The dynamical Borel–Cantelli lemma and the waiting time problems

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ABSTRACT

We investigate the connection between the dynamical Borel–Cantelli and waiting time results. We prove that if a system has the dynamical Borel–Cantelli property, then the time needed to enter for the first time in a sequence of small balls scales as the inverse of the measure of the balls. Conversely if we know the waiting time behavior of a system we can prove that certain sequences of decreasing balls satisfies the Borel–Cantelli property. This allows to obtain Borel–Cantelli like results in systems like axiom A and generic interval exchanges.

1. INTRODUCTION

Let $\{A_n\}$ be a sequence of subsets in a probability space (X, μ). The classical Borel–Cantelli lemma states that:

(1) If $\sum \mu(A_n) < \infty$, then $\mu(\limsup A_n) = 0$, that is, the set of points which are contained in infinitely many A_n has zero measure.
(2) Moreover, if the sets A_n are independent, then $\sum \mu(A_n) = \infty$ implies that $\mu(\limsup A_n) = 1$.

Now, let us consider a dynamical system (X, T, μ) and suppose that $T : X \to X$ preserves μ. In this case, if the sets $T^{-n} A_n$ are independent and $\sum \mu(A_n) = \infty$,

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then the set of points such that $T^n x \in A_n$ infinitely many times as n increases has full measure.

In a chaotic, mixing measure preserving dynamical system, sets of the form A and $T^{-n}A$ tend to "behave" as independent in a certain sense, as $n \to \infty$. By this it is reasonable to ask if a statement like point 2 above is valid. The answer is that it is not always valid. In [7] it is shown an example of a mixing system, where the BC property does not hold even for nice set sequences (decreasing sequences of balls with the same center). Hence some stronger requirements are needed (some stronger form of mixing or some stronger constraint in the sequence of sets).

In this context a decreasing sequence A_n is said to be a Borel–Cantelli sequence (BC) if $\sum \mu(A_n) = \infty$ and $\mu(\limsup T^{-n} A_n) = 1$ (here $\limsup S_n$ is the set of points which belongs to infinitely many S_n). In a mixing dynamical system, thus, the "abundance" of BC sequences can be interpreted as an aspect of strong chaos and stochastic behavior of the system. Indeed is proved (see, e.g., [5,20,6,15,8,22,11]) that in many kind of (more or less) hyperbolic or "fast" mixing systems, various sequences of geometrically nice sets have the BC property. The kind of sets which are interesting to be considered in this kind of problems are usually decreasing sequences of balls with the same center (these are also called shrinking targets, this approach has relations with the theory of approximation speed, see [13,18]) or cylinders.

Let us consider another concept which as we will see is closely related to the Borel–Cantelli property: the waiting time. Let A be a subset of X and

$$\tau_A(x) = \min\{n \in \mathbf{N}\colon\ T^n(x) \in A\}$$

be the time needed for $x \in X$ to enter for the first time in A. It is clear that in an ergodic system (when A has positive measure) $\tau_A(x)$ is almost everywhere finite. Intuitively, when A is smaller and smaller, then $\tau_A(x)$ is bigger and bigger. If the behavior of the system is chaotic enough, one could expect that for most points $\tau_A(x) \sim \frac{1}{\mu(A)}$. More precisely, let $B(y, r_n)$ be a sequence of balls with center y and radius r_n. We say that x satisfies the waiting time problem (with respect to the sequence of sets $B(y, r_n)$) if

$$\lim_{n\to\infty} \frac{\log \tau_{B(y,r_n)}(x)}{-\log \mu(B(y, r_n))} = 1. \tag{1.1}$$

In this case if the local dimension[2] of μ at y is $d_\mu(y)$, then the measure of balls scales, for small r as $\mu(B(y, r_n)) \sim r_n^{d_\mu(y)}$ and then

$$\tau_{B(y,r_n)}(x) \sim r_n^{-d_\mu(y)}. \tag{1.2}$$

[2] If X is a metric space and μ is a measure on X the upper local dimension at $x \in X$ is defined as

$$\bar{d}_\mu(x) = \limsup_{r\to 0} \frac{\log(\mu(B(x,r)))}{\log(r)} = \limsup_{k\in\mathrm{N}, k\to\infty} \frac{-\log(\mu(B(x, 2^{-k})))}{k}.$$

The lower local dimension $\underline{d}_\mu(x)$ is defined in an analogous way by replacing $\limsup$ with $\liminf$. If $\bar{d}_\mu(x) = \underline{d}_\mu(x)$ we denote with $d_\mu(x)$ the local dimension at x.

A result of this kind has been proved in various kind of chaotic systems ([10,9, 11,14]; see also [17] for cases where it does not hold). Moreover this problem is related to the distribution of return times (the property holds when the distribution of return times in small balls tends to be exponential, see [9]; see also [12] for other general relations between waiting time and recurrence time distribution). While in the literature results on Borel–Cantelli and Waiting time are somewhat similar (and sometime used together, as in [8]), as far as we know, no explicit general relations about these two concepts are stated.

In this note we show that the Borel–Cantelli property and the waiting time problem are in general strictly connected: in Section 2 we show that in systems where decreasing sequences of balls have the BC property, then the waiting time problem is satisfied for almost all points of the space (Theorem 2.4 or in a different point of view, Theorem 2.7). In Section 3 we see that there are examples of systems where the waiting time problem is satisfied but certain decreasing sequences of balls does not satisfy the BC condition. This says that this kind of Borel–Cantelli condition is stronger than the one imposed by the waiting time problem. However, we see in Section 3 that if we impose further conditions on the sequences of balls we consider, making radius to decrease in a "controlled" way, then we have converse statements (Theorem 3.4 and following). This allows to use results on the waiting time problem to obtain Borel–Cantelli like results on certain decreasing sequences of balls in systems like axiom A and generic interval exchanges.

2. BOREL–CANTELLI IMPLIES WAITING TIME

We assume that T is a measure preserving transformation on a metric space (X, μ, d). We will prove a general result about the waiting time problem which generalizes an inequality between waiting time and measure of sets proved in [9], allowing sets which are not necessarily balls. Then we prove that in system where decreasing sequences of balls have the BC property the inequality becomes equality, and then in such systems the scaling behavior of the waiting time is the same as the scaling behavior of the measure of small balls (1.1).

Proposition 2.1. *Let B_n be a decreasing sequence of measurable subsets in X with $\lim_n \mu(B_n) = 0$. Then we have*

$$\liminf_{n\to\infty} \frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} \geqslant 1 \quad \textit{for a.e. } x.$$

Proof. Choose a subsequence n_i as $n_i = \min\{n \geqslant 1\colon\ \mu(B_n) < 2^{-i}\}$. If $n_i \leqslant n < n_{i+1}$, then we have $\tau_{B_n}(x) \geqslant \tau_{B_{n_i}}(x)$ for every x and $2^{-i} > \mu(B_{n_i}) \geqslant \mu(B_n) \geqslant 2^{-i-1}$. Therefore, if $n_i \leqslant n < n_{i+1}$, for every x

$$\frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} \geqslant \frac{\log \tau_{B_{n_i}}(x)}{-\log \mu(B_{n_i})} \cdot \frac{-\log \mu(B_{n_i})}{-\log \mu(B_n)} > \frac{\log \tau_{B_{n_i}}(x)}{-\log \mu(B_{n_i})} \cdot \frac{i}{i+1},$$

which implies that

$$\liminf_{i\to\infty} \frac{\log \tau_{B_{n_i}}(x)}{-\log \mu(B_{n_i})} = \liminf_{n\to\infty} \frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} \quad \text{for every } x.$$

Thus we may assume that $\mu(B_n) \leqslant 2^{-n}$.

Let

$$E_n = \left\{ x\colon \frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} < 1-\delta \right\}$$

for some $\delta > 0$. Then we have

$$\begin{aligned} \mu(E_n) &= \mu\big(\big\{x\colon \tau_{B_n}(x) < \mu(B_n)^{-(1-\delta)}\big\}\big) \\ &= \sum_{1\leqslant i<\mu(B_n)^{-(1-\delta)}} \mu\big(\{x\colon \tau_{B_n}(x) = i\}\big) \\ &\leqslant \sum_{1\leqslant i<\mu(B_n)^{-(1-\delta)}} \mu\big(T^{-i}B_n\big) \\ &\leqslant \mu(B_n)^{-(1-\delta)}\mu(B_n) = \mu(B_n)^{\delta} < 2^{-n\delta}. \end{aligned} \tag{2.1}$$

Hence $\sum_n \mu(E_n) < \infty$ and by the Borel–Cantelli lemma we have

$$\mu(\limsup E_n) = 0.$$

Since δ is arbitrary we have for almost every x

$$\liminf_n \frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} \geqslant 1. \qquad \square$$

Remark 2.2. In the above proof (in Eq. (2.1)), for each $\epsilon > 0$, we can replace δ with a sequence $\delta_n \to 0$ such that $\delta_n \leqslant \frac{(1+\epsilon)\log n}{-\log \mu(B_n)}$. The proof is still valid and we have an estimation on the "speed of convergence" to this limit inequality. Indeed we obtain that if B_n is a decreasing sequence and $\mu(B_n) \leqslant 2^{-n}$, typical points will eventually satisfy

$$\frac{\log \tau_{B_n}(x)}{-\log \mu(B_n)} \geqslant 1 - \frac{(1+\epsilon)\log n}{-\log \mu(B_n)}$$

for each $\epsilon > 0$. This is interesting when waiting time is used to give numerical estimations on the local dimension of attractors. Indeed, by the above result, working like in (1.2) we see that in general systems the scaling behavior of the waiting time gives an upper bound to the local dimension. This can suggest a numerical method to estimate such a dimension (see [9,4]). This remark, hence, gives also an estimation on the speed this upper bound is approached. This is very general and does not require assumptions on the system we consider.

A sequence of sets A_n is said to be strongly Borel–Cantelli if in some sense the preimages $T^{-n}A_n$ covers the space uniformly:

Definition 2.3. Let 1_A be the indicator function of the set A. The sequence of subsets $A_n \subset X$ is said to be a strongly Borel–Cantelli sequence (SBC) if for μ-a.e. $x \in X$ we have as $N \to \infty$

$$\frac{\sum_{n=1}^{N} 1_{T^{-n}A_n}(x)}{\sum_{n=1}^{N} \mu(A_n)} \to 1.$$

As mentioned in the introduction, next theorem says that in Borel–Cantelli systems we have a relation between waiting time and scaling behavior of the measure of small balls (as in the waiting time problem $\tau_{B(y,r_n)}(x) \sim r_n^{-d_\mu(y)}$).

Theorem 2.4. *Assume that there is no atom in X.*

(i) *If every decreasing sequence of balls in X with the same center is BC, then for every y we have*

$$\liminf_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} = 1 \quad \textit{for a.e. } x.$$

(ii) *Suppose that $B(y,r) = \{x\colon\ d(x,y) \leqslant r\}$ is the closed ball. If every decreasing sequence of balls in X with the same center is SBC, then for every y we have*

$$\lim_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} = 1 \quad \textit{for a.e. } x.$$

Proof. (i) Fix $y \in X$. Since y is not an atom, we have $\mu(B(y,r)) \downarrow 0$ as $r \to 0$.

For each positive integer i define $m(i)$ by the smallest positive integer such that

$$\left\{ r > 0 : \frac{1}{m(i)+1} \leqslant \mu\big(B(y,r)\big) < \frac{1}{i} \right\} \neq \emptyset.$$

Then $m(1) \leqslant m(2) \leqslant \cdots$. Choose r_i' as

$$\frac{1}{m(i)+1} \leqslant \mu\big(B\big(y,r_i'\big)\big) < \frac{1}{m(i)}, \qquad i = 1, 2, \ldots.$$

Then there is a sequence i_k' such that $m(i_k') = i_k'$ and $i_k' \geqslant 2i_{k-1}'$. Hence we have

$$\sum_{i=1}^{\infty} \mu\big(B\big(y,r_i'\big)\big) \geqslant \sum_{i=1}^{\infty} \frac{1}{m(i)+1} \geqslant \sum_{k=1}^{\infty} \frac{i_k' - i_{k-1}'}{i_k' + 1} \geqslant \sum_{k=1}^{\infty} \frac{i_k'/2}{i_k' + 1} = \infty.$$

By the BC assumption for almost every x, $T^i x \in B(y, r_i')$ for infinitely many i's. Therefore, for almost every x we have $\tau_{B(y,r_i')}(x) \leqslant i \leqslant 1/\mu(B(y,r_i'))$ infinitely many i's. Hence for almost every x

$$\liminf_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} \leqslant 1$$

for infinitely many i's. The other inequality is obtained by Proposition 2.1.

(ii) Suppose that there is a $y \in X$ such that

$$\limsup_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} > 1$$

for x ranging in a positive measure set. Choose $m(i)$ as in (i). Then there is a strictly increasing sequence $\{i_n\}_{n\geqslant 1}$ such that $m(j) = i_n$ for $i_{n-1} < j \leqslant i_n$. Let

$$r_n = \inf\left\{r > 0 : \frac{1}{i_n + 1} \leqslant \mu\big(B(y,r)\big) < \frac{1}{i_n}\right\}, \quad i = 1, 2, \ldots.$$

Note that

$$\left\{r > 0\colon \frac{1}{i_{n+1}} \leqslant \mu\big(B(y,r)\big) < \frac{1}{i_n + 1}\right\} = \emptyset.$$

Since

$$B(y, r_n) = \{x\colon d(x,y) \leqslant r_n\} = \bigcap_{m\geqslant 1} B\left(y, r_n + \frac{1}{m}\right),$$

we have

$$\mu\big(B(y,r_n)\big) = \lim_{m\to\infty} \mu\left(B\left(y, r_n + \frac{1}{m}\right)\right) \geqslant \frac{1}{i_n + 1}.$$

If $r_n \leqslant r < r_{n-1}$, then we have

$$\frac{1}{i_n + 1} \leqslant \mu\big(B(y,r_n)\big) \leqslant \mu\big(B(y,r)\big) < \frac{1}{i_n}.$$

Let $A_k = B(y, r_1)$ for k, $1 \leqslant k < i_1$ and $A_k = B(y, r_n)$ for $k = i_n$, $n = 1, 2, \ldots$. If $i_n < k < i_{n+1}$ for some $n \geqslant 1$, let

$$A_k = \begin{cases} B(y, r_n) & \text{if } k \leqslant i_n \log(i_{n+1}/i_n), \\ B(y, r_{n+1}) & \text{if } k > i_n \log(i_{n+1}/i_n). \end{cases}$$

Then for $\log(i_{n+1}/i_n) < 1$ we have

$$\text{(2.2)} \qquad \sum_{k=i_n+1}^{i_{n+1}} \mu(A_k) = \sum_{k=i_n+1}^{i_{n+1}} \mu\big(B(y, r_{n+1})\big) < \frac{i_{n+1} - i_n}{i_{n+1}} = 1 - \frac{i_n}{i_{n+1}} < \log \frac{i_{n+1}}{i_n},$$

and for $\log(i_{n+1}/i_n) \geqslant 1$ we have

$$(2.3)\quad \sum_{k=i_n+1}^{i_{n+1}} \mu(A_k) = \sum_{k=i_n+1}^{\lfloor i_n \log(i_{n+1}/i_n)\rfloor} \mu\big(B(y,r_n)\big) + \sum_{k=\lfloor i_n \log(i_{n+1}/i_n)\rfloor+1}^{i_{n+1}} \mu\big(B(y,r_{n+1})\big)$$
$$< \left(\left\lfloor i_n \log\frac{i_{n+1}}{i_n}\right\rfloor - i_n\right)\frac{1}{i_n} + \left(i_{n+1} - \left\lfloor i_n \log\frac{i_{n+1}}{i_n}\right\rfloor\right)\frac{1}{i_{n+1}}$$
$$\leqslant \left(1 - \frac{i_n}{i_{n+1}}\right)\log\frac{i_{n+1}}{i_n} < \log\frac{i_{n+1}}{i_n},$$

where $\lfloor t\rfloor$ is the floor of t.

On the opposite side, if $\log(i_{n+1}/i_n) < 1$, for any $i_n < j \leqslant i_{n+1}$ we have

$$(2.4)\quad \sum_{k=i_n+1}^{j} \mu(A_k) \geqslant \frac{j-i_n}{i_{n+1}+1} = \frac{1-i_n/i_{n+1}}{1+1/i_{n+1}} - \frac{i_{n+1}-j}{i_{n+1}+1}$$
$$> \frac{1-1/e}{1+1/i_{n+1}}\log\frac{i_{n+1}}{i_n} - \frac{i_{n+1}-j}{i_{n+1}+1}$$
$$> \left(1 - \frac{1}{e} - \frac{1}{i_{n+1}}\right)\log\frac{j}{i_n} - \frac{i_{n+1}-j}{i_{n+1}+1}.$$

The second inequality comes from $(1-1/t) > (1-1/e)\log t$ for $0<t<e$. In case of $\log(i_{n+1}/i_n) \geqslant 1$, for any j with $i_n \log(i_{n+1}/i_n) < j \leqslant i_{n+1}$ we have

$$(2.5)\quad \sum_{k=i_n+1}^{j} \mu(A_k) \geqslant \left(\left\lfloor i_n \log\frac{i_{n+1}}{i_n}\right\rfloor - i_n\right)\frac{1}{i_n+1} + \left(j - \left\lfloor i_n \log\frac{i_{n+1}}{i_n}\right\rfloor\right)\frac{1}{i_{n+1}+1}$$
$$\geqslant \left(\frac{i_n}{i_n+1} - \frac{i_n}{i_{n+1}+1}\right)\log\frac{i_{n+1}}{i_n} - \frac{i_{n+1}-j}{i_{n+1}+1}$$
$$> \left(1 - \frac{1}{i_n+1} - \frac{1}{e}\right)\log\frac{j}{i_n} - \frac{i_{n+1}-j}{i_{n+1}+1}.$$

If $\log(i_{n+1}/i_n) \geqslant 1$ and $i_n < j \leqslant i_n \log(i_{n+1}/i_n)$, then we have

$$(2.6)\quad \sum_{k=i_n+1}^{j} \mu(A_k) \geqslant \frac{j-i_n}{i_n+1} = \frac{j/i_n - 1}{1+1/i_n} > \frac{\log(j/i_n)}{1+1/i_n} > \left(1-\frac{1}{i_n}\right)\log\frac{j}{i_n}.$$

Pick x as $\limsup_{r\to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log\mu(B(y,r))} > 1$. Then for some $\delta > 0$ we have infinitely many n's such that there exists an r with $r_n \leqslant r < r_{n-1}$ satisfying

$$\frac{\log \tau_{B(y,r)}(x)}{-\log\mu(B(y,r))} > 1+\delta.$$

As noticed above

$$\frac{1}{i_n+1} \leqslant \mu\big(B(y,r_n)\big) \leqslant \mu\big(B(y,r)\big) < \frac{1}{i_n} \quad \text{for } r_n \leqslant r < r_{n-1}.$$

We have

$$1+\delta < \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} \leqslant \frac{\log \tau_{B(y,r_n)}(x)}{\log i_n}$$

which implies that $\tau_{B(y,r_n)}(x) > {i_n}^{1+\delta}$. Note that if $\tau_{A_{i_n}}(x) = \tau_{B(y,r_n)}(x) > {i_n}^{1+\delta}$, since the A_j are decreasing then $T^j x \notin A_j$ for $i_n \leqslant j \leqslant {i_n}^{1+\delta}$. Thus there are infinitely many n's such that

$$S_{i_n}(x) = S_{\lfloor {i_n}^{1+\delta} \rfloor}(x), \tag{2.7}$$

where

$$S_N(x) = \sum_{n=1}^{N} 1_{A_n} \circ T^n(x) = \sum_{n=1}^{N} 1_{T^{-n}A_n}(x).$$

By the SBC assumption we have

$$\lim_N \frac{S_N(x)}{\sum_{n=1}^{N} \mu(A_n)} = 1 \quad \text{a.e.} \tag{2.8}$$

But we have by (2.2) and (2.3)

$$\sum_{k=1}^{i_n} \mu(A_k) < i_1 \frac{1}{i_1} + \sum_{\ell=1}^{n-1} \log \frac{i_{\ell+1}}{i_\ell} = \log \frac{i_n}{i_1} + 1$$

and by (2.4), (2.5) and (2.6) for $i_m < \lfloor {i_n}^{1+\delta} \rfloor \leqslant i_{m+1}$ we have

$$\begin{aligned}
\sum_{k=i_n+1}^{\lfloor {i_n}^{1+\delta} \rfloor} \mu(A_k) &> \sum_{\ell=n}^{m-1} \left(1 - \frac{1}{i_\ell} - \frac{1}{e}\right) \log \frac{i_{\ell+1}}{i_\ell} + \left(1 - \frac{1}{i_m} - \frac{1}{e}\right) \log \frac{\lfloor {i_n}^{1+\delta} \rfloor}{i_m} \\
&\quad - \frac{i_{m+1} - \lfloor {i_n}^{1+\delta} \rfloor}{i_{m+1} + 1} \\
&> \left(1 - \frac{1}{i_n} - \frac{1}{e}\right) \log \frac{\lfloor {i_n}^{1+\delta} \rfloor}{i_n} - 1 \\
&> \left(1 - \frac{1}{i_n} - \frac{1}{e}\right) \left(\delta \log i_n - \frac{1}{\lfloor {i_n}^{1+\delta} \rfloor}\right) - 1,
\end{aligned}$$

which contradicts (2.7) and (2.8). By SBC assumption, the set of x contradicting (2.7) and (2.8) must have zero measure, hence $\limsup_{r \to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log \mu(B(y,r))} \leqslant 1$ for almost each x. Combining this result with Proposition 2.1 where the opposite inequality is proved, the proof is complete. □

Corollary 2.5. *If every centered, decreasing sequence of balls in X is SBC. Then for every y we have*

$$\liminf_{r \to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log r} = \underline{d}_\mu(y), \quad \limsup_{r \to 0} \frac{\log \tau_{B(y,r)}(x)}{-\log r} = \bar{d}_\mu(y) \quad \textit{for a.e. } x.$$

If every decreasing sequence of balls in X is BC and $d_\mu(y)$ exists, then

$$\liminf_{r\to 0}\frac{\log \tau_{B(y,r)}(x)}{-\log r}=d_\mu(y)\quad \textit{for a.e. } x.$$

The waiting time describes the speed the orbit of a certain point x approaches another point y. Another way to consider this kind of questions is to consider the behavior of limits of the form $\liminf_{n\geqslant 1} n^\beta \cdot d(y, T^n(x))$. Under this approach, in [2], Boshernitzan showed the following quantitative recurrence theorem.

Fact 2.6. *Let (X, Φ, μ, d, T) be a metric measure preserving system. Assume that for some $\alpha > 0$, the Hausdorff α-measure H_α is σ-finite on $X = (X, d)$. Then for almost all $x \in X$ we have*

$$\liminf_{n\geqslant 1} n^\beta \cdot d\big(x, T^n(x)\big) < \infty,\quad \textit{with } \beta=\frac{1}{\alpha}.$$

If, moreover, $H_\alpha(X) = 0$, then for almost all $x \in X$

$$\liminf_{n\geqslant 1} n^\beta \cdot d\big(x, T^n(x)\big) = 0.$$

By the BC properties of the balls we have an analogous quantitative approximation theorem for the waiting time. In [1], for general measure preserving transformations, it is proved that for μ-almost all $x \in X$ one has

$$\liminf_{n\geqslant 1} n^\beta \cdot d\big(T^n x, y\big) = \infty \quad \text{with } \beta > \frac{1}{\underline{d}_\mu(y)}.$$

In the case of Borel–Cantelli systems we have

Theorem 2.7. *Let (X, Φ, μ, d, T) be a metric measure preserving system. If every decreasing sequence of balls in X is BC, then for μ-almost all $x \in X$ one has*

$$\liminf_{n\geqslant 1} n^\beta \cdot d\big(T^n x, y\big) = 0 \quad \textit{with } \beta < \frac{1}{\underline{d}_\mu(y)}.$$

Proof. Fix a $y \in X$. By the definition of $\underline{d}_\mu(y)$, if $\beta < \frac{1}{\underline{d}_\mu(y)}$, then there are infinitely many n_i's such that for any $C > 0$

$$\mu\left(B\left(y, \frac{C}{n_i^\beta}\right)\right) \geqslant \frac{1}{n_i}.$$

Assume that $n_i > 2n_{i-1}$. Then we have

$$\sum_{n=1}^{\infty}\mu\left(B\left(y,\frac{C}{n^{\beta}}\right)\right) \geqslant \sum_{i=1}^{\infty}(n_i - n_{i-1})\mu\left(B\left(y,\frac{C}{n_i^{\beta}}\right)\right)$$
$$\geqslant \sum_{i=1}^{\infty}(n_i - n_{i-1})\frac{1}{n_i}$$
$$\geqslant \sum_{i=1}^{\infty}\frac{1}{2} = \infty.$$

The BC condition implies that $T^n x \in B(y, \frac{C}{n^{\beta}})$ for infinitely many n's. Hence, we have for any $C > 0$

$$\liminf_{n\geqslant 1} n^{\beta} \cdot d\left(T^n x, y\right) \leqslant C. \qquad \square$$

3. WAITING TIME AND SHRINKING TARGETS

In the previous section we supposed that sequences of nested balls have the BC or SBC property and we have seen that this implies strong properties about waiting time behavior. In this section we will see that the two concepts are not equivalent and in some sense "waiting time is weaker than Borel–Cantelli property". We also will see in which direction it is possible to weaken the BC property to have some converse implication. This direction is very natural: indeed we have to consider sequences of nice sets, as balls with the same center (shrinking targets) and such that the sequence of radii decreases in a controlled way. We remark that this kind of general philosophy (weaker mixing assumption, stronger requirements on the sets) is similar to the one which is present in the results of [22]. This remark allows to use results on waiting time to obtain some Borel–Cantelli results in systems like typical Interval Exchanges or Axiom A systems.

Definition 3.1. We say that a system (X, T, μ) has the shrinking target property (STP) if for any $x_0 \in X$ any sequences of balls centered at x_0 has the BC property. Moreover we say that a system has the monotone shrinking target property (MSTP) if any *decreasing* sequences of balls centered at x_0 has the BC property.

In [7] (see also [19]) it is proved that no rotations on the d-dimensional torus have the STP property and moreover, only rotations having some particular arithmetical property have the MSTP.

More precisely, let us introduce some notation: consider $\alpha \in \mathbf{R}^d$, and consider the sup norm $|\alpha| = \sup(|\alpha_1|, \ldots, |\alpha_d|)$. Moreover, for $\alpha \in \mathbf{R}$ let us consider the distance to the closest integer $\|\alpha\| = \inf_{p\in Z} |\alpha - p|$ and its generalization on $\mathbf{R}^d$: $\|\alpha\| = \sup_i \|\alpha_i\|$.

Definition 3.2. Let $d \geqslant 1$, the set

$$\Omega^d = \left\{\alpha \in \mathbf{R}^d\colon \exists C > 0 \text{ s.t. } \forall Q \in \mathbf{Z} - \{0\},\ \|Q\alpha\| \geqslant C|Q|^{-\frac{1}{d}}\right\}$$

is called set of constant type vectors in $\mathbf{R}^d$.

Theorem 3.3. ([7]) *Let $\mathbb{T}^d$ be the d-dimensional torus. Let us consider the system $(\mathbb{T}^d, T_\alpha, \mu)$ where T_α is the translation by a vector α and μ is the Haar measure on $\mathbb{T}^d$. Then we have that $\forall\alpha$ $(\mathbb{T}^d, T_\alpha, \mu)$ does not have the STP, moreover $(\mathbb{T}^d, T_\alpha, \mu)$ has the MSTP if and only if α is of constant type.*

It is known [17] that in dimension 1, for almost every α we have

$$\lim_{r\to 0}\frac{\log \tau_{B(y,r)}(x)}{-\log\mu(B(y,r))}=1 \quad \text{for a.e. } x,$$

moreover for every α

$$\liminf_{r\to 0}\frac{\log \tau_{B(y,r)}(x)}{-\log\mu(B(y,r))}=1 \quad \text{for a.e. } x.$$

Thus the converse of Theorem 2.4 (in particular point (i)) does not hold, even if we restrict to decreasing families of balls having the same centers, as in the shrinking targets framework. (See also [16].)

One of the reasons why not all translations have the STP is that the radii of the balls can decrease in any way. Putting some restriction on this decreasing rate the STP become equivalent to the waiting time problem.

Theorem 3.4. *Let $\{B(y,r_n)\}$ be a decreasing sequence of centered balls such that*

$$\limsup_{n\to\infty}\frac{\log r_n}{-\log n}<\frac{1}{\underline{d}_\mu(y)}.$$

If

$$\liminf_{r\to 0}\frac{\log \tau_{B(y,r)}(x)}{-\log r}=\underline{d}_\mu(y) \tag{3.1}$$

then $x\in\limsup T^{-n}(B(y,r_n))$.

We remark that condition (3.1) above is implied by the waiting time problem and if Eq. (3.1) holds for almost each x, then

$$\mu\big(\limsup T^{-n}\big(B(y,r_n)\big)\big)=1$$

and then such a $\{B(y,r_n)\}$ has the BC property.

Proof. If $\liminf_{r\to 0}\frac{\log \tau_{B(y,r)}(x)}{-\log r}=\underline{d}_\mu(y)$, then there is a sequence $\rho_n\downarrow 0$ such that for each small $\epsilon>0$,

$$x\in\bigcup_{i\leqslant\rho_n^{-\underline{d}_\mu(y)-\epsilon}} T^{-i}\big(B(y,\rho_n)\big) \quad \text{for each } n.$$

If $\limsup_{n\to\infty} \frac{-\log r_n}{\log n} = \frac{1}{d} < \frac{1}{\underline{d}_\mu(y)}$ then, when m is big enough $r_m \geqslant m^{-1/(d-\epsilon)}$. Therefore, if ϵ is small enough such that $\epsilon \leqslant (d - \underline{d}_\mu)/2$, then $\rho_n \leqslant r_{\lfloor \rho_n^{-\underline{d}_\mu - \epsilon} \rfloor}$ eventually, with respect to n. Hence, since r_m is decreasing, we have

$$x \in \bigcup_{i \leqslant \rho_n^{-\underline{d}_\mu - \epsilon}} T^{-i}\big(B(y, \rho_n)\big) \subset \bigcup_{m \leqslant \rho_n^{-\underline{d}_\mu - \epsilon}} T^{-m}\big(B(y, r_m)\big).$$

This is true for infinitely many n and thus $x \in \limsup T^{-n}(B(y, r_n))$. $\square$

Since in Axiom A systems it holds that (3.1) is verified for typical points [10], this implies that in such systems, decreasing sequences of balls, verifying the assumptions in Theorem 3.4 have the BC property. This extend a result of [6] (Theorem 7) which requires the invariant measure to have a smooth density (but has milder requirements on the hyperbolicity of the system).

We remark that if we have stronger assumptions on the behavior of $\tau_{B(y,r)}(x)$ we can include other kind of sequences r_n and generalize the above theorem to the following

Proposition 3.5. *If for some $x, y \in X$ there is a sequence $\rho_n \downarrow 0$ such that $\tau_{B(y,\rho_n)}(x) < f(\rho_n)$, with $f: \mathbf{R}^+ \to \mathbf{R}^+$ be invertible and both f and f^{-1} are strictly decreasing. Then for each decreasing sequence r_n with $r_n > f^{-1}(n)$, it holds $x \in \limsup T^{-n}(B(y, r_n))$.*

Proof. The proof of this proposition is similar to the above proof of Theorem 3.4. Indeed we have that $\rho_n \leqslant f^{-1}(f(\rho_n)) < r_{\lfloor f(\rho_n) \rfloor}$ and the proof follows as before. $\square$

Interval Exchanges are particular bijective piecewise isometries which preserve the Lebesgue measure. We refer to [3] for generalities on this important class of maps. We only remark that Interval exchanges are not hyperbolic and never mixing, hence Borel–Cantelli results about this class of systems cannot come from speed of mixing arguments, as in [22]. These results will come from arithmetic arguments like in the rotation case. Let T be some interval exchange. Let $\delta(n)$ be the minimum distance between the discontinuity points of T^n. We say that T has the property $\tilde{P}$ if it is ergodic and there is a constant C and a sequence n_k such that $\delta(n_k) \geqslant \frac{C}{n_k}$.

Lemma 3.6. (By [3].) *The set of interval exchanges having the property $\tilde{P}$ has full measure in the space of ergodic interval exchange maps.*

Now we can apply the above Proposition 3.5 to obtain the following:

Theorem 3.7. *If T has the property $\tilde{P}$, (hence for typical interval exchanges) there is a constant K such that if r_n is a decreasing sequence and $r_n \geqslant \frac{K}{n}$ eventually when n is big enough, then the sequence $\{B(y, r_n)\}$ has the BC property for almost each $y \in [0, 1]$.*

Proof. In [10] (in the proof of Theorem 9), it is proved that if T has property $\tilde{P}$ it holds that there is a sequence $\rho_n \to 0$ such that $\tau_{B(y,\rho_n)}(x) \leqslant \frac{4}{C\rho_n}$ (C is the constant in the definition of property $\tilde{P}$ and may depend on T) for x and y ranging in positive measure sets B and B'. We now remark that, if A_n is a decreasing sequence of sets, then $A = \limsup T^{-n}(A_n)$ is a forward invariant set, hence in an ergodic system this set has either zero or full measure. By Proposition 3.5 we have that if $y \in B'$ and $r_n > \frac{4}{Cn}$ eventually, then $B \subset \limsup T^{-n}(B(y,r_n))$. This implies $\mu(\limsup T^{-n}(B(y,r_n))) = 1$ and that $\{B(y,r_n)\}$ has the BC property. Choosing $K > \frac{4}{C}$ we have the result for $y \in B'$.

Let us consider a sequence r_n such that $r_n \geqslant \frac{K}{n}$ eventually and y such that $T(y) \in B'$. Since r_n is decreasing this implies that the sequence r_{n-1} is such that $r_{n-1} \geqslant \frac{K}{n}$ eventually (here we set $r_{-1} = 1$) and then by what is proved above $\{B(T(y), r_{n-1})\}$ is a BC sequence.

Now, if $\{B(T(y), r_{n-1})\}$ is a BC sequence of decreasing balls and both y and $T(y)$ are not discontinuity points then also $\{B(y,r_n)\}$ is a BC sequence. This is true because T^{-1} is an isometry from a small neighborhood of $T(y)$ to a small neighborhood of y. This proves the required result for each $y \in B' \cup T^{-1}(B')$ and the result follows by the ergodicity of T^{-1}. $\square$

Hence not only in rotations (by the results cited at the beginning of this section), but also in a full measure set of interval exchanges we have that a large class of decreasing sequences of centered balls have the BC property.

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