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An equivalence relation in finite planar spaces ☆

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Abstract

This paper is concerned with quasiparallelism relation in a finite planar space **S**. In particular, we prove that if no plane in **S** is the union of two lines quasiparallelism relation between planes is an equivalence relation. Moreover, three-dimensional affine spaces with a point at infinity and three-dimensional affine spaces with a line at infinity are characterized.

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1. Introduction

A *linear space* **S** is a pair (P, L), where P is a non-empty set of *points* and L is a family of proper subsets of P, called *lines*, such that any two distinct points x and y belong to a unique line xy, every line contains at least two points and there are at least two lines. We denote by v the number of its points and by b the number of its lines.

A *subspace* X of **S** is a subset of P such that the line joining any two distinct points of X is already contained in X. It is easy to see that the empty set, a point, a line and P are subspaces. Moreover, the intersection of any family of subspaces is a subspace. Thus, any subset Y of P spans the subspace $\langle Y\rangle$, intersection of all the subspaces containing Y.

An *independent set* Z is a set of points such that for each $z \in Z$, $z \notin \langle Z\backslash\{z\}\rangle$. A *generator* of **S** is a subset G such that $\langle G\rangle = P$ and a basis of **S** is an independent subset of the points of **S** which generates **S**. Moreover, the *dimension* of **S** is the integer dim $\mathbf{S} = \min(|B| : B$ is a basis for $\mathbf{S})-1$ (see [1]).

Let $\mathbf{S} = (P, L)$ be a finite linear space. The *length* of a line l is the number $|l|$ of points on l. Every line of maximal size will be called a *long* line. All other lines will be called *short* lines. The *degree* of a point x is the number b_x of lines on which it lies. If $n + 1 = \max(b_x, x \in P)$, then the integer n is called the *order* of **S**. Let X be a subspace through x. We denote by $b_x(X)$ the number of lines through x on X. The integer $b_x(X)$ is the degree of x in X.

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A *planar space* $\mathbf{S}$ is a triple (P, L, P^*), where (P, L) is a linear space and P^* is a non-empty family of subspaces, called *planes*, such that:

- every plane contains at least three non-collinear points;
- any three non-collinear points lie in exactly one plane and it is the smallest subspace containing them;
- there are at least two planes.

Let $\mathbf{S} = (P, L, P^*)$ be a finite planar space. If π is a plane of $\mathbf{S}$, we denote by $|\pi|$ the number of its points and by b_π the number of its lines. Consider the family L_π of lines on the plane π. The pair (π, L_π) is a finite linear space on $|\pi|$ points and b_π lines. If $n(\pi)$ denotes the *order* of linear space (π, L_π), the integer $n = \max\{n(\pi) : \pi \in P^*\}$, is called *order* of the planar space.

The subspaces X and X' are *quasiparallel* if $|X \cap l| = |X' \cap l|$ for all lines $l \not\subseteq X \cup X'$ (see [9]). Let X and X' be two quasiparallel subspaces in a finite linear space $\mathbf{S}$. The following results hold (see [9]):

(i) $b_x = b_x(X) + |X'| - |X \cap X'|$ for every $x \in X \backslash X'$.
(ii) $|X| = |X'|$ if $X \cup X' \neq P$.

In the following we shall use the notation $X \sim X'$ to mean that the subspaces X and X' are quasiparallel.

In a plane π of $\mathbf{S}$ the relation of quasiparallelism for lines is an equivalence relation. In the following the equivalence class of quasiparallel lines which contains a line l will be denoted by $[l]$. An equivalence class $[l]$ will be called a *singleton* if $[l]$ contains the unique line l. Let $\mathbf{S} = (P, L)$ be a finite linear space. A family of pairwise intersecting lines, with at least three non-concurrent lines, will be called a *clique* of lines. A *pencil* of lines is a family of lines passing through a common point. Finally, a *spread* of lines is a family of pairwise disjoint lines. An equivalence class $[l]$ of quasiparallel lines, with size at least 2, is one of the following (see [5]):

(I) $[l]$ is a clique, also called a C-*class* (or *clique-class*);
(II) $[l]$ is a pencil of centre x, also called a P-*class* (or *pencil-class*) *of centre* x;
(III) $[l]$ is a spread, also called a S-*class* (or *spread-class*).

In this paper we want to study quasiparallelism relation for planes in a finite planar space $\mathbf{S} = (P, L, P^*)$ with the following property:

$$\text{(A)} \quad \forall \alpha \in P^*, \quad \forall r, s \in L_\alpha, \quad \alpha \neq r \cup s.$$

In particular, we shall prove that for planes in a finite planar space fulfilling the condition (A) the quasiparallelism relation is an equivalence relation. Furthermore, three-dimensional affine spaces with a point at infinity and three-dimensional affine spaces with a line at infinity are characterized.

2. The equivalence relation

Let $\mathbf{S}$ be a finite planar space with property (A). In $\mathbf{S}$ the following condition holds

$$\text{(B)} \quad \forall \alpha, \beta \in P^*, \quad P \neq \alpha \cup \beta.$$

By Theorem 3.1 in [8] we have

$$\pi \sim \pi' \quad \Leftrightarrow \quad |\alpha \cap \pi| = |\alpha \cap \pi'| \tag{2.1}$$

for every plane α such that $\pi \neq \alpha \neq \pi'$. Consider a plane π in $\mathbf{S}$. In [7] we denoted by (π) the subset of P^* which contains π and each plane π' such that $\pi \sim \pi'$. Furthermore, in [7] we showed that, if there exists in (π) a plane π' with $\pi \cap \pi' = \emptyset$, planes in (π) are pairwise disjoint and quasiparallel to each other. Moreover, in [8] we proved that, if π and π' are two quasiparallel planes which intersect in a line, planes in (π) are pairwise quasiparallel and any two distinct planes in (π) intersect in a line of constant length.

Proposition 2.1. *If π' is a plane in (π) such that $\pi \cap \pi'$ is a point x, any two distinct planes in (π) intersect in point x.*

Proof. Let π'' be a plane in (π) with $\pi \neq \pi'' \neq \pi'$. Since π and π'' are quasiparallel, by (2.1) we obtain

$$|\pi \cap \pi'| = |\pi'' \cap \pi'|.$$

Hence, π'' intersects π' in a point y. We suppose $y \neq x$. Every line of π' through x intersects π'' while the unique line on π' containing x and intersecting π'' is the line xy. This forces $x = y$ and $\pi' \cap \pi'' = \{x\}$. Moreover, by $\pi \sim \pi'$ we deduce

$$|\pi \cap \pi''| = |\pi' \cap \pi''|$$

and $\pi \cap \pi'' = \{x\}$. Hence, any two distinct planes in (π) intersect in point x and the proposition is proved. □

Proposition 2.2. *Let π' and π'' two distinct planes of (π). If π' intersects π in a point x, then π' and π'' are quasiparallel.*

Proof. Let π'' be a plane in (π), with $\pi \neq \pi''$. At once (B) and (ii) yield $|\pi| = |\pi'| = |\pi''|$. We want to show that π' and π'' are quasiparallel. In other words, we shall prove that $|\pi' \cap r| = |\pi'' \cap r|$ for every line $r \not\subseteq \pi' \cup \pi''$. By Proposition 2.1 we have $\pi \cap \pi'' = \pi' \cap \pi'' = \{x\}$. If r meets π' in x, r meets both planes, π' and π''. Now, we suppose that r meets π' in a point y, with $y \neq x$. By (i) we deduce that the lines through y outside π' are $b_y - b_y(\pi') = |\pi| - 1$. On the other hand, the lines joining y to points which lie on $\pi'' \backslash \pi'$ are $|\pi''| - |\pi' \cap \pi''| = |\pi| - 1$. Hence, every line which meets π' in y, meets π'', too. Analogously, if r is a line which intersects π'', r meets π', too. The proof is complete. □

Now, we are ready to state our first result.

Theorem I. *Let* **S** *be a finite planar space with property* (A). *For planes quasiparallelism relation is an equivalence relation.*

Proof. Let π and π' be two quasiparallel planes in **S**. If $\pi \cap \pi' = \emptyset$, by Propositions 4.1, 4.2 and 4.3 in [7], planes in (π) are pairwise quasiparallel. Now, we assume that $\pi \cap \pi'$ is a point x. By Proposition 2.2 planes in (π) are pairwise quasiparallel. Finally, if π and π' intersect in a line, from Proposition 4.3 in [8] we deduce that planes in (π) are pairwise quasiparallel. Hence, the desired result follows. □

In the following the equivalence class of quasiparallel planes which contains a plane π will be denoted by $[\pi]$. An equivalence class $[\pi]$ will be called a *singleton* if $[\pi]$ contains the unique plane π. Moreover, a family of pairwise disjoint planes will be called a *spread*, a family of planes such that any two distinct planes intersect in a common point x will be called a *star* of planes of *centre* x. A *pencil* of planes *of axis* l is a family of planes with a common line l. A *clique of centre* x of planes is a family of planes such that point x lies on all planes of the family and any two distinct planes intersect in a line through x of constant length. Finally, a *clique* of planes is a family of planes such that any two distinct planes intersect in a line of constant length and there are at least three planes which have no common point. Therefore, the results in [7,8] and Propositions 2.1, 2.2 imply the following:

Proposition 2.3. *Let* **S** *be a finite planar space having property* (A). *If π is a plane in* **S**, *the equivalence class of quasiparallel planes* $[\pi]$ *is one of the following*:

(a) $[\pi]$ *is a singleton*;
(b) $[\pi]$ *is a spread*, *also called a S-class* (*or spread-class*);
(c) $[\pi]$ *is a star of centre x*, *also called a S^*-class* (*or star-class*) *of centre x*;
(d) $[\pi]$ *is a pencil of axis l*, *also called a P-class* (*or pencil-class*) *of axis l*;
(e) $[\pi]$ *is a clique of centre x*, *also called a C^*-class* (*or clique-star-class*) *of centre x*;
(f) $[\pi]$ *is a clique*, *also called a C-class* (*or clique-class*).

3. Planes which intersect in a line

Let **S** be a finite planar space fulfilling property (A). We state here the results concerning the intersection of two non-quasiparallel planes. In [7] we showed that if $[\alpha]$ and $[\beta]$ are two distinct S-*classes* of quasiparallel planes, every plane in $[\alpha]$ has at least a point in common with every plane in $[\beta]$. Moreover, in [8] we proved that if $[\sigma]$ is a S-*class*, for every plane π with $\pi \notin [\sigma]$, either $\sigma \cap \pi = \emptyset$ or $\sigma \cap \pi$ is a line. Hence, every plane in $[\alpha]$ meets every plane in $[\beta]$ in a line. Furthermore, by Proposition 4.8 in [8] we see that if $[\sigma]$ is a S-*class* and $[\pi]$ is a C-*class* of quasiparallel planes, each plane σ' in $[\sigma]$ meets each plane π' in $[\pi]$ in a line of constant length.

Now, we state the following general results.

Proposition 3.1. *Let* $[\sigma]$ *be a S-class of quasiparallel planes in* **S**. *If* $[\pi]$ *is a class of quasiparallel planes, with size at least two and* $\sigma \notin [\pi]$, *each plane* σ' *in* $[\sigma]$ *meets every plane* π' *in* $[\pi]$ *in a line of constant length.*

Proof. By Proposition 4.7 in [8] either $\sigma \cap \pi = \emptyset$ or $\sigma \cap \pi$ is a line. First,we assume $\sigma \cap \pi = \emptyset$. Let x be a point in σ. From (B), (i) and (ii) we obtain

$$b_x = b_x(\sigma) + |\sigma|.$$

Hence, the number of lines through x outside σ is $|\sigma|$. Since the number of lines joining x to points on π is $|\pi|$, we have

$$|\pi| \leqslant |\sigma|.$$

Now, we denote by y a point on π, with $y \notin \pi'$ and $\pi' \in [\pi]$. We infer

$$b_y = b_y(\pi) + |\pi| - |\pi \cap \pi'|,$$

$$|\sigma| \leqslant |\pi| - |\pi \cap \pi'|.$$

Thus, we find

$$|\sigma| \leqslant |\pi| - |\pi \cap \pi'| \leqslant |\pi| \leqslant |\sigma|,$$

which implies $|\sigma| = |\pi|$ and $\pi \cap \pi' = \emptyset$. Therefore, $[\pi]$ is a S-*class* of quasiparallel planes, also. By Proposition 4.4 in [7] σ has at least a point in common with π, a contradiction. Hence, $\sigma \cap \pi$ is a line. Moreover, from Theorem 3.1 in [8] we obtain

$$|\sigma \cap \pi| = |\sigma \cap \pi'| = |\sigma' \cap \pi'| = |\sigma' \cap \pi|$$

and the proposition is proved. □

Proposition 3.2. *Let* $[\alpha]$ *and* $[\beta]$ *be two distinct equivalence classes of quasiparallel planes in* **S**, *with size at least two. Every plane* α' *in* $[\alpha]$ *meets every plane* β' *in* $[\beta]$ *in a line of constant length.*

Proof. If either $[\alpha]$ or $[\beta]$ is a S-*class* the desired result follows from Proposition 3.1. Now, we suppose that neither $[\alpha]$ nor $[\beta]$ is a S-*class*. Moreover, we suppose $\alpha \cap \beta = \emptyset$. We know that

$$b_y = b_y(\alpha) + |\alpha| - |\alpha \cap \alpha'| \tag{3.1}$$

with $\alpha' \in [\alpha]$ and $y \in \alpha \backslash \alpha'$. Analogously, if β' belongs to $[\beta]$ and z denotes a point on $\beta \backslash \beta'$, we infer that

$$b_z = b_z(\beta) + |\beta| - |\beta \cap \beta'|. \tag{3.2}$$

We see, also, that

$$|\alpha| - |\alpha \cap \alpha'| \geqslant |\beta| \quad \text{and} \quad |\beta| - |\beta \cap \beta'| \geqslant |\alpha|.$$

Hence,

$$|\beta| \leqslant |\alpha| - |\alpha \cap \alpha'| < |\alpha| \leqslant |\beta| - |\beta \cap \beta'| < |\beta|,$$

which is a contradiction.Thus, planes α and β have at least a point in common. We suppose $\alpha \cap \beta = \{x\}$. Since $|\alpha \cap \beta| = |\alpha' \cap \beta|$, α' and β intersect in a unique point x'. From $\alpha \sim \alpha'$ we deduce that each line through x on β contains x', also. This forces $x = x'$. Therefore, by using Theorem 3.1 in [8], we have $|\alpha \cap \beta| = |\alpha \cap \beta'|$. If x'' denotes the unique point which is common to α and β', from the foregoing, we obtain $x = x' = x''$. In order to show the proposition, we consider a point y on α, with $y \notin \alpha'$, and a point z on β, with $z \notin \beta'$. From (3.1) and (3.2) we infer that

$$|\alpha| - |\alpha \cap \alpha'| \geqslant |\beta| - 1 \quad \text{and} \quad |\beta| - |\beta \cap \beta'| \geqslant |\alpha| - 1,$$

$$|\alpha| - 1 \geqslant |\alpha| - |\alpha \cap \alpha'| \geqslant |\beta| - 1 \geqslant |\beta| - |\beta \cap \beta'| \geqslant |\alpha| - 1$$

which means that

$$|\alpha \cap \alpha'| = 1 = |\beta \cap \beta'| \quad \text{and} \quad |\alpha| = |\beta|.$$

Hence, the equivalence classes $[\alpha]$ and $[\beta]$ are S*-*classes of same centre* x. Furthermore, every line which meets α in a point y meets β, also, and each line which intersects β meets α. It follows that α and β are quasiparallel, a contradiction. We conclude that $\alpha \cap \beta$ is a line and by Theorem 3.1 in [8] every plane in $[\alpha]$ has a line of constant length $|\alpha \cap \beta|$ in common with each plane in $[\beta]$. □

4. Three-dimensional affine spaces with a point at infinity

A linear space of dimension two is called an *affine-projective plane* [11] or *linearly h-punctured projective plane* if it is a projective plane deprived of h collinear points. In particular, an *affine plane* of order n is obtained from a projective plane of order n by deleting one line l. The $n+1$ points of l are called the *points at infinity* of the affine plane. An affine plane of order n $(n \geqslant 2)$ is a finite linear space with every line of length n and every point of degree $n+1$. An *affine plane with a point at infinity* of order n is a linear space which is obtained from a projective plane of order n by removing n collinear points. We know that an affine plane of order n with a point x at infinity has n^2+1 points and n^2+n lines, x has degree n, all the other points have degree $n+1$, lines through x have length $n+1$, lines which miss x have length n.

A three-*dimensional affine space* is a three-dimensional projective space deprived of a plane π. In an affine space **S** of order n all lines have the same length n. The points of π are called the *points at infinity* of the affine space **S**. A three-*dimensional affine space* **S** of order *n with a point x at infinity* is a three-dimensional projective space of order n deprived of a plane π passing through x together with its points different from x. All the planes through x are affine planes of order n with point x at infinity, all the other planes are affine planes of order n. It is clear that planes through x are contained in $n+1$ S*-*classes of same centre* x and planes, which miss point x, belong to n^2 S-*classes*.

Proposition 4.1. *There exists no finite linear space where there are two distinct P-classes of quasiparallel lines with same centre x.*

Proof. Let **S** be a finite linear space. We denote by $[r]$ and $[s]$ two distinct P-*classes* of quasiparallel lines in **S**, both *of centre* x. By Proposition 2.2 in [5] lines r and s have same length $n+1$, each point y on r, with $y \neq x$, has degree $n+1$ and each point z on s, with $z \neq x$, has degree $n+1$. Thus, every line which meets r, intersects s, also and vice versa. Hence, r and s are quasiparallel and we have a contradiction. □

Now, we can state our result.

Theorem II. *Let* **S** *be a finite planar space satisfying property* (A) *and such that there is at least a line l of length* $|l| \geqslant 5$. *If equivalence classes of quasiparallel planes in* **S** *are only S*-classes and S-classes, there are* $n+1$ *S*-classes with same centre x and* n^2 *S-classes of quasiparallel planes and S is a three-dimensional affine space of order n with a point x at infinity.*

Proof. We proceed in steps.

Step 1: Let $[\sigma_i]$ be a S-*class* of quasiparallel planes in **S**. If $[\sigma_j]$ denotes a S-*class* of quasiparallel planes, with $[\sigma_i] \neq [\sigma_j]$, all the planes in $[\sigma_j]$ intersect σ_i in quasiparallel disjoint lines which belong to a same equivalence S-*class* $[s_j]$ of quasiparallel lines in σ_i. Now, let $[\alpha_h]$ be a S*-*class* of quasiparallel planes, *of centre* x_h. By Proposition 3.1 all the planes in $[\alpha_h]$ meet σ_i in quasiparallel lines which are contained in a same equivalence class of quasiparallel lines in σ_i, $[a_h]$. If x_h lies on σ_i, $[a_h]$ is a P-*class of centre* x_h. Conversely, if $x_h \notin \sigma_i$, $[a_h]$ is a S-*class*. Let s be a line on σ_i.

We denote by β a plane through s, with $\beta \neq \sigma_i$. By hypothesis, equivalence class $[\beta]$ is either a S-*class* or a S*-*class*. Hence, the equivalence class $[s]$ of quasiparallel lines in σ_i is either a S-*class* or a P-*class*. By Proposition 2.9 in [5] and Proposition 4.1 there exists at most one P-*class* in plane σ_i. Therefore, either each class of quasiparallel lines in σ_i is a S-*class* and σ_i is an affine plane (see Theorem 2 in [5]) or there exists a unique P-*class* and σ_i is an affine plane with a point at infinity [5, Theorem 5]. In particular, if centre x_h of a S^*-*class* $[\alpha_h]$ belongs to σ_i, σ_i is an affine plane with point x_h at infinity. Furthermore, *each plane in a* S-*class of quasiparallel planes is either an affine plane or an affine plane with a point at infinity*.

Step 2: Let $[\sigma_i]$ be a S-*class* of quasiparallel planes in **S**. There are two distinct cases to be considered.

Case 1: We assume that σ_i is an affine plane of order n with point x_h at infinity, where x_h is centre of a S*-*class* $[\alpha_h]$. The lines through x_h on σ_i have length $n+1$ and are long lines of σ_i. Instead, the lines on σ_i which miss x_h have length n and are short lines. Let σ_i' be a plane of order p in $[\sigma_i]$, with $\sigma_i \neq \sigma_i'$. By (B) and (ii) we have $|\sigma_i| = |\sigma_i'|$. Hence, if σ_i' is an affine plane, we obtain $n^2 + 1 = p^2$, a contradiction. Therefore, every plane in $[\sigma_i]$ is an affine plane of order n with a point at infinity. Moreover, since σ_i and σ_i' are disjoint, σ_i' has a point x_k at infinity, with $x_h \neq x_k$. From the foregoing, we see that x_k is centre of a S*-*class* $[\alpha_k]$ of quasiparallel planes. The plane α_h meets σ_i in a long line of length $n+1$. Since σ_i and σ_i' are quasiparallel planes, by Theorem 3.1 in [8] we deduce that $\alpha_h \cap \sigma_i'$ is a long line of length $n+1$, also. Furthermore, since long lines on σ_i' pass through x_k, we obtain that x_k lies on α_h. Let α_h' be a plane in $[\alpha_h]$, with $\alpha_h \neq \alpha_h'$. By Theorem 3.1 in [8] we have that the lines $\alpha_h' \cap \sigma_i$ and $\alpha_h' \cap \sigma_i'$ have length $n+1$. Therefore, x_k belongs to α_h' and α_h meets α_h' in the line $x_h x_k$, a contradiction.

Case 2: We assume that σ_i is an affine plane of order n. All the planes in $[\sigma_i]$ are affine planes of order n. Let $[\sigma_j]$ be a S-*class* of quasiparallel planes, with $[\sigma_i] \neq [\sigma_j]$. From the foregoing, all the planes in $[\sigma_j]$ are affine planes. Moreover, $\sigma_i \cap \sigma_j$ is a line of length n. Hence, *any plane* σ_i *which belongs to a* S-*class of quasiparallel planes is an affine plane of order n and no centre of a* S*-*class lies on* σ_i.

Step 3: Let $[\alpha_h]$ be a S*-*class* of quasiparallel planes *of centre* x_h. Planes in $[\sigma_i]$ intersect α_h in disjoint quasiparallel lines of length n, which belong to a S-*class*. Let $[\alpha_k]$ be a S*-*class of centre* x_k such that $\alpha_h \notin [\alpha_k]$. By Proposition 3.2 planes in $[\alpha_k]$ meet α_h in quasiparallel lines which belong to an equivalence class $[a_k]$. Furthermore, if x_k lies on α_h, $[a_k]$ is a P-*class*. Conversely, if $x_k \notin \alpha_h$, $[a_k]$ is a S-*class*. Since in a plane there exists at most one P-*class*, α_h is either an affine plane or an affine plane with a point x_k at infinity. Let α_h be an affine plane of order p with a point x_k at infinity, where x_k is centre of a S^*-*class* $[\alpha_k]$. The long lines on α_h have length $p+1$ and pass through x_k. The short lines have length p and miss x_k. If σ_i is an affine plane in a S-*class* of quasiparallel planes, we have $x_k \notin \sigma_i$ and $\sigma_i \cap \alpha_h$ is a short line of length $p = n$. Hence, α_h has order n. Therefore, *every plane in a* S*-*class* $[\alpha_h]$ *of quasiparallel planes is either an affine plane of order n or an affine plane of order n with a point* x_k *at infinity*, *where* x_k *is centre of a* S*-*class* $[\alpha_k]$ *of quasiparallel planes*.

Step 4: We suppose that each plane in **S** is an affine plane of order n. Moreover, any line has length n. By hypothesis we have $n = |l| \geqslant 5$ and **S** is a three-dimensional affine space of order n (see [2]). Therefore, each equivalence class of quasiparallel planes is a S-*class* [7, Theorem II]), a contradiction. Thus, *there is at least a plane which is an affine plane with a point at infinity*.

Step 5: Let α_h be an affine plane of order n with a point x_k at infinity. Since $|\alpha_h| = |\alpha_h'|$ for each plane $\alpha_h' \in [\alpha_h]$, every plane α_h' is an affine plane of order n with a point x_k' at infinity. We know that $|\alpha_k \cap \alpha_h| = |\alpha_k \cap \alpha_h'| = n+1$. Thus, the line $\alpha_k \cap \alpha_h'$ contains point at infinity x_k' of α_h'. Let α_k' be a plane in $[\alpha_k]$, with $\alpha_k \neq \alpha_k'$. From $|\alpha_k \cap \alpha_h'| = |\alpha_k' \cap \alpha_h'|$ we have that α_k' meets α_h' in a long line which contains the point x_k'. Hence, both planes α_k and α_k' pass through both points x_k and x_k'. Since α_k' belongs to S*-*class* $[\alpha_k]$ we infer that $x_k = x_k'$. Therefore, points x_h and x_k belong to both planes α_h and α_h'. Thus, $x_k = x_k' = x_h$. We conclude that *every plane in* $[\alpha_h]$ *is an affine plane with a point at infinity of order n and the point at infinity is the centre* x_h *of* $[\alpha_h]$. *Moreover*, α_h *does not contain a centre* x_k *of a* S*-*class* $[\alpha_k]$, *with* $x_k \neq x_h$.

Step 6: Let $[\alpha_k]$ be a S*-*class of centre* x_k, with $[\alpha_h] \neq [\alpha_k]$. We suppose that α_k is an affine plane of order n. The line $\alpha_h \cap \alpha_k$ is a short line and centre x_k of $[\alpha_k]$ does not lie on α_h. Let y be a point on $\alpha_h \cap \alpha_k$. Using a plane α_h' which is quasiparallel to α_h and a plane α_k' which is quasiparallel to α_k, from (i), (ii) and (B) we have

$$b_y = b_y(\alpha_h) + |\alpha_h| - 1 = n + 1 + n^2 + 1 - 1,$$

$$b_y = b_y(\alpha_k) + |\alpha_k| - 1 = n + 1 + n^2 - 1,$$

a contradiction. Hence, *all the planes in* S*-*classes are affine planes of order n with a point at infinity*.

Step 7: Let $[\alpha_h]$ of centre x_h and $[\alpha_k]$ of centre x_k be two different S*-*classes*. Since planes in $[\alpha_h]$ are affine planes with x_h at infinity and planes in $[\alpha_k]$ are affine planes with x_k at infinity we have either $x_h \notin \alpha_k$ and $x_k \notin \alpha_h$ or $x_h = x_k$. First, we suppose $x_h \neq x_k$. Therefore, we find $x_h \notin \alpha_k$ and $x_k \notin \alpha_h$. Consider a plane γ through the line $x_h x_k$. Plane γ is not an affine plane of a S-*class* of quasiparallel planes because γ contains x_h. Therefore, plane γ is an affine plane with a point at infinity and $[\gamma]$ is a S*-*class* of quasiparallel planes. From the foregoing we deduce that γ contains a centre of a S*-*class* which is different from centre of S*-*class* $[\gamma]$. This contradiction proves that *all the S*-classes of quasiparallel planes have the same centre x*.

Step 8: All the lines of length $n+1$ contain x. Moreover, we have $n \geqslant 4$ because in **S** there exists a line of length at least 5. Hence, since any plane of **S** is an affine-projective plane of order $n \geqslant 4$, **S** is obtained from a projective space **P** of order n by removing a part of a plane π of **P** (see [11]). Let s' be the line $\alpha_h \cap \sigma_i$. If we denote by z a point on s', using a plane σ'_i which is quasiparallel to σ_i, we obtain

$$b_z = b_z(\sigma_i) + |\sigma_i| = n + 1 + n^2.$$

The unique line through z of length $n+1$ is the line xz. Hence, we have

$$v = |P| = n + 1 + (n^2 + n)(n-1) = n^3 + 1.$$

Thus, **S** is a three-dimensional projective space **P** of order n which $n^2 + n$ coplanar points have been deleted from. Hence, **S** is space **P** deprived of a plane π passing through x together with its points different from x. All the planes through x belong to $n+1$ S*-*classes* of quasiparallel planes *with same centre x*, all the planes which miss x belong to n^2 S-*classes* of quasiparallel planes. This completes the proof. □

5. Three-dimensional affine spaces with a long line at infinity

A *generalized projective* plane is a two-dimensional linear space in which any two distinct lines have a common point. In particular, a *near-pencil* on v points is a finite linear space with one line of length $v-1$ and $v-1$ lines of length 2. Furthermore, a *projective plane* of order n, with $n \geqslant 2$, is a finite linear space with every line of length $n+1$ and every point of degree $n+1$. It is easy to see that projective planes are generalized projective planes without lines of length two and near-pencils are generalized projective planes with lines of length 2.

A *three-dimensional affine space* **S** *of order n with a long line at infinity* is a three-dimensional projective space **P** of order n deprived of a plane π, together with its lines but one, say l, and its points except those of l. It is clear that planes through l belong to a unique P-*class* of quasiparallel planes *of axis l*. Therefore, each plane which misses l belongs to a S*-*class* of centre on l.

Proposition 5.1. *Let* **S** *be a finite linear space. If l is a line such that every line r in* **S**, *with* $r \neq l$, *meets l in a point* x_i *and r belongs to a pencil of quasiparallel lines of centre* x_i, *all the lines in* **S** *are quasiparallel each other and* **S** *is a generalized projective plane*.

Proof. The class $[r]$ of quasiparallel lines is neither a S-*class* nor a *singleton*. Thus, $[r]$ is either a P-*class of centre* x_i on l or a C-*class* (see [5]). It is clear that there exists at least a line s which misses x_i and intersects l in a point x_j, with $x_i \neq x_j$. The class $[s]$ is either a P-*class of centre* x_j or a C-*class*. If $[r]$ and $[s]$ are two P-*classes*, by Proposition 2.9 in [5], there exists in **S** a *singleton* $[p]$ such that either $x_i \in p$ and $x_j \notin p$ or $x_i \notin p$ and $x_j \in p$. Since points x_i and x_j lie on l and each line which is distinct from l, does not belong to a *singleton*, we have a contradiction. By Proposition 2.1 in [5], there exists at most one C-*class* in **S**. Moreover, by Proposition 2.8 in [5], a linear space **S**, with both equivalence classes of quasiparallel lines, a C-*class* and a P-*class*, does not exist. Hence, r and s are quasiparallel lines and class $[r] = [s]$ is a C-*class*. Furthermore, all the lines which are distinct from l belong to unique C-*class* $[r]$. By Proposition 2.2 in [6], there exists no linear space with only one C-*class* and one *singleton*. Hence, lines in **S** are pairwise quasiparallel and **S** is a generalized projective plane (see [5]). □

Theorem III. *Let* **S** *be a finite planar space satisfying property* (A). *If there is a long line of length* $k \geqslant 5$ *and equivalence classes of quasiparallel planes in* **S** *are only one P-class of axis a line l of length* $n+1$ *and S*-classes of centres on l*, **S** *is a three-dimensional affine space of order n with line l at infinity*.

Proof. We proceed in steps.

Step 1: Let $[\pi]$ be the P-*class of axis l*. If $[\alpha_h]$ is a S*-*class of centre* x_h on l, from Proposition 3.2 we deduce that all the planes in $[\alpha_h]$ intersect π in quasiparallel lines through x_h. Furthermore, the planes of each S*-*class* intersect π in quasiparallel lines through a point on l. Let r be a line on π which misses x_h. A plane β through r, which is distinct from π, belongs to a S*-*class* $[\alpha_k]$ *of centre* $x_k \neq x_h$. Hence, each line r in π, which is distinct from l, meets l and belongs to a pencil of quasiparallel lines of centre on l. By Proposition 5.1, π *is a generalized projective plane*.

Step 2: First, we suppose that π is a near-pencil and l is the long line on π. The planes in $[\alpha_h]$ intersect π in lines of the same length through x_h. So, we have a contradiction. Now, we suppose that l is a line of length 2. Let s be the long line on π. We denote by x the point $l \cap s$ and by α a plane through s, distinct from π. The class $[\alpha]$ is a S*-*class* and planes in $[\alpha]$ meet π in quasiparallel lines of the same length, which is a contradiction. This implies that π is a projective plane. Furthermore, *each plane in* $[\pi]$ *is a projective plane and all the planes in* $[\pi]$ *have the same order n*.

Step 3: Let $[\alpha_h]$ be a S*-*class of centre* x_h on l. We consider a point y in **S**, with $y \notin \pi$, and a line t on π which misses x_h. It is clear that x_h does not lie on the plane $\langle t, y\rangle$. Thus, there exists at least one S*-*class* $[\alpha_k]$ *of centre* $x_k \neq x_h$, such that $\langle t, y\rangle \sim \alpha_k$. Since two planes of $[\alpha_h]$ intersect in the point x_h, there exists at most a plane α'_h in $[\alpha_h]$ such that $l \subset \alpha'_h$. We may assume $l \not\subset \alpha_h$. By Proposition 3.2 planes in $[\pi]$ meet α_h in quasiparallel lines of length $n+1$, which belong to a pencil of centre x_h. If $[\alpha_k]$ is a S*-*class of centre* $x_k \neq x_h$, planes in $[\alpha_k]$ meet α_h in quasiparallel disjoint lines. Finally, let $[\alpha_i]$ be a S*-*class of centre* x_h, such that $[\alpha_i] \neq [\alpha_h]$. The planes in $[\alpha_i]$ meet α_h in quasiparallel lines through x_h. By Proposition 4.1 equivalence classes of quasiparallel lines in α_h are one P-*class of centre* x_h and S-*classes*. By Theorem 5 in [5], α_h is an affine plane with point x_h at infinity. Since the line $\pi \cap \alpha_h$ of length $n+1$ is a long line of α_h, α_h has order n. Hence, if $l \not\subset \alpha_h$, α_h *is an affine plane of order n with point* x_h *at infinity*.

Step 4: Let α'_h be a plane in $[\alpha_h]$, with $\alpha'_h \neq \alpha_h$. By (ii) we have $|\alpha_h| = |\alpha'_h| = n^2+1$. If $l \not\subset \alpha'_h$, α'_h is an affine plane of order n with x_h at infinity. Now, we suppose $l \subset \alpha'_h$. Each plane in $[\pi]$ has line l in common with α'_h. Planes in $[\alpha_k]$ meet α'_h in quasiparallel lines through x_k. Let g be a line in α'_h, with $g \neq l$. We denote by γ a plane through g, with $\gamma \neq \alpha'_h$. Since γ does not belong to $[\pi]$, $[\gamma]$ is a S*-*class of centre* a point x_i on l. Thus, g meets l in x_i and belongs to a pencil of quasiparallel lines of centre x_i. By Proposition 5.1 α'_h is a generalized projective plane. First, we suppose that α'_h is a near-pencil. Since l lies on the projective plane π, l is long line of α'_h. By $|\alpha_h| = |\alpha'_h| = n^2+1$, we have a contradiction. Now, we suppose that α'_h is a projective plane of order n. From $|\alpha_h| = |\alpha'_h| = n^2+1$, we obtain a contradiction. Thus, $l \not\subset \alpha'_h$. Each plane in $[\alpha_h]$ is an affine plane of order n with x_h at infinity. Furthermore, *all the planes in* S*-*classes are affine planes of order n with a point at infinity on l*.

Step 5: Since there is a long line of length $k \geqslant 5$, each plane has order $n \geqslant 4$. Furthermore, each plane is affine-projective. By a result in [11] we deduce that **S** is a projective space which a part of a plane has been deleted from. Let z be a point on π such that $z \notin l$. We denote by δ a plane through z, distinct from π. Plane δ belongs to a S*-*class of centre p* on l. All the lines through z on δ, distinct from line pz, have length n. Let π' be a plane in $[\pi]$ with $\pi' \neq \pi$. By (i) we have

$$b_z = b_z(\pi) + |\pi'| - |\pi \cap \pi'|,$$

$$b_z = (n+1) + (n^2+n+1) - (n+1).$$

Hence, there exist n^2+n+1 lines through z, $n+1$ of length $n+1$ and n^2 of length n. Thus, we obtain

$$v = |P| = 1 + (n+1)n + n^2(n-1) = n^3+n+1. \tag{5.1}$$

Finally, we denote by **P** the three-dimensional projective space from which **S** has been obtained by deleting a part of a plane τ. It is clear that $\mathbf{P}\backslash\tau$ is an affine space and each plane in $\mathbf{P}\backslash\tau$ is an affine plane. Since in **S** a plane α_h (α_k) is an affine plane with a point x_h (x_k) at infinity, in **P** the points x_h and x_k lie on τ and $l \subset \tau$. Furthermore, by (5.1) we delete exactly n^2 points from τ. Hence, since line l has length $n+1$ we obtain **S** deleting $\tau\backslash l$ from **P**. **S** *is a three-dimensional affine space of order n with line l of length* $n+1$ *at infinity*. □

6. Three-dimensional affine spaces with a short line at infinity

A *punctured projective plane* is a projective plane deprived of a point. In the following if r is a line, we denote by $i(r)$ the number of lines which have at least a common point with r and by $\mathrm{ext}(r)$ the number of lines which miss line r.

Proposition 6.1. *There exists no finite linear space* **S** *such that the equivalence classes of quasiparallel lines in* **S** *are only one C-class, one S-class and one singleton.*

Proof. On the contrary, we suppose that **S** is a finite linear space such that equivalence classes of quasiparallel lines are exactly one C-*class* $[r]$, one S-*class* $[s]$ and one *singleton* $[l]$. If each line in $[r]$ has length $n+1$, by Proposition 2.5 in [5], each line in $[s]$ has length n and each point has degree $n+1$. Let x be a point on r which does not lie on l. There exists at most one line s' of $[s]$ such that x belongs to s'. By counting points on lines through x, we have

$$n^2+n \leqslant v \leqslant 1+(n+1)n. \tag{6.1}$$

By Proposition 2.5 in [5], each line in $[s]$ meets every line in $[r]$. Therefore, since $b_y=n+1$ for each point y in **S**, l meets each line in $[r]$ and we obtain

$$b=i(r)=n^2+n+1. \tag{6.2}$$

If $v=n^2+n+1$, **S** is a generalized projective plane and there exists a unique equivalence class of quasiparallel lines, a contradiction. Now, we assume $v=n^2+n$. By Theorem 1.2 in [10] **S** can be embedded into a projective plane of order n. Since $v=n^2+n$, **S** is a punctured projective plane and there exists no *singleton*. So, we have a contradiction. □

Proposition 6.2. *There exists no finite linear space whose equivalence classes of quasiparallel lines are only one P-class of centre x, one singleton $[l]$, with $x \in l$, and S-classes.*

Proof. On the contrary, we suppose that **S** is a finite linear space such that equivalence classes are exactly one P-*class* $[r]$ *of centre* x, h S-*classes* $[s_1],\ldots,[s_h]$ and one *singleton* $[l]$, with $x \in l$. Let $n+1$ be length of r. By (ii) all the lines in $[r]$ have length $n+1$. Each point y on r, with $y \neq x$, has degree $n+1$ (see [5]). From Proposition 2.6 in [5] each line in a S-*class* $[si]$ $(i=1,\ldots,h)$ has length n and every point on a line in a S-*class* has degree $n+1$. The $n+1$ lines through y are the line r and n lines in n *S-classes*. Thus, we have $h \geqslant n \geqslant 2$ and

$$v=n+1+n(n-1)=n^2+1. \tag{6.3}$$

Let z be a point on l, with $z \neq x$. Each line g through z, with $g \neq l$, is a line in a S-*class*. Hence, z has degree $n+1$ and we obtain:

$$v=|l|+n(n-1)=n^2-n+|l|. \tag{6.4}$$

By (6.3) and (6.4) we find:

$$|l|=n+1. \tag{6.5}$$

From a result of Erdös et al. [4] we deduce

$$b \geqslant n^2+n. \tag{6.6}$$

By Propositions 2.4 and 2.6 in [5], each line s_i of a S-*class* of quasiparallel lines meets r. Hence, we have

$$b=i(r)=n^2+b_x \geqslant n^2+n.$$

If $b_x > n+1$, all the lines through x, which are distinct from l, belong to $[r]$ and we obtain

$$v=1+b_x \cdot n=n^2+1,$$

a contradiction. Hence, either $b_x=n$ or $b_x=n+1$. First, we suppose $b_x=n+1$. There exist at least three lines through x of length $n+1$, l and two lines in $[r]$. If $n=2$, we find $v=1+3n$, a contradiction. If $n \geqslant 3$, we have:

$$v \geqslant 1+3n+(n-2)(n-1)=n^2+3,$$

a contradiction. Hence, we deduce $b_x=n$, each line through x has length $n+1$ and $b=n^2+n$. From Lemma 2.6 in [10] **S** can be embedded in a projective plane π of order n. Since $v=n^2+1$ and $b=n^2+n$, we obtain **S** by deleting from π n points $p_1, \dots, p_n$ and one line. Thus points p_i $(i = 1, \dots, n)$ are collinear and **S** is an affine plane with a point at infinity. Furthermore, there exists no *singleton* in **S**. So, we have a contradiction. □

Proposition 6.3. *Let* **S** *be a finite linear space such that there exists a unique singleton* $[l]$ *and, if there exist P-classes of quasiparallel lines, each centre of a P-class lies on* l. **S** *is linearly h-punctured projective plane of order n* $(2 \leqslant h \leqslant n-1)$, *equivalence classes are one C-class* $[r]$, *h S-classes and one singleton* $[l]$, *with* $|l| = |r| - h$, *and there exists no P-class.*

Proof. First, we suppose that in **S** there exist P-*classes* of centres on l. By Proposition 4.1 two distinct P-*classes* of quasiparallel lines have distinct centres. From Proposition 2.9 in [5] we deduce that if there are two distinct P-*classes* $[t_i]$, $[t_j]$ *of centres* x_i, x_j, there exists a *singleton* $[g]$ such that either $x_i \in g$ and $x_j \notin g$ or $x_i \notin g$ and $x_j \in g$. Since there is a unique *singleton* $[l]$ with $x_i \in l$ and $x_j \in l$, there exists at most one P-*class* $[t]$ in **S**. By Proposition 2.8 in [5] there exists no C-*class* in **S**. If equivalence classes are only $[t]$ and $[l]$, all the lines belong to a pencil. So, we have a contradiction. Hence, there are S-*classes* of quasiparallel lines. By Proposition 6.2 there exists no finite linear space with only one P-*class*, one *singleton* and S-*classes*. Hence, there is no P-*class* in **S**. Now, we suppose that equivalence classes are only S-*classes* and unique *singleton* $[l]$. All the lines in *S-classes* have the same length n and by Theorem 2 in [6], either there exists no *singleton* or there exist at least $n+1$ *singletons*. So, we have a contradiction. Finally, we suppose that there exists a C-*class* $[r]$. By Proposition 2.1 in [5], $[r]$ is the unique C-*class*. If there exists a class $[s]$ of quasiparallel lines, with $[r] \neq [s] \neq [l]$, $[s]$ is a S-*class*. Therefore, if there exists no S-*class*, by Proposition 2.2 in [6], the number of *singletons* is at least $k > |r|$. So, we have a contradiction. Thus, there is at least one S-*class* in **S**. We suppose that equivalence classes are exactly one C-*class*, one S-*class* and one *singleton*. By Proposition 6.1, we have a contradiction. Now, we suppose that there are h S-*classes*, with $h \geqslant 2$. By Theorem 4 in [5], **S** is a linearly h-punctured projective plane of order n, with $|r| = n+1$ and $|l| = n+1-h$ $(2 \leqslant h \leqslant n-1)$. □

Let **S** be a three-dimensional affine space of order $n \geqslant 4$ with a line l of length $|l| \leqslant n$ at infinity. It is clear that equivalence classes of quasiparallel planes are only one P-*class of axis* l, S*-*classes of centres* on l and S-*classes* and there exists at least one line of length at least 5.

Theorem IV. *Let* **S** *be a finite planar space with property* (A). *If there is at least one line of length at least* 5 *and equivalence classes of quasiparallel planes are only one P-class of axis a line l, S*-classes of centres on l and at least one S-class,* **S** *is a three-dimensional affine space of order* $n \geqslant 4$ *with line l at infinity. Moreover, if k is the number of S-classes,* $|l| = n+1-k$ *and the number of S*-classes is* $n+1-k$.

Proof. We proceed in steps.

Step 1: Let $[\sigma_h]$ be a S-*class* of quasiparallel planes. We denote by n the order of σ_h. There exists at most one plane $\sigma'_h \in [\sigma_h]$ such that $l \subset \sigma'_h$. Thus, we may suppose $l \not\subset \sigma_h$. There are three distinct cases to be considered.

Case 1: We assume that each plane in $[\sigma_h]$ has no point in common with l. Planes in each class of quasiparallel planes, which is distinct from $[\sigma_h]$, meet σ_h in quasiparallel disjoint lines. Hence, every equivalence class of quasiparallel lines in σ_h is a S-*class* and by Theorem 2 in [5] σ_h is an affine plane. Furthermore, *every plane in* $[\sigma_h]$ *is an affine plane of order n.*

Case 2: We suppose that σ_h has empty intersection with l and there exists a plane σ'_h in $[\sigma_h]$ of order m such that $l \subset \sigma'_h$. From the foregoing, σ_h is an affine plane of order n. Planes in each S-*class* $[\sigma_k]$, with $\sigma_h \notin [\sigma_k]$, intersect σ'_h in quasiparallel disjoint lines. Moreover, planes in P-*class* $[\pi]$ intersect σ'_h in line l and planes in each S*-*class* $[\alpha_i]$ *of centre* x_i on l meet σ'_h in quasiparallel lines through x_i. It is clear that each equivalence class of quasiparallel lines in σ'_h, which is distinct from $[l]$, is not a *singleton*. By Proposition 6.3, if $[l]$ is a *singleton*, σ'_h is a linearly k-punctured projective plane, with $2 \leqslant k \leqslant m-1$. If $[l]$ is not a *singleton*, each equivalence class of quasiparallel lines in σ'_h has size at least 2. By Proposition 2.1 in [8] one of the following cases occurs: σ'_h is a projective plane, σ'_h is a punctured projective plane, σ'_h is an affine plane with a point at infinity, σ'_h is an affine plane. Since in σ'_h there exists at least an equivalence class of quasiparallel lines which is distinct from a S-*class*, σ'_h is not an affine plane. If σ'_h is a projective plane, by (ii) we find $|\sigma_h| = n^2 = |\sigma'_h| = m^2 + m + 1$, a contradiction. If σ'_h is a punctured projective plane, we have $|\sigma_h| = n^2 = |\sigma'_h| = m^2 + m$, a contradiction. Moreover, if σ'_h is an affine plane with a point at infinity, we deduce $n^2 = m^2 + 1$. So, we have a contradiction. Finally, if σ'_h is a k-punctured projective plane, we have $n^2 = m^2 + m + 1 - k$, with $2 \leqslant k \leqslant m-1$, a contradiction.

Case 3: We assume $l \cap \sigma_h = \{x_h\}$. Every plane σ'_h in $[\sigma_h]$ intersects l in a point. By Proposition 3.1 planes in a S-*class* $[\sigma_k]$, with $\sigma_h \notin [\sigma_k]$, meet σ_h in quasiparallel disjoint lines. If $[\alpha_i]$ is a S*-*class of centre* $x_i \neq x_h$, planes in $[\alpha_i]$ meet σ_h in quasiparallel disjoint lines. Let $[\alpha_h]$ be a S*-*class of centre* x_h, planes in $[\alpha_h]$ intersect σ_h in quasiparallel lines through x_h. Finally, planes in $[\pi]$ meet σ_h in quasiparallel lines through x_h. By Proposition 4.1 there exists a unique P-*class of centre* x_h. Hence, the equivalence classes of quasiparallel lines in σ_h are only one P-*class* and S-*classes*. By Theorem 5 in [5], σ_h is an affine plane with point x_h at infinity. Furthermore, *every plane in* $[\sigma_h]$ *is an affine plane with a point at infinity*.

Hence, either each plane in $[\sigma_h]$ *is an affine plane of order n and it is disjoint from l or every plane* σ'_h *in* $[\sigma_h]$ *meets l in a point* x'_h *and* σ'_h *is an affine plane of order n with point* x'_h *at infinity*.

Step 2: Let $[\alpha_i]$ be a S*-*class of centre* $x_i \in l$. There exists at most one plane $\alpha'_i \in [\alpha_i]$ such that $l \subset \alpha'_i$. We may suppose that $l \not\subset \alpha_i$. By Proposition 3.1 planes in a S-*class* $[\sigma]$ intersect α_i in quasiparallel disjoint lines. By Proposition 3.2 if $[\alpha_j]$ is a S*-*class of centre* x_i, planes in $[\alpha_j]$ intersect α_i in quasiparallel lines through x_i, if $[\alpha_j]$ is a S*-*class of centre* $x_j \neq x_i$, planes in $[\alpha_j]$ intersect α_i in quasiparallel disjoint lines. Finally, planes in P-*class* $[\pi]$ intersect α_i in quasiparallel lines through x_i. By Proposition 4.1 in α_i there exists a unique P-*class* of quasiparallel lines *of centre* x_i. Hence, α_i *is an affine plane with point* x_i *at infinity* (see [5]).

Step 3: Let α'_i be a plane in $[\alpha_i]$ such that $l \subset \alpha'_i$. Since planes in a S-*class* $[\sigma]$ intersect α'_i in quasiparallel disjoint lines, there exists at least one S-*class* of quasiparallel lines in α'_i. Planes in a S*-*class* $[\alpha_j]$ *of centre* x_j meet α'_i in quasiparallel lines through x_j and planes in $[\pi]$ meet α'_i in line l. Hence, there is at most one *singleton*, line l. First, we suppose that there exists a *singleton*. By Proposition 6.3, α'_i is a linearly k-punctured projective plane, with $k \geqslant 2$. Now, we suppose that each equivalence class has size at least 2. Since in α'_i there exists at least one S-*class* of quasiparallel lines, α'_i is not a projective plane. Analogously, α'_i is not an affine plane because there exists at least one class of quasiparallel lines which is distinct from a S-*class*. By Proposition 2.1 in [8], α'_i is either a punctured projective plane or an affine plane with a point at infinity. We denote by p the order of α_i and by q the order of α'_i. If α'_i is a linearly k-punctured projective plane, with $k \geqslant 2$, we obtain

$$|\alpha_i| = p^2 + 1 = |\alpha'_i| = q^2 + q + 1 - k.$$

So, we have $p = q = k$ and α'_i is an affine plane with a point at infinity. Now, let α'_i be a punctured projective plane. We deduce

$$|\alpha_i| = p^2 + 1 = |\alpha'_i| = q^2 + q.$$

So, we have a contradiction. Hence, *each plane in a* S*-*class* $[\alpha_i]$ *is an affine plane with a point at infinity and all the planes in* $[\alpha_i]$ *have same order*.

Step 4: We suppose that σ_h is an affine plane of order n with a point x_h at infinity which belongs to a S-*class* and α_i is an affine plane of order p with a point x_i at infinity which belongs to a S*-*class*. We can assume $x_h \neq x_i$ and $l \not\subset \alpha_i$. Hence, we have $x_h \notin \alpha_i$ and $x_i \notin \sigma_h$. We denote by σ'_h a plane in $[\sigma_h]$, with $\sigma_h \neq \sigma'_h$, and by α'_i a plane in $[\alpha_i]$, with $\alpha_i \neq \alpha'_i$. Let r be line $\sigma_h \cap \alpha_i$. Line r is a short line of both planes σ_h and α_i and has length $n = p$. Let y be a point on r. By (i) and (ii) we obtain

$$b_y = b_y(\sigma_h) + |\sigma'_h| = n + 1 + n^2 + 1,$$

$$b_y = b_y(\alpha_i) + |\alpha'_i| - 1 = n + 1 + n^2 + 1 - 1.$$

Hence, we have a contradiction. Thus, σ_h is an affine plane of order n. Furthermore, *all the planes in a* S-*class* $[\sigma]$ *are affine planes of order n and each plane in* $[\sigma]$ *has no point in common with line l*.

Step 5: Let $[\alpha_i]$ be a S*-*class*, with $\alpha_i \cap l = \{x_i\}$. From the foregoing α_i is an affine plane with point x_i at infinity. Line $r = \sigma_h \cap \alpha_i$ of length n is a short line in α_i. Hence, α_i has order n and *all the planes of* S*-*classes have same order n*.

Step 6: By Propositions 3.1 and 3.2 planes in a S-*class* $[\sigma]$ intersect each plane of $[\pi]$ in quasiparallel disjoint lines. Therefore, the line $s = \sigma \cap \pi$ is a line of length n which is disjoint from l. Planes in a S*-*class* $[\alpha_i]$ *of centre* x_i on l meet every plane of $[\pi]$ in quasiparallel lines through x_i. In π there is at most one *singleton* $[l]$. If $[l]$ is a *singleton*, by Proposition 6.3 π is a linearly t-punctured projective plane with $t \geqslant 2$. If $[l]$ is not a *singleton*, by Proposition 2.1 in [8], π is either a punctured projective plane or an affine plane with a point x_i on l at infinity.

There are two distinct cases to be considered.

Case 1: We suppose that π is an affine plane with point x_i at infinity. Since s is a short line of π, π has order n. We consider a plane $\sigma' \in [\sigma]$, with $\sigma' \neq \sigma$, and a plane $\pi' \in [\pi]$, with $\pi' \neq \pi$. If z is a point on s we have

$$b_z = b_z(\sigma) + |\sigma'| = n + 1 + n^2, \tag{6.7}$$

$$b_z = b_z(\pi) + |\pi'| - |\pi \cap \pi'| = n + 1 + n^2 + 1 - |l|.$$

Hence, we have $|l| = 1$, a contradiction.

Case 2: Each plane in $[\pi]$ is either a punctured projective plane or linearly t-punctured projective plane with $t \geqslant 2$. We may state that each plane in $[\pi]$ is a k-punctured projective plane with $k \geqslant 1$. Planes in a S-*class* $[\sigma]$ intersect π in short lines of length n. Hence, π has order n. We obtain

$$b_z = b_z(\sigma) + |\sigma'| = n + 1 + n^2 = b_z(\pi) + |\pi'| - |\pi \cap \pi'| = n + 1 + n^2 + n + 1 - k - |l|$$

for a point z on $s = \pi \cap \sigma$. Thus, we have

$$|l| = n + 1 - k.$$

There are $n + 1$ lines through z on π, k of length n which miss l and $n + 1 - k$ of length $n + 1$ which intersect line l. The plane π is an affine plane of order n with line l at infinity. All the lines through z on a plane σ of a S-*class* of quasiparallel planes have length n. Let x be a point on l. If α is a plane through line xz, with $\alpha \neq \pi$, we have that α belongs to a S*-*class*. Therefore, α is an affine plane with x at infinity. The unique line through z of length $n + 1$ on α is line xz. Thus, the $n + 1 - k$ lines joining z to points on l have length $n + 1$, all the other lines through z have length n. Hence, by (6.7) we obtain

$$v = |P| = 1 + (n + 1 - k)n + (n^2 + k)(n - 1),$$

$$v = n^3 + n + 1 - k = n^3 + |l|. \tag{6.8}$$

All the planes in **S** are affine-projective planes of order n. Since there is a line of length at least 5, we obtain $n \geqslant 4$. By a result in [11] **S** is a projective space **P** of order n deprived of a part of its plane τ. Let X be the set of points on τ which are in **S**. By (6.8) **S** has been obtained from **P** by deleting $n^2 + n + 1 - |l| = n^2 + k$ points of τ and $|X| = |l|$. Hence, **S** is a three-dimensional affine space with line l at infinity. Furthermore, the number of S*-*classes* is exactly $n + 1 - k = |l|$ and the number of S-*classes* is k. This completes the proof. □

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