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Existence results for a fourth-order ordinary differential equation with a four-point boundary condition

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Abstract

The fourth-order differential equation

$$y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \le t \le 1,$$

with the four-point boundary value problem

$$\begin{aligned} &y(0) = y(1) = 0, \\ &ay''(\xi_1) - by'''(\xi_1) = 0, \qquad cy''(\xi_2) + dy'''(\xi_2) = 0 \end{aligned}$$

is studied in this work, where $0 \le \xi_1 < \xi_2 \le 1$. Some results on the existence of at least one positive solution to the above four-point boundary value problem are obtained by using the Krasnoselskii fixed point theorem.

Keywords: Fourth order; Four-point boundary value problem; Sufficient condition; Positive solutions; Fixed point

1. Introduction

It is well known that boundary value problems for ordinary differential equations can be used to describe a large number of physical, biological and chemical phenomena. For example, the deformations of an elastic beam in the equilibrium state can be described as a boundary value problem of some fourth-order differential equations. This has resulted in an extensive study on diverse boundary value problems for second-order and higher order differential equations via many methods ([1–10] and the references therein). Most of these studies are based upon the Leray–Schauder degree theory [5], the fixed point theorem on a cone [2,3,6], or the lower and upper solutions method [4], and mainly studied the multi-point boundary value problem for second-order ordinary differential equations [5,6] or studied the two-point boundary value problem for higher order ordinary differential equations [2,3]. There are very few works on the multi-point boundary value problem for higher order ordinary differential equations.

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For this reason, we are going to investigate the fourth-order nonlinear ordinary differential equation

$$y^{(4)}(t) - f(t, y(t), y''(t)) = 0, \quad 0 \le t \le 1 \tag{1.1}$$

with the following four-point boundary value conditions:

$$\begin{aligned} &y(0) = y(1) = 0, \\ &ay''(\xi_1) - by'''(\xi_1) = 0, \qquad cy''(\xi_2) + dy'''(\xi_2) = 0, \end{aligned} \tag{1.2}$$

where $f \in C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty))$, and a, b, c, d are nonnegative constants, $0 \le \xi_1 < \xi_2 \le 1$.

Problems (1.1) and (1.2) occur in beam theory such as for a beam with small deformation; a beam of a material which satisfies a nonlinear power-like stress law; a beam with two-sided links which satisfies a nonlinear power-like elasticity law; we refer the readers to the work of Zill and Cullen [1, pp. 237–243] for a brief and easily accessible discussion and the physical interpretation for some of the boundary conditions associated with the beam equation. Some existence and multiplicity results for positive solutions for (1.1) and (1.2) have been obtained for $a = 1, b = 0$, $c = 1, d = 0, \xi_1 = 0, \xi_2 = 1$, for which the boundary value problems (1.1) and (1.2) describe the deformations of an elastic beam both of whose ends are simply supported. The aim of the present work is to obtain a sufficient condition for the existence of a positive solution, i.e., a solution $u(t)$ of (1.1) and (1.2) such that $u(t) \geqslant 0$ on (0, 1) and $u(t) > 0$ on some interval in (0, 1).

The remainder of the work is organized as follows. In Section 2, we present some preliminaries and lemmas. Section 3 is devoted to presenting and proving our main results.

2. Preliminaries

In this section, we shall give some preliminary considerations and some lemmas. Let $E = \{y \in C^2[0, 1] | y(0) = y(1) = 0\}$. Then we have the following lemma.

Lemma 2.1. *For $y \in E$, $\|y\|_\infty \leqslant \|y'\|_\infty \leqslant \|y''\|_\infty$, where $\|y\|_\infty = \sup_{t\in[0,1]} |y(t)|$.*

Thus, E is a Banach space when it is endowed with the norm $\|y\| = \|y''\|_\infty$. For convenience we set

$$\begin{aligned} \max f_0 &= \lim_{-v\to 0^+} \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} \frac{f(t,u,v)}{-v}, \\ \min f_0 &= \lim_{-v\to 0^+} \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} \frac{f(t,u,v)}{-v}, \\ \max f_\infty &= \lim_{-v\to +\infty} \max_{t\in[0,1]} \sup_{u\in[0,+\infty)} \frac{f(t,u,v)}{-v}, \\ \min f_\infty &= \lim_{-v\to +\infty} \min_{t\in[0,1]} \inf_{u\in[0,+\infty)} \frac{f(t,u,v)}{-v}. \end{aligned}$$

From the Lemma 2.1 of the reference [4] we have the following lemma.

Lemma 2.2. *If $\alpha = ad + bc + ac(\xi_2 - \xi_1) \neq 0$ and $h(t) \in C[\xi_1, \xi_2]$, then the boundary value problem*

$$\begin{aligned} &u^{(4)}(t) = h(t), \\ &u(0) = u(1) = 0, \\ &au''(\xi_1) - bu'''(\xi_1) = 0, \qquad cu''(\xi_2) + du'''(\xi_2) = 0 \end{aligned} \tag{2.1}$$

has a unique solution

$$y(t) = \int_0^1 G_1(t, s) \int_{\xi_1}^{\xi_2} G_2(s, \tau) h(\tau) \mathrm{d}\tau \mathrm{d}s,$$

where

$$G_1(t, s) = \begin{cases} s(1 - t), & 0 \le s < t \le 1, \\ t(1 - s), & 0 \le t \le s \le 1 \end{cases} \tag{2.2}$$

and

$$G_2(t,s) = \begin{cases} \dfrac{1}{\alpha}(a(s-\xi_1)+b)(d+c(\xi_2-t)), & s < t \le 1,\ \xi_1 \le s \le \xi_2, \\ \dfrac{1}{\alpha}(a(t-\xi_1)+b)(d+c(\xi_2-s)), & 0 \le t \le s,\ \xi_1 \le s \le \xi_2. \end{cases} \tag{2.3}$$

Lemma 2.3. *Suppose $\alpha > 0$; then the following results hold:*

(1) $G_2(t,s) \leqslant G_2(s,s)$, *for* $t \in [0,1]$, $s \in [\xi_1, \xi_2]$,
(2) $\frac{G_2(t,s)}{G_2(s,s)} \geqslant \frac{1}{4}$, *for* $\eta_1 \leqslant t \leqslant \eta_2$, $s \in [\xi_1,\xi_2]$, *where* $\eta_1 = \xi_1 + \frac{1}{4}(\xi_2-\xi_1)$, $\eta_2 = \xi_2 - \frac{1}{4}(\xi_2-\xi_1)$.

Proof. (1) If $s < t$, then $G_2(t,s) = \frac{1}{\alpha}(a(s-\xi_1)+b)(d+c(\xi_2-t)) \leqslant \frac{1}{\alpha}(a(s-\xi_1)+b)(d+c(\xi_2-s)) = G_2(s,s)$ because $\frac{1}{\alpha}(a(s-\xi_1)+b) \geqslant 0$ and $c \geqslant 0$. In the same way we have $G_2(t,s) \leqslant G_2(s,s)$ if $s \geqslant t$.

(2) From Eq. (2.3) we know immediately that

$$\frac{G_2(t,s)}{G_2(s,s)} = \begin{cases} \dfrac{d+c(\xi_2-t)}{d+c(\xi_2-s)}, & s < t \le 1,\ \xi_1 \le s \le \xi_2, \\ \dfrac{a(t-\xi_1)+b}{a(s-\xi_1)+b}, & 1 \le t \le s,\ \xi_1 \le s \le \xi_2. \end{cases}$$

Because $\xi_1 \le s \le \xi_2, \eta_1 \leqslant t \leqslant \eta_2$ and c, d are nonnegative, so when $s < t$ we have $\frac{G_2(t,s)}{G_2(s,s)} = \frac{d+c(\xi_2-t)}{d+c(\xi_2-s)} \geqslant \frac{d+c(\xi_2-\eta_2)}{d+c(\xi_2-\xi_1)} \geqslant \frac{1}{4}$. In the same way we have $\frac{G_2(t,s)}{G_2(s,s)} \geqslant \frac{1}{4}$ if $s \geqslant t$. This completes the proof of the lemma. □

We define the operator $A : E \to E$ by

$$Ay(t) = \int_0^1 G_1(t,s) \int_{\xi_1}^{\xi_2} G_2(s,\tau) f(\tau, y(\tau), y''(\tau))\mathrm{d}\tau \mathrm{d}s. \tag{2.4}$$

Obviously, $u(t)$ is a solution of the four-point boundary problems (1.1) and (1.2) if and only if $u(t)$ solves the operator equation $Au(t) = u(t)$, where the operator A is defined above.

Lemma 2.4. *Suppose $\alpha > 0$ and $-a\xi_1 + b \ge 0$, $c(\xi_2-1)+d \ge 0$; then $A(K) \subset K$, where $\eta_1 = \xi_1 + \frac{1}{4}(\xi_2-\xi_1)$, $\eta_2 = \xi_2 - \frac{1}{4}(\xi_2-\xi_1)$ and*

$$K = \left\{ y \in E | y \geqslant 0, \min_{\eta_1 \leqslant t \leqslant \eta_2} (-y''(t)) \geqslant \frac{1}{4}\|y\| \right\}, \tag{2.5}$$

which is obviously a cone in E.

Proof. For any $y \in K$, we know from (2.4) and Lemma 2.3(1) that

$$0 \leqslant -(Ay)''(t) = \int_{\xi_1}^{\xi_2} G_2(t,s) f(s,y(s),y''(s))\mathrm{d}s \leqslant \int_{\xi_1}^{\xi_2} G_2(s,s) f(s,y(s),y''(s))\mathrm{d}s, \tag{2.6}$$

which implies

$$\| -(Ay)''(t)\|_\infty \leqslant \int_{\xi_1}^{\xi_2} G_2(s,s) f(s,y(s),y''(s))\mathrm{d}s. \tag{2.7}$$

Thus from Lemma 2.3(2) and (2.7), we have

$$\begin{aligned} \min_{\eta_1 \leqslant t \leqslant \eta_2} (-(Ay)''(t)) &= \min_{\eta_1 \leqslant t \leqslant \eta_2} \int_{\xi_1}^{\xi_2} G_2(t,s) f(s,y(s),y''(s))\mathrm{d}s \\ &\geqslant \frac{1}{4} \int_{\xi_1}^{\xi_2} G_2(s,s) f(s,y(s),y''(s))\mathrm{d}s \\ &\geqslant \frac{1}{4}\|(Ay)''\|_\infty. \end{aligned}$$

Noting that $G_1(t,s) \geqslant 0$ and $G_2(t,s) \geqslant 0$, we can conclude that $A(K) \subset K$. □

The key tool in our approach is the following fixed point theorem.

Theorem 2.1 (*Krasnoselskii*). *Let E be a Banach space, and $K \subset E$ be a cone in E. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\bar{\Omega}_1 \subseteq \Omega_2$ and let*

$$A : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either:

(i) $\|Au\| \leqslant \|u\|, u \in K \cup \partial\Omega_1$ *and* $\|Au\| \geqslant \|u\|, u \in K \cup \partial\Omega_2$*; or*
(ii) $\|Au\| \geqslant \|u\|, u \in K \cup \partial\Omega_1$ *and* $\|Au\| \leqslant \|u\|, u \in K \cup \partial\Omega_2$.

Then A has a fixed point in $K \cup (\bar{\Omega}_2 \setminus \Omega_1)$.

3. Main results

We are now in a position to present and prove our main results. Throughout this section we assume that a, b, c, d are nonnegative constants, $0 \leq \xi_1 < \xi_2 \leq 1$ and $\alpha = ad + bc + ac(\xi_2 - \xi_1) > 0$, $-a\xi_1 + b \geq 0$, $c(\xi_2 - 1) + d \geq 0$.

Theorem 3.1. *Assume that $f \in C([0, 1] \times [0, \infty) \times (-\infty, 0], [0, \infty))$ and f is sublinear, i.e.,* $\min f_0 = +\infty$ *and* $\max f_\infty = 0$*; then the boundary value problems (1.1) and (1.2) have at least one positive solution.*

Proof. Since $\min f_0 = +\infty$, then, for any ϵ satisfying $\frac{1}{4}\epsilon \int_{\eta_1}^{\eta_2} G_2(\frac{1}{2}(\eta_1 + \eta_2), s)\mathrm{d}s \geqslant 1$, we can choose $\rho_1 > 0$ such that

$$f(t, u, v) \geqslant \epsilon(-v), \quad \text{for } t \in [0, 1],\ u \in [0, +\infty),\ 0 \leqslant -v \leqslant \rho_1. \tag{3.1}$$

Set $\Omega_{\rho_1} = \{y \in K \mid \|y\| < \rho_1\}$; then, for $y \in \partial\Omega_{\rho_1}$, we can get from (2.5), (2.6) and (3.1)

$$\begin{aligned} -(Ay)''\left(\frac{1}{2}(\eta_1 + \eta_2)\right) &= \int_{\xi_1}^{\xi_2} G_2\left(\frac{1}{2}(\eta_1 + \eta_2), s\right) f(s, y(s), y''(s))\mathrm{d}s \\ &\geqslant \int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1 + \eta_2), s\right) f(s, y(s), y''(s))\mathrm{d}s \\ &\geqslant \epsilon \int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1 + \eta_2), s\right) (-y''(s))\mathrm{d}s \\ &\geqslant \frac{1}{4}\epsilon\|y\| \int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1 + \eta_2), s\right) \mathrm{d}s \\ &\geqslant \|y\|, \end{aligned}$$

which implies

$$\|Ay\| \geqslant \|y\|, \quad \text{for } y \in \partial\Omega_{\rho_1}. \tag{3.2}$$

Next, since $\max f_\infty = 0$, then, for any ϵ^* satisfying $\epsilon^* \int_{\xi_1}^{\xi_2} G_2(s, s)\mathrm{d}s \leqslant 1$, there exists $\rho^* > \rho_1$ such that

$$f(t, u, v) \leqslant \epsilon^*(-v), \quad \text{for } t \in [0, 1],\ u \in [0, +\infty),\ -v \geqslant \rho^*. \tag{3.3}$$

We consider two cases:

Case(i): Suppose that $f(t, u, v)$ is unbounded; then define a function $f^*(r) : [0, \infty) \to [0, \infty)$ by

$$f^*(r) := \max\{f(t, u, v) : t \in [0, 1], 0 \leqslant u \leqslant r, 0 \leqslant -v \leqslant r\}.$$

It is easy to see that $f^*(r)$ is nondecreasing and $\lim_{r\to+\infty} \frac{f^*(r)}{r} = 0$, and

$$f^*(r) \leqslant \epsilon^* r, \quad \text{for } r > \rho^*. \tag{3.4}$$

Taking $\rho_2 > \rho^*$, then, from (3.3) and (3.4) we can get

$$f(t, u, v) \leqslant f^*(\rho_2) \leqslant \epsilon^*\rho_2, \quad \text{for } t \in [0, 1],\ 0 \leqslant u \leqslant \rho_2,\ 0 \leqslant -v \leqslant \rho_2. \tag{3.5}$$

On the other hand, for $y \in K$, and $\|y\| = \rho_2$, from Lemma 2.1 we know that

$$\|y\|_\infty \leqslant \rho_2. \tag{3.6}$$

Thus from (2.6), (3.5) and (3.6) and Lemma 2.4, we have for $y \in K$, and $\|y\| = \rho_2$

$$-(Ay)''(t) \leqslant \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, y(s), y''(s))\mathrm{d}s \leqslant \epsilon^* \rho_2 \int_{\xi_1}^{\xi_2} G_2(s,s)\mathrm{d}s \leqslant \rho_2 = \|y\|.$$

Case(ii): Suppose $f(t,u,v)$ is bounded, i.e., there exists a positive constant L such that $f(t,u,v) \leqslant L$. Take $\rho_2 \geqslant \max\{L \int_{\xi_1}^{\xi_2} G_2(s,s)\mathrm{d}s, \rho_1\}$. For $y \in K$ and $\|y\| = \rho_2$, from (2.6) and Lemma 2.3 we can get

$$-(Ay)''(t) \leqslant \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, y(s), y''(s))\mathrm{d}s \leqslant L \int_{\xi_1}^{\xi_2} G_2(s,s)\mathrm{d}s \leqslant \rho_2^* = \|y\|.$$

Therefore, in either case, we can set $\Omega_{\rho_2} = \{y \in K \,|\, \|y\| < \rho_2\}$ such that

$$\|Ay\| \leqslant \|y\|, \quad \text{for } y \in \partial\Omega_{\rho_2}. \tag{3.7}$$

On the other hand, it is easy to check by the Arzera–Ascoli theorem that the operator A is completely continuous. Hence from (3.2), (3.7) and Theorem 2.1, A has a fixed point y in $\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1}$, y is a positive solution of (1.1) and (1.2) and satisfies $\rho_1 \leqslant \|y\| \leqslant \rho_2$. □

Theorem 3.2. *Assume that $f \in C([0,1] \times [0,\infty) \times (-\infty,0], [0,\infty))$ and f is superlinear, i.e., $\max f_0 = 0$ and $\min f_\infty = +\infty$; then the boundary value problems (1.1) and (1.2) have at least one positive solution.*

Proof. Since $\max f_0 = 0$, then for any ϵ satisfying $\epsilon \int_{\xi_1}^{\xi_2} G_2(s,s)\mathrm{d}s \leqslant 1$, we can choose $\rho_1 > 0$ such that

$$f(t,u,v) \leqslant \epsilon(-v), \quad \text{for } t \in [0,1],\ u \in [0,+\infty),\ 0 \leqslant -v \leqslant \rho_1 \tag{3.8}$$

Set $\Omega_{\rho_1} = \{y \in K \,|\, \|y\| < \rho_1\}$. For $y \in \partial\Omega_{\rho_1}$, we can get from (2.6) and (3.8)

$$\begin{aligned} -(Ay)''(t) &\leqslant \int_{\xi_1}^{\xi_2} G_2(s,s) f(s, y(s), y''(s))\mathrm{d}s \\ &\leqslant \epsilon_1 \int_{\xi_1}^{\xi_2} G_2(s,s)(-y''(s))\mathrm{d}s \\ &\leqslant \|y''\|_\infty = \|y\|, \end{aligned}$$

which implies

$$\|Ay\| \leqslant \|y\|, \quad \text{for } y \in \partial\Omega_{\rho_1}. \tag{3.9}$$

Next, since $\min f_\infty = +\infty$, then for any ϵ^* satisfying $\frac{1}{4}\epsilon^* \int_{\eta_1}^{\eta_2} G_2(\frac{1}{2}(\eta_1+\eta_2), s)\mathrm{d}s \geqslant 1$, there exists $\rho_2 > \rho_1$ such that

$$f(t,u,v) \geqslant \epsilon^*(-v), \quad \text{for } t \in [0,1],\ u \in [0,+\infty),\ -v \geqslant \frac{1}{4}\rho_2. \tag{3.10}$$

Set $\Omega_{\rho_2} = \{y \in K \,|\, \|y\| < \rho_2\}$. For $y \in \partial\Omega_{\rho_2}$, since $y \in K$, we know that $\min_{\eta_1 \leqslant t \leqslant \eta_2}(-y''(t)) \geqslant \frac{1}{4}\|y\|$. Thus, for any $y \in \partial\Omega_{\rho_2}$, we have from (2.6) and (3.10)

$$\begin{aligned} -(Ay)''\left(\frac{1}{2}(\eta_1+\eta_2)\right) &= \int_{\xi_1}^{\xi_2} G_2\left(\frac{1}{2}(\eta_1+\eta_2), s\right) f(s, y(s), y''(s))\mathrm{d}s \\ &\geqslant \int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1+\eta_2), s\right) f(s, y(s), y''(s))\mathrm{d}s \\ &\geqslant \epsilon^* \int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1+\eta_2), s\right)(-y''(s))\mathrm{d}s \end{aligned}$$

$$\begin{aligned}&\geqslant \frac{\epsilon^*\|y\|}{4}\int_{\eta_1}^{\eta_2} G_2\left(\frac{1}{2}(\eta_1+\eta_2), s\right)\mathrm{d}s\\&\geqslant \|y\|,\end{aligned}$$

which implies

$$\|Ay\| \geqslant \|y\|, \quad \text{for } y \in \partial\Omega_{\rho_2}. \tag{3.11}$$

On the other hand, it is easy to check by the Arzera–Ascoli theorem that the operator A is completely continuous. Hence from (3.9), (3.11) and Theorem 2.1, the operator A has a fixed point y in $\bar{\Omega}_{\rho_2} \setminus \Omega_{\rho_1}$, y is a positive solution of (1.1) and (1.2) and satisfies $\rho_1 \leqslant \|y\| \leqslant \rho_2$. $\square$

4. Conclusions

A fourth-order ordinary differential equation with a four-point boundary condition is studied in this work. Some new and simple sufficient conditions for the existence of at least one positive solution are obtained by using the Krasnoselskii fixed point theorem.

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