\partial_3\left(\partial_k\Delta^{-1}\left(\frac{\omega_\theta}{r}\right)[\mathcal{R}_{ij},x_k]\rho\right)=\partial_3\left(\partial_k\Delta^{-1}\left(\frac{\omega_\theta}{r}\right)\mathcal{L}_{ij}^k\rho\right)$$
$$=\mathcal{R}_{3k}\left(\frac{\omega_\theta}{r}\right)\mathcal{L}_{ij}^k\rho+\partial_k\Delta^{-1}\left(\frac{\omega_\theta}{r}\right)\partial_3\mathcal{L}_{ij}^k\rho.$$

It follows that

$$\|\bar{\mathrm{I}}\|_{L^{3,1}}\leqslant\sum_{k=1}^{2}\left(\left\|\mathcal{L}_{ij}^k\rho\right\|_{L^\infty}\left\|\mathcal{R}_{3k}(\omega_\theta/r)\right\|_{L^{3,1}}+\left\|\partial_k\Delta^{-1}(\omega_\theta/r)\right\|_{L^\infty}\left\|\partial_3\mathcal{L}_{ij}^k\rho\right\|_{L^{3,1}}\right)$$
$$\lesssim\|\rho\|_{L^{3,1}}\|\omega_\theta/r\|_{L^{3,1}}$$

thanks to (26), (27). The estimates of the terms $\bar{\mathrm{II}}$ and $\bar{\mathrm{III}}$ are similar to the ones of II and III in (30) (indeed, the operator $\Delta^{-1}\mathcal{R}_{33}=\partial_{33}\Delta^{-2}$ has the same properties as $\mathcal{L}=-2\partial_{13}\Delta^{-2}$ which arises in (30)) consequently, we also get as in (36) and (41) that

$$\|\bar{\mathrm{I}}\|_{L^{3,1}}\leqslant\|\omega_\theta/r\|_{L^3}\|\rho\|_{B_{2,1}^{\frac{1}{2}}},\qquad \|\bar{\mathrm{II}}\|_{L^{3,1}}\lesssim\|\omega_\theta/r\|_{L^{3,1}}\|\rho x_h\|_{B_{\infty,1}^0\cap L^2}.$$

Consequently, we also find that

$$\left\|\partial_3\left(\left[\mathcal{R}_{ij},v^3\right]\rho\right)\right\|_{L^{3,1}}\lesssim\|\omega_\theta/r\|_{L^{3,1}}\left(\|\rho x_h\|_{B_{\infty,1}^0\cap L^2}+\|\rho\|_{B_{2,1}^{\frac{1}{2}}}\right).$$

This ends the proof of Theorem 3.1. □

3.2. *Commutation between the advection operator and Δ_q*

The last commutator estimate which is needed in the proof of our main result is the following.

Proposition 3.2. *Let v be an axisymmetric divergence free vector field without swirl and ρ a smooth scalar function. Then there exists $C>0$ such that for every $q\in\mathbb{N}\cup\{-1\}$ we have*

$$\left\|[\Delta_q,v\cdot\nabla]\rho\right\|_{L^2}\leqslant C\|\omega_\theta/r\|_{L^{3,1}}\left(\|\rho x_h\|_{L^6}+\|\rho\|_{L^2}\right).$$

Proof. From the incompressibility of the velocity we have

$$[\Delta_q,v\cdot\nabla]\rho=\sum_{i=1}^{3}\partial_i\left(\left[\Delta_q,v^i\right]\rho\right)=\mathrm{I}+\mathrm{II}+\mathrm{III}.\tag{43}$$

The first and the second terms can be handled in the same way, so we shall only detail the proof of the estimate of the first one. Thanks to (29), we have that

$$v^1(x)=x_1\Delta^{-1}\partial_3(\omega_\theta/r)+\mathcal{L}(\omega_\theta/r),\quad\text{with }\mathcal{L}=-2\mathcal{R}_{13}\Delta^{-1}$$

and hence we get

$$\begin{aligned}\mathrm{I} &= \partial_1\big(\big[\Delta_q, x_1\Delta^{-1}\partial_3(\omega_\theta/r)\big]\rho\big) + \partial_1\big(\big[\Delta_q, \mathcal{L}(\omega_\theta/r)\big]\rho\big)\\ &= \mathrm{I}_1 + \mathrm{I}_2.\end{aligned}$$

The estimate of the second term in the right-hand side is again a direct consequence of Lemma 2.8 and (24). Indeed, we write

$$\|\mathrm{I}_2\|_{L^2} \lesssim \big\|\nabla\mathcal{L}(\omega_\theta/r)\big\|_{L^\infty}\|\rho\|_{L^2} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.$$

The first term I_1 in the right-hand side can be expanded under the form

$$\mathrm{I}_1 = \partial_1\big(\big[\Delta_q, \Delta^{-1}\partial_3(\omega_\theta/r)\big](x_1\rho)\big) + \partial_1\big(\Delta^{-1}\partial_3(\omega_\theta/r)[\Delta_q, x_1]\rho\big) = \mathrm{I}_{11} + \mathrm{I}_{12}.$$

We start with the estimate of I_{12}. By definition of Δ_q, we have

$$\begin{aligned}x_1\Delta_q\rho &= x_1 2^{3q}\int_{\mathbb{R}^3}\varphi\big(2^q(x-y)\big)\rho(y)\,dy\\ &= 2^{3q}\int_{\mathbb{R}^3}\varphi\big(2^q(x-y)\big)y_1\rho(y)\,dy + 2^{3q}\int_{\mathbb{R}^3}\varphi\big(2^q(x-y)\big)(x_1-y_1)\rho(y)\,dy\\ &= \Delta_q(x_1\rho) + 2^{-q}2^{3q}\,\varphi_1\big(2^q\cdot\big)\star\rho,\end{aligned}$$

where $\varphi_1(x) = x_1\varphi(x) \in \mathcal{S}(\mathbb{R}^3)$. Consequently we get the expression of the commutator:

$$[\Delta_q, x_1]\rho = -2^{-q}2^{3q}\varphi_1\big(2^q\cdot\big)\star\rho. \tag{44}$$

This yields

$$\mathrm{I}_{12} = -\big(\mathcal{R}_{13}(\omega_\theta/r)\big)2^{2q}\varphi_1\big(2^q\cdot\big)\star\rho - \big\{\Delta^{-1}\partial_3(\omega_\theta/r)\big\}2^{3q}(\partial_1\varphi_1)\big(2^q\cdot\big)\star\rho.$$

Therefore we get by using again the Hölder inequality, the continuity of the Riesz transform, (18) and the Young inequality for convolutions that:

$$\begin{aligned}\|\mathrm{I}_{12}\|_{L^2} &\leqslant \big\|\mathcal{R}_{13}(\omega_\theta/r)\big\|_{L^3}2^{2q}\big\|\varphi_1\big(2^q\cdot\big)\star\rho\big\|_{L^6} + \big\|\Delta^{-1}\partial_3(\omega_\theta/r)\big\|_{L^\infty}2^{3q}\big\|(\partial_1\varphi_1)\big(2^q\cdot\big)\star\rho\big\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^3}\|\varphi_1\|_{L^{\frac{3}{2}}}\|\rho\|_{L^2} + \|\omega_\theta/r\|_{L^{3,1}}\|\partial_1\varphi_1\|_{L^1}\|\rho\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.\end{aligned}$$

Note that we have also used the embedding (15).

To estimate I_{11} we use again Lemma 2.8:

$$\|\mathrm{I}_{11}\|_{L^2} \lesssim \big\|\nabla\Delta^{-1}\partial_3(\omega_\theta/r)\big\|_{L^3}\|x_1\rho\|_{L^6} \lesssim \|\omega_\theta/r\|_{L^3}\|x_1\rho\|_{L^6} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_1\rho\|_{L^6}.$$

We have thus shown that

$$\|\mathrm{I}\|_{L^2} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|\rho\|_{L^2} + \|x_1\rho\|_{L^6}\big). \tag{45}$$

In the same way, we obtain that

$$\|\mathrm{II}\|_{L^2} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|\rho\|_{L^2} + \|x_2\rho\|_{L^6}\big). \tag{46}$$

It remains to estimate the last term III. By using (42), we get

$$\begin{aligned}-\mathrm{III} &= \partial_3\big\{\big[\Delta_q, \nabla_h\Delta^{-1}(\omega_\theta/r)\big](x_h\rho)\big\} + \partial_3\big\{\nabla_h\Delta^{-1}(\omega_\theta/r)[\Delta_q, x_h]\rho\big\}\\ &\quad + 2\partial_3\big\{\big[\Delta_q, \Delta^{-1}\mathcal{R}_{33}(\omega_\theta/r)\big]\rho\big\}\\ &= \mathrm{III}_1 + \mathrm{III}_2 + \mathrm{III}_3.\end{aligned}$$

The estimates of the first and last terms follow again from Lemma 2.8: we write that

$$\|\mathrm{III}_1\|_{L^2} \leqslant C\big\|\nabla^2\Delta^{-1}(\omega_\theta/r)\big\|_{L^3}\|x_h\rho\|_{L^6} \lesssim \|\omega_\theta/r\|_{L^3}\|x_h\rho\|_{L^6} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\|x_h\rho\|_{L^6}$$

and that

$$\|\mathrm{III}_3\|_{L^2} \leqslant C\big\|\nabla\Delta^{-1}\mathcal{R}_{33}(\omega_\theta/r)\big\|_{L^\infty}\|\rho\|_{L^2} \lesssim \big\|\mathcal{R}_{33}(\omega_\theta/r)\big\|_{L^{3,1}}\|\rho\|_{L^2} \lesssim C\|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.$$

Note that we have used again the estimate (18).

Finally, to estimate the second term III_2 we can use the expression of the commutator $[\Delta_q, x_h]$ given by (44):

$$\begin{aligned}\mathrm{III}_2 &= 2^{-q}\partial_3\big(\big(\nabla_h\Delta^{-1}(\omega_\theta/r)\big)2^{3q}\varphi_h\big(2^q\cdot\big)\star\rho\big)\\ &= 2^{-q}\big(\partial_3\nabla_h\Delta^{-1}(\omega_\theta/r)\big)\big(2^{3q}\varphi_h\big(2^q\cdot\big)\star\rho\big) + \nabla_h\Delta^{-1}(\omega_\theta/r)\big(2^{3q}(\partial_3\varphi_h)\big(2^q\cdot\big)\star\rho\big),\end{aligned}$$

with $\varphi_h(x) = -x_h\varphi(x)$. It follows as before that

$$\begin{aligned}\|\mathrm{III}_2\|_{L^2} &\lesssim 2^{-q}\|\omega_\theta/r\|_{L^3}2^{3q}\big\|\varphi_h\big(2^q\cdot\big)\star\rho\big\|_{L^6} + \big\|\nabla_h\Delta^{-1}(\omega_\theta/r)\big\|_{L^\infty}2^{3q}\big\|(\partial_3\varphi_h)\big(2^q\cdot\big)\star\rho\big\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^3}\|\varphi_h\|_{L^{\frac{3}{2}}}\|\rho\|_{L^2} + \|\omega_\theta/r\|_{L^{3,1}}\|\partial_3\varphi_h\|_{L^1}\|\rho\|_{L^2}\\ &\lesssim \|\omega_\theta/r\|_{L^{3,1}}\|\rho\|_{L^2}.\end{aligned}$$

Gathering these estimates we also find that

$$\|\mathrm{III}\|_{L^3} \lesssim \|\omega_\theta/r\|_{L^{3,1}}\big(\|\rho\|_{L^p} + \|x_h\rho\|_{L^6}\big).$$

In view of (43), (45), (46) and the last estimate, this ends the proof of Proposition 3.2. □

4. A priori estimates

In this section we intend to establish the global a priori estimates needed for the proof of Theorem 1.1. We shall first prove some basic weak estimates that can be obtained easily through energy type estimates. In a second step, we shall prove the control of some stronger norms such as $\|\omega(t)\|_{L^\infty}$ and $\|\nabla v(t)\|_{L^\infty}$. This part requires more refined analysis: we use the special structure of the Boussinesq model combined with the previous commutator estimates.

4.1. Energy estimates

We start with some elementary energy estimates.

Proposition 4.1. *Let* (v, ρ) *be a smooth solution of* (1) *then*

(1) *For* $p \in]1, \infty[$, $q \in [1, \infty]$ *and* $t \in \mathbb{R}_+$, *we have*

$$\|\rho\|_{L_t^\infty L^2}^2 + 2\|\nabla\rho\|_{L_t^2 L^2}^2 \leqslant \|\rho_0\|_{L^2}^2, \qquad \|\rho\|_{L_t^\infty L^{p,q}} \leqslant C\|\rho_0\|_{L^{p,q}}.$$

(2) *For* $v_0 \in L^2$, $\rho_0 \in L^2$ *and* $t \in \mathbb{R}_+$ *we have*

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2} + t\|\rho_0\|_{L^2}.$$

(3) *For* $\rho_0 \in L^2$ *we have the dispersive estimate*

$$\|\rho(t)\|_{L^\infty} \leqslant C\left(1 + \frac{1}{t^{\frac{3}{4}}}\right)\|\rho_0\|_{L^2}.$$

The constant C *is absolute.*

Note that the axisymmetric assumption is not needed in this proposition.

Proof. (1) By taking the L^2-scalar product of the second equation of (1) with ρ and integrating by parts, we get since v is divergence free that

$$\frac{1}{2}\frac{d}{dt}\|\rho(t)\|_{L^2}^2 + \int_{\mathbb{R}^3} |\nabla\rho(t,x)|^2\, dx = 0.$$

Integrating in time this differential inequality gives the desired result.

Let us now move to the estimate of the density in Lorentz spaces. First, the same argument yields that for every $p \in [1, \infty]$, we have

$$\|\rho(t)\|_{L^p} \leqslant \|\rho_0\|_{L^p}.$$

It suffices now to use the interpolation result of Theorem 2.4.

(2) We take the L^2-scalar product of the velocity equation with v and we integrate by parts

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{L^2}^2 \leqslant \|v(t)\|_{L^2}\|\rho(t)\|_{L^2}$$

and this implies that

$$\frac{d}{dt}\|v(t)\|_{L^2} \leqslant \|\rho(t)\|_{L^2}.$$

Thus, integrating in time gives

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2} + \int_0^t \|\rho(\tau)\|_{L^2}\, d\tau.$$

Since $\|\rho(t)\|_{L^2} \leqslant \|\rho_0\|_{L^2}$, we infer

$$\|v(t)\|_{L^2} \leqslant \|v_0\|_{L^2} + t\|\rho_0\|_{L^2}.$$

(3) The estimate is a direct consequence of Lemma A.1 The proof of the proposition is now achieved. □

4.2. Estimates of the moments of ρ.

We have seen in Section 3.1 and Section 3.2 that the estimates of the commutators involve some moments of the density. Thus we aim in this paragraph at giving suitable estimates for the moments that will be needed later when we shall perform our diagonalization of the Boussinesq system. Two types of estimates are discussed: the energy estimates of the horizontal moments $|x_h|^k\rho$, with $k = 1, 2$ and some dispersive estimates. More precisely we prove the following.

Proposition 4.2. *Let v be a vector field with zero divergence and satisfying the energy estimate of Proposition* 4.1. *Let ρ be a solution of the transport-diffusion equation*

$$\partial_t\rho + v\cdot\nabla\rho - \Delta\rho = 0, \quad \rho(0, x) = \rho_0.$$

Then we have the following estimates.

(1) *For $\rho_0 \in L^2$ and $x_h\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t \geqslant 0$*

$$\|x_h\rho\|_{L_t^\infty L^2} + \|x_h\rho\|_{L_t^2\dot{H}^1} \lesssim C_0\big(1 + t^{\frac{5}{4}}\big).$$

(2) *For $\rho_0 \in L^2 \cap L^m$, $m > 6$ and $x_h\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t > 0$*

$$\|x_h\rho(t)\|_{L^\infty} \leqslant C_0\big(t^{\frac{1}{4}} + t^{-\frac{3}{4}}\big).$$

(3) *For $\rho_0 \in L^2$ and $|x_h|^2\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t \geqslant 0$*

$$\||x_h|^2\rho\|_{L_t^\infty L^2} + \||x_h|^2\rho\|_{L_t^2\dot{H}^1} \leqslant C_0\big(1 + t^{\frac{5}{2}}\big).$$

(4) *For $\rho_0 \in L^2 \cap L^6$ and $|x_h|^2\rho_0 \in L^2$, there exists $C_0 > 0$ such that for every $t > 0$*

$$\||x_h|^2\rho(t)\|_{L^6} \leqslant C_0\big(t^{\frac{13}{6}} + t^{-\frac{1}{2}}\big).$$

Remark 4.3. Note that when $\rho_0 \in L^2$ and $|x_h|^2\rho_0 \in L^2$ then automatically the moment of order one belongs to L^2, that is $x_h\rho_0 \in L^2$. This is an easy consequence of the Hölder inequality

$$\|x_h\rho\|_{L^2} \leqslant \|\rho\|_{L^2}^{\frac{1}{2}} \big\||x_h|^2\rho\big\|_{L^2}^{\frac{1}{2}}.$$

Proof. (1) Setting $f = x_h\rho$, we can easily check that f solves the equation

$$\partial_t f + v\cdot\nabla f - \Delta f = v^h\rho - 2\nabla_h\rho \tag{47}$$

with the notations $v^h = (v^1, v^2)$ and $\nabla_h = (\partial_1, \partial_2)$. Now, taking the L^2-scalar product with f, integrating by parts and using the Hölder inequality

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\big\|f(t)\big\|_{L^2}^2 + \big\|\nabla f(t)\big\|_{L^2}^2 &= \int_{\mathbb{R}^3} v^h\rho f\,dx - 2\int_{\mathbb{R}^3}\nabla_h\rho f\,dx\\ &\leqslant \|v\|_{L^2}\|\rho\|_{L^3}\|f\|_{L^6} + 2\|\rho\|_{L^2}\|\nabla f\|_{L^2}.\end{aligned}$$

By using the Sobolev embedding $\dot{H}^1 \hookrightarrow L^6$ combined with the Young inequality, we obtain

$$\frac{d}{dt}\big\|f(t)\big\|_{L^2}^2 + \big\|\nabla f(t)\big\|_{L^2}^2 \lesssim \|v\|_{L^2}^2\|\rho\|_{L^3}^2 + \|\rho\|_{L^2}^2.$$

Since the Gagliardo–Nirenberg inequality gives that

$$\|\rho\|_{L^3}^2 \leqslant \|\rho\|_{L^2}\|\nabla\rho\|_{L^2},$$

we infer

$$\frac{d}{dt}\big\|f(t)\big\|_{L^2}^2 + \big\|\nabla f(t)\big\|_{L^2}^2 \lesssim \|v\|_{L^2}^2\|\rho\|_{L^2}\|\nabla\rho\|_{L^2} + \|\rho\|_{L^2}^2. \tag{48}$$

Integrating in time and using the energy estimate of Proposition 4.1(1), we thus obtain

$$\begin{aligned}\big\|f(t)\big\|_{L^2}^2 + \big\|\nabla f(t)\big\|_{L_t^2L^2}^2 &\lesssim \|f_0\|_{L^2}^2 + \|v\|_{L_t^\infty L^2}^2\|\rho_0\|_{L^2}\|\nabla\rho\|_{L_t^1L^2} + \|\rho_0\|_{L^2}^2 t\\ &\lesssim \|f_0\|_{L^2}^2 + C_0\big(1+t^2\big)t^{\frac{1}{2}} + \|\rho_0\|_{L^2}^2 t\\ &\leqslant C_0\big(1+t^{\frac{5}{2}}\big).\end{aligned}$$

(2) We shall apply Lemma A.1 to (47) with $F = \rho\,e_i$ and $G = v_i\rho$. First, we observe that we have obviously from the Hölder inequality combined with Proposition 4.1(1) that for $m \geqslant 2$

$$\begin{aligned}\|G\|_{L_t^\infty L^{\frac{2m}{m+2}}} &\leqslant \|v\|_{L_t^\infty L^2}\|\rho\|_{L_t^\infty L^m}\\ &\leqslant C_0(1+t)\end{aligned}$$

and

$$\|F\|_{L_t^\infty L^6} \leqslant \|\rho_0\|_{L^6}.$$

Consequently, we get from Lemma A.1 and Proposition 4.1 that for $m > 6$ and for $t > 0$,

$$\begin{aligned}\|f(t)\|_{L^\infty} &\leqslant C\big(1+t^{-\frac{3}{4}}\big)\|f_0\|_{L^2} + C_0\big(1+t^{\frac{1}{4}-\frac{3}{2m}}\big) + \big(1+t^{\frac{1}{4}}\big)\|\rho_0\|_{L^6}\\ &\leqslant C_0\big(t^{-\frac{3}{4}}+t^{\frac{1}{4}}\big).\end{aligned}$$

(3) The second moment $g = |x_h|^2\rho$ solves the following equation

$$\begin{aligned}\partial_t g + v\cdot\nabla g - \Delta g &= 2v^h(x_h\rho) - 2\nabla_h\rho - 4\,\mathrm{div}_h(x_h\rho)\\ &= v_h f - 2\nabla_h\rho - 4\,\mathrm{div}_h f.\end{aligned}$$

By using again an L^2 energy estimate, we find that

$$\frac{d}{dt}\|g(t)\|_{L^2}^2 + \|\nabla g(t)\|_{L^2}^2 \lesssim \|v(t)\|_{L^2}^2\|f(t)\|_{L^2}\|\nabla f(t)\|_{L^2} + \|\rho(t)\|_{L^2}^2 + \|f(t)\|_{L^2}^2.$$

Thus, by integrating in time and using the energy estimates for ρ, v and f we get

$$\begin{aligned}\|g(t)\|_{L^2} + \|\nabla g\|_{L_t^2L^2} &\lesssim \|g_0\|_{L^2} + \|v\|_{L_t^\infty L^2}\|f\|_{L_t^\infty L^2}^{\frac{1}{2}}\|\nabla f\|_{L_t^2L^2}^{\frac{1}{2}}t^{\frac{1}{4}} + \|\rho_0\|_{L^2}t^{\frac{1}{2}} + \|f\|_{L_t^\infty L^2}t^{\frac{1}{2}}\\ &\leqslant C_0\big(1+t^{\frac{5}{2}}\big),\end{aligned}$$

where C_0 is a constant depending on the quantities $\||x_h|^k\rho_0\|_{L^2}$ for $k = 0, 1, 2$.

(4) By setting $g_1(t,x) = tg(t,x)$, we have that

$$\partial_t g_1 + v\cdot\nabla g_1 - \Delta g_1 = g + 2v^h(tx_h\rho) - 2t\nabla_h\rho - 4\,\mathrm{div}_h(tx_h\rho).$$

Multiplying this equation by $|g_1|^4 g_1$, integrating by parts and the obvious inequality $|x_h\rho| \leqslant |\rho|^{\frac{1}{2}}|g|^{\frac{1}{2}}$, we thus get

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\|g_1\|_{L^6}^6 + 5\int_{\mathbb{R}^3}|\nabla g_1|^2|g_1|^4\,dx &\lesssim \|g\|_{L^6}\|g_1\|_{L^6}^5 + t^{\frac{1}{2}}\int_{\mathbb{R}^3}|v||\rho|^{\frac{1}{2}}|g_1|^{\frac{11}{2}}\,dx\\ &\quad + t\int_{\mathbb{R}^3}|\rho||\nabla g_1||g_1|^4\,dx + t^{\frac{1}{2}}\int_{\mathbb{R}^3}|\rho|^{\frac{1}{2}}|g_1|^{\frac{9}{2}}|\nabla g_1|\,dx.\end{aligned}$$

It follows from Hölder inequality that

$$t^{\frac{1}{2}}\int_{\mathbb{R}^3}|\rho|^{\frac{1}{2}}|g_1|^{\frac{9}{2}}|\nabla g_1|\,dx \leqslant t^{\frac{1}{2}}\|v\|_{L^2}\|g_1\|_{L^{18}}^{\frac{11}{2}}\|\rho\|_{L^{\frac{18}{7}}}^{\frac{1}{2}}.$$

Consequently

$$\begin{aligned}\frac{1}{6}\frac{d}{dt}\|g_1\|_{L^6}^6 + 5\int_{\mathbb{R}^3} |\nabla g_1|^2|g_1|^4\,dx &\lesssim \|g\|_{L^6}\|g_1\|_{L^6}^5 + t^{\frac{1}{2}}\|v\|_{L^2}\|g_1\|_{L^{18}}^{\frac{11}{2}}\|\rho\|_{L^{\frac{18}{7}}}^{\frac{1}{2}} \\ &\quad + \big(t\|\rho\|_{L^6}\|g_1\|_{L^6}^2 + t^{\frac{1}{2}}\|\rho\|_{L^6}^{\frac{1}{2}}\|g_1\|_{L^6}^{\frac{5}{2}}\big)\bigg(\int_{\mathbb{R}^3} |\nabla g_1|^2|g_1|^4\,dx\bigg)^{\frac{1}{2}}.\end{aligned}$$

Now we can use the Young inequality combined with the following Sobolev inequality

$$\|g_1\|_{L^{18}}^{18} \lesssim \big\|\nabla\big(g_1^3\big)\big\|_{L^2}^2 = 9\int_{\mathbb{R}^3} |\nabla g_1|^2|g_1|^4\,dx$$

to obtain that

$$\begin{aligned}\frac{d}{dt}\|g_1\|_{L^6}^6 + c\|g_1\|_{L^{18}}^6 + c\int_{\mathbb{R}^3} |\nabla g_1|^2|g_1|^4\,dx &\lesssim \|g\|_{L^6}\|g_1\|_{L^6}^5 + t^6\|v\|_{L^2}^{12}\|\rho\|_{L^{\frac{18}{7}}}^6 \\ &\quad + t^2\|\rho\|_{L^6}^2\|g_1\|_{L^6}^4 + t\|\rho\|_{L^6}\|g_1\|_{L^6}^5.\end{aligned}$$

By using Proposition 4.1, we deduce that

$$\begin{aligned}\frac{d}{dt}\|g_1\|_{L^6}^6 + c\|g_1\|_{L^{18}}^6 + c\int_{\mathbb{R}^3} |\nabla g_1|^2|g_1|^4\,dx &\lesssim \|g\|_{L^6}\|g_1\|_{L^6}^5 + C_0t^6\big(1+t^{12}\big) \\ &\quad + t^2\|\rho_0\|_{L^6}^2\|g_1\|_{L^6}^4 + t\|\rho_0\|_{L^6}\|g_1\|_{L^6}^5.\end{aligned}$$

Next, by using again the Young inequality, we infer

$$\frac{d}{dt}\big\|g_1(t)\big\|_{L^6}^6 \leqslant C_0\big(t^6+t^{18}\big) + C_0\big(t + \big\|g(t)\big\|_{L^6}\big)\big\|g_1(t)\big\|_{L^6}^5.$$

By integrating in time this differential inequality, we obtain that

$$\big\|g_1(t)\big\|_{L^6}^6 \leqslant C_0\big(t^7+t^{19}\big) + C_0\int_0^t \big(\tau + \big\|g(\tau)\big\|_{L^6}\big)\big\|g_1(\tau)\big\|_{L^6}^5\,d\tau.$$

Therefore we get from Proposition 4.2(3) combined with the Sobolev embedding $\dot{H}^1 \subset L^6$ that

$$\begin{aligned}\big\|g_1(t)\big\|_{L^6} &\leqslant C_0\big(t^{\frac{7}{6}}+t^{\frac{19}{6}}\big) + C_0\|g\|_{L_t^1L^6} \\ &\leqslant C_0\big(t^{\frac{7}{6}}+t^{\frac{19}{6}}\big) + t^{\frac{1}{2}}\|\nabla g\|_{L_t^2L^2} \\ &\leqslant C_0\big(t^{\frac{7}{6}}+t^{\frac{19}{6}}\big) + C_0\big(1+t^{\frac{5}{2}}\big)t^{\frac{1}{2}}.\end{aligned}$$

Therefore, we obtain that

$$\big\| |x_h|^2 \rho(t) \big\|_{L^6} \leqslant C_0 \big(t^{\frac{13}{6}} + t^{-\frac{1}{2}} \big).$$

This ends the proof of Proposition 4.2. □

4.3. Strong estimates

As in the study of the axisymmetric Euler equation, the main important quantity that one should estimate in order to get the global existence of smooth solutions is $\|\frac{\omega}{r}(t)\|_{L^{3,1}}$. Indeed, this will enable us to bound stronger norms such as $\|\omega(t)\|_{L^\infty}$ and $\|\nabla v(t)\|_{L^\infty}$ which are the significant quantities to propagate higher regularities.

4.3.1. Estimate of $\|\frac{\omega}{r}(t)\|_{L^\infty}$

First, we will introduce the following notation: we denote by Φ_k any function of the form

$$\Phi_k(t) = C_0 \underbrace{\exp\big(\dots \exp}_{k \text{ times}}\big(C_0 t^{\frac{19}{6}} \big) \dots \big),$$

where C_0 depends on the involved norms of the initial data and its value may vary from line to line up to some absolute constants. We will make an intensive use (without mentioning it) of the following trivial facts

$$\int_0^t \Phi_k(\tau)\, d\tau \leqslant \Phi_k(t) \quad \text{and} \quad \exp\bigg(\int_0^t \Phi_k(\tau)\, d\tau \bigg) \leqslant \Phi_{k+1}(t).$$

We first establish the following result.

Proposition 4.4. *Let $v_0 \in L^2$ be an axisymmetric vector field such that $\frac{\omega_0}{r} \in L^{3,1}$ and $\rho_0 \in L^2 \cap L^m$, for $m > 6$, axisymmetric and such that $|x_h|^2 \rho_0 \in L^2$. Then, we have for every $t \in \mathbb{R}_+$*

$$\bigg\| \frac{\omega}{r}(t) \bigg\|_{L^{3,1}} + \bigg\| \frac{v^r}{r}(t) \bigg\|_{L^\infty} \leqslant \Phi_2(t),$$

where C_0 is a constant depending on the norms of the initial data.

Proof. Recall that the equation of the scalar component of the vorticity $\omega = \omega_\theta e_\theta$ is given by

$$\partial_t \omega_\theta + v \cdot \nabla \omega_\theta = \frac{v^r}{r} \omega_\theta - \partial_r \rho. \tag{49}$$

It follows that the evolution of the quantity $\frac{\omega_\theta}{r}$ is governed by the equation

$$(\partial_t + v \cdot \nabla) \frac{\omega_\theta}{r} = -\frac{\partial_r \rho}{r} \cdot \tag{50}$$

By applying the operator $\frac{\partial_r}{r}\Delta^{-1}$ to the equation of the density in (1), we obtain that

$$(\partial_t + v\cdot\nabla)\left(\frac{1}{r}\partial_r\Delta^{-1}\rho\right) - \frac{\partial_r\rho}{r} = -\left[\frac{1}{r}\partial_r\Delta^{-1}, v\cdot\nabla\right]\rho.$$

By setting $\Gamma := \frac{\omega_\theta}{r} + \frac{\partial_r}{r}\Delta^{-1}\rho$, we infer

$$(\partial_t + v\cdot\nabla)\Gamma = -\left[\frac{1}{r}\partial_r\Delta^{-1}, v\cdot\nabla\right]\rho.$$

Observe that the incompressibility of the velocity field allows us to get that for every $p\in[1,\infty]$

$$\big\|\Gamma(t)\big\|_{L^p} \leqslant \|\Gamma_0\|_{L^p} + \int_0^t \left\|\left[\frac{1}{r}\partial_r\Delta^{-1}, v\cdot\nabla\right]\rho\right\|_{L^p} d\tau.$$

Therefore we get by the interpolation result of Theorem 2.4 that for $1<p<\infty$ and $q\in[1,\infty]$

$$\big\|\Gamma(t)\big\|_{L^{p,q}} \leqslant \|\Gamma_0\|_{L^{p,q}} + \int_0^t \left\|\left[\frac{1}{r}\partial_r\Delta^{-1}, v\cdot\nabla\right]\rho(\tau)\right\|_{L^{p,q}} d\tau.$$

In particular, we have

$$\big\|\Gamma(t)\big\|_{L^{3,1}} \leqslant \|\Gamma_0\|_{L^{3,1}} + \int_0^t \left\|\left[\frac{1}{r}\partial_r\Delta^{-1}, v\cdot\nabla\right]\rho(\tau)\right\|_{L^{3,1}} d\tau.$$

Applying Theorem 3.1 we find

$$\big\|\Gamma(t)\big\|_{L^{3,1}} \leqslant \|\Gamma_0\|_{L^{3,1}} + \int_0^t \big\|(\omega_\theta/r)(\tau)\big\|_{L^{3,1}}\big(\big\|x_h\rho(\tau)\big\|_{B^0_{\infty,1}\cap L^2} + \big\|\rho(\tau)\big\|_{B^{\frac{1}{2}}_{2,1}}\big)\,d\tau.$$

Moreover, thanks to Proposition 2.9 and Proposition 4.1 we have

$$\big\|(\omega_\theta/r)(t)\big\|_{L^{3,1}} \leqslant \big\|\Gamma(t)\big\|_{L^{3,1}} + \left\|\frac{1}{r}\partial_r\Delta^{-1}\rho(t)\right\|_{L^{3,1}} \leqslant \big\|\Gamma(t)\big\|_{L^{3,1}} + C\|\rho_0\|_{L^{3,1}}.$$

The combination of these last estimates yield

$$\begin{aligned}\big\|(\omega_\theta/r)(t)\big\|_{L^{3,1}} &\leqslant C\big(\|\omega_0/r\|_{L^{3,1}} + \|\rho_0\|_{L^{3,1}}\big)\\ &\quad + \int_0^t \big\|(\omega_\theta/r)(\tau)\big\|_{L^{3,1}}\big(\big\|x_h\rho(\tau)\big\|_{B^0_{\infty,1}\cap L^2} + \big\|\rho(\tau)\big\|_{B^{\frac{1}{2}}_{2,1}}\big)\,d\tau.\end{aligned}$$

Thus we get by the Gronwall inequality that

$$\big\|(\omega_\theta/r)(t)\big\|_{L^{3,1}} \leqslant C\big(\|\omega_0/r\|_{L^{3,1}} + \|\rho_0\|_{L^{3,1}}\big)\exp\big(C\|x_h\rho\|_{L^1_t(B^0_{\infty,1}\cap L^2)} + C\|\rho\|_{L^1_t B^{\frac{1}{2}}_{2,1}}\big). \tag{51}$$

The term $\|\rho\|_{L^1_t B^{\frac{1}{2}}_{2,1}}$ will be controlled only by energy estimates. Indeed, the interpolation estimate

$$\|\rho\|_{B^{\frac{1}{2}}_{2,1}} \lesssim \|\rho\|^{\frac{1}{2}}_{L^2}\|\nabla\rho\|^{\frac{1}{2}}_{L^2}$$

combined with Proposition 4.1 and the Hölder inequality give

$$\|\rho\|_{L^1_t B^{\frac{1}{2}}_{2,1}} \lesssim \|\rho\|^{\frac{1}{2}}_{L^\infty_t L^2} t^{\frac{3}{4}} \|\nabla\rho\|^{\frac{1}{2}}_{L^2_t L^2} \lesssim t^{\frac{3}{4}}\|\rho_0\|_{L^2}.$$

To control the term $\|x_h\rho\|_{L^1_t L^2}$ in the right-hand side of (51), we can use Proposition 4.2:

$$\|x_h\rho\|_{L^1_t L^2} \leqslant t\|x_h\rho\|_{L^\infty_t L^2} \leqslant C_0\big(1 + t^{\frac{9}{4}}\big).$$

Consequently we obtain in view of (51)

$$\left\|\frac{\omega}{r}(t)\right\|_{L^{3,1}} \leqslant C_0 e^{C_0 t^{\frac{9}{4}}} e^{C\|\rho x_h\|_{L^1_t B^0_{\infty,1}}}. \tag{52}$$

Now it remains to estimate the right term of (52) inside the exponential. Let us first sketch the strategy of our approach. We will introduce an integer $N(t) \in \mathbb{N}$ that will be chosen in an optimal way in the end and we will split in frequency the involved quantity into two parts: low frequencies corresponding to $q \leqslant N(t)$ and high frequencies associated to $q > N(t)$. To estimate the low frequencies we use the dispersive result of Proposition 4.2(2). The estimate of high frequencies is based on a smoothing effect.

By using Proposition 4.2(2) and the Bernstein inequality, we find that

$$\begin{aligned}
\|x_h\rho\|_{L^1_t B^0_{\infty,1}} &= \int_0^t \sum_{q\leqslant N(\tau)} \big\|\Delta_q(x_h\rho)(\tau)\big\|_{L^\infty}\,d\tau + \int_0^t \sum_{q>N(\tau)} \big\|\Delta_q(x_h\rho)(\tau)\big\|_{L^\infty}\,d\tau \\
&\leqslant C_0\int_0^t \big(\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\big)N(\tau)\,d\tau + C\int_0^t \sum_{q>N(\tau)} 2^{q\frac{3}{2}}\big\|\Delta_q(x_h\rho)(\tau)\big\|_{L^2}\,d\tau.
\end{aligned} \tag{53}$$

Now we intend to estimate the last sum in the above inequality. For this purpose we localize in frequency the equation for $f = x_h\rho$ which is

$$\partial_t f + v\cdot\nabla f - \Delta f = v^h\rho - 2\nabla_h\rho := F.$$

By setting $f_q := \Delta_q f$, we infer

$$\partial_t f_q + v \cdot \nabla f_q - \Delta f_q = -[\Delta_q, v \cdot \nabla] f + F_q.$$

From an L^2 energy estimate, we obtain that

$$\frac{1}{2}\frac{d}{dt}\|f_q(t)\|_{L^2}^2 - \int_{\mathbb{R}^3} (\Delta f_q) f_q \, dx \leqslant \|f_q\|_{L^2}\big(\big\|[\Delta_q, v \cdot \nabla] f\big\|_{L^2} + \|F_q\|_{L^2}\big).$$

Since the Bessel identity yields

$$c2^{2q}\|f_q\|_{L^2}^2 \leqslant -\int_{\mathbb{R}^3} (\Delta f_q) f_q \, dx,$$

it follows that

$$\frac{d}{dt}\big\|f_q(t)\big\|_{L^2} + c2^{2q}\big\|f_q(t)\big\|_{L^2} \leqslant C\big(\big\|[\Delta_q, v \cdot \nabla] f\big\|_{L^2} + \|F_q\|_{L^2}\big).$$

Therefore we obtain by integration in time that

$$\big\|f_q(t)\big\|_{L^2} \lesssim e^{-ct2^{2q}}\big\|f_q(0)\big\|_{L^2} + \int_0^t e^{-c(t-\tau)2^{2q}}\big(\big\|[\Delta_q, v \cdot \nabla] f\big\|_{L^2} + \|F_q\|_{L^2}\big)\, d\tau.$$

To estimate the commutator in the right-hand side, we can use Proposition 3.2 and Proposition 4.2,

$$\begin{aligned}\big\|[\Delta_q, v \cdot \nabla] f(\tau)\big\|_{L^2} &\leqslant C\big\|(\omega_\theta/r)(\tau)\big\|_{L^{3,1}}\big(\big\||x_h|^2\rho(\tau)\big\|_{L^6} + \big\|x_h\rho(\tau)\big\|_{L^2}\big)\\ &\leqslant C_0\big\|(\omega_\theta/r)(\tau)\big\|_{L^{3,1}}\big(\tau^{\frac{13}{6}} + \tau^{-\frac{1}{2}}\big).\end{aligned}$$

Hence we get

$$\begin{aligned}\big\|f_q(t)\big\|_{L^2} \lesssim{}& e^{-ct2^{2q}}\big\|f_q(0)\big\|_{L^2} + \int_0^t e^{-c(t-\tau)2^{2q}}\big\|F_q(\tau)\big\|_{L^2}\, d\tau\\ &+ C_0\int_0^t e^{-c(t-\tau)2^{2q}}\big\|(\omega_\theta/r)(\tau)\big\|_{L^{3,1}}\big(\tau^{\frac{13}{6}} + \tau^{-\frac{1}{2}}\big)\, d\tau. \qquad (54)\end{aligned}$$

Let us set $\mathcal{K}(\tau) = \tau^{\frac{13}{6}} + \tau^{-\frac{1}{2}}$, then (54) and convolution inequalities yield

$$\int_0^t \sum_{q>N(\tau)} 2^{q\frac{3}{2}} \big\| \Delta_q(x_h\rho)(\tau)\big\|_{L^2}\,d\tau$$
$$\lesssim \sum_{q\geqslant -1} 2^{-\frac{1}{2}q}\big(\big\|f_q(0)\big\|_{L^2} + \|F_q\|_{L^1_tL^2}\big)$$
$$+ C_0\int_0^t \sum_{q>N(\tau)} 2^{q\frac{3}{2}} \int_0^\tau e^{-c(\tau-\tau')2^{2q}}\mathcal{K}\big(\tau'\big)\big\|(\omega_\theta/r)\big(\tau'\big)\big\|_{L^{3,1}}\,d\tau'$$
$$\lesssim \|f_0\|_{L^2} + \|F\|_{L^1_tL^2} + C_0\int_0^t \|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\left(\sum_{q>N(\tau)} 2^{q\frac{3}{2}}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\mathcal{K}\big(\tau'\big)\,d\tau'\right)d\tau.$$

Moreover, from Proposition 4.1 and Lemma A.1, we also have

$$\begin{aligned}\|F\|_{L^1_tL^2} &\leqslant t^{\frac{1}{2}}\|\nabla\rho\|_{L^2_tL^2} + \|v\|_{L^\infty_tL^2}\|\rho\|_{L^1_tL^\infty}\\ &\leqslant C\|\rho_0\|_{L^2}t^{\frac{1}{2}} + C_0(1+t)\|\rho_0\|_{L^2}\left(\int_0^t \tau^{-\frac{3}{4}}\,d\tau + t\right)\\ &\leqslant C_0\big(1+t^2\big).\end{aligned}$$

Inserting these estimates into (53) yields

$$\begin{aligned}&\|x_h\rho\|_{L^1_tB^0_{\infty,1}}\\ &\quad\leqslant C_0\big(1+t^2\big) + C_0\int_0^t \big(\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\big)N(\tau)\,d\tau\\ &\qquad + C_0\int_0^t \|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\left(\sum_{q>N(\tau)} 2^{q\frac{3}{2}}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\big(\big\{\tau'\big\}^{\frac{13}{6}} + \big\{\tau'\big\}^{-\frac{1}{2}}\big)\,d\tau'\right)d\tau\\ &\quad\leqslant C_0\big(1+t^2\big) + C_0\int_0^t \big(\tau^{\frac{1}{4}} + \tau^{-\frac{3}{4}}\big)N(\tau)\,d\tau\\ &\qquad + C_0\int_0^t \|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\left(\tau^{\frac{13}{6}}2^{-\frac{1}{2}N(\tau)} + \sum_{q>N(\tau)} 2^{q\frac{3}{2}}\int_0^\tau e^{-c(\tau-\tau')2^{2q}}\big\{\tau'\big\}^{-\frac{1}{2}}\,d\tau'\right)d\tau. \qquad (55)\end{aligned}$$

By a change of variables we get

$$\sum_{q>N(\tau)} 2^{q\frac{3}{2}} \int_0^{\tau} e^{-c(\tau-\tau')2^{2q}} \{\tau'\}^{-\frac{1}{2}}\, d\tau' = \sum_{q>N(\tau)} 2^{q\frac{1}{2}} e^{-c\tau 2^{2q}} \int_0^{2^{2q}\tau} e^{c\tau'} \{\tau'\}^{-\frac{1}{2}} d\tau'$$
$$= \sum_{q\in\Lambda_1(\tau)} 2^{q\frac{1}{2}} e^{-c\tau 2^{2q}} \int_0^{2^{2q}\tau} e^{c\tau'} \{\tau'\}^{-\frac{1}{2}} d\tau'$$
$$+ \sum_{q\in\Lambda_2(\tau)} 2^{q\frac{1}{2}} e^{-c\tau 2^{2q}} \int_0^{2^{2q}\tau} e^{c\tau'} \{\tau'\}^{-\frac{1}{2}}\, d\tau'$$
$$:= \mathrm{I}(\tau) + \mathrm{II}(\tau).$$

with

$$\Lambda_1(\tau) = \{q > N(\tau) \text{ and } \tau 2^{2q} \geqslant 1\} \quad \text{and} \quad \Lambda_2(\tau) = \{q > N(\tau) \text{ and } \tau 2^{2q} \leqslant 1\}.$$

To estimate the first term we use the following inequality which can, be proven by integration by parts: there exists $C > 0$ such that for every $x \geqslant 1$

$$\int_0^x y^{-\frac{1}{2}} e^{cy}\, dy \leqslant C x^{-\frac{1}{2}} e^{cx}.$$

It follows that

$$\mathrm{I}(\tau) \lesssim \tau^{-\frac{1}{2}} \sum_{q>N(\tau)} 2^{-\frac{1}{2}q} \lesssim 2^{-\frac{1}{2}N(\tau)} \tau^{-\frac{1}{2}}.$$

To estimate the second term, we observe that the integral is bounded by a fixed number and hence, we find that

$$\mathrm{II}(\tau) \lesssim \sum_{q\in\Lambda_2(\tau)} 2^{\frac{1}{2}q} \lesssim 2^{-\frac{1}{2}N(\tau)} \sum_{2^{2q}\leqslant\tau^{-1}} 2^q \lesssim 2^{-\frac{1}{2}N(\tau)}\big(1 + \tau^{-\frac{1}{2}}\big).$$

Gathering these estimates, we obtain

$$\sum_{q>N(\tau)} 2^{q\frac{3}{2}} \int_0^{\tau} e^{-c(\tau-\tau')2^{2q}} \{\tau'\}^{-\frac{1}{2}}\, d\tau' \leqslant 2^{-\frac{1}{2}N(\tau)}\big(1 + \tau^{-\frac{1}{2}}\big).$$

By plugging this estimate into (55), we get

$$\|x_h \rho\|_{L^1_t B^0_{\infty,1}} \leqslant C_0\big(1 + t^2\big) + C_0 \int_0^t \big(\tau^{\frac{13}{6}} + \tau^{-\frac{3}{4}}\big)\big(N(\tau) + \|\omega_\theta / r\|_{L^\infty_\tau L^{3,1}} 2^{-\frac{1}{2}N(\tau)}\big)\, d\tau.$$

We choose N such that

$$N(\tau)=2\big[\log_2\big(2+\|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\big)\big]$$

and then we find

$$\|x_h\rho\|_{L^1_t B^0_{\infty,1}} \leqslant C_0\big(1+t^{\frac{19}{6}}\big)+C_0\int_0^t \big(\tau^{\frac{13}{6}}+\tau^{-\frac{3}{4}}\big)\log\big(2+\|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\big)\,d\tau. \tag{56}$$

Putting together (52), (56) and Proposition 4.2(2), we find that

$$\log\big(2+\|\omega_\theta/r\|_{L^\infty_t L^{3,1}}\big) \leqslant C_0\big(1+t^{\frac{19}{6}}\big)+C_0\int_0^t \big(\tau^{\frac{13}{6}}+\tau^{-\frac{3}{4}}\big)\log\big(2+\|\omega_\theta/r\|_{L^\infty_\tau L^{3,1}}\big)\,d\tau.$$

From the Gronwall inequality, we infer

$$\log\big(2+\|\omega_\theta/r\|_{L^\infty_t L^{3,1}}\big) \leqslant C_0\big(1+t^{\frac{19}{6}}\big)e^{C_0(t^{\frac{19}{6}}+t^{\frac{1}{4}})} \leqslant \Phi_1(t).$$

Therefore we get by using again (56) that

$$\left\|\frac{\omega_\theta}{r}(t)\right\|_{L^{3,1}} \leqslant \Phi_2(t).$$

Since $\frac{\omega}{r}=\frac{\omega_\theta}{r}e_\theta$, (22) implies that

$$\left\|\frac{\omega}{r}(t)\right\|_{L^{3,1}} \leqslant \Phi_2(t). \tag{57}$$

Finally, thanks to (19), we obtain

$$\left\|\frac{v^r}{r}(t)\right\|_{L^\infty} \leqslant C\left\|\frac{\omega}{r}(t)\right\|_{L^{3,1}} \leqslant \Phi_2(t).$$

This ends the proof of Proposition 4.4. □

4.3.2. Estimate of $\|\omega(t)\|_{L^\infty}$

Our purpose now is to bound the vorticity.

Proposition 4.5. *Let $v_0\in L^2$ be an axisymmetric divergence free vector field without swirl such that $\omega_0\in L^\infty$, $\frac{\omega_0}{r}\in L^{3,1}$. Let ρ_0 be axisymmetric scalar function, belonging to $L^2\cap L^m$, $m>6$ and such that $|x_h|^2\rho_0\in L^2$. Then we have for every $t\in\mathbb{R}_+$*

$$\big\|\omega(t)\big\|_{L^\infty}+\|\nabla\rho\|_{L^1_t L^\infty} \leqslant \Phi_4(t).$$

Proof. From the maximum principle for Eq. (49), we obtain that

$$\big\|\omega(t)\big\|_{L^\infty} \leqslant \|\omega_0\|_{L^\infty} + \int_0^t \big\|v^r/r(\tau)\big\|_{L^\infty}\big\|\omega(\tau)\big\|_{L^\infty}\,d\tau + \int_0^t \big\|\nabla\rho(\tau)\big\|_{L^\infty}\,d\tau.$$

By combining Proposition 4.4 and the Gronwall inequality, this yields

$$\big\|\omega(t)\big\|_{L^\infty} \leqslant \Phi_3(t)\left(1 + \int_0^t \|\nabla\rho\|_{L^\infty}\,d\tau\right).$$

Now we claim that,

$$\|\nabla\rho\|_{L^1_t L^\infty} \leqslant C_0\left(1 + t^2 + \int_0^t \big\|\omega(\tau)\big\|_{L^\infty}\,d\tau\right). \tag{58}$$

Let us first finish the proof by using this estimate. We deduce that

$$\big\|\omega(t)\big\|_{L^\infty} \leqslant \Phi_3(t)\left(1 + \int_0^t \big\|\omega(\tau)\big\|_{L^\infty}\,d\tau\right)$$

and thanks to the Gronwall inequality that

$$\big\|\omega(t)\big\|_{L^\infty} \leqslant \Phi_4(t).$$

This gives in turn

$$\|\nabla\rho\|_{L^1_t L^\infty} \leqslant \Phi_4(t).$$

Let us now come back to the proof of (58). For $q \in \mathbb{N}$ we set $\rho_q := \Delta_q\rho$, then

$$\partial_t\rho_q + v\cdot\nabla\rho_q - \Delta\rho_q = -[\Delta_q, v\cdot\nabla]\rho.$$

Let $p \geqslant 2$ then multiplying this equation by $|\rho_q|^{p-2}\rho_q$ and using Hölder inequality

$$\frac{1}{p}\frac{d}{dt}\big\|\rho(t)\big\|_{L^p}^p - \int_{\mathbb{R}^3} (\Delta\rho_q)|\rho_q|^{p-2}\rho_q\,dx \leqslant \|\rho_q\|_{L^p}^{p-1}\big\|[\Delta_q, v\cdot\nabla]\rho\big\|_{L^p}.$$

Now we use the generalized Bernstein inequality, see [24],

$$\frac{1}{p}2^{2q}\|\rho_q\|_{L^p}^p \leqslant -\int_{\mathbb{R}^3} (\Delta\rho_q)|\rho_q|^{p-2}\rho_q\,dx.$$

Hence we get

$$\frac{d}{dt}\big\|\rho(t)\big\|_{L^p}+c_p2^{2q}\|\rho_q\|_{L^p}\lesssim\big\|[\Delta_q,v\cdot\nabla]\rho\big\|_{L^p}.$$

This gives

$$\big\|\rho_q(t)\big\|_{L^p}\leqslant e^{-c_pt2^{2q}}\|\Delta_q\rho_0\|_{L^p}+\int_0^t e^{-c_p2^{2q}(t-\tau)}\big\|[\Delta_q,v\cdot\nabla]\rho\big\|_{L^p}\,d\tau. \tag{59}$$

Integrating in time implies that

$$\|\rho_q\|_{L^1_tL^p}\lesssim 2^{-2q}\|\rho_0\|_{L^p}+2^{-2q}\big\|[\Delta_q,v\cdot\nabla]\rho\big\|_{L^1_tL^p}.$$

According to Proposition 2.3 [18] and Proposition 4.1 we have

$$\begin{aligned}\big\|[\Delta_q,v\cdot\nabla]\rho\big\|_{L^p}&\leqslant C\|\rho\|_{L^p}\big((q+1)\|\omega\|_{L^\infty}+\|\nabla\Delta_{-1}v\|_{L^\infty}\big)\\&\leqslant C\|\rho_0\|_{L^p}\big((q+1)\|\omega\|_{L^\infty}+\|v\|_{L^2}\big)\\&\leqslant C\|\rho_0\|_{L^p}\big((q+1)\|\omega\|_{L^\infty}+C_0(1+t)\big).\end{aligned}$$

It follows that

$$\|\rho_q\|_{L^1_tL^p}\leqslant C_0\big(1+t^2\big)2^{-2q}+C\|\rho_0\|_{L^p}(q+1)2^{-2q}\|\omega\|_{L^1_tL^\infty}.$$

By using the Bernstein inequality, we find for $p>3$ that

$$\begin{aligned}\|\nabla\rho\|_{L^1_tL^\infty}&\leqslant Ct\|\rho_0\|_{L^2}+C\sum_{q\in\mathbb{N}}2^{q(1+\frac{3}{p})}\|\rho_q\|_{L^1_tL^p}\\&\leqslant C_0\big(1+t^2\big)\sum_{q\in\mathbb{N}}2^{q(-1+\frac{3}{p})}+C_0\|\omega\|_{L^1_tL^\infty}\sum_{q\in\mathbb{N}}2^{q(-1+\frac{3}{p})}(q+1)\\&\leqslant C_0\big(1+t^2\big)+C_0\|\omega\|_{L^1_tL^\infty}.\end{aligned}$$

This ends the proof of the desired inequality. □

4.3.3. Lipschitz bound of the velocity

We shall now deal with the global propagation of the sub-critical Sobolev regularities. This is basically related to the control of the Lipschitz norm of the velocity.

Proposition 4.6. *Let $\frac{5}{2}<s<3$ and $(v_0,\rho_0)\in H^s\times H^{s-2}$ and (v,ρ) be a solution of the Boussinesq system* (1). *Then we have for every $t\geqslant 0$*

$$\|v\|_{\widetilde{L}^\infty_tH^s}+\|\rho\|_{\widetilde{L}^\infty_tH^{s-2}}+\|\rho\|_{\widetilde{L}^1_tH^s}\leqslant C_0(1+t)e^{C\|\nabla v\|_{L^1_tL^\infty}}.$$

If in addition $\rho_0 \in L^m$ *with* $m > 6$ *and* $|x_h|^2\rho_0 \in L^2$*, then we get for every* $t \geqslant 0$

$$\|\nabla v(t)\|_{L^\infty} \leqslant \Phi_5(t); \qquad \|v\|_{\widetilde{L}^\infty_t H^s} + \|\rho\|_{\widetilde{L}^\infty_t H^{s-2}} + \|\rho\|_{\widetilde{L}^1_t H^s} \leqslant \Phi_6(t).$$

Remark 4.7. We point out that we can extend the results of Proposition 4.6 to higher regularities $s \geqslant 3$ but for the sake of simplicity we restrict ourselves here to the case of $s < 3$.

Proof. We localize in frequency the equation of the velocity. For $q \in \mathbb{N} \cup \{-1\}$ we set $v_q := \Delta_q v$ and $\rho_q := \Delta_q \rho$.

$$\partial_t v_q + v \cdot \nabla v_q + \nabla \pi_q = \rho_q e_z - [\Delta_q, v \cdot \nabla]v.$$

Thus taking the L^2-scalar product with v_q and using the incompressibility of v and v_q we get

$$\frac{d}{dt}\|v_q(t)\|_{L^2} \leqslant \|\rho_q\|_{L^2} + \|[\Delta_q, v \cdot \nabla]v\|_{L^2}.$$

Integrating in time we obtain

$$\|v_q(t)\|_{L^2} \leqslant \|v_q(0)\|_{L^2} + \|\rho_q\|_{L^1_t L^2} + \|[\Delta_q, v \cdot \nabla]v\|_{L^1_t L^2}.$$

Thus we get

$$\|v\|_{\widetilde{L}^\infty_t H^s} \leqslant \|v_0\|_{H^s} + \|\rho\|_{\widetilde{L}^1_t H^s} + \left\|\left(2^{qs}\|[\Delta_q, v \cdot \nabla]v\|_{L^1_t L^2}\right)_q\right\|_{\ell^2}.$$

We will use the commutator estimate, see for instance Lemma B.5 of [10],

$$\left\|\left(2^{qs}\|[\Delta_q, v \cdot \nabla]v\|_{L^1_t L^2}\right)_q\right\|_{\ell^2} \leqslant C\int_0^t \|\nabla v(\tau)\|_{L^\infty}\|v(\tau)\|_{H^s}\,d\tau.$$

Putting together these estimates and using Gronwall inequality yield

$$\|v\|_{\widetilde{L}^\infty_t H^s} \leqslant \left(\|v_0\|_{H^s} + \|\rho\|_{\widetilde{L}^1_t H^s}\right)e^{C\|\nabla v\|_{L^1_t L^\infty}}. \tag{60}$$

Using the estimate (59) we get for $q \in \mathbb{N}$

$$\|\rho_q\|_{L^\infty_t L^2} + 2^{2q}\|\rho_q\|_{L^1_t L^2} \leqslant C\|\rho_q(0)\|_{L^2} + \|[\Delta_q, v \cdot \nabla]\rho\|_{L^1_t L^2}.$$

Therefore we find

$$\begin{aligned}\|\rho\|_{\widetilde{L}^\infty_t H^{s-2}} + \|\rho\|_{\widetilde{L}^1_t H^s} &\leqslant \|\Delta_{-1}\rho\|_{L^1_t L^2} + \|\rho_0\|_{H^{s-2}} + \left\|\left(2^{q(s-2)}\|[\Delta_q, v \cdot \nabla]\rho\|_{L^1_t L^2}\right)_q\right\|_{\ell^2} \\ &\leqslant Ct\|\rho_0\|_{L^2} + \|\rho_0\|_{H^{s-2}} + \left\|\left(2^{q(s-2)}\|[\Delta_q, v \cdot \nabla]\rho\|_{L^1_t L^2}\right)_q\right\|_{\ell^2}.\end{aligned}$$

Since $-1 < s-2 < 1$ then we have the estimate, see [10],

$$\left\| \left(2^{q(s-2)} \left\| [\Delta_q, v \cdot \nabla]\rho \right\|_{L^1_t L^2}\right)_q \right\|_{\ell^2} \leqslant C \int_0^t \left\| \nabla v(\tau) \right\|_{L^\infty} \left\| \rho(\tau) \right\|_{H^{s-2}} d\tau.$$

Consequently,

$$\|\rho\|_{\widetilde{L}^\infty_t H^{s-2}} + \|\rho\|_{\widetilde{L}^1_t H^s} \leqslant C_0(1+t) + C \int_0^t \left\| \nabla v(\tau) \right\|_{L^\infty} \left\| \rho(\tau) \right\|_{H^{s-2}} d\tau.$$

By Gronwall inequality

$$\|\rho\|_{\widetilde{L}^\infty_t H^{s-2}} + \|\rho\|_{\widetilde{L}^1_t H^s} \leqslant C_0(1+t) e^{C\|\nabla v\|_{L^1_t L^\infty}}. \tag{61}$$

Combining this estimate with (60) gives the desired estimates.

Now to get a global bound for Lipschitz norm of the velocity we use the classical logarithmic estimate: for $s > \frac{5}{2}$

$$\|\nabla v\|_{L^\infty} \lesssim \|v\|_{L^2} + \|\omega\|_{L^\infty} \log\left(e + \|v\|_{H^s}\right).$$

Combining this estimate with the first result of Proposition 4.6 and Proposition 4.5

$$\|\nabla v\|_{L^\infty} \leqslant \Phi_4(t)\left(1 + \int_0^t \left\| \nabla v(\tau) \right\|_{L^\infty} d\tau\right).$$

It follows from Gronwall inequality that

$$\left\| \nabla v(t) \right\|_{L^\infty} \leqslant \Phi_5(t).$$

Plugging this estimate into (60) and (61) gives

$$\|v\|_{\widetilde{L}^\infty_t H^s} + \|\rho\|_{\widetilde{L}^\infty_t H^{s-2}} + \|\rho\|_{\widetilde{L}^1_t H^s} \leqslant \Phi_6(t).$$

This ends the proof of the proposition. □

5. Proof of the main result

The proof of the existence part of Theorem 1.1 can be done in a classical way by smoothing out the initial data as follows

$$v_{0,n} = S_n v_0 = 2^{3n} \chi\left(2^n \cdot\right) \star v_0, \qquad \rho_{0,n} = S_n \rho_0 = 2^{3n} \chi\left(2^n \cdot\right) \star \rho_0$$

where S_n is the cut-off in frequency defined in the preliminaries. Since χ is radial then the functions $v_{0,n}$ and $\rho_{0,n}$ remain axisymmetric. Moreover this family is uniformly bounded in the space of initial data: this is obvious in Sobolev and Lebesgue spaces but it remains to check the uniform boundedness of the horizontal moment of the density. We will show that

$$\sup_{n\in\mathbb{N}}\big\||x_h|^2\rho_{n,0}\big\|_{L^2}\leqslant C\big(\|\rho_0\|_{L^2}+\big\||x_h|^2\rho_0\big\|_{L^2}\big). \tag{62}$$

For this purpose we write

$$\begin{aligned}|x_h|^2\big|\rho_{n,0}(x)\big|&=|x_h|^2\bigg|\int_{\mathbb{R}^3}\chi\big(2^n(x-y)\big)\rho(y)\,dy\bigg|\\&\leqslant 2\int_{\mathbb{R}^3}|x_h-y_h|^2|\chi|\big(2^n(x-y)\big)|\rho|(y)\,dy+2\bigg|\int_{\mathbb{R}^3}|\chi|\big(2^n(x-y)\big)|y_h|^2\rho(y)\,dy\bigg|\\&\leqslant 2\,2^{-2n}\big(2^{3n}\chi_1\big(2^n\cdot\big)\star\rho\big)(x)+2\big(|\chi|\big(2^n\cdot\big)\star\big(|y_h|^2\rho\big)\big)(x)\end{aligned}$$

with $\chi_1(x)=|x_h|^2|\chi(x)$. From convolution laws we get

$$\big\||x_h|^2|\rho_{n,0}\big\|_{L^2}\leqslant C2^{-2n}\|\rho\|_{L^2}+C\big\||x_h|^2\rho_0\big\|_{L^2}.$$

This achieves the proof of (62). Now, by using standard arguments based on the a priori estimates described in Proposition 4.6, Proposition 4.5 and Proposition 4.2 we can construct a unique global solution (v_n,ρ_n) in the following space

$$v_n\in\mathcal{C}\big(\mathbb{R}_+;H^s\big)\cap L^1\big(\mathbb{R}_+;W^{1,\infty}\big)\quad\text{and}\quad\rho_n\in\mathcal{C}\big(\mathbb{R}_+;H^{s-2}\big)\cap L^1\big(\mathbb{R}_+;W^{1,\infty}\big).$$

The control is uniform with respect to the parameter n. Therefore we can prove the strong convergence of a subsequence of $(v_n,\rho_n)_{n\in\mathbb{N}}$ to some (v,ρ) belonging to the same space and satisfying the initial value problem. It remains to prove the uniqueness problem. This gives the existence of a solution.

The uniqueness will be proven in the following space

$$(v,\rho)\in\mathcal{X}:=\big(\mathcal{C}\big(\mathbb{R}_+;L^2\big)\cap L^1\big(\mathbb{R}_+;W^{1,\infty}\big)\big)^2.$$

Let $(v^i,\rho^i)\in\mathcal{X}$, $1\leqslant i\leqslant 2$ be two solutions of the system (1) with the same initial data (v_0,θ_0) and denote $\delta v=v^2-v^1$, $\delta\rho=\rho^2-\rho^1$. Then

$$\begin{cases}\partial_t\delta v+v^2\cdot\nabla\delta v+\nabla\Pi=-\delta v\cdot\nabla v^1+\delta\rho e_z,\\ \partial_t\delta\rho+v^2\cdot\nabla\delta\rho-\Delta\delta\rho=-\delta v\cdot\nabla\rho^1,\\ \operatorname{div}v^i=0.\end{cases} \tag{63}$$

Taking the L^2-scalar product of the first equation with δv and integrating by parts gives

$$\frac{1}{2}\frac{d}{dt}\big\|\delta v(t)\big\|_{L^2}^2\leqslant\big\|\nabla v^1\big\|_{L^\infty}\|\delta v\|_{L^2}^2+\|\delta\rho\|_{L^2}\|\delta v\|_{L^2}.$$

Consequently,

$$\frac{d}{dt}\left\|\delta v(t)\right\|_{L^2} \leqslant \left\|\nabla v^1\right\|_{L^\infty}\|\delta v\|_{L^2} + \|\delta\rho\|_{L^2}.$$

Using the Gronwall inequality yields

$$e^{-\|\nabla v^1\|_{L^1_t L^\infty}}\left\|\delta v(t)\right\|_{L^2} \leqslant \left(\|\delta v_0\|_{L^2} + \int_0^t e^{-\|\nabla v^1\|_{L^\tau_t L^\infty}}\left\|\delta\rho(\tau)\right\|_{L^2}\,d\tau\right).$$

By the same computations we get

$$\left\|\delta\rho(t)\right\|_{L^2} \leqslant \|\delta\rho_0\|_{L^2} + \int_0^t \left\|\nabla\rho^1(\tau)\right\|_{L^\infty}\left\|\delta v(\tau)\right\|_{L^2}\,d\tau.$$

It suffices now to put together these estimates and to use the Gronwall inequality.

Appendix A. De Giorgi–Nash–Moser estimates for convection–diffusion equations

We intend to prove some dispersive estimates for a parabolic equation. We use De Giorgi–Nash–Moser method similarly to [6].

Lemma A.1. *Consider the equation*

$$\partial_t f + u\cdot\nabla f - \Delta f = \nabla\cdot F + G,\quad t>0,\quad x\in\mathbb{R}^3,\quad f(0,x) = f_0(x). \tag{64}$$

Consider $p, q, p_1, q_1, \in [1,+\infty]$, $r\in[2,+\infty]$ *with*

$$\frac{2}{p}+\frac{3}{q}<1, \qquad \frac{2}{p_1}+\frac{3}{q_1}<2.$$

There exists $C>0$ *such that for every smooth divergence free vector field* u *and every* $T>0$, *if* $F\in L^p_T L^q$ *and* $f_0\in L^r$, *the solution of* (64) *satisfies the estimate*:

$$\begin{aligned}\left\|f(T)\right\|_{L^\infty} \leqslant{}& C\left(1+\frac{1}{T^{\frac{3}{2r}}}\right)\|f_0\|_{L^r} + C\left(1+\sqrt{T}^{1-(\frac{2}{p}+\frac{3}{q})}\right)\|F\|_{L^p_T L^q}\\ &+ C\left(1+\sqrt{T}^{2-(\frac{2}{p_1}+\frac{3}{q_1})}\right)\|G\|_{L^{p_1}_T L^{q_1}}.\end{aligned} \tag{65}$$

Proof. Since the equation is linear, we can study separately the three problems

$$\begin{cases}\mathcal{P}f = \nabla\cdot F,\\ f(0,x)=0,\end{cases}\qquad \begin{cases}\mathcal{P}f = G,\\ f(0,x)=0,\end{cases}\qquad \begin{cases}\mathcal{P}f = 0,\\ f(0,x)=f_0(x),\end{cases} \tag{66}$$

where we have set $\mathcal{P}f = \partial_t f + u\cdot\nabla f - \Delta f$.

Let us start with the first problem in (66). We shall prove that there exists $C > 0$ such that for every F with $\|F\|_{L^p_1 L^q} \leqslant 1$, we have the estimate

$$\|f\|_{L^\infty_1 L^\infty} \leqslant C. \tag{67}$$

Once this estimate, is proven, the estimate involving F in (65) will just follow by a scaling argument.

The first step is to use the standard L^q a priori estimate (obtained by multiplying by $\mathcal{P}f$ by $|f|^{q-1}\text{sign } f$). Since u is divergence free, we have that

$$\frac{d}{dt}\left(\frac{1}{q}\|f(t)\|^q_{L^q}\right) + (q-1)\int_{\mathbb{R}^2} |\nabla f|^2|f|^{q-2}\,dx \leqslant (q-1)\int_{\mathbb{R}^3} |F||\nabla f||f|^{q-2}\,dx.$$

From the Young inequality and the Holder inequality (note that since $2/p + 3/q < 1$, we necessarily have that $q > 3$ and $p > 2$) this yields

$$\frac{d}{dt}\left(\frac{1}{q}\|f(t)\|^q_{L^q}\right) \leqslant (q-1)\|F\|^2_{L^q}\|f\|^{q-2}_{L^q}$$

and hence by integration in time, we obtain since $q > 3$ that

$$\|f\|_{L^\infty_1 L^q} \leqslant C_q\left(\int_0^1 \|F(t)\|^2_{L^q}\,dt\right)^{\frac{1}{2}} \leqslant C_q\|F\|_{L^p_1 L^q} \leqslant C_q \tag{68}$$

where C_q depends only on q. To improve this estimate that is to go from the above L^q estimate to an L^∞ estimate, we shall follow the De Giorgi, Nash iteration argument. For $M > 0$ to be chosen, let us take a positive increasing sequence $(M_k)_{k\geqslant 0}$ such that $M_k \leqslant M$ and M_k converges towards M. A good choice is for example

$$M_k = M\left(1 - \frac{1}{k+1}\right), \quad k \geqslant 0. \tag{69}$$

We shall use the standard notation $x_+ = \max(x, 0)$. Since u is divergence free, we obtain the level set energy estimate

$$\begin{aligned}\frac{d}{dt}\left(\frac{1}{2}\|(f-M_k)_+(t)\|^2_{L^2}\right) + \|\nabla(f-M_k)_+\|^2_{L^2} &\leqslant \int_{f\geqslant M_k} |F||\nabla f|\,dx \\ &\leqslant \left(\int_{f\geqslant M_k} |F|^2\right)^{\frac{1}{2}}\|\nabla(f-M_k)_+\|_{L^2}\end{aligned}$$

where the last inequality comes from Cauchy–Schwarz. By using the Young inequality, we thus obtain

$$U_k \leqslant \int_0^1 \int_{f \geqslant M_k} |F|^2 \, dx \, dt \tag{70}$$

where

$$U_k = \left\| (f - M_k)_+ \right\|_{L_1^\infty L^2}^2 + \left\| \nabla (f - M_k)_+ \right\|_{L_1^2 L^2}^2.$$

The main idea is to prove that the right-hand side of (70) can estimated by a power of U_{k-1} strictly larger than 1. By using the Hölder inequality, we first get that

$$U_k \leqslant \int_0^1 \left\| F(t) \right\|_{L^q}^2 m_k(t)^{1-\frac{2}{q}} \, dt \leqslant \|F\|_{L_1^p L^q}^2 \left(\int_0^1 m_k(t)^{(1-\frac{2}{q})(\frac{p}{p-2})} \, dt \right)^{1-\frac{2}{p}}, \tag{71}$$

where $m_k(t) = |\{x, f(t) \geqslant M_k\}|$. To estimate $m_k(t)$, we note that if $f(t,x) \geqslant M_k$ then

$$f(t,x) - M_{k-1} \geqslant M_k - M_{k-1} \geqslant 0$$

and thus we have

$$\mathbf{1}_{f(t,x) \geqslant M_k} \leqslant \frac{(k+1)^2}{M} \big(f(t,x) - M_{k-1} \big)_+. \tag{72}$$

This yields

$$m_k(t) \leqslant \frac{(k+1)^{2m}}{M^m} \left\| \big(f(t) - M_{k-1} \big)_+ \right\|_{L^m}^m \tag{73}$$

for every $m \geqslant 1$. We shall choose m carefully below. By plugging this last estimate in (71), we get

$$U_k \leqslant \|F\|_{L_1^p L^q}^2 \left(\frac{(k+1)^2}{M} \right)^{m(1-\frac{2}{q})} \left(\int_0^1 \left\| \big(f(t) - M_{k-1} \big)_+ \right\|_{L^m}^{m(1-\frac{2}{q})(\frac{p}{p-2})} \right)^{1-\frac{2}{p}}. \tag{74}$$

Now let us notice that if $\alpha \geqslant 1$ and $\beta \in [2,6]$ are such that $\frac{2}{\alpha} + \frac{3}{\beta} \geqslant \frac{3}{2}$ then we have

$$\left\| \big(f(t) - M_{k-1} \big)_+ \right\|_{L_1^\alpha L^\beta}^2 \leqslant U_{k-1}.$$

Indeed the control of U_{k-1} gives a control of the $L_1^\infty L^1$ and the $L_1^2 H^1$ norm. By Sobolev embedding this gives a control of the $L_1^2 L^6$ norm and then the inequality follows by standard interpolation in Lebesgue spaces. Consequently, to achieve our program, we need to choose $m \in [2, 6]$ such that

$$m\left(1-\frac{2}{q}\right) > 2, \quad 2\frac{1-\frac{2}{p}}{m\left(1-\frac{2}{q}\right)} + \frac{3}{m} \geqslant \frac{3}{2}. \tag{75}$$

The first constraint can be satisfied as soon as $2/(1-2/q) < 6$ which is equivalent to $q > 3$ while the second constraint can be satisfied as soon as

$$2\left(1-\frac{2}{p}\right) + 3\left(1-\frac{2}{q}\right) \geqslant m\left(1-\frac{2}{q}\right) > \frac{3}{2}\cdot 2 = 3$$

which is equivalent to

$$\frac{2}{p} + \frac{3}{q} < 1.$$

Consequently, since we have $q > 3$ and $2/p + 3/q < 1$ by assumption we can choose m such that the constraint (75) are matched. This yields that there exists $\gamma > 1$ such that

$$U_k \leqslant \|F\|^2_{L^p_1 L^q}\left(\frac{(k+1)^2}{M}\right)^{m(1-\frac{2}{p})} U^\gamma_{k-1} \leqslant \left(\frac{(k+1)^2}{M}\right)^{m(1-\frac{2}{p})} U^\gamma_{k-1}, \quad \forall k \geqslant 2.$$

If U_1 is sufficiently small, this yields that $\lim_{k\to+\infty} U_k = 0$. Since, we have from (71), the Tchebychev inequality and the energy inequality (68) that

$$U_1 \leqslant \left(\int_0^1 m_1(t)^{(1-\frac{2}{q})(\frac{p}{p-2})}\,dt\right)^{1-\frac{2}{p}} \leqslant \left(\frac{4\|f\|_{L^\infty_1 L^q}}{M}\right)^{q-2} \leqslant \left(\frac{4C_q}{M}\right)^{q-2},$$

we can indeed make U_0 arbitrarily small by taking M sufficiently large and thus $\lim_{k\to+\infty} U_k = 0$. From Fatou's Lemma, we obtain that for every $t \in [0, 1]$

$$\int_{\mathbb{R}^3} \big(f(t,x) - M\big)_+\,dx \leqslant 0$$

and therefore that almost everywhere

$$f(t,x) \leqslant M.$$

By changing f into $-f$, we obtain in a similar way that $f \geqslant -M$ almost everywhere and thus (67) is proven. To obtain the part of estimate (65) involving F if $T \geqslant 1$ we can use a change of scale argument. Let us set $K\tilde{f}(\tau, X) = f(T\tau, \sqrt{T}X)$ for $K > 0$ to be chosen. Then we have $K\|\tilde{f}\|_{L^\infty_1 L^\infty} = \|f\|_{L^\infty_T L^\infty}$ and $\tilde{f}(\tau, X)$ solves the equation

$$\partial_\tau \tilde{f} + \tilde{u}\cdot\nabla_X \tilde{f} - \Delta_X \tilde{f} = \nabla_X \cdot \tilde{F}$$

where $\tilde{u}$ is still divergence free and

$$\tilde{F}(\tau, X) = \frac{\sqrt{T}}{K} F(\tau T, \sqrt{T} X).$$

In particular, with the choice

$$K = \sqrt{T}^{1-(\frac{2}{p}+\frac{3}{q})} \|F\|_{L_T^p L^q},$$

we get that $\|\tilde{F}\|_{L_1^p L^q} = 1$ and thus that

$$\|f\|_{L_T^\infty L^\infty} = K \|\tilde{f}\|_{L_1^\infty L^\infty} \leqslant MK = M\sqrt{T}^{1-(\frac{2}{p}+\frac{3}{q})} \|F\|_{L_T^p L^q}.$$

This gives the part of the estimate (65) involving F.

Let us turn to the study of the second problem in (66) in order to get the part of the estimate (65) involving G. The estimate can be deduced from the previous one when $q_1 < +\infty$ which is the interesting case (when $q_1 = \infty$, the estimate is a direct consequence of the Maximum principle). Indeed, if $G \in L_T^{p_1} L^{q_1}$, we can write $G = \nabla \cdot F$ with $F \in L_T^{p_1} W^{1,q_1} \subset L_T^{p_1} L^{q_1^*}$ by Sobolev embedding (where $q_1^* = \frac{3q_1}{3-q_1}$) and moreover, we have

$$\|F\|_{L_T^{p_1} L^{q_1^*}} \leqslant \|G\|_{L_T^{p_1} L^{q_1}}.$$

Consequently, by using the estimate that we have already proven, we get that

$$\|f\|_{L_T^\infty L^\infty} \leqslant M\sqrt{T}^{1-(\frac{2}{p_1}+\frac{3}{q_1^*})} \|F\|_{L_T^{p_1} L^{q_1^*}}$$

if $2/p_1 + 3/q_1^* < 1$. This gives the claimed estimate.

It remains to study the third problem in (66) that is the problem with no source term but a nontrivial initial data. Again, we shall first prove that there exists $M > 0$ such that for every $f_0 \in L^r$ with $\|f_0\|_{L^2} \leqslant 1$, we have the estimate

$$\sup_{t \geqslant 1} \| f(t) \|_{L^\infty} \leqslant M. \tag{76}$$

The standard energy estimate gives that

$$\|f\|_{L^\infty L^2}^2 + \|f\|_{L^2 L^2}^2 \leqslant \|f_0\|_{L^2}. \tag{77}$$

To improve this estimate, we shall also use the De Giorgi–Nash iteration method. We take a sequence M_k as previously, and we also choose a sequence of times $T_k = 1 - \frac{1}{k+1}$ which tends to 1. The energy estimate for $(f - M_k)_+$ yields that for every t, s with $t \geqslant T_k \geqslant s$, we have

$$\sup_{t \geqslant T_k} \| (f - M_k)_+(t) \|_{L^2}^2 + \int_s^{+\infty} \| \nabla (f - M_k)_+)(\tau) \|_{L^2}^2 \, d\tau \leqslant 2 \| (f - M_k)_+(s) \|_{L^2}^2$$

and hence, by integrating in s for $T_{k-1} \leqslant s \leqslant T_k$, we obtain that

$$U_k \leqslant 2(k+1)^2 \int_{T_{k-1}}^{+\infty} \left\| (f - M_k)_+(s) \right\|_{L^2}^2 \, ds \tag{78}$$

with

$$U_k = \sup_{t \geqslant T_k} \left\| (f - M_k)_+(t) \right\|_{L^2}^2 + \int_{T_k}^{+\infty} \left\| \nabla (f - M_k)_+)(\tau) \right\|_{L^2}^2 \, d\tau.$$

The aim is again to estimate the right-hand side of (78) by a power of U_{k-1} strictly greater than 1. By using the same notations as previously, we get from (72) that

$$\begin{aligned} U_k &\leqslant 2(k+1)^2 \left(\frac{(k+1)^2}{M} \right)^{\frac{4}{3}} \int_{T_{k-1}}^{+\infty} \int_{\mathbb{R}^3} (f - M_{k-1})_+^{\frac{10}{3}} \, dt \, dx \\ &\leqslant 2(k+1)^2 \left(\frac{(k+1)^2}{M} \right)^{\frac{4}{3}} U_{k-1}^{\frac{5}{3}}. \end{aligned} \tag{79}$$

Since we have from (77) that $U_0 \leqslant \|(f_0)_+\|_{L^2}^2 \leqslant \|f_0\|_{L^2}^2$, U_1 can be made arbitrarily small by taking M sufficiently large and hence we get from (79) that $\lim_{k \to +\infty} U_k = 0$. This proves that $f(t,x) \leqslant M$ for $t \geqslant 1$ and then we get (76) by changing f into $-f$. We have thus proven that for every $t \geqslant 1$ the linear operator $f \mapsto f(t, \cdot)$ is bounded from L^2 into L^∞ with norm smaller than M. Since by the standard maximum principle, it is also bounded from L^∞ to L^∞ with norm 1, we get by interpolation that it also maps L^r to L^∞ for every $r \geqslant 2$. To get the claimed estimate in (65) for $t \leqslant 1$, it suffices to use again a scaling argument.

This ends the proof. □

Appendix B. Proof of Lemma 2.7

Proof. Set $\phi = \mathcal{F}^{-1} h$, then we have by definition and from Taylor formula

$$\begin{aligned} \big[h(\mathrm{D}), f\big] g &= \int_{\mathbb{R}^d} \phi(x-y) g(y) \big(f(y) - f(x)\big) \, dy \\ &= \int_0^1 \int_{\mathbb{R}^d} g(y) \Phi(x-y) \cdot \nabla f\big(x + t(y-x)\big) \, dy \, dt, \end{aligned}$$

with $\Phi(x) = x\phi(x)$. Let $\alpha, \beta \in]0,1[$ with $\alpha + \beta = 1$. Using Hölder inequality and a change of variables we get with $\Phi_t = t^{-3} \Phi(\frac{x}{t})$

$$\big|\big[h(\mathrm{D}),f\big]g(x)\big| \leqslant \int_0^1\int_{\mathbb{R}^d}\big(\big|g(y)\big|\big|\Phi(x-y)\big|^{\alpha}\big)\big(\big|\nabla f\big(x+t(y-x)\big)\big|\big|\Phi(x-y)\big|^{\beta}\big)\,dy\,dt$$

$$\leqslant \bigg(\int_{\mathbb{R}^3}\big(\big|\Phi(x-y)\big|\big|g(y)\big|^{\frac{1}{\alpha}}\big)\,dy\bigg)^{\alpha}\int_0^1\bigg(\int_{\mathbb{R}^3}\big|\Phi_t(y)\big|\big|\nabla f(x-y)\big|^{\frac{1}{\beta}}\,dy\bigg)^{\beta}$$

$$\leqslant \big(|\Phi|\star|g|^{\frac{1}{\alpha}}\big)^{\alpha}(x)\int_0^1\big(|\Phi_t|\star|\nabla f|^{\frac{1}{\beta}}\big)^{\beta}(x)\,dt.$$

Let $p_1,p_2\in[1,\infty]$ such

$$\alpha p_1,\beta p_2\geqslant 1\quad\text{and}\quad \frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}. \tag{80}$$

Then by Hölder inequality

$$\big\|\big[h(\mathrm{D}),f\big]g\big\|_{L^p}\leqslant\big\||\Phi|\star|g|^{\frac{1}{\alpha}}\big\|^{\alpha}_{L^{\alpha p_1}}\int_0^1\big\||\Phi_t|\star|\nabla f|^{\frac{1}{\beta}}\big\|^{\beta}_{L^{\beta p_2}}\,dt.$$

We choose p_1,p_2,α and β such that

$$\alpha m\geqslant 1,\quad \beta\rho\geqslant 1,\quad 1+\frac{1}{\alpha p_1}=\frac{1}{r}+\frac{1}{\alpha m}\quad\text{and}\quad 1+\frac{1}{\beta p_2}=\frac{1}{r}+\frac{1}{\beta\rho} \tag{81}$$

then the classical convolution laws give

$$\big\|\big[h(\mathrm{D}),f\big]g\big\|_{L^p}\leqslant\|\Phi\|^{\alpha}_{L^r}\|g\|_{L^m}\|\nabla f\|_{L^\rho}\int_0^1\|\Phi_t\|^{\beta}_{L^r}\,dt$$

$$\leqslant\|\Phi\|_{L^r}\|g\|_{L^m}\|\nabla f\|_{L^\rho}\int_0^1 t^{3\beta(-1+\frac{1}{r})}\,dt.$$

This last integral is finite provided that

$$\beta<\frac{1}{3}\frac{r}{r-1}. \tag{82}$$

Now let us check that the set given by conditions (80), (81) and (82) is not empty. First for the case $r=1$ we choose $p_1=m$, $p_2=\rho$, $\alpha=\frac{1}{m}$ and $\beta=1-\alpha$. Let us now discuss the case $r>1$. From (81)

$$\alpha=\frac{r}{r-1}\bigg(\frac{1}{m}-\frac{1}{p_1}\bigg).$$

To get $\alpha \in [\frac{1}{m}, 1[$ we must choose p_1 such that

$$\frac{1}{p} - \frac{1}{\rho} < \frac{1}{p_1} \leqslant \frac{1}{rm}. \tag{83}$$

The condition $\beta\rho \geqslant 1$ is equivalent, by the use of $1 + \frac{1}{p} = \frac{1}{m} + \frac{1}{\rho} + \frac{1}{r}$, to

$$\frac{1}{p} - \frac{1}{r\rho} \leqslant \frac{1}{p_1}. \tag{84}$$

The condition $\alpha p_1 \geqslant 1$ is automatically satisfied from (83) since

$$\alpha p_1 \geqslant r\alpha m \geqslant r \geqslant 1.$$

The condition $\beta p_2 \geqslant 1$ is also a consequence of (83) and (84). Indeed, from the value of α and $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, this condition is equivalent to

$$\frac{1}{p} - \frac{r}{2r-1}\frac{1}{\rho} \leqslant \frac{1}{p_1}.$$

This condition is weaker than (84). We can easily check that (83) and (84) are equivalent to

$$\frac{1}{p} - \frac{1}{r\rho} \leqslant \frac{1}{p_1} \leqslant \frac{1}{rm}.$$

The set of p_1 described by the above condition is nonempty if

$$\frac{1}{rm} - \frac{1}{p} + \frac{1}{r\rho} \geqslant 0.$$

Using the identity $1 + \frac{1}{p} = \frac{1}{m} + \frac{1}{\rho} + \frac{1}{r}$, this is satisfied under the condition $p \geqslant r$. The condition (82) is equivalent to

$$\frac{1}{p_1} < \frac{1}{3} + \frac{1}{p} - \frac{1}{\rho}.$$

Now there is a compatibility between this condition and (84) if

$$3\left(1 - \frac{1}{r}\right) < \rho.$$

This ends the proof of Lemma 2.7. □

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