ELSEVIER
Available online at www.sciencedirect.com
ScienceDirect
Topology and its Applications 154 (2007) 1581–1591
Topology and its Applications
www.elsevier.com/locate/topol

Filament sets and homogeneous continua

Janusz R. Prajs [a,b,1], Keith Whittington [c,*]

[a] *California State University Sacramento, Department of Mathematics and Statistics, 6000 J street, Sacramento, CA 95819, USA*
[b] *University of Opole, Institute of Mathematics, ul. Oleska 48, 45-052 Opole, Poland*
[c] *University of the Pacific, Stockton, CA 95211, USA*

Received 26 January 2005; accepted 4 December 2006

Abstract

New tools are introduced for the study of homogeneous continua. The subcontinua of a given continuum are classified into three types: *filament*, *non-filament*, and *ample*, with ample being a subcategory of non-filament. The richness of the collection of ample subcontinua of a homogeneous continuum reflects where the space lies in the gradation from being locally connected at one extreme to indecomposable at another. Applications are given to the general theory of homogeneous continua and their hyperspaces.

MSC: primary 54F15; secondary 54H15

Keywords: Ample; Continuum; Homogeneous; Filament

This is the introductory paper in a series of articles [18,17,19,16] which develop and apply new tools in continuum theory based on the concepts of *filament* and *ample* subcontinua. These techniques are especially suited to studying homogeneous continua as well as a larger, well known, class of spaces, which we refer to as Kelley continua.

The topic of homogeneous continua has been a classic since 1920, when Knaster and Kuratowski asked about such continua in the plane [9]. Homogeneous continua form a natural class of spaces, with remarkable examples such as the pseudo-arc, solenoids, and the Menger universal curve, to mention just a few. Interest in homogeneous continua over the decades has opened new areas of mathematical research going far beyond homogeneity. Despite the obvious attention, progress in classifying homogeneous continua could be characterized as slow with sporadic bursts of activity resulting from the introduction of new tools such as the aposyndetic decomposition theorem of Jones [7], its improvements by Rogers [20,21,23], and the Theorem of Effros [4, Theorem 2]. The present case appears to be another such event. The tools which we offer also connect with another, quite different, area of intense interest in topology: the hyperspace of subcontinua of a given continuum.

In a continuum X, the subcontinua fall naturally into two mutually exclusive types which we call *filament* and *non-filament*. A subcontinuum F is filament if it has a neighborhood in which the component containing F has empty

* Corresponding author.
E-mail addresses: prajs@csus.edu (J.R. Prajs), kwhittin@pacific.edu (K. Whittington).
[1] The first named author was supported by the National Science Foundation grant DMS-0405374, and an RCA assigned time award at California State University, Sacramento.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2006.12.005

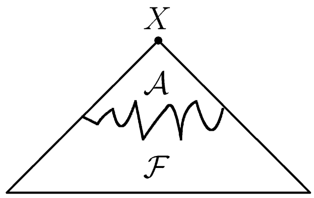

Fig. 1.

interior. For instance, in the cone over the Cantor set, the filament subcontinua are precisely those which do not contain the vertex.

We call the union of all filament continua containing a point a *filament composant* of the space. There is no distinction between composants and filament composants when X is indecomposable, but for continua which are decomposable yet not locally connected, the difference is important. The cone over the Cantor set has only one composant, yet each fiber (minus the vertex) is a filament composant. We can see that the filament composants in this example somewhat resemble the composants of an indecomposable continuum. In particular, each is first-category in the space, being a countable union of filament continua. As another example, the composants of a solenoid embed naturally into the filament composants of the product of the solenoid with a circle, yet again, the latter has only one composant.

In the examples given above, the non-filament proper subcontinua have a stronger property: each is contained in the interior of slightly larger continua. We call such continua (as well as the entire space) *ample*. In Kelley continua (i.e. continua with property 3.2 of [8]), and in particular, homogeneous continua, a subcontinuum is ample if and only if it is non-filament. In such spaces, the richness of the collection of ample subcontinua reflects where the space lies in the gradation from being locally connected at one extreme to indecomposable at another. The space is indecomposable if and only if the entire space is the only ample subcontinuum, and locally connected if and only if every subcontinuum is ample.

The duality between filament and ample is particularly striking in a homogeneous continuum X. Fig. 1 is a schematic picture of the hyperspace, $C(X)$, of subcontinua of X. The symbol $\mathcal{A}$ denotes the set of ample continua of X, and $\mathcal{F}$ the filament continua. Every subcontinuum of a filament continuum is filament, and every continuum containing an ample is ample. Thus, if X is non-locally connected, and we follow an order arc in $C(X)$ from a singleton up to X, the points will at first all be filament continua contained in a filament composant of X and then, beginning from some point, all become ample. The set $\mathcal{A}$ is a compact absolute retract, and $\mathcal{F}$ is a connected open subset of $C(X)$. The space X is locally connected if and only if $\mathcal{A}$ consumes the entire figure. The space is indecomposable if and only if $\mathcal{F}$ consumes everything except X itself. The space is non-locally connected if and only if every singleton is an element of $\mathcal{F}$. The elements of $\mathcal{A}$ are precisely the points at which $C(X)$ is locally connected. Every ample continuum contains a minimal ample continuum, and the latter lie in the boundary of $\mathcal{F}$.

The paper is organized into three parts going from general to specific, starting with considerations for all continua, then proceeding to Kelley continua, and concluding with homogeneous continua.

1. Filament sets

A *continuum* is a compact, connected, nonempty metric space. If X is a continuum, $C(X)$ and 2^X will denote the hyperspaces consisting, respectively, of all subcontinua of X and all nonempty compact subsets of X, under the Hausdorff metric.

All metrics will be denoted by d, with context determining whether it is the Hausdorff metric. If K is a subset of X and ε is a positive number, $N(K,\varepsilon)$ is the open set consisting of all points of distance less than ε from some point of K. A *neighborhood* of K is any subset of X containing K in its interior.

Our definition of local connectedness follows the terminology of Kuratowski [11, p. 227]. A space is *locally connected* at a point p if every neighborhood of p contains a connected neighborhood of p. (Since we have made no requirement that the neighborhood be open, many authors would refer to this as being *connected im kleinen* at p.) The space is *locally arcwise connected* at p if every neighborhood of p contains an arcwise connected neighborhood of p [11, p. 252]. A space is *locally connected* if it is locally connected at each of its points.

Let X be a topological space and A and B subsets of X such that $A \subset B$ and A is connected. The symbol $\mathrm{Cnt}_B(A)$ denotes the component of B containing A. If $A = \{a\}$, a singleton, we use $\mathrm{Cnt}_B(a)$ for $\mathrm{Cnt}_B(\{a\})$. Similarly, if A is a continuum, $\mathrm{Ctu}_B(A)$ will denote the *constituant* of B containing A; $\mathrm{Ctu}_B(A)$ equals the union of all continua in B that meet A.

If X is a homogeneous continuum, then for every positive ε, there is a number δ, called an *Effros number* for ε, such that for each pair of points with $d(x, y) < \delta$, there is a homeomorphism $f : X \to X$ that carries x to y and such that $d(z, f(z)) < \varepsilon$ for each $z \in X$. This is called the *Effros Theorem*. It follows from the more general statement that for each $x \in X$, the evaluation map, $g \mapsto gx$, from the homeomorphism group onto X is open. The latter follows from [4, Theorem 2]. (See also [25, Theorem 3.1].)

Definition 1. If X is a continuum, a subcontinuum K is called *filament* (in X) provided there exists a neighborhood N of K such that $\mathrm{Cnt}_N(K)$ has empty interior. A set $S \subset X$ is called *filament* provided every continuum contained in S is filament in X. The complement of a filament set will be called a *co-filament* set.

Definition 2. If X is a continuum, a subcontinuum K is called *ample* if for every open set U containing K there is a continuum L such that $K \subset \mathrm{Int}(L) \subset L \subset U$.

Obviously, the ample continua and the filament ones form disjoint classes.

Definition 3. If $p \in X$, the *filament composant*[2] of X determined by p, denoted $\mathrm{Fcs}(p)$, is the union of all filament continua in X containing p.

Remark 1.1. The concepts of a filament continuum (set), ample continuum, and filament composant are topological, and thus are preserved by homeomorphisms of the space X.

We begin with three elementary (though fundamental) observations. The second follows easily from the basic fact that in a continuum X, if K is a proper subcontinuum contained in an open set U, then K is a proper subset of $\mathrm{Cnt}_K(U)$ [11, Theorem 2, p.172].

Proposition 1.2. *Let X be a continuum and $K \subset L$ subcontinua. If L is filament, then K is filament.* (*Consequently, if K is non-filament, then L is non-filament.*)

Proposition 1.3. *If K is a filament subcontinuum of a continuum X, then there exists a filament continuum L such that K is a proper subset of L. Moreover, all continua containing K and sufficiently close to K are filament.*

Proposition 1.4. *If X is a continuum and F is a filament subcontinuum, then X is not locally connected at any point of F.*

In a locally connected continuum, there are no filament continua; the filament composants are empty. The next proposition indicates that there is often an abundance of filament continua.

Proposition 1.5. *If X is a continuum which is nowhere locally connected, then the union of all open filaments sets in X is dense.*

Proof. Call this set U. Suppose U is not dense. Let V be a nonempty open set in $X \setminus U$. Then every open set contained in V has a component with interior. Let W_1 be a nonempty open set of diameter < 1 with $\mathrm{Cl}(W_1) \subset V$. Then W_1 has a component K_1 with interior. There is an open set W_2 of diameter $< 1/2$ such that $\mathrm{Cl}(W_2) \subset \mathrm{Int}(K_1)$. Let K_2 be a component of W_2 with interior. Continuing in this manner, we obtain continua $\mathrm{Cl}(K_n)$ of diameter $\leqslant 1/n$ which form a neighborhood base at some point p, contradicting that X is nowhere locally connected. □

[2] Similar to the set M_p in [29].

In a continuum X, if U is the union of a collection $\{U_\alpha\}$ of open filament sets, linearly ordered by inclusion, then every continuum K in U must be contained in some U_α, and is thus filament. Consequently, U is filament. This observation and Zorn's Lemma lead to the following.

Proposition 1.6. *Each open filament set in a continuum X is contained in a maximal open filament set in X. (Consequently, each closed co-filament set contains a minimal closed, co-filament set.)*

It easily follows by definition that a set is co-filament if an only if it meets each non-filament continuum in the space. In the cone over the Cantor set, the singleton $\{p\}$, where p is the vertex, is the only minimal closed co-filament set. The product $S^1 \times S$, where S is a solenoid other than S^1, offers less obvious examples. Here one can show that each set of the form $S^1 \times \{x\}$ is a minimal closed co-filament set.

Co-filament sets are proving a robust tool in the study of homogeneous continua and will be thoroughly discussed in another manuscript.

Proposition 1.7. *If X is a continuum and K is a subcontinuum such that* $\mathrm{Fcs}(x) \subset K$ *for some $x \in X$, then K is non-filament.*

Proof. Suppose K is filament. Then by Proposition 1.3, K is a proper subset of some filament continuum L. However, $L \subset \mathrm{Fcs}(x)$, leading to a contradiction. □

Proposition 1.8. *Let X be a continuum and $p \in X$. If* $\mathrm{Fcs}(p)$ *is nonempty, it is a countable union of filament continua, each containing p. Thus each filament composant is a first-category F_σ subset of X, and its complement is a dense G_δ.*

Proof. [3] Since X has a countable base, it also has a base $U_1, U_2, \ldots$ which is closed with respect to taking finite unions. Let $V_1, V_2, \ldots$ be the elements of this list which contain p and such that the continuum $K_n = \mathrm{Cl}(\mathrm{Cnt}_{V_n}(p))$ is filament. It suffices to show that the K_n cover $\mathrm{Fcs}(p)$.

Let $q \in \mathrm{Fcs}(p)$. Then there is a continuum K containing p and q and an open set U containing K such that $\mathrm{Cnt}_U(p)$ has empty interior. There exist $U_{n_1}, U_{n_2}, \ldots, U_{n_t}$ covering K such that $\mathrm{Cl}(U_{n_i}) \subset U$ for each i. Let $W = U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_t}$. Then $\mathrm{Cl}(\mathrm{Cnt}_W(p)) \subset \mathrm{Cnt}_U(p)$, and is thus filament. Thus W equals some V_j, and $q \in K_j = \mathrm{Cl}(\mathrm{Cnt}_W(p))$. □

Proposition 1.9. *The following statements are equivalent for every continuum X:*

(a) *X is indecomposable.*
(b) *The entire space is the only non-filament subcontinuum.*
(c) *A subcontinuum is filament if and only if it is proper.*
(d) *Each singleton is a (minimal closed) co-filament set.*
(e) *There is no distinction between composants and filament composants.*

Proof. (a)⇒(b) Suppose F is a proper non-filament subcontinuum. Say $x \in X - F$. Let U be a neighborhood of F such that $x \notin \mathrm{Cl}(U)$. Then $\mathrm{Cl}(\mathrm{Cnt}_U(F))$ is a proper subcontinuum of X with interior, showing X to be decomposable [11, Theorem 2, p. 272].

(b)⇒(c) and (c)⇒(d) are immediate.

(d)⇒(e) This follows from the definitions of composant and filament composant.

(e)⇒(a) Assume that every composant of X is a filament composant and vise versa. Then for each $x \in X$, $\mathrm{Fcs}(x)$ is nonempty. Since $\mathrm{Fcs}(x)$ is first category by Proposition 1.8, there are infinitely many filament composants, and hence infinitely many composants. Thus X is indecomposable by [14, Theorem 11.13]. □

Proposition 1.10. *Let X be a continuum and K a subcontinuum. The following statements are equivalent:*

[3] Similar to the proofs of [28, Lemma 1] and [29, Theorem 2.1].

(a) *K is non-filament.*
(b) *K is the intersection of a sequence $K_1 \supset K_2 \supset \cdots$ such that each K_n is a continuum with nonempty interior.*
(c) *K is the intersection of a set of non-filament continua which is linearly ordered by inclusion.*

Proof. (a)⇒(b) Assume K is non-filament. For each n, let $U_n = N(K, 1/n)$, and $K_n = \mathrm{Cl}(\mathrm{Cnt}_{U_n}(K))$.

(b)⇒(c) Immediate.

(c)⇒(a) Assume K is the intersection of $\{K_\alpha\}$, non-filament continua linearly ordered by inclusion. Let U be an open set containing K. Then some $K_\alpha \subset U$. Since $\mathrm{Cnt}_U(K) = \mathrm{Cnt}_U(K_\alpha)$, $\mathrm{Cnt}_U(K)$ has nonempty interior. Thus K is non-filament. □

The following result of Goodykoontz is used to show a connection between non-filament continua in X and local connectedness at points of the hyperspace $C(X)$.

Lemma 1.11. *(See [5, Theorem 2].) Let X be a continuum and K a subcontinuum. Then $C(X)$ is locally connected at K if and only if for each open set U in X containing K, each sequence $\{K_n\}$ of continua converging to K is eventually in* $\mathrm{Cnt}_U(K)$.

It easily follows from Lemma 1.11 that $C(X)$ is locally connected at each element that has interior in X. Thus Proposition 1.10 yields the following.

Corollary 1.12. *Let X be a continuum and K a subcontinuum. If K is non-filament, then there are continua $K_1 \supset K_2 \supset \cdots$ whose intersection is K and such that $C(X)$ is locally connected at each K_n.*

By Zorn's Lemma and (c)⇒(a) of Proposition 1.10, we have the following.

Corollary 1.13. *Let X be a continuum. Every non-filament continuum in X contains a minimal non-filament continuum.*

Proposition 1.14. *Every minimal non-filament continuum is either indecomposable or has empty interior.*

Proof. Let X be a continuum and A a minimal non-filament subcontinuum which decomposes into proper subcontinua: $A = K \cup L$. Then K and L are filament continua and thus must have empty interiors. It follows that A also has empty interior. □

We conclude this section with a preliminary result further relating the hyperspace $C(X)$ with our new theory. This result concerns nonempty sets $\mathcal{D} \subset C(X)$ with the property:

$$K \in \mathcal{D} \text{ implies } L \in \mathcal{D} \text{ for each } L \in C(X) \text{ with } L \supset K. \tag{1.14.1}$$

Proposition 1.15. *If X is a continuum and $\mathcal{D} \subset C(X)$ with property* (1.14.1)*, then $\mathcal{D}$ is arcwise connected. Moreover, if $C(X)$ is locally connected at every element of $\mathcal{D}$, then $\mathcal{D}$ is locally arcwise connected.*

Proof. For every $K \in \mathcal{D}$, there is an order arc in $\mathcal{D}$ from K to X [13, Theorem 1.12]. Hence $\mathcal{D}$ is arcwise connected.

Now assume that $C(X)$ is locally connected at every element of $\mathcal{D}$. To prove that $\mathcal{D}$ is locally arcwise connected, let $L \in \mathcal{D}$, and $\mathcal{N}$ an ϵ-neighborhood of L in $\mathcal{D}$. Since $C(X)$ is locally connected at L, L has a closed, connected neighborhood $\mathcal{M}$ in $C(X)$ of diameter $< \epsilon/2$. The set $M = \bigcup \mathcal{M}$ is a subcontinuum of X containing L, and $d(L, M) < \epsilon/2$. For some $\delta > 0$, every element of $C(X)$ within δ of L is an element $\mathcal{M}$. Let K be any element of $\mathcal{N}$ with $d(K, L) < \delta$. Then $K \in \mathcal{M}$. Hence, $K \subset M$ and $d(K, M) < \epsilon/2$. Let $\mathcal{P}$ and $\mathcal{Q}$ be order arcs in $C(M)$ from K to M and L to M, respectively. Let $\mathcal{R} = \mathcal{P} \cup \mathcal{Q}$. Then every element of $\mathcal{R}$ is in $\mathcal{D}$ since it contains either K or L. Furthermore, each is within ϵ of L and hence lies in $\mathcal{N}$. It follows that $\mathcal{R}$ is a path in $\mathcal{N}$ from L to K. Thus $\mathcal{D}$ is locally path-connected. Hence, it is locally arcwise connected. □

2. Filament sets in Kelley continua

A metric space is called a *Kelley space*[4] (or *Kelley continuum* when applicable) provided that whenever a sequence of points x_n converges to an element x contained in a continuum $C \subset X$, there are continua C_n such that $x_n \in C_n \subset X$ converging to C under the Hausdorff metric. Every homogeneous continuum is a Kelley space [27]. Kelley spaces have proven important in continuum theory, especially in the study of hyperspaces. See [13, pp. 538ff] for a survey of early results. The current interest arose from the observation that many of our initial results on homogeneous continua, the main focus of our study, extend to Kelley continua.

Lemma 2.1. *(See [27].) Every homogeneous continuum is a Kelley space.*

Our thanks to Stephen Watson, York University, for pointing out that the following holds in Kelley continua.[5]

Lemma 2.2. *Let X be a Kelley continuum and K a non-filament subcontinuum. Then for each open set U containing K, $\mathrm{Ctu}_U(K)$ is open.*

Proof. Let $x \in \mathrm{Ctu}_U(K)$ and $\{x_n\}$ a sequence converging to x. To conclude that $\mathrm{Ctu}_U(K)$ is open it suffices to show that $\{x_n\}$ is eventually in the latter.

Let V be an open set containing K such that $\mathrm{Cl}(V) \subset U$. Since K is not filament, $\mathrm{Cnt}_V(K)$ has nonempty interior. Since $\mathrm{Cl}(\mathrm{Cnt}_V(K)) \subset \mathrm{Ctu}_U(K)$, there is a continuum L in U containing both x and $\mathrm{Cl}(\mathrm{Cnt}_V(K))$. Notice that L has nonempty interior. Since X is a Kelley space, there are continua L_n converging to L with $x_n \in L_n$. The L_n are eventually in U, and must eventually meet L since L has interior. Thus $\{L_n\}$ is eventually in $\mathrm{Ctu}_U(K)$, and likewise for $\{x_n\}$. Thus $\mathrm{Ctu}_U(K)$ is open. □

Proposition 2.3. *Let X be a Kelley continuum and K a subcontinuum. The following statements are equivalent*:

(a) *K is ample.*
(b) *K is not filament.*
(c) *For every open set U containing K, the constituant $\mathrm{Ctu}_U(K)$ is open.*
(d) *$C(X)$ is locally connected at K.*

Proof. (a)⇒(b) Obvious by the definitions.

(b)⇒(c) Proved in Lemma 2.2.

(c)⇒(d) This follows from Lemma 1.11.

(d)⇒(a) Assume that $C(X)$ is locally connected at K. Let $\mathcal{N}$ be a small connected neighborhood of K in $C(X)$. Let $U = \bigcup \mathcal{N}$. Clearly U is connected. To conclude the proof, it suffices to show that $K \subset \mathrm{Int}(U)$.

Let $k \in K$ and $\{x_n\}$ a sequence in X converging to k. Since X is a Kelley space, there is a sequence of continua K_n in X converging to K such that $x_n \in K_n$. The K_n are eventually in $\mathcal{N}$, so x_n is eventually in U. It follows that $K \subset \mathrm{Int}(U)$. □

Corollary 2.4. *If X is a Kelley continuum and $p \in X$, then X is locally connected at p if and only if the singleton $\{p\}$ is non-filament.*

Please note that we have understated by a small amount our conclusions on local connectedness. In Corollary 2.4, we could have added: *if and only if X is locally connected at p by open connected sets.* Also by the results in [5,6], we could have added the following statements to the list of equivalences in Proposition 2.3: 2^X *is locally connected at K*; 2^X *is locally connected at K by open connected sets*; $C(X)$ *is locally connected at K by open arcwise connected sets.*

[4] Historically the name has varied. Originally called a *space with property* 3.2 in [8], one also finds the expressions: *space with the property of Kelley*, *space with property* [k], et al.

[5] Originally given for homogeneous polish spaces in [29, Corollary 3.3].

Corollary 2.5. *In a Kelley continuum, every ample continuum contains a minimal ample continuum.*

Proof. This follows from Corollary 1.13 and Proposition 2.3. □

Corollary 2.6. *Let X be a Kelley continuum. The following conditions are equivalent*:

(a) *X is not locally connected.*
(b) *Some singleton in X is filament.*
(c) *X has a nondegenerate filament subcontinuum.*

Proof. This follows from Corollary 2.4 and Proposition 1.3. □

Corollary 2.7. *Let X be a Kelley continuum. For each $x \in X$, $X \setminus \mathrm{Fcs}(x)$ equals the set of all points y in X such that every continuum containing x and y is ample. Thus by Proposition* 1.8*, the set of all such y is a dense G_δ subset of X.*

In [6, Theorem 5] Goodykoontz shows that if X is the only point at which $C(X)$ is locally connected, then X is indecomposable. An example is also given showing the converse to be false. The following corollary shows that this example must not be a Kelley space.

Corollary 2.8. *The following statements are equivalent for a Kelley continuum X*:

(a) *X is indecomposable.*
(b) *X is the only point at which $C(X)$ is locally connected.*
(c) *The entire space is the only ample subcontinuum.*

Proof. This follows from Propositions 1.9 and 2.3. □

Proposition 2.9. *Let X be a Kelley continuum and K a subcontinuum. If $\mathrm{Fcs}(x) \subset K$ for some $x \in X$, then K is ample.*

Proof. This follows from Propositions 1.7 and 2.3. □

Proposition 2.10. *Let X be a Kelley continuum and $\mathcal{D} \subset C(X)$. If $\mathcal{D}$ has property* (1.14.1)*, and each element of $\mathcal{D}$ is ample, then $\mathcal{D}$ is arcwise connected and locally arcwise connected.*

Proof. This follows from Propositions 1.15 and 2.3. □

We will show that if X is a Kelley continuum, a set $\mathcal{D} \subset C(X)$ with property (1.14.1) which is closed and consists only of ample continua must be an absolute retract. The proof follows the method of Kelley [8] as is outlined in the following series of lemmas. We make use of the map $\sigma : C(C(X)) \to C(X)$ given by $\sigma(\mathcal{D}) = \bigcup_{D\in\mathcal{D}} D$.

Lemma 2.11. *(See [8].) If X is a Kelley continuum, then $C(X)$ is contractible in itself (to X). Moreover, the deformation $G : C(X) \times I \to C(X)$ can be chosen such that if $K \in C(X)$ and $0 \leqslant t_1 < t_2 \leqslant 1$, then $G(K, t_1) \subset G(K, t_2)$.*

Proof. This follows from [8, Theorem 3.3] and [8, Remark, p. 26]. □

Lemma 2.12. *If $G : C(X) \times I \to C(X)$ is a function as in Lemma* 2.11*, define $G_1 : X \times I \to C(X)$ by $G_1(x,t) = G(\{x\}, t)$. Then G_1 is continuous and for each $x \in X$, $G_1(x, 0) = \{x\}$ and $G_1(x, 1) = X$.*

Lemma 2.13. *(See [8, Lemma 4.2].) Let X be a continuum and $\mathcal{D}$ a subset of $C(X)$ with property* (1.14.1)*. If $\mathcal{S}$ is a locally connected subcontinuum of $C(X)$ contained in $\mathcal{D}$, then $\mathcal{S}$ is contractible over a subset $\mathcal{T}$ of $\mathcal{D}$ such that the diameter of $\mathcal{S}$ equals the diameter of $\mathcal{T}$.*

Proof. Since a locally connected continuum is a Kelley continuum, let $G : C(\mathcal{S}) \times I \to C(\mathcal{S})$ be the function given by Lemma 2.11 deforming $C(\mathcal{S})$ to the point $\mathcal{S}$. The function G_1 given in Lemma 2.12 maps $\mathcal{S} \times I$ to $C(\mathcal{S})$. Let $\mathcal{T} = \sigma(C(\mathcal{S}))$. Notice that $\mathcal{T} \subset \mathcal{D}$. The function $\sigma \circ G_1 : \mathcal{S} \times I \to \mathcal{T}$ is the desired deformation. Since $C(\mathcal{S})$ has the same diameter as $\mathcal{S}$, and σ is a contraction (meaning it cannot increase distance) [8, Lemma 1.1], it follows that $\mathcal{T}$ has the same diameter as $\mathcal{S}$. □

Lemma 2.14. *(See [8, Theorem 4.3].) Let X be a continuum and $\mathcal{D} \subset C(X)$ with property* (1.14.1). *Let K be a finite complex, K_1 a subcomplex including all* 1*-dimensional simplices of K, and $f : K_1 \to \mathcal{D}$ a continuous mapping such that the partial image under f of any simplex of K has diameter less than ϵ. Then f may be extended to a map from K into $\mathcal{D}$ such that the diameter of the image of any simplex is less than ϵ.*

Proof. The proof of [8, Theorem 4.3] may now be followed exactly, with $\mathcal{D}$ taking the role of $C(X)$. The key observation is that if $g : S^n \to \mathcal{D}$, $n \geqslant 1$, is a map of the surface of an $n+1$ cell, I^{n+1}, into $\mathcal{D}$, then $g(S^n)$ is locally connected, so by Lemma 2.13, g can be extended to a map of I^{n+1} into $\mathcal{D}$ without increasing the diameter of the image. □

Proposition 2.15. *If X is a Kelley continuum and $\mathcal{D} \subset C(X)$ has property* (1.14.1)*, then $\mathcal{D}$ is contractible in itself* (*to X*).

Proof. If G is the function given in Lemma 2.11, then $G|_{\mathcal{D} \times I}$ is a deformation of $\mathcal{D}$ in $\mathcal{D}$ to the point X. □

Proposition 2.16. *If X is a Kelley continuum and $\mathcal{D} \subset C(X)$ is closed, has property* (1.14.1)*, and consists only of ample continua, then $\mathcal{D}$ is an absolute retract.*

Proof. The proof is patterned after the proof of [8, Theorem 4.4].

By 2.10, $\mathcal{D}$ is a locally arcwise connected subcontinuum of $C(X)$. It follows that for each $\epsilon > 0$, there is a number $\delta > 0$ such that if $K, L \in \mathcal{D}$ with $d(K, L) < \delta$, there is an arc from K to L in $\mathcal{D}$ of diameter less than ϵ.

Now let K be a finite complex, K_0 a subcomplex including all of the vertices of K, and f a map from K_0 into $\mathcal{D}$ such that the partial image under f of any simplex of K is of diameter less than δ. By the previous paragraph, f can be extended to a subcomplex K_1 of K containing all of the vertices and edges of K, and such that the partial image under this map of any simplex of K has diameter less than ϵ. By Lemma 2.14, f can be extended further to all of K such that the diameter of the image of any simplex is less than ϵ. Thus by Lefschetz' criterion [12, Theorem II], $\mathcal{D}$ is an ANR. Since $\mathcal{D}$ is contractible by Proposition 2.15, it is an AR [26, Theorem 5.2.15]. □

3. Filament sets in homogeneous continua

All of the results of the paper apply to homogeneous continua, which we consider further in this section. As a first remark, notice that by Corollary 2.7 and Remark 1.1, it is transparent why 2-homogeneous continua must be locally connected [25], for no homeomorphism of a space can carry two points that lie in a filament continuum to two points that do not. [6]

Theorem 3.1. *(See [29, 3.7].) If X is a homogeneous continuum, the set of all ordered pairs (p, q) such that only ample continua contain both p and q is a dense G_δ subset of $X \times X$.*

In Theorem 3.1, the points p and q satisfy $p \notin \mathrm{Fcs}(q)$ (and $q \notin \mathrm{Fcs}(p)$). Each continuum containing such a pair of points has to be ample by Proposition 2.3.

Question 1. If A is an ample continuum in a homogeneous continuum X, do there exist points $p, q \in A$ such that $q \notin \mathrm{Fcs}(p)$?

[6] See [28] for a self-contained exposition.

Theorem 3.2. *If X is a homogeneous continuum and K is a filament subcontinuum of X, there is an open filament set U in X containing K.*

Proof. The existence of U under more general hypotheses can be found in [29, Lemma 3.6]. We include a proof here for completeness.

By definition, there is an open set V containing K such that $\mathrm{Cnt}_V(K)$ has empty interior. Let $\varepsilon = \inf\{d(k, x) \mid k \in K,\ x \in X \setminus V\}$. Let δ be an Effros number for $\varepsilon/2$. Let $\alpha > 0$ be less than both $\varepsilon/2$ and δ. Let $U = N(K, \alpha)$. Suppose U has a component C with nonempty interior. By the Effros theorem, there is a homeomorphism f of X with itself such that $f(C)$ meets K and moves each point of X less than $\varepsilon/2$. Thus each point of $f(C)$ has distance less than ε from K, so $f(C) \subset V$. Thus $f(C) \subset \mathrm{Cnt}_V(K)$, contradicting that the latter has empty interior. □

Corollary 3.3. *Let X be a homogeneous continuum. The following conditions are equivalent*:

(a) *X is not locally connected.*
(b) *Each singleton in X is filament.*
(c) *Sufficiently small subsets of X are filament.*
(d) *The topology on X has a base of open filament sets.*

Proof. This follows easily from Corollary 2.6 and Theorem 3.2. □

Corollary 3.4. *In a homogeneous continuum X, the collection $\mathcal{F}$ of all filament subcontinua of X is a connected open subset of $C(X)$.*

Proof. Indeed, for a filament continuum K, the collection of subcontinua of the open set U guaranteed by Theorem 3.2 is a neighborhood of K in $C(X)$ composed of only filament continua.

All that remains is to show that $\mathcal{F}$, when nonempty, is connected. However, this easily follows since the subcontinuum of $C(X)$ consisting of all singletons is contained in $\mathcal{F}$, and each element of $\mathcal{F}$ is connected by an order arc in $\mathcal{F}$ to a point in this subcontinuum [13, Theorem 1.12]. □

In any locally connected continuum X, the set $\mathcal{A}$ of all ample subcontinua is a compact, absolute retract. Indeed, $\mathcal{A} = C(X)$ in this case, and $C(X)$ is an absolute retract by [8, Theorem 4.4]. By the following theorem, if X is not locally connected, yet homogeneous, $\mathcal{A}$ will again be an absolute retract (though $C(X)$ will not).

Theorem 3.5. *Let X be a homogeneous continuum, $p \in X$, $\mathcal{A}$ the set of all ample subcontinua of X, and $\mathcal{A}_p$ the set of all ample continua containing p. Then $\mathcal{A}$ and $\mathcal{A}_p$ are compact absolute retracts.*

Proof. By Proposition 2.3, the complement of $\mathcal{A}$ in $C(X)$ is the set of filament subcontinua of X, and the latter is open by Corollary 3.4. Hence, $\mathcal{A}$ is compact, and $\mathcal{A}_p$ as well. By Proposition 2.16, $\mathcal{A}$ and $\mathcal{A}_p$ are absolute retracts. □

Corollary 3.6. *If X is a homogeneous continuum, the set $\mathcal{A}_0$ of all minimal ample subcontinua of X is a G_δ subset of $C(X)$.*

Proof. Let $\mathcal{A}$ be the set of all ample subcontinua of X. Define $F_n = \{L \in \mathcal{A} \mid \text{there exists } L_1 \in \mathcal{A} \text{ with } L_1 \subset L \text{ and } d(L_1, L) \geqslant 1/n\}$. Since $\mathcal{A}$ is closed by Theorem 3.5, it follows that each F_n is closed. The union, $\bigcup_n F_n$, is the complement of $\mathcal{A}_0$ in $\mathcal{A}$, and thus $\mathcal{A}_0$ is a G_δ in $\mathcal{A}$. Since $\mathcal{A}$ is compact, $\mathcal{A}_0$ is a G_δ in $C(X)$. □

Proposition 3.7. *If A is a minimal ample subcontinuum of a homogeneous continuum X, then A is either indecomposable or has empty interior.*

Proof. This follows from Propositions 1.14 and 2.3. □

In all examples known to the authors, minimal ample subcontinua of decomposable homogeneous continua have empty interiors. If an example with nonempty interior exists, then by Proposition 3.7, homogeneity, and compactness, the whole space must be a union of finitely many indecomposable subcontinua.

Question 2. Does there exist a decomposable homogeneous continuum with a minimal ample subcontinuum having nonempty interior?

Question 3. Does there exist a decomposable homogeneous continuum that is the union of finitely many indecomposable subcontinua?

Question 4. If A is a minimal ample continuum in a homogeneous continuum X, do there exist points $p, q \in A$ such that A is irreducible between p and q?

Question 5. Let $\mathcal{A}_0$ be the set of minimal ample subcontinua of a homogeneous continuum X. If F is a filament subcontinuum of X, does there exist a continuum $A \in \mathcal{A}_0$ that contains F?

Question 6. If X is a homogeneous continuum and $\mathcal{A}_0$ is the set of all minimal ample subcontinua of X, is $\mathcal{A}_0$ a connected subspace of $C(X)$? Is the closure of $\mathcal{A}_0$ connected?

Remark 3.8. In forthcoming papers, the authors will present applications of the material thus developed. Notably, we give: a scheme to categorize homogeneous continua into subclasses defined in terms of filament and ample subcontinua which compares interestingly to those given by Rogers in [22,24]; aposyndetic decompositions of Kelley continua extending Jones' decomposition of homogeneous continua [7]; newly defined decompositions of homogeneous continua related to those given in [7,3,20,21,23]; and applications to questions of aposyndesis, local connectedness, and indecomposability.

In these manuscripts, the filament and ample subcontinua of several spaces are identified. Sometimes, the result is quite intuitive. For instance, in the product space $S^1 \times S$, where S^1 is the unit circle and S is a solenoid different from S^1, the filament composants are the sets $S^1 \times C$, where C is a composant of S. Furthermore, a subcontinuum of the space is filament if and only if it is contained in one of the filament composants.

In an arcwise connected, homogeneous, non-locally connected continuum (see [15] for existence and [30,29,1–3, 10] for history and additional properties), the filament composants have a richer structure than the ordinary composants of a continuum. Similar to the case for composants in an indecomposable continuum, they are uncountable in number and cover the space; however, they do not form a partition. If we travel along an arc from a point p to a point $q \notin \mathrm{Fcs}(p)$, there will be a first point x such that the subarc $[p, x]$ is ample. If y is any point strictly between p and x on this arc, then $[p, y]$ will be filament. If we choose y sufficiently close to x, then $[y, x]$ will also be filament.

The space $P \times P$, where P is the pseudo-arc, offers more accessible examples of interesting phenomena. Here, the set $(\{p\} \times P) \cup (P \times \{q\})$ is a minimal ample subcontinuum of $P \times P$ contained in $\mathrm{Fcs}(p, q)$. Thus, in this example, and the previous one as well, the filament composants, unlike what is found in the product of a circle with a solenoid, are not filament sets. The reader may now sense the beginning of a classification scheme.

References

[1] D.P. Bellamy, Arcwise connected homogeneous metric continua are colocally arcwise connected, Houston J. Math. 11 (1985) 277–281.
[2] D.P. Bellamy, Short paths in homogeneous continua, Topology Appl. 26 (1987) 287–291.
[3] D.P. Bellamy, L. Lum, The cyclic connectivity of homogeneous arcwise connected continua, Trans. Amer. Math. Soc. 266 (1981) 389–396.
[4] E.G. Effros, Transformation groups and C^*-algebras, Ann. of Math. (2) 81 (1965) 38–55.
[5] J.T. Goodykoontz Jr., Connectedness im kleinen and local connectedness in 2^X and $C(X)$, Pacific J. Math. 53 (1974) 387–397.
[6] J.T. Goodykoontz Jr., More on connectedness im kleinen and local connectedness in $C(X)$, Proc. Amer. Math. Soc. 65 (1977) 357–364.
[7] F.B. Jones, On a certain type of homogeneous plane continuum, Proc. Amer. Math. Soc. 6 (1955) 735–740.
[8] J.L. Kelley, Hyperspaces of a continuum, Trans. Amer. Math. Soc. 52 (1942) 22–36.
[9] B. Knaster, K. Kuratowski, Problem 2, Fund. Math. 1 (1920) 223.
[10] K. Kuperberg, A locally connected, micro-homogeneous, non-homogeneous continuum, Bull. Acad. Polon. Sci. Ser. Sci. Math. 28 (1980) 627–630.

[11] K. Kuratowski, Topology, vol. II, in: New edition, revised and augmented, Academic Press, New York, 1968. Translated from French by A. Kirkor.
[12] S. Lefschetz, On locally connected and related sets, Ann. of Math. 35 (1934) 118–129.
[13] S.B. Nadler, Hyperspaces of Sets, Marcel Dekker, New York, 1978.
[14] S.B. Nadler, Continuum Theory, Marcel Dekker, New York, 1992.
[15] J.R. Prajs, A homogeneous arcwise connected non-locally-connected curve, Amer. J. Math. 124 (2002) 649–675.
[16] J.R. Prajs, Filament local product structures in homogeneous continua, Preprint, 12 pp.
[17] J.R. Prajs, K. Whittington, Filament sets, aposyndesis, and the decomposition theorem of Jones, Trans. Amer. Math. Soc., in press.
[18] J.R. Prajs, K. Whittington, Filament additive homogeneous continua, Indiana Univ. Math. J. 1 (2007).
[19] J.R. Prajs, K. Whittington, Filament sets and decompositions of homogeneous continua, Preprint, 12 pp.
[20] J.T. Rogers, Completely regular mappings and homogeneous, aposyndetic continua, Canad. J. Math. 33 (1981) 450–453.
[21] J.T. Rogers, Decompositions of homogeneous continua, Pacific J. Math. 99 (1982) 137–144.
[22] J.T. Rogers, Homogeneous continua, Top. Proc. 8 (1983) 213–233.
[23] J.T. Rogers, Cell-like decompositions of homogeneous continua, Proc. Amer. Math. Soc. 87 (1983) 375–377.
[24] J.T. Rogers, Classifying homogeneous continua, Topology Appl. 44 (1992) 341–352.
[25] G.S. Ungar, On all kinds of homogeneous spaces, Trans. Amer. Math. Soc. 212 (1975) 393–400.
[26] J. Van Mill, Infinite Dimensional Topology, Prerequisites and Introduction, North-Holland, Amsterdam, 1989.
[27] R.W. Wardle, On a property of J.L. Kelley, Houston J. Math. 3 (1977) 291–299.
[28] K. Whittington, A generalization of 2-homogeneous continua being locally connected, Proc. Amer. Math. Soc. 126 (1998) 3131–3132.
[29] K. Whittington, Open connected sets in homogeneous spaces, Houston J. Math. 27 (2001) 523–531.
[30] K. Whittington, Local connectedness in transformation groups, Proc. Amer. Math. Soc. 130 (2001) 903–907.