Remarks on the Schwarzian derivatives and the invariant quantization by means of a Finsler function

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ABSTRACT

Let (M, F) be a Finsler manifold. We construct a 1-cocycle on Diff(M) with values in the space of differential operators acting on sections of some bundles, by means of the Finsler function F. As an operator, it has several expressions: in terms of the Chern, Berwald, Cartan or Hashiguchi connection, although its cohomology class does not depend on them. This cocycle is closely related to the conformal Schwarzian derivatives introduced in our previous work. The second main result of this paper is to discuss some properties of the conformally invariant quantization map by means of a Sazaki (type) metric on the slit bundle $TM \setminus 0$ induced by F.

1. INTRODUCTION

The notion of *equivariant quantization* has been recently introduced by Duval, Lecomte and Ovsienko in the papers [11,12,17]. The aim is to seek for an equivariant isomorphism between the space of differential operators and the corresponding space of symbols, intertwining the action of a Lie group G acting locally on a manifold M—see also [3,6,10,14,18] for related works. The computation was carried out for the projective group $G = \mathrm{SL}(n + 1, \mathbb{R})$ in [17], and for the conformal group $G = \mathrm{O}(p + 1, q + 1)$, where $p + q = \dim M$, in [11,12]. It turns out that the projectively/conformally equivariant quantization maps make sense on any manifold, not necessarily flat, as shown in [4,6,12]. For instance, the conformally equivariant map has the property that it does not depend on the rescaling of the

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(not necessarily conformally flat) pseudo-Riemannian metric. The existence of such maps induces naturally cohomology classes on the group Diff(M) with values in the space of differential operators acting on the space of tensor fields on M of appropriate type. These classes were given explicitly in [5,7,8], and interpreted as *projective* and *conformal* multi-dimensional analogous to the famous *Schwarzian derivative* (see [5,7,8] for more details).

A Riemannian metric is a particular case of more general functions called *Finsler functions*. A Finsler function, whose definition seems to go back to Riemann, is closely related to the calculus of variation; it arises naturally in many context: Physics, Mathematical ecology, ... (see [1]). The first main result of this paper is to extend one of the two 1-cocycles introduced by the author in [5] as conformal Schwarzian derivatives, to the more general framework of Finsler structures. The 1-cocycle can be built in terms of the Chern, Berwald, Cartan or Hashiguchi connection. All of these connections are considered as generalizations of the well-known Levi-Civita connection for Riemannian structures, thereby the 1-cocycle in question coincide with the conformal 1-cocycle introduced in [5] when the Finsler function F is Riemannian. This 1-cocycle, thus, can be exhibit in four ways accordingly to the used connection. This property contrasts sharply with the case of projective structures where the projectively invariant 1-cocycle has a unique expression (cf. [7]).

The second part of this paper deals with the conformally invariant quantization procedure. As the Finsler function gives rise to a Riemannian metric, say **m**, on the slit bundle $TM \setminus 0$, we shall apply the Duval–Ovsienko's quantization procedure through the metric **m**. That means that we associate with functions on the cotangent bundle of the manifold $TM \setminus 0$, differential operators acting on the space of λ-densities on $TM \setminus 0$. The second main result of this paper is to prove that, for almost all λ, this map cannot descend as an operator acting on the space of λ-densities on M even though the Finsler function is Riemannian.

2. INTRODUCTION TO FINSLER STRUCTURES

We will follow verbatim the notation of [2]. Let M be a manifold of dimension n. A local system of coordinates (x^i), $i = 1, \ldots, n$, on M gives rise to a local system of coordinates (x^i, y^i) on the tangent bundle TM through[2]

$$y = y^i \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n.$$

Definition 2.1. A Finsler structure on M is a function $F : TM \to [0, \infty)$ satisfying the following conditions:

(i) the function F is differentiable away from the origin;
(ii) the function F is homogeneous of degree one in y, viz. $F(x, \lambda y) = \lambda F(x, y)$ for all $\lambda > 0$;

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[2] We will use the convention of summation on repeated indices.

(iii) the $n \times n$ matrix

$$g_{ij} := \frac{1}{2} \frac{\partial^2}{\partial y^i \partial y^j} (F^2),$$

is positive-definite at every point of $TM \setminus 0$.

Example 2.2. (i) Let (M, a) be a Riemannian manifold. The function $F :=$ $\sqrt{a_{ij} y^i y^j}$ satisfies the conditions (i)–(iii). In this case, the Finsler function is called *Riemannian*.

(ii) Let (M, a) be a Riemannian manifold, and α be a closed 1-form on M. We put $F := \sqrt{a_{ij} y^i y^j} + \alpha_i y^i$. One can prove that F satisfies (i)–(iii) if and only if $\|\alpha \cdot \alpha\|_a < 1$ (see, e.g., [2]). In this case, F is Riemannian if and only if the 1-form α is identically zero.

Denote by π the natural projection $TM \setminus 0 \to M$. The pull-back bundle of T^*M with respect to π is denoted by $\pi^*(T^*M)$. The universal property implies that one has a commutative diagram

(2.1)

$$\begin{array}{ccc}
T^*(TM \setminus 0) \xrightarrow{\,u\,} \pi^*(T^*M) & \longrightarrow & T^*M \\
\searrow & \downarrow & \downarrow \\
& TM \setminus 0 \xrightarrow{\,\pi\,} & M
\end{array}$$

The components g_{ij} in (iii) of the definition above are actually the components of a section of the pulled-back bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$.

The geometric object g in (iii) is called *fundamental* tensor; it depends on x and on y as well. The fundamental tensor is nothing but the Riemannian metric if F is Riemannian.

The tensor

(2.2) $A := A_{ijk} dx^i \otimes dx^j \otimes dx^k,$

where $A_{ijk} := F/2 \cdot \partial g_{ij}/\partial y^k$, is called *Cartan* tensor. It is symmetric on its three indices, and defines a section of the pulled-back bundle $(\pi^*(T^*M))^{\otimes 3}$. The Cartan tensor measures whether the Finsler function F is Riemannian or not.

The tensor

(2.3) $\omega := \omega_i dx^i,$

where $\omega_i := \partial F/\partial y^i$, is called *Hilbert* form; it defines a section of the pulled-back bundle $\pi^*(T^*M)$.

Throughout this paper, indices are lowered or raised with respect to the fundamental tensor g. For instance, the tensor whose components are A^l_{ij} stands for the tensor whose components are $A_{ijk} g^{kl}$.

We will also use the following notation: on the manifold $TM \setminus 0$, the index i runs with respect to the basis dx^i or $\partial/\partial x^i$, and the index $\bar{i}$ runs with respect to the basis dy^i or $\partial/\partial y^i$.

3. THE SPACE OF DENSITIES, THE SPACE OF LINEAR DIFFERENTIAL OPERATORS AND THE SPACE OF SYMBOLS

Let M be an oriented manifold of dimension n. Some backgrounds are needed here to present our results. A thorough description of the forthcoming definitions can be found in [11,12].

3.1. The space of densities and the space of linear differential operators

Let (E, M) be a vector bundle over M of rank p. We define the space of λ-densities of (E, M) as the space of sections of the line bundle $| \bigwedge^p E|^{\otimes\lambda}$. Denote by $\mathcal{F}_\lambda(M)$ the space of λ-densities associated with the bundle $T^*M \to M$ and denote by $\mathcal{F}_\lambda(\pi^*(T^*M))$ the space of λ-densities associated with the bundle $\pi^*(T^*M) \to TM \setminus 0$ (see (2.1)). Both $\mathcal{F}_\lambda(M)$ and $\mathcal{F}_\lambda(\pi^*(T^*M))$ are modules over the group of diffeomorphisms $\text{Diff}(M)$: for $f \in \text{Diff}(M)$, $\phi \in \mathcal{F}_\lambda(M)$ and $\varphi \in \mathcal{F}_\lambda(\pi^*(T^*M))$, the actions are given in local coordinates (x, y) by

(3.1) $\qquad f^*\phi = \phi \circ f^{-1} \cdot (J_{f^{-1}})^\lambda,$

(3.2) $\qquad f^*\varphi = \varphi \circ \tilde{f}^{-1} \cdot (J_{f^{-1}})^\lambda,$

where $\tilde{f}$ is a lift of f to TM and $J_f = |Df/Dx|$ is the Jacobian of f.

It is worth noticing that the formlæ above do not depend on the choice of the system of coordinates.

By differentiating these actions, one can obtain the actions of the Lie algebra of vector fields $\text{Vect}(M)$.

Consider now $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$, the space of linear differential operators

(3.3) $\qquad T: \mathcal{F}_\lambda(M) \to \mathcal{F}_\mu(M).$

The action of $\text{Diff}(M)$ on $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ depends on the two parameters λ and μ; it is given by the equation

(3.4) $\qquad f_{\lambda,\mu}(T) = f^* \circ T \circ f^{*-1},$

where f^* is the action (3.1) of $\text{Diff}(M)$ on $\mathcal{F}_\lambda(M)$.

Denote by $\mathcal{D}^2(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ the space of second-order linear differential operators with the $\text{Diff}(M)$-module structure given by (3.4). The space $\mathcal{D}^2(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ is in fact a $\text{Diff}(M)$-submodule of $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$.

Example 3.1. The space of Sturm–Liouville operators $\frac{d^2}{dx^2} + u(x): \mathcal{F}_{-1/2}(S^1) \to \mathcal{F}_{3/2}(S^1)$ on S^1, where $u(x) \in \mathcal{F}_2(S^1)$ is the potential, is a submodule of $\mathcal{D}^2_{-1/2,3/2}(S^1)$ (see [20]).

Likewise, we define $\mathcal{D}(\mathcal{F}_\lambda(\pi^*(T^*M)), \mathcal{F}_\mu(\pi^*(T^*M)))$, the space of linear differential operators

$$(3.5) \qquad U : \mathcal{F}_\lambda(\pi^*(T^*M)) \to \mathcal{F}_\mu(\pi^*(T^*M)),$$

with the action

$$(3.6) \qquad f_{\lambda,\mu}(U) = f^* \circ U \circ f^{*-1},$$

where f^* is the action (3.2) of $\mathrm{Diff}(M)$ on $\mathcal{F}_\lambda(\pi^*(T^*M))$.

By differentiating the actions (3.4), (3.6), one can obtain the actions of the Lie algebra $\mathrm{Vect}(M)$.

The formulæ (3.4) and (3.6) do not depend on the choice of the system of coordinates.

3.2. The space of symbols

The space of symbols $\mathrm{Pol}(T^*M)$ is defined as the space of functions on the cotangent bundle T^*M that are polynomial on fibers. This space is naturally isomorphic to the space $\bigoplus_{p \geqslant 0} S\Gamma(TM^{\otimes p})$ of symmetric contravariant tensor fields on M.

We define a one parameter family of $\mathrm{Diff}(M)$-modules on the space of symbols by

$$\mathrm{Pol}_\delta(T^*M) := \mathrm{Pol}(T^*M) \otimes \mathcal{F}_\delta(M).$$

For $f \in \mathrm{Diff}(M)$ and $P \in \mathrm{Pol}_\delta(T^*M)$, in local coordinates (x^i), the action is defined by

$$(3.7) \qquad f_\delta(P) = f^* P \cdot (J_{f^{-1}})^\delta,$$

where $J_f = |Df/Dx|$ is the Jacobian of f, and f^* is the natural action of $\mathrm{Diff}(M)$ on $\mathrm{Pol}(T^*M)$.

We then have a graduation of $\mathrm{Diff}(M)$-modules given by

$$\mathrm{Pol}_\delta(T^*M) = \bigoplus_{k=0}^{\infty} \mathrm{Pol}_\delta^k(T^*M),$$

where $\mathrm{Pol}_\delta^k(T^*M)$ is the space of polynoms of degree k endowed with the $\mathrm{Diff}(M)$-module structure (3.7).

Remark 3.2. As $\mathrm{Diff}(M)$-modules, the spaces $\mathrm{Pol}_\delta(T^*M)$ and $\mathcal{D}(\mathcal{F}_\lambda(M), \mathcal{F}_\mu(M))$ are not isomorphic (cf. [11,12]).

4. SCHWARZIAN DERIVATIVE FOR FINSLER STRUCTURES

Let (M, F) be a Finsler manifold of dimension n.

4.1. The Chern connection

There exists a unique symmetric connection $D : \Gamma(\pi^*(T^*M)) \to \Gamma(\pi^*(T^*M) \otimes T^*(TM \setminus 0))$ whose Christoffel symbols are given by

$$\gamma_{ij}^k = \frac{1}{2} g^{ks} \left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right) - g^{ks} \left(\frac{N_j^m}{F} A_{msi} + \frac{N_i^m}{F} A_{msj} - \frac{N_s^m}{F} A_{sij} \right),$$

where A_{ijk} are the components of the Cartan tensor (2.2), g_{ij} are the components of the fundamental tensor and the components N_m^k are given by

$$(4.1) \qquad N_m^k := \frac{1}{4} \frac{\partial}{\partial y^m} \left(g^{ks} \left(\frac{\partial g_{si}}{\partial x^j} + \frac{\partial g_{sj}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^s} \right) y^j y^i \right).$$

This connection is called *Chern connection* and has the following properties (see [2]):

(i) the connection 1-forms have no dy dependence;
(ii) the connection D is almost g-compatible, in the sense that $D_s(g_{ij}) = 0$ and $D_{\bar{s}}(g_{ij}) = 2A_{ijs}$;
(iii) in general, the Chern connection is not a connection on M; however, the Chern connection can descend to a connection on M when F is Riemannian. In that case, it coincides with the Levi-Civita connection associated with the metric g.

4.2. A 1-cocycle as a Schwarzian derivative

Since the connection 1-forms of the Chern connection have no dy dependance, the difference between the two connections

$$(4.2) \qquad \ell(f) := f^* \gamma - \gamma,$$

where $f \in \mathrm{Diff}(M)$, transforms under coordinates change as a section of the bundle $\pi^*(T^*M)^{\otimes 2} \otimes \pi^*(TM)$. From the construction of the tensor (4.2), one can easily seen that the map

$$f \mapsto \ell(f^{-1})$$

defines a non-trivial 1-cocycle on $\mathrm{Diff}(M)$ with values in $S\Gamma(\pi^*(T^*M)^{\otimes 2}) \otimes \Gamma(\pi^*(TM))$.

Our main definition is the linear differential operator $\mathcal{A}(f)$ acting from $S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M)$ to $\Gamma(\pi^*(TM)) \otimes \mathcal{F}_\delta(\pi^*(T^*M))$ defined by

$$(4.3) \qquad \mathcal{A}(f)_{ij}^k := f^{*-1} \left(g^{sk} g_{ij} D_s \right) - g^{sk} g_{ij} D_s$$

$$+ (2 - \delta n) \left(\ell(f)_{ij}^k - \frac{1}{n} \mathrm{Sym}_{i,j} \, \delta_i^k \ell(f)_{tj}^t \right)$$

$$+ g^{kl} \left(\mathrm{Sym}_{i,j} \, g_{sj} B_{li}^s - \delta g_{ij} B_{lt}^t \right)$$

$$- f^{-1*} \left(g^{kl} \left(\mathrm{Sym}_{i,j} \, g_{sj} B_{li}^s - \delta g_{ij} B_{lt}^t \right) \right)$$

$$- (2 - \delta n)\left(f^{-1^*} B^k_{ij} - B^k_{ij} - \frac{1}{n} \operatorname{Sym}_{i,j} \delta^k_i \big(f^{-1^*} B^t_{jt} - B^t_{jt} \big) \right),$$

where we have put

$$B^k_{ij} := \big(A^{kr}_i \omega_j + A^{kr}_j \omega_i - A^k_{ij} \omega^r - A^r_{ij} \omega^k - A^k_{is} A^{sr}_j$$
$$- A^k_{js} A^{sr}_i + A^{rk}_u A^u_{ij} \big) u \circ d_r (\log F),$$

to avoid clutter; D is the Chern connection, A_{ijk} are the components of the Cartan tensor (2.2), ω_i are the components of the Hilbert form (2.3), $\ell(f)^k_{ij}$ are the components of the tensor (4.2) and u is the map as in (2.1).

Theorem 4.1. (i) *The map* $f \mapsto A(f^{-1})$, *defines a 1-cocycle on* $\operatorname{Diff}(M)$, *non-trivial for all* $\delta \neq 2/n$, *with values in the space* $\mathcal{D}(S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M), \Gamma(\pi^*(TM))$ $\otimes \mathcal{F}_\delta(\pi^*(T^*M)))$;

 (ii) *The operator* (4.3) *does not depend on the rescaling of the Finsler function* F *by any non-zero positive function on* M;

 (iii) *If* $M := \mathbb{R}^n$ *and* F *is Riemannian such that the metric* g *is the flat metric, this operator vanishes on the conformal group* $O(n + 1, 1)$.

Proof. Let us first explain how the contraction between the tensor $D(P)$ and the tensor g^{-1} is permitted, for all $P \in S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M)$. Indeed, the tensor $D(P)$ should take its values in $S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M) \otimes \Gamma(T^*(TM \setminus 0))$, from the definition of the Chern connection. But taking into account that the tensor P lives in $S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M)$, the components $D_{\bar{s}}(P^{ij}) = 0$, and the components $D_s(P^{ij})$ behave under coordinates change as components of a tensor in $\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M) \otimes \Gamma(\pi^*(T^*M))$. It follows therefore that the contraction between $D_s(P^{ij})$ and g^{sk} makes a sense.

To prove (i) we have to verify the 1-cocycle condition

$$A\big((f \circ h)^{-1} \big) = f^* A\big(h^{-1} \big) + A\big(f^{-1} \big) \quad \text{for all } f, h \in \operatorname{Diff}(M),$$

where f^* is the natural action on

$$\mathcal{D}\big(S\Gamma\big(TM^{\otimes 2} \big) \otimes \mathcal{F}_\delta(M), \Gamma\big(\pi^*(TM) \big) \otimes \mathcal{F}_\delta\big(\pi^*(T^*M) \big) \big).$$

This condition holds because, in the expression of the operator (4.3), ℓ is a 1-cocycle and the rest is a coboundary.

Let us proof that this 1-cocycle is not trivial for $\delta \neq 2/n$. Suppose that there is a first-order differential operator $A^k_{ij} = u^{sk}_{ij} D_s + v^k_{ij}$ such that

(4.4) $A\big(f^{-1} \big) = f^* A - A.$

It follows, by a direct computation, that

$$f^* v^k_{ij} - v^k_{ij} = (2 - \delta n)\left(\ell\big(f^{-1} \big)^k_{ij} - \frac{1}{n} \operatorname{Sym}_{i,j} \delta^k_i \ell\big(f^{-1} \big)^t_{tj} \right)$$

$$+ g^{kl}\left(\text{Sym}_{i,j}\, g_{sj} B^s_{li} - \delta g_{ij} B^t_{lt}\right) - f^*\left(g^{kl}\left(\text{Sym}_{i,j}\, g_{sj} B^s_{li} - \delta g_{ij} B^t_{lt}\right)\right)$$

$$- (2 - \delta n)\left(f^* B^k_{ij} - B^k_{ij} - \frac{1}{n}\text{Sym}_{i,j}\, \delta^k_i \left(f^* B^t_{jt} - B^t_{jt}\right)\right).$$

The right-hand side of this equation depends on the second jet of the diffeomorphism f, while the left-hand side depends on the first jet of f, which is absurd.

For $\delta = 2/n$, one can easily see that the 1-cocycle (4.3) is a coboundary.

Let us prove (ii). Consider a Finsler function $\widetilde{F} = \sqrt{\psi} \cdot F$, where ψ is a non-zero positive function on M. Denote by $\tilde{A}(f)$ the operator (4.3) written by means of the function $\widetilde{F}$. To prove that $\tilde{A}(f) = A(f)$ we proceed as follows: we write down the tensors $\widetilde{D}(P)$ and $\tilde{\ell}(f)$ associated with the Finsler function $\widetilde{F}$ in terms of the tensor $D(P)$ and $\ell(f)$ associated with the Finsler function F, then by replacing their expressions into the explicit formula of the operator (4.3) we show that the constants arising in the expression of the operator (4.3) will annihilate the non-desired terms.

Let us first compare the Chern connections associated with the functions F and $\widetilde{F}$, namely

$$(4.5) \qquad \tilde{\gamma}^k_{ij} = \gamma^k_{ij} + \frac{1}{2\psi}\left(\psi_i \delta^k_j + \psi_j \delta^k_i - \psi_t g^{tk} g_{ij}\right)$$

$$+ \frac{1}{2\psi}\left(A^{kr}_i \omega_j + A^{kr}_j \omega_i - A^k_{ij}\omega^r - A^r_{ij}\omega^k - A^k_{is} A^{sr}_j\right.$$

$$\left. - A^k_{js} A^{sr}_i + A^{rk}_u A^u_{ij}\right)\psi_r,$$

where $\psi_r = \partial\psi/\partial x^r$.

From (4.5), a direct computation gives

$$(4.6) \qquad \widetilde{D}_k P^{ij} = D_k P^{ij} + \frac{1}{2\psi}\left(\text{Sym}_{i,j}\, P^{mi}\left(\psi_m \delta^j_k - \psi_t g^{tj} g_{km}\right) + (2 - n\delta) P^{ij}\psi_k\right),$$

$$+ \text{Sym}_{i,j}\, P^{sj} C^i_{ks} - \delta P^{ij} C^t_{tk},$$

$$\tilde{\ell}(f)^k_{ij} = \ell(f)^k_{ij} + f^{-1*}\left(\frac{1}{2\psi}\left(\text{Sym}_{i,j}\, \psi_i \delta^k_j - \psi_t g^{tk} g_{ij}\right)\right)$$

$$- \frac{1}{2\psi}\left(\text{Sym}_{i,j}\, \psi_i \delta^k_j - \psi_t g^{tk} g_{ij}\right) + f^{-1*} C^k_{ij} - C^k_{ij},$$

where we have put

$$C^k_{ij} := \frac{1}{2\psi}\left(A^{kr}_i \omega_j + A^{kr}_j \omega_i - A^k_{ij}\omega^r - A^r_{ij}\omega^k - A^k_{is} A^{sr}_j - A^k_{js} A^{sr}_i + A^{rk}_u A^u_{ij}\right)\psi_r,$$

to avoid clutter; where P^{ij} are the components of the tensor $P \in S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M)$. By substituting the formulæ (4.6) into (4.3) we get by straightforward computation that $\mathcal{A}(f) = \tilde{\mathcal{A}}(f)$.

Let us prove (iii). Suppose that F is Riemannian, namely $F = \sqrt{g_{ij} y^i y^j}$. In that case, the Cartan tensor A is identically zero. The Chern connection D can descend

to a connection on M and coincides with Levi-Civita connection associated with g. It follows that the operator (4.3) turns into the form

$$(4.7) \qquad \mathcal{A}(f)_{ij}^k := f^{*-1}\big(g^{sk}g_{ij}D_s\big) - g^{sk}g_{ij}D_s$$

$$+ (2 - \delta n)\Big(\ell(f)_{ij}^k - \frac{1}{n}\,\mathrm{Sym}_{i,j}\,\delta_i^k \ell(f)_{tj}^t\Big).$$

As the Chern connection and the fundamental tensor has no y dependance, the operator (4.7) will take its values in $\Gamma(TM) \otimes \mathcal{F}_\delta(M)$, instead of $\Gamma(\pi^*(TM)) \otimes \mathcal{F}_\delta(\pi^*(T^*M))$. The operator (4.7) is nothing but one of the conformally invariant operators introduced in [6]. If, furthermore, M is $\mathbb{R}^n$ and g is the flat metric then the operator (4.7) vanishes on the conformal group $O(n+1,1)$ (cf. [6]). $\quad\square$

Remark 4.2. (i) When the Finsler function F is Riemannian, the formula (4.7) assures that the 1-cocycle $\mathcal{A}$ defined here coincides with one of the conformally invariant operators introduced in [5] as multi-dimensional conformal Schwarzian derivatives. We refer to [5,7,8] for more explanations and details concerning the relation between the classical Schwarzian derivative and the projectively/conformally invariant operators introduced in [5,7,8].

(ii) The tensor whose components are B_{ij}^{kr} coming out in the formula (4.3) is only identically zero for Finsler functions that are Riemannian. Indeed, if $B_{ij}^{kr} \equiv 0$ then a contraction by the inverse of the Hilbert form will give the equality $nA_{ij}^k \equiv 0$.

Now, how can we adjust the 1-cocycle $\mathcal{A}$ in order to take its values in the space $\mathcal{D}(S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M), \Gamma(TM) \otimes \mathcal{F}_\delta(M))$, as for the projectively/conformally invariant 1-cocycles of [5,7,8]? A positive answer to this question can be given by demanding an extra condition on the topology of M. More precisely, suppose that M admits a non-zero vector fields—which is true when the Euler characteristic of M is zero (cf. [9]). One has

Proposition 4.3. *Let $\mathcal{X}$ be a fixed non-zero vector field on M and denote by $\breve{\mathcal{A}}(f)$ the operator obtained by substituting the vector fields $\mathcal{X}$ into $\mathcal{A}(f)$ on the vertical coordinates (namely y). The map*

$$f \mapsto \breve{\mathcal{A}}\big(f^{-1}\big),$$

defines a 1-cocycle on $\mathrm{Diff}(M)$ *with values in the space* $\mathcal{D}(S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M), \Gamma(TM) \otimes \mathcal{F}_\delta(T^*M))$.

Proof. Since the connection 1-forms of the Chern connection do not depend on the direction of dy, the evaluation by the vector fields does not affect the 1-cocycle condition. $\quad\square$

Remark 4.4. As a 1-cocycle, the cohomology class of $\breve{\mathcal{A}}$ does not depend on the chosen vector fields. However, one has a family of operators indexed by a family of non-vanishing vector fields on M.

4.3. The 1-cocycle $\mathcal{A}(f)$ in terms of the Berwald connection

There exists another connection on the bundle $\pi^*(TM) \to TM \setminus 0$ called *Berwald* connection. Its Christoffel symbols are given, in local coordinates, by

$$(4.8) \qquad {}^{\flat}\gamma^i_{jk} = \frac{\partial N^i_j}{\partial y^k},$$

where the components N^i_j are given as in (4.1). Like the Chern connection, this connection has no torsion (see [19]).

As in Section 4.2, we define the following object

$$(4.9) \qquad {}^{\flat}\ell(f) = f^{*\flat}\gamma - {}^{\flat}\gamma,$$

where $f \in \text{Diff}(M)$. As the connection 1-forms of the Berwald connection have no dy dependance, this object is actually a section of the bundle $\pi^*(T^*M)^{\otimes 2} \otimes \pi^*(TM)$.

The 1-cocycle $\mathcal{A}(f)$ can be expressed in terms of the Berwald connection as follows.

(4.10)

$$
\begin{aligned}
{}^{\flat}\mathcal{A}(f)^k_{ij} :=\ & f^{*-1}\left(g^{sk}g_{ij}{}^{\flat}D_s\right) - g^{sk}g_{ij}{}^{\flat}D_s + (2 - \delta n)\left({}^{\flat}\ell(f)^k_{ij} - \frac{1}{n}\,\text{Sym}_{i,j}\,\delta^k_i{}^{\flat}\ell(f)^t_{tj}\right) \\
&+ g^{kl}\left(\text{Sym}_{i,j}\,g_{sj}{}^{\flat}B^s_{li} - \delta g_{ij}{}^{\flat}B^t_{lt}\right) - f^{-1*}\left(g^{kl}\left(\text{Sym}_{i,j}\,g_{sj}{}^{\flat}B^s_{li} - \delta g_{ij}{}^{\flat}B^t_{lt}\right)\right) \\
&- (2 - \delta n)\left(f^{-1*\flat}B^k_{ij} - {}^{\flat}B^k_{ij} - \frac{1}{n}\,\text{Sym}_{i,j}\,\delta^k_i\left(f^{-1*\flat}B^t_{jt} - {}^{\flat}B^t_{jt}\right)\right),
\end{aligned}
$$

where we have put

$$
{}^{\flat}B^k_{ij} := \left({}^{\flat}D_j\left(A^{kr}_i\right)F + A^{kr}_i\omega_j + 2A^{sk}_j\omega_i\right)u \circ d_r(\log F),
$$

to avoid clutter; ${}^{\flat}D$ is the covariant derivative associated with the Berwald connection, A_{ijk} are the components of the Cartan tensor (2.2), ω_i are the components of the Hilbert form (2.3), ${}^{\flat}\ell(f)^k_{ij}$ are the components of the tensor (4.9) and u is the map as in (2.1).

Theorem 4.1 still holds for the operator ${}^{\flat}\mathcal{A}(f)$. For the proof we proceed as in Theorem (4.1). Part (i) is obvious form the construction of the operator. Part (ii) lead us to compare the Berwald connections associated with the Finsler function F and $\sqrt{\psi}F$, respectively. The proof then is a direct computation. Part (iii) results from the fact that, as for the Chern connection, the Berwald connection coincides with the Levi-Civita connection associated with a Riemannian metric when F is Riemannian.

It is worth noticing that, viewed as operators, the operator $\mathcal{A}(f)$ and ${}^{\flat}\mathcal{A}(f)$ are not equal; however, they can be compared via the following definition.

Definition 4.5 (*see* [2]). A Finsler manifold is called a *Landsberg* space if the tensor whose components

$$\dot{A}_{ijk} := -\frac{1}{2} y_l \frac{\partial^2 N_i^l}{\partial y^j \partial y^k},$$

where the components N_i^l are as in (4.1), is identically zero.

Proposition 4.6. *If the Finsler manifold is a Landsberg space, the operators*

$$\mathcal{A}(f) \equiv {}^{\flat}\mathcal{A}(f),$$

for all $f \in \mathrm{Diff}(M)$.

Proof. The proof results from the fact that the Berwald connection and the Chern connection coincides when the Finsler manifold is a Landsberg space (cf. [2, 19]). □

Remark 4.7. An obvious example of a Landsberg space is a Riemannian manifold. More general examples of Finsler manifolds that are Landsberg spaces but not Riemannian are provided in [2]. Proposition 4.6 shows that the operator $\mathcal{A}(f)$ and ${}^{\flat}\mathcal{A}(f)$ does not coincide only for Riemannian manifolds but also for some manifolds little more general.

4.4. The 1-cocycle $\mathcal{A}(f)$ in terms of the Cartan connection

There exists another connection on the bundle $\pi^*(TM) \to TM \setminus 0$ called *Cartan* connection. Its Christoffel symbols are given, in local coordinates, by

$$(4.11) \qquad {}^{\natural}\gamma_{jk}^i = \gamma_{jk}^i + A_{jt}^i \frac{N_k^t}{F}, \qquad {}^{\natural}\gamma_{j\bar{k}}^i = \frac{A_{jk}^i}{F}, \qquad {}^{\natural}\gamma_{\bar{j}\bar{k}}^i = 0, \qquad {}^{\natural}\gamma_{\bar{j}k}^i = \frac{A_{jk}^i}{F},$$

where γ_{jk}^i are the Christoffel symbols of the Chern connection, the components N_j^i are as in (4.1) and A_{jt}^i are defined in the Section 2. In contradistinction with the Chern or Berwald connection, this connection has the properties (cf. [19]):

 (i) it has torsion;
 (ii) the connection 1-forms do depend on the direction of dy.

As in Section 4.2, we define the following object

$$(4.12) \qquad {}^{\natural}\ell(f) = \tilde{f}^* {}^{\natural}\gamma - {}^{\natural}\gamma,$$

where $\tilde{f}$ is a natural lift of $f \in \mathrm{Diff}(M)$. This object takes its values in $\Gamma(\pi^*(TM) \otimes T^*(TM \setminus 0) \otimes T^*(TM \setminus 0))$, in contrast with the previous object $\ell(f)$ defined by means of the Chern or Berwald connection. One takes the image of ${}^{\natural}\ell(f)$ by the

map $\text{Id} \otimes u \otimes u$, where Id is the identity map and u is the map defined in the diagram (2.1). Let us still denote this tensor by $^\natural\ell(f)$.

The 1-cocycle $\mathcal{A}(f)$ can be expressed in terms of the Cartan connection as follows.

$$(4.13) \quad {}^\natural\mathcal{A}(f)^k_{ij} := f^{*-1}\big(g^{sk}g_{ij}u \circ {}^\natural D_s\big) - g^{sk}g_{ij}u \circ {}^\natural D_s$$

$$+ (2 - \delta n)\left({}^\natural\ell(f)^k_{ij} - \frac{1}{n}\,\text{Sym}_{i,j}\,\delta^k_i\,{}^\natural\ell(f)^t_{tj}\right)$$

$$+ g^{kl}\big(\text{Sym}_{i,j}\,g_{sj}{}^\natural B^s_{li} - \delta g_{ij}{}^\natural B^t_{lt}\big)$$

$$- f^{-1*}\big(g^{kl}\big(\text{Sym}_{i,j}\,g_{sj}{}^\natural B^s_{li} - \delta g_{ij}{}^\natural B^t_{lt}\big)\big)$$

$$- (2 - \delta n)\left(f^{-1*\natural}B^k_{ij} - {}^\natural B^k_{ij} - \frac{1}{n}\,\text{Sym}_{i,j}\,\delta^k_i\big(f^{-1*\natural}B^t_{jt} - {}^\natural B^t_{jt}\big)\right),$$

where we have put

$${}^\natural B^k_{ij} := \big(A^{kr}_j\omega_i - A^r_{ij}\omega^k - A^k_{sj}A^{sr}_i + A^{rk}_u A^u_{ij}\big)u \circ d_r(\log F),$$

to avoid clutter; $^\natural D$ is the covariant derivative associated with the Cartan connection, A_{ijk} are the components of the Cartan tensor (2.2), ω_i are the components of the Hilbert form (2.3), $^\natural\ell(f)^k_{ij}$ are the components of the tensor above and u is the map as in diagram (2.1).

Theorem 4.1 still holds for the operator $^\natural\mathcal{A}(f)$.

Remark 4.8. (i) The operator $^\natural\mathcal{A}(f)$ written by means of the Cartan connection coincides with the operator $\mathcal{A}(f)$ written by means of the Chern connection only and only when the Finsler function F is Riemannian. Indeed, the Christoffel symbols of the Cartan connection as defined in (4.11) coincide with the Christoffel symbols of the Chern connection only and only when the components $A^i_{jk} \equiv 0$.

(ii) The operator $\mathcal{A}$ can be expressed in terms of the Hashiguchi connection as well. We omit here its explicit expression. Its worth noticing that the operator $\mathcal{A}$ written by means of the Cartan connection coincides with the operator $\mathcal{A}$ written by means of the Hashiguchi connection when the Finsler manifold is a Landsberg space.

5. CONFORMALLY INVARIANT QUANTIZATION BY MEANS OF SAZAKI TYPE METRIC

5.1. Conformally invariant quantization

Let (N, a) be a Riemannian manifold of dimension m. Denote by ∇ the Levi-Civita connection associated with the metric a. We recall the following theorem.

Theorem 5.1 [12]. *For $m > 2$ and for all $\delta := \mu - \lambda \notin \{2/m, (m+2)/(2m), (m+1)/m, (m+2)/m\}$, there exists an isomorphism*

$$Q^a_{\lambda,\mu} : \text{Pol}^2_\delta(T^*N) \to \mathcal{D}^2\big(\mathcal{F}_\lambda(N), \mathcal{F}_\mu(N)\big),$$

given as follows: for all $P \in \text{Pol}_\delta^2(T^*N)$, *one can associate a linear differential operator given by*

$$(5.1) \quad \mathcal{Q}_{\lambda,\mu}^a(P) = P^{ij}\nabla_i\nabla_j + \left(\beta_1\nabla_iP^{ij} + \beta_2a^{ij}\nabla_i\left(a_{kl}P^{kl}\right)\right)\nabla_j$$

$$+ \beta_3\nabla_i\nabla_jP^{ij} + \beta_4a^{st}\nabla_s\nabla_t\left(a_{ij}P^{ij}\right) + \beta_5R_{ij}P^{ij} + \beta_6Ra_{ij}P^{ij},$$

where P^{ij} *are the components of* P *and* R_{ij} *(respectively* R) *are the Ricci tensor components (respectively the scalar curvature) of the metric* a; *constants* $\beta_1, \ldots, \beta_6$ *are given by*

$$(5.2)$$
$$\beta_1 = \frac{2(m\lambda + 1)}{2 + m(1 - \delta)},$$

$$\beta_2 = \frac{m(2\lambda + \delta - 1)}{(2 + m(1 - \delta))(2 - m\delta)},$$

$$\beta_3 = \frac{m\lambda(m\lambda + 1)}{(1 + m(1 - \delta))(2 + m(1 - \delta))},$$

$$\beta_4 = \frac{m\lambda(m^2\mu(2 - 2\lambda - \delta) + 2(m\lambda + 1)^2 - m(m + 1))}{(1 + m(1 - \delta))(2 + m(1 - \delta))(2 + m(1 - 2\delta))(2 - m\delta)},$$

$$\beta_5 = \frac{m^2\lambda(\lambda + \delta - 1)}{(m - 2)(1 + m(1 - \delta))},$$

$$\beta_6 = \frac{(m\delta - 2)}{(m - 1)(2 + m(1 - 2\delta))}\beta_5.$$

The quantization map $\mathcal{Q}_{\lambda,\mu}^a$ *has the following properties*:

(i) *it does not depend on the rescaling of the metric* a;
(ii) *if* $N = \mathbb{R}^m$ *and it is endowed with a flat conformal structure, this map is unique, equivariant with respect to the action of the group* $O(p + 1, q + 1) \subset \text{Diff}(\mathbb{R}^m)$, *where* $p + q = m$.

5.2. A Sazaki type metric on $TM \setminus 0$

Let (M, F) be a Finsler manifold of dimension n. The Finsler function F gives rise to a Sazaki (type) metric **m** on the manifold $TM \setminus 0$, given in local coordinates (x^i, y^i) by

$$(5.3) \quad \mathbf{m} := \left(g_{ij} + g_{st}\frac{N_i^s}{F}\frac{N_j^t}{F}\right)dx^i \otimes dx^j + g_{is}\frac{N_j^s}{F^2}dy^i \otimes dx^j$$

$$+ g_{is}\frac{N_j^s}{F^2}dx^j \otimes dy^i + \frac{g_{ij}}{F^2}dy^i \otimes dy^j,$$

where g_{ij} are the components of the fundamental tensor and N_j^i are given as in (4.1).

Remark 5.2. Let us emphasize the difference between the geometric objects **m** and g. The metric **m** is a Riemannian metric on the bundle $TM \setminus 0$, whereas g

defines a section of the bundle $\pi^*(T^*M) \otimes \pi^*(T^*M)$. When (and only when) F is Riemannian, g has no y dependence and can then descend to a metric on the manifold M.

Lemma 5.3. *Any tensor P on $S\Gamma(TM^{\otimes 2})$ can be extended to a tensor on $S\Gamma(T(TM \setminus 0)^{\otimes 2})$, given in local coordinates (x^i, y^i) by*

$$(5.4) \qquad \widetilde{P} = P^{ij} \frac{\partial}{\partial x^i} \frac{\partial}{\partial x^j} - P^{it} N_t^j \frac{\partial}{\partial x^i} \frac{\partial}{\partial y^j} - P^{js} N_s^i \frac{\partial}{\partial y^i} \frac{\partial}{\partial x^j}$$
$$+ \left(P^{st} N_s^i N_t^j + P^{ij} F^2 \right) \frac{\partial}{\partial y^i} \frac{\partial}{\partial y^j},$$

where P^{ij} are the components of the tensor P and N_j^i are given as in (4.1).

Proof. The objects N_j^i, $\partial/\partial x^i$ and $\partial/\partial y^i$, behave under coordinates change as follows: for local changes on M, say (x^i) and their inverses $(\tilde{x}^i)$, one has

$$\widetilde{N}_j^i = \frac{\partial x^t}{\partial \tilde{x}^j} \frac{\partial \tilde{x}^i}{\partial x^s} N_t^s + \frac{\partial \tilde{x}^i}{\partial x^t} \frac{\partial^2 x^t}{\partial \tilde{x}^s \partial \tilde{x}^j} \tilde{y}^s,$$

$$(5.5) \qquad \frac{\partial}{\partial \tilde{x}^i} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial}{\partial x^p} + \frac{\partial^2 x^j}{\partial \tilde{x}^i \partial \tilde{x}^s} \tilde{y}^s \frac{\partial}{\partial y^j},$$

$$\frac{\partial}{\partial \tilde{y}^i} = \frac{\partial x^p}{\partial \tilde{x}^i} \frac{\partial}{\partial y^p}.$$

By substituting these formulas into (5.4) we see that the geometrical object $\widetilde{P}$ behaves under coordinates change as a symmetric twice-contravariant tensor field on $TM \setminus 0$. □

Lemma 5.4. *The space $\mathcal{F}_\lambda(M)$ can be identified with the subspace of $\mathcal{F}_{\lambda/2}(TM \setminus 0)$ with elements of the form*

$$(5.6) \qquad \phi(x) \left(dx^1 \wedge \cdots \wedge dx^n \wedge dy^1 \wedge \cdots \wedge dy^n \right)^{\lambda/2}.$$

Proof. The 1-forms dx^i and dy^i behave under coordinates change in $TM \setminus 0$ as follows: for local coordinates change on M, say (x^i) and their inverses $(\tilde{x}^i)$, one has

$$d\tilde{y}^i = \frac{\partial \tilde{x}^i}{\partial x^p} dy^p + \frac{\partial^2 \tilde{x}^i}{\partial x^s \partial x^t} y^s dx^t,$$

$$d\tilde{x}^i = \frac{\partial \tilde{x}^i}{\partial x^p} dx^p.$$

By substituting these formulas into (5.6) we see that the geometrical object (5.6) behaves under coordinates change as a tensor density of degree $\lambda/2$ on $TM \setminus 0$. □

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The second main result of this paper is to give some properties of the conformally invariant quantization map (5.1) by means of the Sazaki (type) metric (5.3). Namely

Theorem 5.5. *For any lift of* $P \in \text{Pol}^2_{2\delta}(T^*M)$, *the quantization map*

$$(5.7) \qquad \mathcal{Q}^{\mathbf{m}}_{\lambda,\mu} : \text{Pol}^2_\delta\big(T^*(TM \setminus 0)\big) \to \mathcal{D}^2\big(\mathcal{F}_\lambda(TM \setminus 0), \mathcal{F}_\mu(TM \setminus 0)\big),$$

has the property that the operator $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})$ *cannot descend as an operator acting on the space of differential operators on tensor densities on* M. *However, when the Finsler function* F *is Riemannian, viz.* $F = \sqrt{\mathrm{g}_{ij} y^i y^j}$, *three properties are distinguished:*

(i) *If* $\lambda \neq 0$ *and* $\mu \neq 1$, *the operator* $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})$ *cannot descend;*

(ii) *If* $\lambda = 0$ *and* $\mu \neq 1$ *(or* $\lambda \neq 0$ *and* $\mu = 1$), *the operator* $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})$ *can descend only if* g *is the Euclidean metric;*

(iii) *If* $(\lambda, \mu) = (0, 1)$, *the operator* $\mathcal{Q}^{\mathbf{m}}_{0,1}(\widetilde{P})$ *can descend, given explicitly in terms of the metric* g *by*

$$P^{ij} \, {}^{\mathrm{g}}\nabla_i \, {}^{\mathrm{g}}\nabla_j + {}^{\mathrm{g}}\nabla_i\big(P^{ij}\big) \, {}^{\mathrm{g}}\nabla_j,$$

for all $P \in \text{Pol}^2_{2\delta}(T^*M)$.

Proof. First, Lemmas 5.3 and 5.4 assure that the operator $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})|_{\mathcal{F}_{2\lambda}(M)}$ is a well-defined operator.

Suppose that F is not Riemannian. In that case, the metric $\mathbf{m}$ (5.3) depends, in any local coordinates (x^i, y^i), on x and on y as well. It is easy to see from the map (5.1) that $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})|_{\mathcal{F}_{2\lambda}(M)}$ depends on y. The crucial point of the proof is when $(\lambda, \mu) = (0, 1)$. In that case, let us exhibit the operator $\mathcal{Q}^{\mathbf{m}}_{0,1}(\widetilde{P})|_{\mathcal{F}_0(M)}$ in local coordinates (x^i, y^i), namely

$$(5.8) \qquad \mathcal{Q}^{\mathbf{m}}_{0,1}(\widetilde{P})|_{\mathcal{F}_0(M)} = P^{ij} \partial_{x^i} \partial_{x^j} + \big(\partial_{x^i} P^{ij} - P^{sj} \partial_{y^i}(N^i_s)\big)\partial_{x^j},$$

where the components N^i_s are given as in (4.1). Since F is not Riemannian, the components $\partial_{y^i}(N^i_s)$ still have y dependance.

Suppose now that F is Riemannian. Let us prove (i) and (ii) simultaneously. Suppose, without lost of generality, that $M = \mathbb{R}^n$ and g is the Euclidean metric. Namely, $\mathrm{g} := \delta_{ij} \, dx^i \otimes dx^j$, where (x^i) are local coordinates on $\mathbb{R}^n$. To achieve the proof, we will express the quantization map (5.7) in these local coordinates, and prove that it has y dependance.

In the coordinates mentioned above, the Christoffel symbols of the Levi-Civita connection associated with the metric **m** (5.3) are given by

$$\Gamma^k_{ij}{}^{\mathbf{m}} = 0, \qquad \Gamma^k_{i\bar{j}}{}^{\mathbf{m}} = 0,$$

$$(5.9) \qquad \Gamma^k_{\bar{i}\bar{j}}{}^{\mathbf{m}} = 0, \qquad \Gamma^{\bar{k}}_{ij}{}^{\mathbf{m}} = 0,$$

$$\Gamma^{\bar{k}}_{i\bar{j}}{}^{\mathbf{m}} = 0, \qquad \Gamma^{\bar{k}}_{\bar{i}\bar{j}}{}^{\mathbf{m}} = -\frac{1}{F}\left(\omega_j \delta^k_i + \omega_i \delta^k_j - \omega^k \delta_{ij}\right),$$

where ω_i are the components of the Hilbert form (2.3).

It follows that the contraction between the tensor $\widetilde{P} \in \mathrm{Pol}_\delta(T^*(TM \setminus 0))$ with the Ricci tensor Ric of the metric **m** is given by the equations

$$(5.10) \qquad \begin{array}{ll} R_{ij}P^{ij} = 0, & R_{\bar{i}j}P^{\bar{i}j} = 0, \\[6pt] R_{i\bar{j}}P^{i\bar{j}} = 0, & R_{\bar{i}\bar{j}}P^{\bar{i}\bar{j}} = (n-2)\omega_i\omega_j P^{ij} + (2-n)g_{ij}P^{ij}, \end{array}$$

where R_{ij} are the components of the Ricci tensor Ric.

A direct computation proves that the scalar curvature of the metric **m** is equal to

$$(5.11) \qquad 3n - n^2 - 2.$$

Now we are in position to express the quantization map (5.7) in local coordinates. Using the explicit formulæ of the connection (5.9), the formulæ (5.10) and (5.11), we will see that the quantization map $\mathcal{Q}_{\lambda,\mu}(\widetilde{P})$ restricted to $\mathcal{F}_{2\lambda}(M)$ turns out to be of the form

$$P^{ij}\partial_{x^i}\partial_{x^j} + \left(\beta_1\partial_{x^i}P^{ij} + 2\beta_2 g^{ji}g_{st}\partial_{x^i}P^{st}\right)\partial_{x^j}$$

$$+ \beta_3\partial_{x^i}\partial_{x^j}P^{ij} + 2\beta_4 g_{ij}\partial_{x^s}\partial_{x^t}P^{ij} + \left((3n-n^2-2)\beta_6 + (2-n)\beta_5\right)g_{ij}P^{ij}$$

$$+ 3\frac{n^2\lambda(\mu-1)}{1+2n}\omega_i\omega_j P^{ij},$$

where constants $\beta_1, \ldots, \beta_6$ are given as in (5.2). As the constant $n^2\lambda(\mu-1)/(1+2n)$ does not vanish when $\lambda \neq 0$ and $\mu \neq 1$, the quantization map still have y dependence, and then does not take its values in $\mathcal{F}_{2\mu}(M)$. Part (i) is proven. To achieve the proof of part (ii), let us consider a non-Euclidean metric g. In local coordinates (x^i, y^i), the operator $\mathcal{Q}^{\mathbf{m}}_{\lambda,\mu}(\widetilde{P})$ restricted to $\mathcal{F}_{2\lambda}(M)$ will have a component of the form

$$(1-\beta_1)\left(\frac{1}{F^2}\partial_{x^j}(N^u_s)N^v_i g_{uv}g^{ks}P^{ij}\right)\partial_{x^k},$$

which has y dependance. The constant $1 - \beta_1$ does not vanish under the condition $\lambda = 0$ and $\mu \neq 1$ (or $\lambda \neq 0$ and $\mu = 1$). Part (ii) is proven.

Let us prove part (iii). For $(\lambda, \mu) = (0, 1)$, the quantization map, written in any local coordinates (x^i, y^i), has the form

$$(5.12) \qquad P^{ij} \partial_{x^i} \partial_{x^j} + \left(\partial_{x^i} P^{ij} - \frac{1}{2} g^{uv} \partial_{x^i} g_{uv} P^{ij} \right) \partial_{x^j}.$$

Using the following formulæ:

$$P^{ij} \partial_{x^i} \partial_{x^j} = P^{ij}\, {}^g\nabla_i\, {}^g\nabla_j + {}^g\Gamma^i_{jk} P^{jk}\, {}^g\nabla_i,$$

$$\partial_{x^i} P^{ij} \partial_{x^j} = {}^g\nabla_i P^{ij}\, {}^g\nabla_j + {}^g\Gamma^i_{jk} P^{jk}\, {}^g\nabla_i + P^{ij}\, {}^g\Gamma_j\, {}^g\nabla_i,$$

we see that the formula (5.12) turns into the form

$$P^{ij}\, {}^g\nabla_i\, {}^g\nabla_j + {}^g\nabla_i P^{ij}\, {}^g\nabla_j.$$

The formula above has certainly no y dependance. Part (iii) is proven. $\quad\square$

Remark 5.6. Theorem above shows that the quantization map $\mathcal{Q}^{\mathrm{m}}_{\lambda,\mu}(\widetilde{P})|_{\mathcal{F}_{2\lambda}(M)}$ does not reproduce the quantization map $\mathcal{Q}^{\mathrm{g}}_{2\lambda,2\mu}(P)$ even if F is Riemannian.

6. OPEN PROBLEMS

(1) Following [5], there exist two 1-cocycles on the group Diff(M), say c_1, c_2, with values in $\mathcal{D}(S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M), \Gamma(TM) \otimes \mathcal{F}_\delta(M))$ and $\mathcal{D}(S\Gamma(TM^{\otimes 2}) \otimes \mathcal{F}_\delta(M), \mathcal{F}_\delta(M))$, respectively, that are conformally invariant; namely, they depend only on the conformal class of the Riemannian metric. These 1-cocycles were introduced in [5] as conformal multi-dimensional Schwarzian derivatives. In this paper, we have introduce the 1-cocycle $\mathcal{A}$ (see (4.3)) as the Finslerian analogous of the 1-cocycle c_1; however, the computation to extend the 1-cocycle c_2 seems to be more intricate.

(2) We ask the following question: Is there a map

$$\mathcal{Q} : \mathrm{Pol}\big(T^*(TM \setminus 0)\big) \otimes \mathcal{F}_{\mu-\lambda}\big(\pi^*(T^*M)\big) \to \mathcal{D}\big(\mathcal{F}_\lambda\big(\pi^*(T^*M)\big), \mathcal{F}_\mu\big(\pi^*(T^*M)\big)\big),$$

having the following properties:
   (i) it does not depend on the rescaling of the Finsler function by a non-zero positive function on M;
   (ii) it coincides with the Duval–Ovsienko's conformally invariant map when F is Riemannian?

We believe that a positive answer to this question will probably produce the 1-cocycle c_2 discussed in part (1). It should be stressed, however, that the quantization map and the 1-cocycle c_2 may not exist in the *generic* Finsler setting. Such a situation happened in Conformal Geometry where a large number of invariant differential operators do not generalize to arbitrarily "curved" manifolds. For example, the power of the Laplacian, Δ^k, where k is an integer, are the unique

differential operators acting on the space of tensor densities of appropriate weights, that are invariant under the action of the conformal group (see [13,16]); however, their curved analogues do not exist when the dimension of the manifold $\dim M$ is greater than 4 and even, and $k > \dim M/2$, as recently proven in [15].

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