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Invariants and Related Liapunov Functions for Difference Equations

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Abstract—Consider the difference equation

$$\mathbf{x}_{n+1} = \mathbf{f}(\mathbf{x}_n),$$

where $\mathbf{x}_n$ is in R^k and $\mathbf{f} : D \to D$ is continuous where $D \subset R^k$. Suppose that $I : R^k \to R$ is a continuous invariant, that is, $I(\mathbf{f}(\mathbf{x})) = I(\mathbf{x})$ for every $\mathbf{x} \in D$. We will show that if I attains an isolated minimum or maximum value at the equilibrium (fixed) point $\mathbf{p}$ of this system, then there exists a Liapunov function, namely $\pm(I(\mathbf{x}) - I(\mathbf{p}))$ and so the equilibrium $\mathbf{p}$ is stable. This result is then applied to some difference equations appearing in different fields of applications.

Keywords—Invariants, Stability, Lyness' equation.

1. INTRODUCTION

Consider Lyness' equation

$$x_{n+1} = \frac{a + x_n}{x_{n-1}},$$

where $a > 0$ is a parameter and $x_{-1} > 0$, $x_0 > 0$. Recently, this equation attracted a lot of attention, see [1,2], and its stability has been proved showing that it can be transformed to an area preserving map, see [3]. Also KAM theory has been applied to provide another proof of stability, see [3]. However, no Liapunov function has been found so far. Here we will construct the Liapunov function for this equation, using the well-known invariant:

$$I(x_n, x_{n-1}) = \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)(a + x_n + x_{n-1}).$$

In fact, we will show that the Liapunov function $V(x, y)$ is given by

$$V(x, y) = I(x, y) - I(p, p) = I(x, y) - \frac{(p+1)^3}{p},$$

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where

$$p = \frac{1 \pm \sqrt{1+4a}}{2}$$

is the equilibrium of this equation.

One of the possible third-order generalizations of Lyness' equation is Todd's equation which has the form:

$$x_{n+1} = \frac{a + x_n + x_{n-1}}{x_{n-2}},$$

where $a > 0$ is a parameter and $x_{-1}, x_0, x_1 > 0$. Here we will construct the Liapunov function for this equation using well-known invariant:

$$I(x_n, x_{n-1}, x_{n-2}) = \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_{n-2}}\right)(a + x_n + x_{n-1} + x_{n-2}).$$

In fact, we will show that the Liapunov function $V(x, y, z)$ is given by

$$V(x,y,z) = I(x,y,z) - I(p,p,p) = I(x,y,z) - \frac{(p+1)^4}{p^2},$$

where p is the equilibrium of this equation.

2. MAIN RESULT

For the sake of completeness, we provide the definition of stability that will be used in this paper.

DEFINITION. *(See [4, p. 171].) Let* $\mathbf{u}$ *belong to* $\mathbf{R}^n$ *and* $r > 0$. *The open ball centered at* $\mathbf{u}$ *with radius* r *is the set*

$$B(\mathbf{u}, r) = \{\mathbf{v} \in \mathbf{R}^n : |\mathbf{v} - \mathbf{u}| < r\}.$$

Let $\mathbf{p}$ *be a fixed point of* $\mathbf{f}$. *Then* $\mathbf{p}$ *is stable provided that, given any ball* $B(\mathbf{p}, \epsilon)$, *there is a ball* $B(\mathbf{p}, \delta)$ *so that if* $\mathbf{u}$ *is in* $B(\mathbf{p}, \epsilon)$, *then* $\mathbf{f}^k(\mathbf{p})$ *is in* $B(\mathbf{p}, \epsilon)$ *for* $k = 1, 2, \ldots$. *Here* $\mathbf{f}^k(\mathbf{p})$ *denotes* k^{th} *iterate of* $\mathbf{p}$.

Our main result is the following.

THEOREM. *Consider the difference equation*

$$\mathbf{x_{n+1}} = \mathbf{f}(\mathbf{x_n})$$

where $\mathbf{x_n}$ *is in* R^k *and* $\mathbf{f} : D \to D$ *is continuous, where* $D \subset R^k$. *Suppose that* $I : R^k \to R$ *is a continuous invariant that is* $I(\mathbf{f}(\mathbf{x})) = I(\mathbf{x})$ *for every* $\mathbf{x} \in D$. *If* I *attains an isolated local minimum or maximum value at the equilibrium (fixed) point* $\mathbf{p}$ *of this system, then there exists a Liapunov function equal to* $\pm(I(\mathbf{x}) - I(\mathbf{p}))$ *and so the equilibrium* $\mathbf{p}$ *is stable.*

PROOF. Assume that I attains a local isolated minimum at $\mathbf{p}$. Then there exists some neighborhood D_p of $\mathbf{p}$ such that $I(\mathbf{x}) > I(\mathbf{p})$ for every $\mathbf{x} \in D_p$. In view of this, we have

(1) $V(\mathbf{p}) = I(\mathbf{p}) - I(\mathbf{p}) = 0$,
(2) $V(\mathbf{x}) = I(\mathbf{x}) - I(\mathbf{p}) > 0$, for $\mathbf{x}$ in D_p,
(3) $(V\mathbf{f})(\mathbf{x}) = I(\mathbf{f}(\mathbf{x})) - I(\mathbf{p}) = I(\mathbf{x}) - I(\mathbf{p}) = V(\mathbf{x})$,

where in Point (3), we use the fact that I is an invariant. Thus, all three conditions in [4, p. 177] are satisfied and so V is Liapunov function. Consequently, $\mathbf{p}$ is stable.

The proof in the case where I attains its maximum follows immediately from the result that was just proved by observing the simple fact that negative of an invariant is also an invariant.

REMARK 1. This theorem shows that whenever certain conditions are satisfied, there exists a Liapunov function which is actually a special invariant of considered equation. As is well known, see [5,6], the invariant is not uniquely determined, but rather if one invariant is given, then any appropriate function (continuous, analytic) of this invariant is invariant itself. Furthermore, the difference equation can have several "independent" invariants (e.g., two invariants neither of which can be derived from the other through homeomorphic changes of variable), see [7]. As we will mention later, we have developed *Mathematica* package that can almost automatically find the rational invariant for a given equation (if there is one) and using our result here construct the corresponding Liapunov function, if such function exists.

REMARK 2. An important question that could be raised here is what are the implications of this theorem on the linearized stability theory for this difference equation. In particular, is it true that all the eigenvalues of the linearized system (assuming that $\mathbf{f}$ is differentiable)

$$\mathbf{y_{n+1}} = \mathbf{f}'(\mathbf{p})\mathbf{y_n}$$

necessarily lie on the unit circle? The long standing conjecture is that the difference equation

$$\mathbf{x_{n+1}} = \mathbf{f}(\mathbf{x_n})$$

has an invariant if and only if the corresponding linearized equation has an invariant, which in turn is equivalent to the fact that the matrix $\mathbf{f}'(\mathbf{p})$ does not have all eigenvalues inside or outside unit circle.

3. APPLICATIONS

Here, we will give several equations with known invariants and using the above result we will obtain the Liapunov function, and prove the stability of their equilibria. We note that the positive definiteness of the Hessian at the critical point is verified by checking that all the principal minors are positive.

APPLICATION 1. Lyness' equation (see [1,3]):

$$x_{n+1} = \frac{a + x_n}{x_{n-1}},$$

where $a > 0$ is a parameter and $x_1 > 0$, $x_0 > 0$.

Invariant:

$$I(x_n, x_{n-1}) = \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)(a + x_n + x_{n-1}).$$

Equilibrium: $p = (1 \pm \sqrt{1+4a})/2$ satisfies $p^2 - p - a = 0$.

The necessary conditions for the extremum give:

$$\frac{\partial I}{\partial x} \equiv \left(1 + \frac{1}{y}\right)\left(1 - \frac{y+a}{x^2}\right) = 0,$$

$$\frac{\partial I}{\partial y} \equiv \left(1 + \frac{1}{x}\right)\left(1 - \frac{x+a}{y^2}\right) = 0,$$

which leads to $x = y$ and to the equation $x^2 - x - a = 0$. This shows that the critical points are exactly the equilibrium points. Now, we will check the Hessian at a positive critical point $p_+ = (1 + \sqrt{1+4a})/2$, which will be denoted as p

$$H = \begin{pmatrix} A & B \\ B & C \end{pmatrix},$$

where

$$A = \frac{\partial^2 I}{\partial x^2}(p,p) = 2\,\frac{p+1}{p^3} > 0,$$

$$B = \frac{\partial^2 I}{\partial x \partial y}(p,p) = -\frac{p+1}{p^3},$$

and

$$C = \frac{\partial^2 I}{\partial y^2}(p,p) = 2\,\frac{p+1}{p^3}.$$

Now,

$$\det H = AC - B^2 = \frac{3(p+1)^2}{p^6} > 0,$$

which implies that the invariant I attains its minimum at (p,p). Thus,

$$\min\{I(x,y) : (x,y) \in D\} = I(p,p) = \frac{(p+1)^3}{p},$$

and so by our theorem $V(x,y) = I(x,y) - (p+1)^3/p$, which shows that p is stable.

APPLICATION 2. Todd's equation (see [1]):

$$x_{n+1} = \frac{a + x_n + x_{n-1}}{x_{n-2}},$$

where $a > 0$ is a parameter, and $x_1 > 0$, $x_0 > 0$, $x_1 > 0$.

Invariant:

$$I(x_n, x_{n-1}, x_{n-2}) = \left(1 + \frac{1}{x_n}\right)\left(1 + \frac{1}{x_{n-1}}\right)\left(1 + \frac{1}{x_{n-2}}\right)(a + x_n + x_{n-1} + x_{n-2}).$$

Equilibrium: $p = 1 \pm \sqrt{1+a}$ satisfies $p^2 - 2p - a = 0$.

The necessary conditions for the extremum give:

$$\frac{\partial I}{\partial x} \equiv \left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right)\left(1 - \frac{a+y+z}{x^2}\right) = 0,$$

$$\frac{\partial I}{\partial y} \equiv \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{z}\right)\left(1 - \frac{a+x+z}{y^2}\right) = 0,$$

and

$$\frac{\partial I}{\partial z} \equiv \left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 - \frac{a+x+y}{z^2}\right) = 0.$$

This leads to $x = y = z$, and to the equation $x^2 - 2x - a = 0$, which shows that the critical points are exactly the equilibrium points. Now, we will check the Hessian at a positive critical point (p,p,p) where $p = 1 + \sqrt{1+a}$.

$$H = \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix},$$

where

$$A = \frac{\partial^2 I}{\partial x^2}(p,p) = \frac{\partial^2 I}{\partial y^2}(p,p) = \frac{\partial^2 I}{\partial z^2}(p,p) = 2\,\frac{p+1^2}{p^3} > 0$$

and

$$B = \frac{\partial^2 I}{\partial x \partial y}(p,p) = \frac{\partial^2 I}{\partial y \partial z}(p,p) = \frac{\partial^2 I}{\partial x \partial z}(p,p) = -\frac{p+1^2}{p^4}.$$

Now,

$$\det H_1 = A > 0, \qquad \det H_2 = \det\begin{pmatrix} A & B \\ B & A \end{pmatrix} = \frac{(p+1)^4}{p^8}(4p^2 - 1) > 0,$$

and

$$\det H = \det \begin{pmatrix} A & B & B \\ B & A & B \\ B & B & A \end{pmatrix} = (A-B)^2(A+2B) = 2\sqrt{1+a}\,(2p+1)^2\,\frac{(p+1)^6}{p^{12}} > 0,$$

which shows, that the invariant I attains its minimum at (p,p,p). Thus,

$$\min\{I(x,y,z) : (x,y,z) \in D\} = I(p,p,p) = \frac{(p+1)^4}{p^2},$$

and so by our theorem $V(x,y,z) = I(x,y,z) - (p+1)^4/p^2$, which shows, that p is stable.

APPLICATION 3. $k+2^{\text{th}}$-order Lyness' equation (see [1]):

$$x_{n+1} = \frac{a + x_n + x_{n-1} + \cdots + x_{n-k}}{x_{n-k-1}},$$

where $a > 0$ is a parameter, and $x\_k > 0, \ldots, x\_1 > 0$.

Invariant:

$$I(x_n, x_{n-1}, \ldots, x_{n-k-1}) = \left(1+\frac{1}{x_n}\right)\left(1+\frac{1}{x_{n-1}}\right)\cdots\left(1+\frac{1}{x_{n-k-1}}\right)$$
$$(a + x_n + x_{n-1} + \cdots + x_{n-k-1}).$$

Equilibrium: $p = (k+1 \pm \sqrt{(k+1)^2+4a})/2$ satisfies $p^2 - (k+1)p - a = 0$. The necessary conditions for the extremum give:

$$\frac{\partial I}{\partial x_{n-i}} \equiv \prod_{j=0,j\neq i}^{k+1}\left(1+\frac{1}{x_{n-j}}\right)\left(1 - \frac{a + \sum\limits_{j=0,j\neq i}^{k+1} x_{n-j}}{x_{n-i}^2}\right) = 0, \qquad i = 1,\ldots,k+1,$$

which gives

$$x_{n-i}^2 - a - \sum_{j=0,j\neq i}^{k+1} x_{n-j} = 0, \qquad i = 1,\ldots,k+1.$$

Subtracting the consecutive equations we get $x_n = x_{n-1} = \cdots = x_{n-k-1} = p$, which shows that the critical point is $(p,p,\ldots,p)$.

Now, we will check the Hessian at a positive critical point $(p_+,\ldots,p_+)$, which for simplicity will be denoted by $(p,\ldots,p)$. Computing the second order derivatives of I with respect to x_{n-i} we get

$$\frac{\partial^2 I}{\partial x_{n-i}^2} = \frac{2}{x_n^3}\left(a + \sum_{j=0,j\neq i}^{k+1} x_{n-j}\right)\prod_{j=0,j\neq i}^{k+1}\left(1+\frac{1}{x_{n-j}}\right),$$

and

$$\frac{\partial^2 I}{\partial x_{n-i}\partial x_{n-j}} = \prod_{p=0,p\neq i,j}^{k+1}\left(1+\frac{1}{x_{n-p}}\right)\left(\frac{1}{x_{n-i}^2} + \frac{1}{x_{n-j}^2} - \frac{a + \sum\limits_{p=0,p\neq i,j}^{k+1} x_{n-p}}{x_{n-i}^2 x_{n-j}^2}\right);$$

$i = 1,\ldots,k+1$ at the equilibrium $(p,p,\ldots,p)$, we get

$$A = \frac{\partial^2 I}{\partial x_{n-i}^2} \mid (p,p,\ldots,p) = 2\,\frac{(p+1)^{k+1}}{p^{k+2}} > 0$$

and

$$B = \frac{\partial^2 I}{\partial x_{n-i}\partial x_{n-j}} \mid (p, p, \ldots, p) = -\frac{(p+1)^{k+1}}{p^{k+3}}.$$

Thus, the Hessian takes the form

$$H = \begin{pmatrix} A & B & \ldots & B \\ B & A & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & A \end{pmatrix}.$$

Now,

$$\begin{aligned} \det H_1 &= A > 0, \\ \det H_m &= \det \begin{pmatrix} A & B & \ldots & B \\ B & A & \ldots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \ldots & A \end{pmatrix} = (A-B)^{m+1}[A + (m-1)B] \\ &= \frac{(p+1)^{(k+1)(k+2)}}{p^{(k+3)(k+2)}} (2p+1)^{k+1} \left(2 + \sqrt{(k+1)^2 + 4a}\right) > 0, \end{aligned}$$

where H_m is m^{th}-order principal minor of the Hessian H.

This shows that the invariant I attains its minimum at $(p, p, \ldots, p)$. Thus,

$$\min\{I(x_n, x_{n-1}, \ldots, x_{n-k-1}) : (x_n, x_{n-1}, \ldots, x_{n-k-1}) \in D\} = I(p, p, \ldots, p) = \frac{(p+1)^{k+3}}{p^{k+1}}$$

and so, by our theorem

$$V(x_n, x_{n-1}, \ldots, x_{n-k-1}) = I(x_n, x_{n-1}, \ldots, x_{n-k-1}) - \frac{(p+1)^{k+3}}{p^{k+1}},$$

which shows, that p is stable.

Obviously, for $k = 0$ and $k = 1$ we get Lyness' and Todd's equation, respectively, and so this example generalizes the Examples 1 and 2. Another generalization of Lyness' equation is given in the following example.

APPLICATION 4. Generalized Lyness' equation (see [1,2,8]):

$$x_{n+1} = \frac{ax_n + b}{(cx_n + d)x_{n-1}},$$

where a, b, c, d are positive parameters and $x_{-1} > 0$, $x_0 > 0$.

Invariant:

$$\begin{aligned} I(x_n, x_{n-1}) = \frac{ab}{x_{n-1}x_n} + (a^2 + bd)\left(\frac{1}{x_n} + \frac{1}{x_{n-1}}\right) + ad\left(\frac{x_{n-1}}{x_n} + \frac{x_n}{x_{n-1}}\right) \\ + (ac + d^2)(x_n + x_{n-1}) + cdx_{n-1}x_n. \end{aligned}$$

Equilibrium: p satisfies $cp^3 + dp^2 - ap - b = 0$. Obviously, this equation has at least one positive solution and we will investigate the stability of this equilibrium. The necessary conditions for the extremum give:

$$\frac{\partial I}{\partial x} \equiv -\frac{ab}{x^2 y} + cdy - \frac{a^2 + bd}{x^2} + ad\left(\frac{1}{y} - \frac{y}{x^2}\right) + ac + d^2 = 0$$

and

$$\frac{\partial I}{\partial y} \equiv -\frac{ab}{xy^2} + cdx - \frac{a^2+bd}{y^2} + ad\left(\frac{1}{x} - \frac{x}{y^2}\right) + ac + d^2 = 0,$$

which after some manipulations leads to $x = y$ and to the equation

$$cdx^4 + \left(ac + d^2\right)x^3 - \left(a^2 + bd\right)x - ab = (dx + a)\left(cx^3 + dx^2 - ax - b\right) = 0.$$

This shows that the critical points are exactly the equilibrium points. Now, we will check the Hessian at a positive critical point (p_+, p_+) which will be denoted simply as (p, p):

$$H = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \frac{\partial^2 I}{\partial x^2}(p,p) = \frac{2}{p^4}\left[adp^2 + \left(a^2 + bd\right)p + ab\right] > 0$$

and

$$B = \frac{\partial^2 I}{\partial x \partial y}(p,p) = \frac{cdp^4 - 2adp^2 + ab}{p^4}.$$

Now,

$$\det\ H = A^2 - B^2 = (A-B)(A+B) = \frac{1}{p^8}\left[cdp^4 + 2\left(a^2+bd\right)p + 3ab\right]E,$$

where by using the equilibrium equation

$$E = -cdp^4 + 4adp^2 + 2\left(a^2+bd\right)p + ab = d^2p^3 + 3adp^2 + 2a^2p + ab > 0,$$

which shows that the Hessian H is positive definite at the point (p,p), which means that the invariant I attains its minimum at (p,p). Thus,

$$\min\{I(x,y) : (x,y) \in D\} = I(p,p),$$

and so by our theorem $V(x,y) = I(x,y) - I(p,p)$ which shows that p is stable.

APPLICATION 5. Gumovski-Mira equation (see [5,6,9]):

$$x_{n+1} = \frac{2ax_n}{1 + x_n^2} - x_{n-1},$$

where $a > 1$ is a parameter.

Invariant:

$$I(x_n, x_{n-1}) = {x_n}^2{x_{n-1}}^2 + {x_n}^2 + {x_{n-1}}^2 - 2ax_nx_{n-1}.$$

Equilibrium: $p = \sqrt{a-1} > 0$ satisfies $p^2 = a - 1$.

The necessary conditions for the extremum give:

$$\frac{\partial I}{\partial x} \equiv 2xy^2 + 2x - 2ay = 0$$

and

$$\frac{\partial I}{\partial y} \equiv 2x^2y + 2y - 2ax = 0,$$

which immediately leads to $x = y = p$. This shows that the critical point is exactly the equilibrium point. Now, we will check the Hessian at a positive critical point (p,p).

$$H = \begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

where

$$A = \frac{\partial^2 I}{\partial x^2}(p,p) = 2p^2 + 2 > 0$$

and

$$B = \frac{\partial^2 I}{\partial x \partial y}(p,p) = 4p^2 - 2a.$$

Now, by using the equilibrium equation we get

$$\det H = A^2 - B^2 = (A-B)(A+B) = 4\left(a+1-p^2\right)\left(3p^2+1-a\right) = 16(a-1) > 0.$$

This shows that the Hessian H is positive definite at the point (p,p) which means that the invariant I attains its minimum at (p,p). Thus,

$$\min\{I(x,y) : (x,y) \in D\} = I(p,p) = -(a-1)^2,$$

and so by our theorem $V(x,y) = I(x,y) + (a-1)^2$, which shows, that p is stable.

REMARK 3. It is worth mentioning that all equations with their invariants were known, and that we have proved the stability of the equilibrium point using our result. The result that we have proved emphasizes the importance of the invariants and of finding some general methods for their construction. Especially we are interested in automatic construction of the invariants and corresponding Liapunov functions, (if they exist) that can be implemented by using some of CAS (Computer Algebra System). In this regard, we have developed *Mathematica* package, see [10], which was successfully tested on all applications of this paper. More applications of this package will be presented in the coming communications.

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