Asymptotics of Matrix Coefficients and Closures of Fourier–Stieltjes Algebras

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The vanishing theorems of R. Howe and C. C. Moore are interpreted in a topological way. This yields a generalization of their results to connected locally compact totally minimal groups. The first section gives the complete classification of these groups and shows a lot of examples. It is proven that, among the connected groups, the totally minimal groups are exactly those groups $G$ whose Fourier–Stieltjes algebras are uniformly dense in the $C^*$-algebra of all weakly almost periodic functions $\text{WAP}(G)$. This solves a problem which was intensely discussed by C. Chou. Furthermore it is shown that $\text{WAP}(G)$ is generated as a $C^*$-algebra by the continuous functions on $G/N$ which vanish at infinity, where $N$ ranges over all closed normal subgroups.

1. INTRODUCTION

The purpose of this article is to analyze the asymptotic behaviour of strongly continuous unitary representations by studying topological properties of the underlying group. This is motivated by the famous result of R. Howe and C. C. Moore [13] that any matrix coefficient on a simple Lie group with finite center is a sum of a constant and a function which vanishes at infinity.

This fact forces the group topology to be very coarse since every continuous homomorphism onto another topological Hausdorff group is required to be open. Groups with this property are called *totally minimal*. We will prove an analogous asymptotic behaviour for general totally minimal locally compact groups.

Furthermore we will reveal these groups to be exactly the connected ones which enjoy the Eberlein-property, i.e. the uniform closure of the Fourier–Stieltjes algebra coincides with the algebra of all weakly almost periodic functions. This question has been intensely studied by C. Chou [2, 3, 4].

As yet, the only known noncompact totally minimal analytic groups were the semisimple ones with finite center [18]. However, we give in
Section 2 a complete classification modulo compact normal subgroups of all connected totally minimal groups which yields a lot of new examples.

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2. TOTALLY MINIMAL GROUPS

In this paper \((G, \tau)\) means a group \(G\) furnished with a topology \(\tau\) making it a locally compact topological group. Furthermore all topologies are assumed to be Hausdorff.

2.1. Definition. A locally compact group \((G, \tau)\) is called minimal if every continuous one-to-one homomorphism onto a (not necessarily locally compact) topological group \(H\) is open. The group \((G, \tau)\) is called totally minimal if this is true for every (not necessarily one-to-one) continuous homomorphism.

A maximal almost periodic group is minimal if and only if it is compact. Any finite dimensional continuous unitary representation of a totally minimal group has cocompact kernel.

Although there has been a lot of research on minimal groups in the last twenty years (for an overview, see [7]), the most important results for the Lie group case were worked out earlier by W. van Est [21], M. Goto [11] and M. Omori [17]. Here we need the following lemma (see [11]):

2.2. Lemma. For an analytic group \((G, \tau)\) the following are equivalent:

(i) \((G, \tau)\) is minimal;

(ii) every continuous one-to-one homomorphism into a locally compact group \((H, \sigma)\) has closed image;

(iii) \((G, \tau)\) has compact center and is a (CA) group. This means that the adjoint group \(\text{Ad} G\) is a closed subgroup of \(\text{GL}(g)\), where \(g\) denotes the Lie algebra of \(G\).

Condition (ii) together with the Howe–Moore theorem shows the minimality of simple groups with finite center and it follows from (iii) that all semisimple groups with finite center are minimal. A nilpotent group is totally minimal if it is compact, but the classical Weyl–Heisenberg group is an example of a noncompact minimal nilpotent group. The Mautner group is obviously not minimal, hence it is not (CA). To characterize totally minimal groups, we need

2.3. Lemma. Let \((G, \tau)\) be a locally compact group and \(N \triangleleft G\) a closed normal subgroup. Then \((G, \tau)\) is (totally) minimal if \(N\) and the factor group
G/N are both (totally) minimal in the relative and in the quotient topology, respectively.

Proof. This follows easily from [7, 7.3.1].

Using Montgomery–Zippin's solution of Hilbert's fifth problem we get the following useful criterion:

2.4. Proposition. For a connected locally compact group the following are equivalent:

(i) $(G, \tau)$ is totally minimal;

(ii) for every closed normal subgroup $N \triangleleft G$ the center $Z(G/N)$ of the factor group $G/N$ is compact in the quotient topology.

Proof. (i) $\Rightarrow$ (ii) follows from (2.2.iii).

(ii) $\Rightarrow$ (i) By [16] there exists a compact normal subgroup $K \triangleleft G$ such that the quotient $G/K$ is an analytic group. Thus, using (2.3), we may assume $(G, \tau)$ to be analytic. In addition (ii) is inherited by all factor groups. So, by (2.2.iii) again, it is enough to show the (CA) property of $(G, \tau)$.

Let $F$ be the closure of $Ad G$ in $GL(g)$. This is a linear group, thus the commutator subgroup $F'$ is closed [12, III.10.9] and by density $F' = (Ad G)' [12, III.8.10]$. Hence the Lie group $H := F((Ad G)')$ is abelian, and (ii) implies that the dense subgroup $Ad G((Ad G)')$ is compact and coincides with $H$. Thus $F = Ad G$.

A slightly weaker version of this proposition was stated as a part of a theorem in [20, III.8.6]. But there is a gap in its proof and another statement (needed in the proof) of this theorem is not true.

Proposition (2.2.iii) shows that a totally minimal analytic $(G, \tau)$ group is a central compact extension of the closed linear group $Ad G$. Hence there is a simply connected solvable group $B$ and a reductive group $H$ in $GL(g)$ with

$$Ad G \xrightarrow{\top} B \rtimes \phi H,$$

where $\phi: H \to \text{Aut}(B)$ denotes a continuous homomorphism and $\rtimes$ the associated semidirect product [12, III.10.8]. If $\phi$ is faithful, we identify $H$ with $\phi(H)$. Now we are able to prove the main result of this section:

2.5. Theorem. Let $(G, \tau)$ be a locally compact connected group. Then $(G, \tau)$ is totally minimal if and only if there exist a compact normal subgroup $K \triangleleft G$, a nilpotent simply connected group $N$ and a reductive group $H$ such that $G/K$ is topologically isomorphic to a semidirect product $N \rtimes \phi H$ and $H$ acts on $N$ via $\phi$ without nontrivial fixed points.
Proof. Let \((G, \tau)\) be totally minimal. By (2.3) and (2.4.ii) we may assume that \((G, \tau)\) is linear and \(G = B \rtimes H\) as above. Then the commutator \(N := [B, G]\) is a nilpotent \([12, II.7.3]\) analytic subgroup of \(B\), hence closed and simply connected \([12, III.3.17]\). But \(G/N\) splits as the direct product \(B/N \times H\). Thus \(B/N\) is central, hence compact (2.4.ii). On the other hand, \(B/N\) is simply connected \([12, III.3.17]\) and abelian, hence trivial. So we have proved

\[G = N \rtimes H.\]

Now \(H\) has no nontrivial fixed point on \(N\). We will show this by induction on the length of the central series of \(N\). If \(N\) is a vector group, the fix space \(\text{FIX}(H)\) is a vector space which is topologically isomorphic to a central subgroup of \(G\). By the compactness of \(\mathcal{X}(G)\), the dimension of \(\text{FIX}(H)\) has to be zero. Now consider the nonabelian case, i.e. the central series

\[N \supseteq N^1 := [N, N] \supseteq N^2 := [N^1, N] \supseteq \cdots \supseteq N^n \supseteq N^{n+1} = 1_N\]

has strictly positive length. Let \(x \in N\) be a fixed point. The characteristic subgroup \(N^n\) is analytic \([12, III.3.19]\). Hence the induction hypothesis applies to \(G/N^n\) and implies \(x \in N^n\). But \(N^n\) is central in \(N\), hence it is a vector group \([12, III.3.17]\) and the above argument yields \(x = 1_N\).

Let us prove the converse. By (2.3), it is enough to show the total minimality of the factor group \(\hat{G} := G/K\). We do this by induction on \(n := \dim N\). For \(n = 0\), the group \(G/K := H\) is reductive, its center \(\mathcal{Z}(H)\) is compact by definition and the factor group \(H/\mathcal{Z}(H)\) is a linear semisimple group. Therefore its center is automatically finite \([12, III.6.16]\) and the total minimality follows with (2.3). Now assume that \(n > 0\). It is easily seen that a closed normal subgroup of a reductive group is totally minimal in the relative topology. Hence, by (2.3), \(\hat{G}\) is totally minimal if \(G/\ker \varphi\) is totally minimal and we may w.l.o.g. assume \(\varphi\) to be faithful. So the center of \(\hat{G}\) is trivial. Furthermore we identify the simply connected group \(N\) with its Lie algebra \(n\) and view \(H\) as subgroup of \(\text{Aut}(n)\). This enables us to use that \(H\) is reductive. We want to apply (2.4), i.e. to show the compactness of the center \(\mathcal{X}(\hat{G}/M)\) for any closed normal subgroup \(M \triangleleft \hat{G}\). Now if the connected component \(M_0\) of \(M\) is trivial, \(M\) has to be central, hence trivial. In case \(M_0 \neq 1_G\), the canonical mapping \(p: \hat{G}/M_0 \to \hat{G}/M\) is a covering map \([12, III.3.6]\), hence \(\mathcal{X}(\hat{G}/M) = p(\mathcal{X}(\hat{G}/M_0))\) \([12, III.3.2]\). So let us assume that \(M\) is connected. The intersection \(M \cap N\) is a closed normal subgroup of \(\hat{G}\). If its connected component \((M \cap N)_0\) is not trivial, the induction hypothesis applies to \(\hat{G}/(M \cap N)_0\), since \(H\) is reductive. Thus the homomorphic image \(\hat{G}/M\) is also totally minimal, hence its center is compact (2.4.ii). However, in case \((M \cap N)_0 = 1_G\), the closed normal subgroup \(M \cap N\) is central, hence trivial. We will show that this implies \(M = 1_G\):
Since $M$ and $N$ are analytic the commutator $[M, N]$ is contained in $(M \cap N)_{0} = 1_{G}$, and the subgroups commute. Hence for all $(a, k) \in M$, $(y, \text{id}_{N}) \in N$ it follows that

$$(a, k)(y, \text{id}_{N})(a, k)^{-1} = (ak(y) a^{-1}, \text{id}_{N}) = (y, \text{id}_{N}).$$

This implies

$$M \subseteq \{(a, \text{Inn}(a^{-1})) | a \in N\}, \quad (1)$$

where $\text{Inn}(a)$ denotes the inner automorphism on $N$ defined by $a \in N$. But then the canonical projection $p_{N} : \hat{G} \to H$ maps $M$ onto a nilpotent, hence central normal subgroup of $H$ and for $(a, \text{Inn}(a^{-1})) \in M$, $h \in H$ it follows that

$$M \ni (1_{N}, h)(a, \text{Inn}(a^{-1}))(1_{N}, h)^{-1} = (h(a), \text{Inn}(a^{-1})).$$

Now (1) implies that $H$ stabilizes the coset $a \cdot \mathcal{Z}(N)$. Since $H$ is reductive, $H$ acts without non trivial fixed points on $N/\mathcal{Z}(N)$. Thus $a \in \mathcal{Z}(N)$. This yields together with (1) that $M \subseteq N$, hence $M = \{(1_{N}, \text{id}_{N})\}$, by assumption. Thus, for every closed normal subgroup $M$ the center of $G/M$ is compact in the quotient topology and $(G, \tau)$ is totally minimal (2.4).

2.6. Examples. (i) Theorem (2.5) shows the total minimality of the Lorentz groups $\mathbb{R}^{n} \ltimes \text{SL}(n, \mathbb{R})$ and the euclidean motion groups $M(n) := \mathbb{R}^{n} \ltimes \text{SO}(n, \mathbb{R})$. Other examples are easily constructed in this way as $G := (\mathbb{R}^{2} \otimes \mathbb{R}^{3}) \rtimes (\text{SL}(2, \mathbb{R}) \times \text{SO}(3, \mathbb{R})).$

(ii) Let $\Sigma$ be the space of all three dimensional real antisymmetric matrices and $n := \mathbb{R}^{3} \ltimes \Sigma$ the two-step nilpotent Lie algebra with the bracket:

$$[(u, U), (v, V)] := (0, u^{t}v - v^{t}u).$$

Here $\mathbb{R}^{3}$ is viewed as the space of row vectors. Identifying $n$ with the associated simply connected group, we form the semidirect product

$$G := n \ltimes \text{SO}(3),$$

where $\varphi(D)(u, U) := (uD, DU D^t)$. Theorem (2.5) shows the total minimality.

(iii) It is not easy to construct low-dimensional examples where $N$ is not a vector group. This is because $H$ has to stabilize each characteristic subgroup of $N$. Thus, a Heisenberg group will never be the nilpotent factor of a totally minimal group.
(iv) Note that the proof of (2.5) implies that any non trivial closed normal subgroup which is not contained in $H$ has non trivial intersection with $N$.

3. VANISHING THEOREMS

We now consider some consequences of total minimality for representation theory.

Let $(G, \pi)$ be a locally compact group. In this paper we assume all representations $(\pi, \mathcal{H})$ to be strongly continuous and unitary on some complex Hilbert space $\mathcal{H}$. The central idea is to consider an enveloping semigroup $S_\pi$ associated to $(\pi, \mathcal{H})$ as it is done in topological dynamics (see for example [8, 9]).

Let $\mathcal{F}(\pi)$ be the uniform closure of the conjugation invariant unital algebra which is generated by all matrix coefficients of $\pi$. By the Stone-Weierstraß theorem, the maximal ideal space $\mathcal{M}(\mathcal{F}(\pi))$ of $\mathcal{F}(\pi)$ is canonically isomorphic to $S_\pi$, the weak operator closure of $\pi(G)$ (cf. [1, VI.2.12]).

The pair $(S_\pi, \pi)$ is a weakly almost periodic compactification of $G$, in the sense that $\pi$ is a continuous homomorphism onto a dense subgroup of the compact semitopological semigroup $S_\pi$ (i.e. the multiplication on $S_\pi$ is separately continuous).

A continuous bounded function on $(G, \pi)$ is called weakly almost periodic if the set of left (or equivalently right) translates is relatively compact in the weak topology. The weakly almost periodic functions endowed with the norm of uniform convergence form a C*-algebra, denoted by $\text{WAP}(G)$. The group multiplication can be extended to a separately continuous multiplication on the maximal ideal space $\text{wG}$ of $\text{WAP}(G)$. This is the largest weakly almost periodic compactification of $(G, \pi)$ in the following sense: Every weakly almost periodic compactification $(S, \varphi)$ of $(G, \pi)$ lifts in a unique way to a continuous semigroup homomorphism $\hat{\varphi}: \text{wG} \to S$ such that

$$\hat{\varphi} \circ \varepsilon_{\text{WAP}(G)} = \varphi.$$

Here and for the remainder of the paper $\varepsilon_\varphi: G \to \mathcal{A}(\mathcal{F})$ denotes the canonical mapping of $G$ into the maximal ideal space of a C*-subalgebra $\mathcal{F}$ of $\text{WAP}(G)$.

Now consider a weakly almost periodic compactification $(S, \varphi)$. The Ryll–Nardzewski theorem guarantees that a unique minimal ideal $K(S)$ exists in the semigroup $S$, and that $K(S)$ is a compact subgroup of $S$. The neutral element $u \in K(S)$ is a central idempotent in $S$. If $(S, \varphi)$ arises from
a unitary representation \((\pi, \mathcal{H})\) this idempotent \(u\) is equal to the orthogonal projection onto the subspace spanned by the finite dimensional subrepresentations of \(\pi\). The orthogonal complement is called the space of dissipative vectors. There exists a net on \(G\) converging to infinity on which all matrix coefficients of dissipative vectors approach zero. This is essentially the content of the Jacobs–de Leeuw splitting theorem (see [1, Chapter 6]).

Now W. Ruppert has shown a generalization which is very useful in view of representation theory. We will cite his theorem in the following form:

3.1. Lemma. Let \((G, \tau)\) be a connected locally compact group, \(\mathcal{F}\) be a two-sided translation invariant \(\mathbb{C}^*\)-subalgebra of \(\text{WAP}(G)\) and \(\mathcal{A}(\mathcal{F})\) be the maximal ideal space of \(\mathcal{F}\). Then the multiplication on the group extends to \(\mathcal{A}(\mathcal{F})\) such as to satisfy

\[ s \cdot e_\varphi(G) = e_\varphi(G) \cdot s \]

for all \(s \in \mathcal{A}(\mathcal{F})\).

Proof. Since on the set of the left (right) translates of \(f \in \mathcal{F}\) the weak topology coincides with the topology of pointwise convergence, [1, II.2.3.vii] shows that \(\mathcal{F}\) is \(m\)-admissible in the sense of [1, II.2.10]. Hence \((e_\varphi, \mathcal{A}(\mathcal{F}))\) is a weakly almost periodic compactification (cf. [1, II.2.11] and [1, IV.2.7]) and Ruppert's theorem [20, III.5.1] applies.

The following facts are easily derived from (3.1):

(i) every left ideal of the semigroup \(\mathcal{A}(\mathcal{F})\) is also a right ideal;

(ii) every idempotent in \(\mathcal{A}(\mathcal{F})\) is central;

(iii) for every \(s \in \mathcal{A}(\mathcal{F})\) the isotropy group \(F(s) := \{g \in G \mid e_\varphi(g) s = s\}\) is normal.

An immediate application is the following:

3.2. Proposition. Let \((\pi, \mathcal{H})\) be an irreducible representation of the connected locally compact group \((G, \tau)\). Then all operators in \(S_\pi\) except 0 are of the form \(\lambda \cdot U\), where \(U\) is unitary and \(0 < \lambda \leq 1\). If \(\pi\) is open onto \(\pi(G)\), all unitaries in \(S_\pi\) are in \(\pi(G)\).

Proof. By (3.1) every \(s \in S_\pi \setminus \{0\}\) is one-to-one and has dense image. In addition \(s^* s\) is an intertwining operator of \(\pi\), hence a scalar and the first assertion follows. Since \(S_\pi\) is selfadjoint, all unitaries are invertible in \(S_\pi\). By the Ellis–Lawson theorem [1, I.4.4] the group of invertible elements \(H(\text{id}_{\mathcal{H}_\pi})\) in \(S_\pi\) is topological, whence \(\pi(G)\), being a locally compact subgroup, is closed in \(H(\text{id}_{\mathcal{H}_\pi})\). This shows the second part.
In order to prove the classical vanishing theorem, Mautner's Lemma is needed (see e.g., [22] or [14]). This is rather easily generalized to weakly almost periodic compactifications by the following argument (see [20, III.5.10]): Consider a weakly almost periodic compactification \((S, \varphi)\) of an analytic group \((G, \tau)\), \(s \in S\) and a net \(\{g_s\}_s\) in \(G\) such that \(\lim_s \varphi(g_s) = s\).

If there exists an \(X \in g \backslash \{0\}\) with
\[
\| \text{Ad} g_s X \| \to 0 \quad \text{or} \quad \| \text{Ad} g_s X \| \to \infty,
\]
where \(|\cdot|\) denotes some norm on \(g\), the Lie algebra of \(F(s)\) has strictly positive dimension. Now we can apply minimality:

3.3. **Lemma.** Let \((S, \varphi)\) be a weakly almost periodic compactification of the minimal analytic group \((G, \tau)\) and \(s \in S \backslash \varphi(G)\). Then \(\dim F(s) > 0\).

**Proof.** Let \(\{g_s\}_s\) be a net in \(G\) such that \(\{\varphi(g_s)\}_s\) converges to \(s\). Because the center \(Z(G)\) is compact we may assume that \(g_s \not\in Z(G)\) for all \(s\). Moreover one of the conditions in (2) is satisfied for some \(X \in g \backslash \{0\}\) because of the (CA) property of \((G, \tau)\), and the assertion follows.

3.4. **Proposition.** Let \((\pi, \mathcal{H})\) be a faithful irreducible representation of the minimal noncompact analytic group \((G, \tau)\). Then all matrix coefficients vanish at infinity. In particular, given an irreducible representation of a totally minimal group, then either it is finite dimensional or all its matrix coefficients vanish at infinity modulo its kernel.

**Proof.** Let \(s \in S \backslash \pi(G)\). In view of (3.3), we can find a \(g \in G \backslash \{1_G\}\) with \(\pi(g) s = s\). But we know from (3.1) that if \(s \neq 0\), \(s\) would have dense image, hence \(\pi(g)\) would have to be the identity and \(\pi\) would fail to be faithful, a contradiction.

The second part follows from the first and the Montgomery–Zippin approximation theorem.

Observe that the use of direct integrals yields the vanishing theorems on semisimple groups. In [15] we show how these results are generalized to vanishing theorems on any connected totally minimal group.

4. **EBERLEIN GROUPS**

Let \(B(G)\) be the Fourier–Stieltjes algebra of \((G, \tau)\) as described by Eymard [10]. Then \(B(G)\) is the space of all matrix coefficients of unitary representations \((\pi, \mathcal{H})\):
\[
G \ni g \mapsto \langle \pi(g) \xi, \eta \rangle, \quad \xi, \eta \in \mathcal{H}.
\]
It is closed under complex conjugation and pointwise multiplication. Since \( B(G) \) is canonically identified with the dual space of the group C*-algebra of \( (G, \tau) \), \( B(G) \) is a normed space and forms a semisimple involutive Banach algebra. If \( G \) is abelian, Bochner's theorem identifies \( B(G) \) with the measure algebra on the dual group.

The norm on \( B(G) \) is finer than the uniform norm and \( B(G) \) is uniformly closed if and only if \( G \) is finite. Here we discuss the question for which groups the Fourier–Stieltjes algebra is uniformly dense in \( \text{WAP}(G) \).

4.1. **Definition.** A locally compact group \( (G, \tau) \) is called an **Eberlein group** if \( E(G) \), the closure of the Fourier–Stieltjes algebra of \( (G, \tau) \) with respect to the norm of the uniform convergence, coincides with \( \text{WAP}(G) \).

The von Neumann approximation theorem for almost periodic functions motivated the conjecture that any locally compact group might be Eberlein. But W. Rudin [19] found a weakly almost periodic function on the integers which is not a uniform limit of Fourier–Stieltjes transforms of regular measures on the torus. This was generalized to all noncompact locally compact abelian groups by C. Dunkl and D. Ramirez [6]. In addition C. Chou [4] proved that noncompact \([1N]\)-groups and noncompact nilpotent groups are not Eberlein groups.

A result of W. A. Veech [22] shows that semisimple groups with finite center are Eberlein groups. Chou proved the Eberlein property of the euclidean motion groups [3]. It was also Chou [2] who found the following important lemma:

4.2. **Lemma.** Let \( (G, \tau) \) be a locally compact group, \( N \lhd G \) be a closed normal subgroup and \( p_N: G \to G/N \) be the canonical projection. Then

\[
p_N^*(E(G/N)) = p_N^*(C(G/N)) \cap E(G), \tag{3}
\]

where \( p_N^* \) denotes the transposed mapping and \( C(G/N) \) denotes the continuous functions on \( G/N \).

The next observation establishes the connection with totally minimal groups:

4.3. **Proposition.** A connected locally compact Eberlein group is totally minimal.

**Proof.** Suppose the converse. According to (2.4), there would exist a closed normal subgroup \( N \lhd G \) such that the center \( Z(G/N) \) would not be compact in the quotient topology. This together with the above remarks would imply the existence of a weakly almost periodic function on \( Z(G/N) \) which cannot be uniformly approximated by matrix coefficients. But both...
weakly almost periodic and positive definite functions can be extended from the center to the whole group \([5, \text{Theorem 1}]\). Therefore \(G/N\) would fail to be an Eberlein group. Now (3) and the Eberlein property of \((G, \tau)\) would imply

\[
p_N^\tau(\operatorname{E}(G/N)) = p_N^\tau(C(G/N)) \cap \operatorname{WAP}(G) = p_N^\tau(\operatorname{WAP}(G/N)),
\]
a contradiction.

The remainder of the paper is devoted to proving the converse of (4.3) by means of some deeper results of semigroup theory. It is well known (see e.g. \([20, \text{III.1.10}]\)) that the weakly almost periodic compactification of a factor group \(G/N\) (\(N \lhd G\) closed normal subgroup) in the quotient topology is a homomorphic image of \(wG = \mathcal{A}(\operatorname{WAP}(G))\). Explicitly it is computed in the following way: Let \(\varepsilon_{\operatorname{WAP}(G)}; G \to wG\) be the canonical embedding and \(M\) be the minimal ideal in the closure of \(\varepsilon_{\operatorname{WAP}(G)}(N)\), which exists by the Ryll–Nardzewski Theorem cf. Section 3. Then \(M\) is a compact subgroup of \(wG\) with neutral element \(u_N\) and satisfies the following “normality” properties:

- for all \(s \in wG\): \(s \cdot M = M \cdot s\);
- the multiplication on \(wG \times M\) is jointly continuous.

We define an equivalence relation on \(wG\) by putting for \(s, t \in wG\):

\[
s \simeq_M t \iff s \cdot M = t \cdot M \quad (s, t \in wG).
\]

Then the classes \([s]_M \text{ s.t. } s \in wG\] form in a canonical way a compact semi-topological semigroup, denoted by \((wG)_M\). Now the homomorphism

\[
\varepsilon_{\operatorname{WAP}(G/N)}(g \cdot N) \mapsto [\varepsilon_{\operatorname{WAP}(G)}(g)]_M
\]
is well-defined and extends to an isomorphism between \(w(G/N)\) and \((wG)/M\).

Also, observe that for an idempotent \(v \in wG\) the set \(\varepsilon_{\operatorname{WAP}(G)}(G) \cdot v\) is a group which is topological because of the Ellis–Lawson continuity theorem \([1, I.4.4]\) and so, by total minimality of \(G\), topologically isomorphic to the factor group \(G/F(v)\). Furthermore for different idempotents \(v \neq v'\) the subgroups \(\varepsilon_{\operatorname{WAP}(G)}(G) \cdot v\) and \(\varepsilon_{\operatorname{WAP}(G)}(G) \cdot v'\) have empty intersection.

**4.4. Lemma.** Let \((G, \tau)\) be an analytic totally minimal group and let \(J(wG)\) denote the set of idempotents in \(wG\). Then

\[
wG = \bigcup_{v \in J(wG)} \varepsilon_{\operatorname{WAP}(G)}(G) \cdot v.
\]
Proof. We use induction on \( n := \dim G \). The case \( n = 0 \) is clear, thus consider \( n > 0 \). If \( s \in wG \backslash \text{WAP}(G) \), (3.3) implies \( \dim F(s) > 0 \) and the induction hypothesis shows that \( w(G/F(s)) \) is a disjoint union of groups. If \( M \) is the minimal ideal in the closure of \( w_{\text{WAP}}(G(F(s))) \), then \( M \) is a subgroup, too, and the above remarks show that \( s \) is contained in a subgroup of \( wG \). Therefore it is contained in a maximal subgroup \( U \) which is the union of all subgroups with the neutral element \( 1_U \) [1, I.1.12]. Hence \( 1_U \cdot w_{\text{WAP}}(G) \leq U \subseteq 1_U \cdot S \). But again by Ellis–Lawson, \( U \) is a topological group, whence the dense subgroup \( 1_U \cdot w_{\text{WAP}}(G) \) is locally compact (by total minimality) and coincides with \( U \).

It is well-known that if \( u \) is the minimal idempotent in \( wG \), then the minimal ideal \( K(wG) = u \cdot wG \) is isomorphic to the Bohr compactification \( bG \) of \( (G, \tau) \). In the totally minimal case, \( bG \) is isomorphic to the compact group \( G/F(u) \) which coincides with \( w(G/F(u)) \). By Lemma (4.4), for any idempotent \( v \in J(wG) \) the subsemigroup \( v \cdot wG \) of \( wG \) can be identified with the weakly almost periodic compactification of \( G/F(v) \) via the extension of the homomorphism

\[
\iota: \text{WAP}(G(F(v))) \ni gF(v) \mapsto \text{WAP}(G) \ni g.
\]

This is easily derived from the universal property of the weakly almost periodic compactification. Thus the transposed mapping

\[
\iota^t: \text{WAP}(G) \ni g \mapsto \text{WAP}(G(F(v))) \ni \iota(gF(v)) = f(gv) \ni \text{WAP}(G).
\]

is onto and we have for \( f \in \text{WAP}(G) \), \( g \in G \):

\[
\iota^t(f)(gF(v)) = f(\iota(gF(v))) = f(gv) := L_v f(g).
\]

In particular the weakly almost periodic functions \( h \) which are constant on the cosets of \( F(v) \) are exactly those of the form

\[
h(g) = f(gv) \quad \text{for} \quad f \in \text{WAP}(G).
\]

Here, as usual, we make no distinction between a function and its Gelfand transform.

Now we can show in which respect the total minimality is exactly the right generalization of compactness:

4.5. Theorem. For a connected locally compact group the following are equivalent:

(i) \( (G, \tau) \) is totally minimal;

(ii) \( (G, \tau) \) is an Eberlein group.
Furthermore $\text{WAP}(G)$ is the $C^*$-algebra generated by the functions in
\[ \mathcal{F}_v := C_0(G/F(v)), \]
where $v$ ranges over the set of idempotents.

Proof. We only need to show (i) $\Rightarrow$ (ii). Let $\mathcal{F}$ be the $C^*$-algebra generated by the $\mathcal{F}_v$ which are subalgebras of $E(G)$ via the regular representations of $G/F(v)$ (cf. [10, 3.7]). We assume at first that $(G, \tau)$ is analytic and use induction on $n := \dim G$.

The case $n = 0$ is trivial, so let $n > 0$. Now let $s_1, s_2 \in wG$ with $f(s_1) = f(s_2)$ for all $f \in \mathcal{F}$. By (4.4) we find $g_i \in G$ and $u_i \in J(wG)$ with
\[ s_i = e_{\text{WAP}(G)}(g_i) u_i, \quad i = 1, 2. \]
We see that $s_1, s_2 \notin e_{\text{WAP}(G)}(G)$ because the functions in $C_0(G)$ separate the points in $G$ and are zero elsewhere on $wG$. Observe that on account of (3.1), $F(s_i) = F(u_i)$ and $\dim F(u_i) > 0$ by (3.3). Thus the induction hypothesis applies to $G/F(u_i)$ and the above remarks show $u_i s_1 = u_1 s_2$, hence
\[ s_1 = u_1 e_{\text{WAP}(G)}(g_1) u_1 = u_1 s_1 = u_1 s_2 = u_1 u_2 e_{\text{WAP}(G)}(g_2). \]
Therefore $u_1 u_2 \in J(wG) \cap H(u_1)$, where $H(u_1)$ denotes the maximal subgroup containing $u_1$ ([1, I.1.12]) and thus $u_1 u_2 = u_1$. By exchanging the roles of $s_1$ and $s_2$ the commutativity of $J(wG)$ shows $u_1 = u_2$, whence the induction hypothesis implies $s_1 = s_2$. Now $\text{WAP}(G) = \mathcal{F}$, by the Stone-Weierstraß theorem.

If $(G, \tau)$ is general locally compact, there is a filtered set of compact normal subgroups $K$ such that $G$ is the projective limit of the factor groups $G/K$, and the quotients are analytic. Let $\mu_*$ be the normalized Haar measure of $K$. Then for $\phi \in \text{WAP}(G)$ the convolutions $\phi * \mu_*$ are constant on the $K$-coset and converge uniformly to $\phi$. Now the above shows the assertion.

4.6. Remarks. (i) It is surprising that only the most simple classes of weakly almost periodic functions, namely those which vanish at infinity modulo a closed normal subgroup, occur in the totally minimal case.

(ii) In a subsequent paper [15] we will show that $\text{WAP}(G)$ is generated by the functions $C_0(G/N)$ ($N \triangleleft G$ closed normal subgroup) if and only if this is the case for $E(G)$. Furthermore we will discuss the exact structure of a weakly almost compactification of a totally minimal group and apply this to representation and ergodic theory.

In [4] Chou asked whether the product of two Eberlein groups is an Eberlein group. This is now easily answered in case the groups are connected:
4.7. Corollary. The extension of a connected Eberlein group by a connected Eberlein group is an Eberlein group.

Proof. Follows from (4.5) and (2.3).

REFERENCES