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Travelling Wave Solutions For A Generalized Fisher Equation

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1. INTRODUCTION

The scalar nonlinear diffusion equation in one space variable

$$u_t = au_{xx} + f(u), \tag{1}$$

where $u = u(x, t)$, a is a constant, and $f(u)$ a nonlinear function, arises in several applications; see [1] and [2] for information and references. For a model in population genetics, Fisher in 1936 proposed the function

$$f(u) = b(u - u^2), \tag{2}$$

where b is a constant. By changing the scales of t and x, Eq. (1), in which $f(u)$ is given by (2), may be written in the parameter-free form

$$u_t = u_{xx} + u - u^2. \tag{3}$$

A travelling wave solution of (3), of interest in applications, is obtained by setting $u(x, t) = u(z)$, where $z = x - ct$, the constant c being the wave speed. The wave profile $u(z)$ satisfies the ordinary nonlinear differential equation

$$u'' + cu' + u - u^2 = 0, \qquad u' = du/dz. \tag{4}$$

Ablowitz and Zeppetella have obtained in [2] the exact general solution of (4) for the wave speed $c = 5/\sqrt{6}$. This particular value was determined when a Laurent expansion for the solution of (4) was tried. The general solution for this speed was then obtained by transforming (4) into an autonomous equation in which the first derivative is missing, so that the equation is immediately integrable.

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2. The Generalized Equation

As the function $f(u)$ given by (2) has not been derived rigorously, but merely chosen as being the simplest nonlinear function such that $f(0) = f(1) = 0$, it is quite possible that, if the second term of $f(u)$ be replaced by u^n, where $n > 1$, a more suitable representation of the process modelled by (1) is thereby obtained. Thus, in place of (3) we shall here consider the more general diffusion equation

$$u_t = u_{xx} + u - u^n, \qquad n \in (1, \infty). \tag{5}$$

The wave profile $u(z)$ is then a solution of

$$u'' + cu' + u - u^n = 0. \tag{6}$$

Our sole object in this paper is to present the exact general solution by quadrature for a certain class of (6), namely, that for which the wave speed c is related to the exponent n by means of the relation

$$c = (n + 3)/(2n + 2)^{1/2}, \qquad c \in (2, \infty). \tag{7}$$

For $n = 2$, this yields the value of c for which Ablowitz and Zeppetella gave the solution of (4), so that their solution is a special case of ours.

The general solution of the special class of (6) being considered will be given in terms of a hyperelliptic integral depending on the value of n; for certain values of n (such as 2 and 3), this integral can be evaluated in terms of elliptic integrals.

3. The Solution

It may be readily verified that a first integral of Eq. (6) when (7) is fulfilled is given in terms of a parameter ξ by

$$(n + 1)\, u^m(\xi^2 - 1)^p + (n - 1)\, \mathscr{I}(\xi, n) = A \tag{8}$$

and

$$u^\nu \xi + \left\{\frac{n+1}{2}\right\}^{1/2} u' + u = 0, \tag{9}$$

where

$$\mathcal{I}(\xi, n) = \int (\xi^2 - 1)^\sigma \, d\xi, \tag{10}$$

$$m = \frac{n-1}{2}, \qquad \rho = \frac{n-1}{2n+2}, \qquad \nu = \frac{n+1}{2}, \qquad \sigma = \frac{-n-3}{2n+2},$$

and A is a constant of integration.

The method of determining relation (7) and the above first integral was indicated at the end of Section 4 of [3].

Equations (8) and (9) give the family of trajectories in the (u', u) phase plane in terms of ξ. Integrating now Eqs. (8) and (9) parametrically, we ultimately find for the complete integral of Eq. (6) when (7) is fulfilled the expressions

$$u(\xi) = (\xi^2 - 1)^s \left\{ \frac{A - (n-1)\,\mathcal{I}(\xi, n)}{n+1} \right\}^r,$$

and

$$z(\xi) = B - \frac{(2n+2)^{1/2}}{(n-1)} \log |A - (n-1)\,\mathcal{I}(\xi, n)|,$$

where

$$r = \frac{2}{n-1}, \qquad s = -\frac{1}{n+1},$$

and B is a constant of integration.

Now, in order to facilitate the evaluation of the integral $\mathcal{I}(\xi, n)$, we transform it into a standard hyperelliptic form by making in (10) the substitution

$$\xi = \pm \{v^{2n+2} + 1\}^{1/2},$$

which yields

$$\mathcal{I}(\xi, n) = \pm (n+1) \int v^{n-2} \{v^{2n+2} + 1\}^{-1/2} \, dv.$$

For $n = 2$, this integral is evaluable in terms of Legendre's elliptic integral of the first kind, and the resulting solution is of course equivalent to that given in [2] in terms of the Weierstrass $\mathscr{P}$-function.

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