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Asymptotic stability of strong rarefaction waves for the compressible fluid models of Korteweg type

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ABSTRACT

This paper is concerned with the time-asymptotic behavior toward strong rarefaction waves of solutions to one-dimension compressible fluid models of Korteweg type, which governs the motions of the compressible fluids with internal capillarity. Assume that the corresponding Riemann problem to the compressible Euler system can be solved by rarefaction waves $(V^R, U^R)(t,x)$. If the initial data is a small perturbation of an approximate rarefaction wave for $(V^R, U^R)(t,x)$, we show that the corresponding Cauchy problem admits a unique global smooth solution which tends to $(V^R, U^R)(t,x)$ time asymptotically. Since we do not require the strength of the rarefaction waves to be small, this result gives the nonlinear stability of strong rarefaction waves for the one-dimensional compressible fluid models of Korteweg type. The analysis is based on the elementary L^2 energy method together with continuation argument.

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1. Introduction

The motion of the compressible isothermal viscous capillary fluids is described by the Korteweg-type model:

$$\begin{cases} \rho_t + \nabla \cdot (\rho u) = 0, \\ (\rho u)_t + \nabla \cdot \left(\rho u \bigotimes u\right) + \nabla p(\rho) = \mu \Delta u + (\nu + \mu)\nabla(\nabla \cdot u) + \kappa \rho \nabla \Delta \rho, \end{cases} \quad t > 0,\ x \in \mathbb{R}^3. \tag{1.1}$$

Here the unknown functions are the density $\rho > 0$, the velocity u, and the pressure $p = p(\rho)$ of the fluids respectively. $\mu > 0$, $\nu \geq 0$ are the viscosity coefficients and $\kappa > 0$ is the capillary coefficient. Notice that when $\kappa = 0$, system (1.1) is reduced to the compressible Navier–Stokes system.

The formulation of the theory of capillarity with diffuse interface was first introduced by Korteweg [1], and derived rigorously by Dunn and Serrin [2]. Recently, many efforts were made on the compressible Navier–Stokes–Korteweg system (1.1). We refer to [3–10] and the references therein. However, most of these results are concentrated on the case when the far-fields of the initial data are equal (consequently the large time behaviors of its global solutions are described by the constant states) and, to the best of our knowledge, fewer results have been obtained for the case when the far-fields of the initial data are different (in this case, its large time behaviors are described by some nonlinear elementary waves, such as rarefaction waves, viscous shock profiles, etc.). The main purpose of this manuscript is devoted to this problem and as a first step toward this goal, we consider the following one-dimension compressible Navier–Stokes–Korteweg system in the Lagrange coordinates:

$$\begin{cases} v_t - u_x = 0, \\ u_t + p(v)_x = \mu \left(\dfrac{u_x}{v}\right)_x + \kappa \dfrac{1}{v}\left(\dfrac{1}{v}\left(\dfrac{1}{v}\left(\dfrac{1}{v}\right)_x\right)_x\right)_x \end{cases} \tag{1.2}$$

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with the initial data

$$(v(t,x),u(t,x))|_{t=0}=(v_0(x),u_0(x))\to(v_\pm,u_\pm),\quad \text{as } x\to\pm\infty, \tag{1.3}$$

where $v>0$ denotes the specific volume, $v_\pm>0$ and $u_\pm$ are constants. The physical meaning of the other variables in (1.2) are the same as those of (1.1).

We are concerned with the global existence of solutions of the initial value problem (1.2), (1.3) and their large time behavior in relation with the expansive waves. For expansive waves, the right hand side of (1.2) decays faster than each individual term on the left hand side. Therefore, the compressible Navier–Stokes–Korteweg system (1.2) may be approximated, time asymptotically, by the Riemann problem of the compressible Euler equations

$$\begin{cases} v_t-u_x=0,\\ u_t+p(v)_x=0\end{cases} \tag{1.4}$$

with the Riemann data

$$(v(t,x),u(t,x))|_{t=0}=(v_0^R,u_0^R)(x)=\begin{cases}(v_-,u_-), & x<0,\\ (v_+,u_+), & x>0.\end{cases} \tag{1.5}$$

We consider the case when the Riemann problem (1.4), (1.5) admits a unique global weak solution $(V^R(t,x),U^R(t,x))$ consisting of a rarefaction wave of the first family, denoted by $(V_1^R(\frac{x}{t}),U_1^R(\frac{x}{t}))$, and another of the second family denoted by $(V_2^R(\frac{x}{t}),U_2^R(\frac{x}{t}))$. That is, there exists a unique constant state $(\bar v,\bar u)\in\mathbb{R}^2$ with $\bar v>0$ such that $(\bar v,\bar u)\in R_1(v_-,u_-)$ and $(v_+,u_+)\in R_2(\bar v,\bar u)$. Here

$$\begin{cases} R_1(v_-,u_-)=\left\{(v,u)\middle| u=u_-+\displaystyle\int_{v_-}^{v}\sqrt{-p'(z)}dz,\ u\ge u_-\right\},\\ R_2(\bar v,\bar u)=\left\{(v,u)\middle| u=\bar u-\displaystyle\int_{\bar v}^{v}\sqrt{-p'(z)}dz,\ u\ge \bar u\right\}.\end{cases} \tag{1.6}$$

Namely,

$$(V^R(t,x),U^R(t,x))=\left(V_1^R\left(\frac{x}{t}\right)+V_2^R\left(\frac{x}{t}\right)-\bar v,\ U_1^R\left(\frac{x}{t}\right)+U_2^R\left(\frac{x}{t}\right)-\bar u\right) \tag{1.7}$$

where $(V_i^R,U_i^R)(\frac{x}{t})$, $i=1,2$ are defined by

$$(V_1^R,U_1^R)\left(\frac{x}{t}\right)=\begin{cases}(v_-,u_-), & -\infty\le\frac{x}{t}\le\lambda_1(v_-),\\ \left(\lambda_1^{-1}\left(\frac{x}{t}\right),u_--\displaystyle\int_{v_-}^{\lambda_1^{-1}(\frac{x}{t})}\lambda_1(s)\,ds\right), & \lambda_1(v_-)\le\frac{x}{t}\le\lambda_1(\bar v),\\ (\bar v,\bar u), & \lambda_1(\bar v)\le\frac{x}{t}\le+\infty,\end{cases} \tag{1.8}$$

and

$$(V_2^R,U_2^R)\left(\frac{x}{t}\right)=\begin{cases}(\bar v,\bar u), & -\infty\le\frac{x}{t}\le\lambda_2(\bar v),\\ \left(\lambda_2^{-1}\left(\frac{x}{t}\right),\bar u-\displaystyle\int_{\bar v}^{\lambda_2^{-1}(\frac{x}{t})}\lambda_2(s)\,ds\right), & \lambda_2(\bar v)\le\frac{x}{t}\le\lambda_2(v_+),\\ (v_+,u_+), & \lambda_2(v_+)\le\frac{x}{t}\le+\infty\end{cases} \tag{1.9}$$

with $\lambda_1(v)=-\sqrt{-p'(v)}$ and $\lambda_2(v)=\sqrt{-p'(v)}$.

The main purpose of this paper is to show that the Cauchy problem (1.2), (1.3) admits a unique global smooth solution which tends to the rarefaction wave $(V^R(t,x),u^R(t,x))$ defined by (1.7), time asymptotically. That is, we want to study the nonlinear stability of the rarefaction wave $(V^R(t,x),u^R(t,x))$. Since rarefaction waves are only Lipschitz continuous, to study the stability problem, we need to construct a smooth approximation to the above Riemann solution $(V^R(t,x),u^R(t,x))$. As in [11], let $w_i(t,x)$, $i=1,2$ be the unique global smooth solution to the following Cauchy problem

$$\begin{cases} w_{it}+w_iw_{ix}=0,\\ w_i(t,x)|_{t=0}=w_{i0}(x)=\dfrac{w_{i+}+w_{i-}}{2}+\dfrac{w_{i+}-w_{i-}}{2}\tanh(\epsilon x)\end{cases} \tag{1.10}$$

where $\epsilon > 0$ is a given sufficiently small constant, which will be determined later, and $w_{1-} = \lambda_1(v_-)$, $w_{1+} = \lambda_1(\bar{v})$, $w_{2-} = \lambda_2(\bar{v})$, $w_{2+} = \lambda_2(v_+)$. Then the smooth approximate solution $(V, U)(t, x)$ of $(V^R(t, x), u^R(t, x))$ is constructed as follows:

$$(V(t,x), U(t,x)) = (V_1(t+t_0,x) + V_2(t+t_0,x) - \bar{v}, U_1(t+t_0,x) + U_2(t+t_0,x) - \bar{u}), \tag{1.11}$$

where $t_0 = \frac{1}{\epsilon^2}$ with ϵ given in (1.10), and $(V_i(t,x), U_i(t,x))$, $i = 1, 2$ are defined as follows

$$\begin{cases} \lambda_i(V_i(t,x)) = w_i(t,x), \qquad \lambda_i(v) = (-1)^i\sqrt{-p'(v)}, \quad i = 1, 2, \\ U_1(t,x) = u_- + \displaystyle\int_{v_-}^{V_1(t,x)} \sqrt{-p'(z)}dz, \\ U_2(t,x) = \bar{u} - \displaystyle\int_{\bar{v}}^{V_2(t,x)} \sqrt{-p'(z)}dz. \end{cases} \tag{1.12}$$

Throughout this paper, we assume that the coefficients of viscosity and capillary, μ and κ, are positive constants and the pressure $p(v)$ is a positive smooth function for $v > 0$ and satisfies

$$p'(v) < 0, \qquad p''(v) > 0, \quad \text{for } v > 0, \tag{1.13}$$

which means the Euler system (1.4) is strictly hyperbolic and both characteristic fields are genuinely nonlinear. Our main result can be stated as follows.

Theorem 1.1. *Suppose that $p(v)$ satisfies (1.13) and the solution $(V^R, U^R)(t, x)$ to the Riemann problem (1.4), (1.5) is given by (1.7). Let $(V, U)(t, x)$ be a smooth approximation of the Riemann solution $(V^R, U^R)(t, x)$ constructed by (1.11), (1.12). Then if*

$$N(0) = \|v_0 - V(0,x)\|_{H^3(\mathbb{R})} + \|u_0 - U(0,x)\|_{H^2(\mathbb{R})}$$

is sufficiently small, the Cauchy problem (1.2), (1.3) admits a unique global smooth solution $(v, u)(t, x)$ satisfying

$$\begin{cases} (v-V)(t,x) \in C(0,+\infty; H^4(\mathbb{R})) \cap C^1(0,+\infty; H^2(\mathbb{R})), \\ (u-U)(t,x) \in C(0,+\infty; H^3(\mathbb{R})) \cap C^1(0,+\infty; H^1(\mathbb{R}^n)), \\ (v-V, u-U)_x(t,x) \in L^2(0,+\infty; H^3(\mathbb{R})), \end{cases} \tag{1.14}$$

and

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \left\{ \left| \left(v(t,x) - V^R(t,x), u(t,x) - U^R(t,x) \right) \right| \right\} = 0. \tag{1.15}$$

Remark 1.1. A similar argument can be applied to the full (non-isentropic) compressible Navier–Stokes–Korteweg system.

Remark 1.2. Since no smallness condition is required for the strength of the rarefaction wave in Theorem 1.1, we obtain the stability of strong rarefaction for the compressible Navier–Stokes–Korteweg system. However, we need the initial perturbation to be small, thus another interesting problem is how to get a similar result for the isentropic compressible Navier–Stokes–Korteweg system under large initial perturbation, which is left for the future.

Now we recall some related work and make some comments on the analysis in this paper. It is well known that there are many interesting results on the large time behavior of solutions to the compressible Navier–Stokes system, which are characterized by those of solutions of the corresponding Riemann problems for the Euler system. Here, we only mention the rarefaction wave case for the Euler system. The study on the stability of rarefaction waves for the compressible Navier–Stokes system was started by Matsumura and Nishihara [12], where the nonlinear stability of weak rarefaction wave for the isentropic compressible Navier–Stokes system under small perturbations was proved. Since then, many interesting results in this aspect were obtained, see [13–18,11,19,20] and the references therein. Compared with the case of the compressible Navier–Stokes system, the main difficulty we encounter here is due to the appearance of the Korteweg tensor $\kappa\frac{1}{v}(\frac{1}{v}(\frac{1}{v}(\frac{1}{v})_x)_x)_x$ in the Eq. $(1.2)_2$, which results in more regularity for the density than the momentum (see (1.14) for details). In order to show the stability of the rarefaction wave defined in (1.7), we need to prove the global existence of solutions to problem (3.1)–(3.2). The key step is to derive the a priori estimates (3.6). To achieve this, we frequently use the Eq. $(3.1)_1$ and the a priori assumption (3.5). Then Theorem 1.1 follows by the standard continuation argument based on the local existence and the a priori estimate.

The reminder of this paper is organized as follows. After stating some notations, we recall in Section 2 the properties of a smooth approximate solution of the Riemann problem for later use. The proof of Theorem 1.1 will be given in Section 3.

Notations: Throughout this paper, for simplicity, we will omit the variables t, x of functions if it does not cause any confusion. C denotes a generic constant which may vary in different estimates. If the dependence need to be explicitly pointed out, the notations C_i $(i \in \mathbb{N})$ are used. $H^l(\mathbb{R}^n)$ denotes the usual lth order Sobolev Space with its norm

$$\|f\|_l = \left(\sum_{i=0}^{l} \|\partial_x^i f\|^2 \right)^{\frac{1}{2}} \quad \text{with } \|\cdot\| \triangleq \|\cdot\|_{L^2}.$$

2. Properties of smooth approximate solution of the Riemann problem

As [15], we start with the Riemann problem for the typical Burgers equation:

$$\begin{cases} w_{it}^R + w_i^R w_{ix}^R = 0, \\ w_i^R(x,0) = w_{i0}^R(x) = \begin{cases} w_{i-}, & x < 0, \\ w_{i+}, & x > 0 \end{cases} \end{cases} \tag{2.1}$$

for each $i \in \{1, 2\}$. If $w_{i-} < w_{i+}$, then the above Riemann problem (2.1) admits a unique rarefaction wave solution

$$w_i^R(x,t) = w_i^R\left(\frac{x}{t}\right) = \begin{cases} w_{i-}, & x \le w_{i-}t, \\ \dfrac{x}{t}, & w_{i-}t < x < w_{i+}t, \\ w_{i+}, & x \ge w_{i+}t. \end{cases} \tag{2.2}$$

We approximate $w_i^R(\frac{x}{t})$ by the solution $w_i(t,x)$ of (1.10). Then by the method of characteristic, $w_i(t,x)$ has the following properties, cf. [11,13].

Lemma 2.1. *For each $i \in \{1, 2\}$, the Cauchy problem (1.10) admits a unique global smooth solution $w_i(t,x)$ which satisfies*

(i) $w_{i-} < w_i(t,x) < w_+$, $w_{ix} > 0$ *for* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

(ii) *For any $1 \le p \le +\infty$, there exists a constant C_p depending only on p such that for $t \ge 0$,*

$$\|w_{ix}(t)\|_{L^p} \le C_p \min\{\tilde{w}_i \epsilon^{1-\frac{1}{p}}, \tilde{w}_i^{\frac{1}{p}} t^{-1+\frac{1}{p}}\},$$

$$\left\| \frac{\partial^j}{\partial x^j} w_i(t) \right\|_{L^p} \le C_p \min\{\tilde{w}_i \epsilon^{j-\frac{1}{p}}, \epsilon^{j-1-\frac{1}{p}} t^{-1}\}, \quad j \ge 2.$$

(iii) $\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left| w_i(t,x) - w_i^R(t,x) \right| = 0$.

Notice that (V, U) satisfies

$$\begin{cases} V_t - U_x = 0, \\ U_t + p(V)_x = g(V)_x, \end{cases} \tag{2.3}$$

where $g(V)_x = p(V) - p(V_1) - p(V_2) + p(\bar{v})$. Based on this and Lemma 2.1, from (1.11), (1.12), we have the following lemma.

Lemma 2.2. *Let $\tilde{\delta} = |v_+ - v_-| + |u_+ - u_-|$, the smooth approximations $(V, U)(t,x)$ constructed in (1.11), (1.12) have the following properties:*

(i) $V_t = U_x > 0$, $|(V_t, U_t)| \le C|(V_x, U_x)|$, $\forall (t,x) \in \mathbb{R}^+ \times \mathbb{R}$.

(ii) *For any $1 \le p \le +\infty$, there exists a constant C_p depending only on p such that for $t \ge 0$,*

$$\|(V_x, U_x)(t)\|_{L^p} \le C_p \min\left\{\tilde{\delta}\epsilon^{1-\frac{1}{p}}, \tilde{\delta}^{\frac{1}{p}} (t+t_0)^{-1+\frac{1}{p}}\right\},$$

$$\left\| \frac{\partial^j}{\partial x^j}(V, U)(t) \right\|_{L^p} \le C_p \min\left\{\tilde{\delta}\epsilon^{j-\frac{1}{p}}, \epsilon^{j-1-\frac{1}{p}} (t+t_0)^{-1}\right\}, \quad j \ge 2.$$

(iii) *There exists a positive constant α depending only on $w_{i\pm}$ $(i = 1, 2)$ given in (1.10) such that*

$$|g(V)_x(t)| \le C\epsilon \exp\{-\alpha\epsilon(|x| + t + t_0)\}.$$

Furthermore,

$$\int_0^{+\infty} \|g(V)_x(t)\|_{L^p} dt \le C\epsilon^{-\frac{1}{p}} \exp\{-\alpha\epsilon t_0\} \le C\alpha^n \epsilon^{n-\frac{1}{p}}, \quad \forall n \in \mathbb{Z}^+.$$

(iv) $\lim_{t\to+\infty} \sup_{x\in\mathbf{R}} \left| (V, U)(t,x) - (V^R, U^R)(t,x) \right| = 0$.

3. Proof of Theorem 1.1

This section is devoted to proving Theorem 1.1. First, define

$$\varphi(t,x) = v(t,x) - V(t,x), \qquad \psi(t,x) = u(t,x) - U(t,x).$$

Then the problem (1.2), (1.3) is reformulated as

$$\begin{cases} \varphi_t - \psi_x = 0, \\ \psi_t + [p(\varphi + V) - p(V)]_x = \mu \left(\dfrac{u_x}{v} - \dfrac{U_x}{V} \right)_x + F \end{cases} \tag{3.1}$$

with initial data

$$(\varphi, \psi)|_{t=0} = (\varphi_0, \psi_0)(x) = (v - V, u - U)(0, x), \tag{3.2}$$

where

$$F=\kappa\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x+\mu\left(\frac{U_x}{V}\right)_x-g(V)_x. \tag{3.3}$$

We look for the solution (φ,ψ) in the solution space $X_M(0,+\infty)$ for some positive constant $M>0$, where for $0\le T\le+\infty$, we define

$$X_M(0,T)=\left\{(\varphi,\psi)(t,x)\left|\begin{array}{c}\varphi(t,x)\in C(0,T;H^4(\mathbb{R}))\cap C^1(0,T;H^2(\mathbb{R})),\\ \psi(t,x)\in C(0,T;H^3(\mathbb{R}))\cap C^1(0,T;H^1(\mathbb{R}^n)),\\ (\varphi_x,\psi_x)(t,x)\in L^2(0,T;H^3(\mathbb{R})),\\ \sup\limits_{t\in[0,T]}\{\|\varphi(t)\|_4+\|\psi(t)\|_3\}\le M,\end{array}\right.\right\}. \tag{3.4}$$

Under the assumptions listed in Theorem 1.1, we have the following local existence result.

Proposition 3.1 (*Local Existence*). *Under the assumptions of Theorem 1.1, suppose that $\|\varphi_0\|_4+\|\psi_0\|_3\le M$, then there exists a positive constant t_1 depending only on M such that the Cauchy problem (3.1)–(3.2) admits a unique smooth solution $(\varphi,\psi)(t,x)\in X_{2M}(0,t_1)$.*

The proof of Proposition 3.1 can be done by using the dual argument and iteration technique, which is similar to that of Theorem 1.1 in [6], the details are omitted here.

In order to get the global existence of the Cauchy problem (3.1), (3.2), we need to establish the following a priori estimates.

Proposition 3.2 (*A Priori Estimates*). *Under the assumptions of Theorem 1.1, suppose that $(\varphi,\psi)(t,x)\in X_M(0,T)$ is a solution of the Cauchy problem (3.1), (3.2) for some positive constants T and M, and satisfies*

$$N(T):=\sup_{t\in[0,T]}\{\|\varphi(t)\|_3+\|\psi(t)\|_2\}\le\delta \tag{3.5}$$

for some suitably small constant $\delta>0$, then there exist three positive constants $\epsilon_0\ll 1$, $\delta_0\ll 1$ and C_0 which are independent of T such that for $0<\delta\le\delta_0$ and $0<\epsilon\le\epsilon_0$, it holds that

$$\|\varphi(t)\|_4^2+\|\psi(t)\|_3^2+\int_0^t\left(\|(\sqrt{V_t}\varphi)(\tau)\|^2+\|(\varphi_x,\psi_x)(\tau)\|_3^2\right)d\tau\le C_0\left(\|\varphi_0\|_4^2+\|\psi_0\|_3^2+\epsilon^{\frac{2}{3}}\right) \tag{3.6}$$

for all $t\in[0,T]$.

Proposition 3.2 can be proved by a series of lemmas below. Before proving the a priori estimate (3.6), we have from the a priori assumption (3.5) and the Sobolev inequality

$$\|f(t)\|_{L^\infty}\le\|f(t)\|_1,\quad \forall f(t)\in H^1(\mathbb{R})$$

that

$$\|(\varphi,\varphi_x,\varphi_{xx},\psi,\psi_x)(t)\|_{L^\infty}\le\delta,\quad \forall t\in[0,T]. \tag{3.7}$$

Furthermore, by the smallness of δ,

$$v(t,x)=\varphi(t,x)+V(t,x)\ge\frac{1}{2}\min\{v_-,v_+\}>0. \tag{3.8}$$

With (3.7) and (3.8) in hand, we now give the following key lemma.

Lemma 3.1. *Under the assumptions of Proposition 3.2, there exists a positive constant $C>0$ such that for $0\le t\le T$, it holds*

$$\|(\varphi,\psi,\varphi_x)(t)\|^2+\int_0^t\left\|(\sqrt{V_t}\varphi,\psi_x)(\tau)\right\|^2d\tau\le C\left(\|(\varphi_0,\psi_0,\varphi_{0x})\|^2+\epsilon\int_0^t\|\varphi_x(\tau)\|^2\,d\tau+\epsilon^{\frac{2}{3}}\right) \tag{3.9}$$

provided that ϵ and δ are suitably small.

Proof. Multiplying $(3.1)_1$ by $[p(V)-p(V+\varphi)]$, $(3.1)_2$ by ψ, then combining the two resulting equations, we have

$$\begin{aligned}&\left(\Phi(v,V)+\frac{\psi^2}{2}\right)_t+V_t\left\{p(V+\varphi)-p(V)-p'(V)\varphi\right\}+H_x+\mu\frac{\psi_x^2}{v}\\&=\kappa\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x\psi+\mu\frac{U_x\psi_x\varphi}{vV}+\mu\psi\left(\frac{U_x}{V}\right)_x-g(V)_x\psi,\end{aligned} \tag{3.10}$$

where

$$\begin{aligned} &\Phi(v,V) = p(V)\varphi - \int_V^v p(s)\,ds, \\ &H = [p(V+\varphi) - p(V)]\psi - \mu\left(\frac{U_x+\psi_x}{v} - \frac{U_x}{V}\right)\psi. \end{aligned} \tag{3.11}$$

Using Eq. (3.1)$_1$, we have

$$\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x \psi = \{\cdots\}_x - \left(\frac{\varphi_x^2}{2v^5}\right)_t + \frac{\psi_x V_{xx}}{v^5} - \frac{5\varphi_x^2 U_x}{2v^6} - \frac{5\psi_x V_x^2}{2v^6}. \tag{3.12}$$

Here and hereafter, $\{\cdots\}_x$ denotes the terms which will disappear after integrating with respect to x.

Putting (3.12) into (3.10) gives rise to

$$\begin{aligned} &\left(\Phi(v,V) + \frac{\psi^2}{2} + \frac{\kappa\varphi_x^2}{2v^5}\right)_t + V_t\left\{p(V+\varphi) - p(V) - p'(V)\varphi\right\} + \mu\frac{\psi_x^2}{v} \\ &\quad = -\frac{5\kappa\varphi_x^2 U_x}{2v^6} + \frac{\kappa\psi_x V_{xx}}{v^5} - \frac{5\kappa\psi_x V_x^2}{2v^6} + \mu\frac{U_x\psi_x\varphi}{vV} + \mu\psi\left(\frac{U_x}{V}\right)_x - g(V)_x\psi + \{\cdots\}_x. \end{aligned} \tag{3.13}$$

Due to [11],

$$\Phi(v,V) \geq C_1\frac{\left(1-\frac{v}{V}\right)^2}{1+\frac{v}{V}} \geq C_2\varphi^2 \tag{3.14}$$

for some constants $C_1 > 0$ and $C_2 > 0$. Integrating (3.13) with respect to t and x over $[0,t]\times\mathbb{R}$, we have from (3.8), (3.14) and the assumption (1.13) that

$$\|(\varphi,\psi,\varphi_x)(t)\|^2 + \int_0^t \|(\sqrt{V_t}\varphi,\psi_x)(\tau)\|^2 d\tau \leq C\left\{\|(\varphi_0,\psi_0,\varphi_{0x})\|^2 + \sum_{i=1}^3 I_i\right\} \tag{3.15}$$

where

$$\begin{aligned} I_1 &= \int_0^t\int_{\mathbb{R}} |\varphi_x^2 U_x|\,dx\,d\tau, \\ I_2 &= \int_0^t\int_{\mathbb{R}} (|U_{xx}| + |V_x U_x| + |g(V)_x|)|\psi|\,dx\,d\tau, \\ I_3 &= \int_0^t\int_{\mathbb{R}} \left(|\psi_x V_{xx}| + |\psi_x V_x^2| + |U_x\psi_x\varphi|\right)dx\,d\tau. \end{aligned} \tag{3.16}$$

By Lemma 2.2, it is easy to see

$$I_1 \leq C\epsilon\int_0^t \|\varphi_x(\tau)\|^2 d\tau. \tag{3.17}$$

It follows from the Sobolev inequality, the Cauchy inequality, the a priori estimates (3.7) and Lemma 2.2 that

$$\begin{aligned} I_2 &\leq \int_0^t \|\psi(\tau)\|^{\frac{1}{2}}\|\psi_x(\tau)\|^{\frac{1}{2}}(\|U_{xx}(\tau)\|_{L^1} + \|(V_xU_x)(\tau)\|_{L^1} + \|g(V)_x(\tau)\|_{L^1})\,d\tau \\ &\leq \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta\int_0^t \|\psi(\tau)\|^{\frac{2}{3}}\left(\|U_{xx}(\tau)\|_{L^1}^{\frac{4}{3}} + \|V_x(\tau)\|_{L^\infty}^{\frac{4}{3}}\|U_x(\tau)\|_{L^1}^{\frac{4}{3}} + \|g(V)_x(\tau)\|_{L^1}^{\frac{4}{3}}\right)d\tau \\ &\leq \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta\delta^{\frac{2}{3}}\int_0^t (\tau+t_0)^{-\frac{4}{3}}\,d\tau \\ &\leq \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta\epsilon^{\frac{2}{3}}, \end{aligned} \tag{3.18}$$

and

$$\begin{aligned} I_3 &\le \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta \int_0^t (\|U_x(\tau)\|_{L^\infty}^2\|\varphi(\tau)\|^2 + \|V_{xx}(\tau)\|^2 + \|V_x(\tau)\|_{L^4}^4) d\tau \\ &\le \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta \int_0^t (\delta^2(\tau+t_0)^{-2} + \epsilon(\tau+t_0)^{-2})\, d\tau \\ &\le \eta\int_0^t \|\psi_x(\tau)\|^2 d\tau + C_\eta\epsilon^2. \end{aligned} \tag{3.19}$$

Here and in the rest of this paper, $\eta > 0$ denotes a suitably small constant. Substituting (3.17)–(3.19) into (3.15), one can immediately get (3.9) by the smallness of η. This completes the proof of Lemma 3.1. □

For the L^2-estimate on $\varphi_x(t,x)$, we have

Lemma 3.2. *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that for $0 \le t \le T$,*

$$\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|_1^2 d\tau \le C\left(\|\varphi_0\|_1^2 + \|\psi_0\|^2 + \epsilon^{\frac{2}{3}}\right) \tag{3.20}$$

provided that ϵ and δ are suitably small.

Proof. From $(3.1)_2$, we have

$$\left(\mu\frac{\varphi_x}{v} - \psi\right)_t = -\mu\left(\frac{V_x}{v}\right)_t - \kappa\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x + [p(\varphi+V) - p(V)]_x + g(V)_x. \tag{3.21}$$

Multiplying (3.21) by $\frac{\varphi_x}{v}$ yields

$$\begin{aligned} &\left(\frac{\mu}{2}\left(\frac{\varphi_x}{v}\right)^2 - \psi\frac{\varphi_x}{v}\right)_t - p'(\varphi+V)\frac{\varphi_x^2}{v} = \frac{\psi_x^2}{v} - \frac{\psi\psi_x V_x}{v^2} + \frac{\psi\varphi_x U_x}{v^2} - \mu\frac{U_{xx}\varphi_x}{v^2} + \mu\frac{V_x\psi_x\varphi_x}{v^3} \\ &\quad + \mu\frac{V_xU_x\varphi_x}{v^3} + \{\cdots\}_x - \kappa\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x\frac{\varphi_x}{v} + [p'(\varphi+V) - p'(V)]\frac{\varphi_x V_x}{v} + g(V)_x\frac{\varphi_x}{v}. \end{aligned} \tag{3.22}$$

By a direct computation, we obtain

$$\begin{aligned} -\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\left(\frac{1}{v}\right)_x\right)_x\right)_x\frac{\varphi_x}{v} &= -\frac{\varphi_{xx}^2}{v^6} - \frac{V_{xx}\varphi_{xx}}{v^6} + \frac{2\varphi_x(\varphi_x+V_x)(\varphi_{xx}+V_{xx})}{v^7} \\ &\quad + \frac{3(\varphi_x+V_x)^2\varphi_{xx}}{v^7} - \frac{6(\varphi_x+V_x)^3\varphi_x}{v^8} + \{\cdots\}_x. \end{aligned} \tag{3.23}$$

Combining (3.22) with (3.23), and integrating the resultant equation in t and x over $[0,t]\times\mathbb{R}$, we have

$$\|\varphi_x(t)\|^2 + \int_0^t \|\varphi_x(\tau)\|_1^2 d\tau \le C\left(\|(\varphi_{0x},\psi_0)\|^2 + \|\psi(t)\|^2 + \int_0^t \|\psi_x(\tau)\|^2 d\tau + \sum_{i=4}^7 I_i\right) \tag{3.24}$$

where

$$\begin{aligned} I_4 &= \int_0^t\int_{\mathbb{R}} (|\psi\psi_xV_x| + |\psi\varphi_xU_x|)\, dxd\tau, \\ I_5 &= \int_0^t\int_{\mathbb{R}} \left(|\varphi_x(\varphi_x+V_x)V_{xx}| + |(\varphi_x+V_x)^3\varphi_x|\right) dxd\tau, \\ I_6 &= \int_0^t\int_{\mathbb{R}} (|\varphi\varphi_xV_x| + |U_{xx}\varphi_x| + |\varphi_xV_x\psi_x| + |\varphi_xU_xV_x| + |\varphi_x g(V)_x|)\, dxd\tau, \\ I_7 &= \int_0^t\int_{\mathbb{R}} \left(|\varphi_{xx}V_{xx}| + |\varphi_x(\varphi_x+V_x)\varphi_{xx}| + |(\varphi_x+V_x)^2\varphi_{xx}|\right) dxd\tau. \end{aligned} \tag{3.25}$$

and we have used the fact that

$$\int_{\mathbb{R}} \frac{\psi\varphi_x}{v}\, dx \le \frac{\mu}{4}\int_{\mathbb{R}}\frac{\varphi_x^2}{v}\, dx + C\int_{\mathbb{R}}\psi^2\, dx. \tag{3.26}$$

By Lemma 2.2, (3.7) and the Cauchy inequality, the terms I_4–I_7 can be estimated as follows:

$$\begin{aligned} I_4 &\leq \eta \int_0^t \|(\psi_x, \varphi_x)(\tau)\|^2 \, d\tau + C_\eta \int_0^t \|(V_x, U_x)(\tau)\|_{L^\infty}^2 \|\psi(\tau)\|^2 \, d\tau \\ &\leq \eta \int_0^t \|(\psi_x, \varphi_x)(\tau)\|^2 \, d\tau + C_\eta \epsilon, \end{aligned} \tag{3.27}$$

$$\begin{aligned} I_5 &\leq C \int_0^t \|(\varphi_x, V_x)(\tau)\|_{L^\infty} \left(\|\varphi_x(\tau)\| \, \|V_{xx}(\tau)\| + \|\varphi_x(\tau)\|_{L^\infty} \|\varphi_x(\tau)\|^2 + \|\varphi_x(\tau)\| \, \|V_x(\tau)\|_{L^4}^2 \right) d\tau \\ &\leq C(\epsilon + \delta) \int_0^t \left(\|\varphi_x(\tau)\|^2 + \|V_{xx}(\tau)\|^2 + \|V_x(\tau)\|_{L^4}^4 \right) d\tau \\ &\leq C(\epsilon + \delta) \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C\epsilon, \end{aligned} \tag{3.28}$$

$$\begin{aligned} I_6 &\leq \eta \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C_\eta \int_0^t \left(\|V_x(\tau)\|_{L^\infty}^2 \|\varphi(\tau)\|^2 + \|U_{xx}(\tau)\|^2 \right. \\ &\quad \left. + \|V_x(\tau)\|_{L^\infty}^2 \|\psi_x(\tau)\|^2 + \|V_x(\tau)\|_{L^\infty}^2 \|U_x(\tau)\|^2 + \|g(V)_x(\tau)\|^2 \right) d\tau \\ &\leq \eta \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C_\eta \epsilon^2 \int_0^t \|\psi_x(\tau)\|^2 \, d\tau + C_\eta \epsilon, \end{aligned} \tag{3.29}$$

and

$$\begin{aligned} I_7 &\leq \eta \int_0^t \|\varphi_{xx}(\tau)\|^2 \, d\tau + C_\eta \int_0^t \left(\|V_{xx}(\tau)\|^2 + \|(\varphi_x + V_x)(\tau)\|_{L^\infty}^2 \|\varphi_x(\tau)\|^2 \right. \\ &\quad \left. + \|\varphi_x(\tau)\|_{L^\infty}^2 \|\varphi_x(\tau)\|^2 + \|V_x(\tau)\|_{L^4}^4 \right) d\tau \\ &\leq \eta \int_0^t \|\varphi_{xx}(\tau)\|^2 \, d\tau + C_\eta(\epsilon + \delta) \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C_\eta \epsilon. \end{aligned} \tag{3.30}$$

Combining (3.24), (3.27)–(3.30), using Lemma 3.1 and the smallness of η, ϵ and δ, we can get (3.20). This completes the proof of Lemma 3.2. □

As a consequence of Lemmas 3.1 and 3.2, we obtain

Lemma 3.3 (*Basic Energy Estimates*)**.** *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that if ϵ and δ are suitably small, it holds for $0 \leq t \leq T$ that*

$$\|(\varphi, \psi, \varphi_x)(t)\|^2 + \int_0^t \left(\|(\sqrt{V_t}\varphi, \psi_x)(\tau)\|^2 + \|\varphi_x(\tau)\|_1^2 \right) d\tau \leq C \left(\|(\varphi_0, \varphi_{0x}, \psi_0)\|^2 + \epsilon^{\frac{2}{3}} \right). \tag{3.31}$$

Next, we derive the higher order energy type estimates on (φ, ψ).

Lemma 3.4. *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that for $0 \leq t \leq T$,*

$$\|(\psi_x, \varphi_{xx})(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 \, d\tau \leq C \left(\|\varphi_0\|_2^2 + \|\psi_0\|_1^2 + \epsilon^{\frac{2}{3}} \right) \tag{3.32}$$

provided that ϵ and δ are suitably small.

Proof. Multiplying $(3.1)_2$ by $-\psi_{xx}$ and using Eq. $(3.1)_1$, we have

$$\begin{aligned} \left(\frac{\psi_x^2}{2} + \frac{\kappa \varphi_{xx}^2}{2v^5} \right)_t + \mu \frac{\psi_{xx}^2}{v} &= [p'(\varphi + V) - p'(V)] V_x \psi_{xx} + p'(\varphi + V) \varphi_x \psi_{xx} - \mu \frac{U_{xx} \psi_{xx}}{v} + \mu \frac{\psi_x \psi_{xx} (\varphi_x + V_x)}{v^2} \\ &\quad + \mu \frac{U_x \psi_{xx} (\varphi_x + V_x)}{v^2} + \kappa \frac{\psi_{xx} V_{xxx}}{v^5} - 5\kappa \frac{\varphi_{xx}^2 (\psi_x + U_x)}{2v^6} \\ &\quad - 5\kappa \frac{(\varphi_x + V_x)(\varphi_{xx} + V_{xx}) \psi_{xx}}{v^6} - 5\kappa \frac{\psi_{xx} V_{xx} (\varphi_x + V_x)}{v^6} \\ &\quad + 15\kappa \frac{(\varphi_x + V_x)^3 \psi_{xx}}{v^7} + g(V)_x \psi_{xx} + \{\cdots\}_x. \end{aligned} \tag{3.33}$$

Integrating (3.33) with respect to t and x over $[0, t] \times \mathbb{R}$, we get

$$\|(\psi_x, \varphi_{xx})(t)\|^2 + \int_0^t \|\psi_{xx}(\tau)\|^2 \, d\tau \le C \left(\|(\varphi_{0xx}, \psi_{0x})\|^2 + \sum_{i=8}^{11} I_i \right), \tag{3.34}$$

where

$$\begin{aligned} I_8 &= \int_0^t \int_{\mathbb{R}} (|U_{xx}\psi_{xx}| + |U_x\psi_{xx}(\varphi_x + V_x)|) \, dxd\tau, \\ I_9 &= \int_0^t \int_{\mathbb{R}} (|\varphi V_x \psi_{xx}| + |\varphi_x\psi_{xx}| + |\psi_x\psi_{xx}(\varphi_x + V_x)| + |g(V)_x|\psi_{xx}) \, dxd\tau, \\ I_{10} &= \int_0^t \int_{\mathbb{R}} \big(|\psi_{xx}V_{xxx}| + |(\varphi_x + V_x)^3\psi_{xx}| + |\psi_{xx}V_{xx}(\varphi_x + V_x)|\big) \, dxd\tau, \\ I_{11} &= \int_0^t \int_{\mathbb{R}} \big(|(\varphi_x + V_x)(\varphi_{xx} + V_{xx})\psi_{xx}| + |\varphi_{xx}^2(\psi_x + U_x)|\big) \, dxd\tau. \end{aligned} \tag{3.35}$$

Similar to the estimates of $I_i,\ i = 1, \dots, 7$ in Lemmas 3.1 and 3.2, we have

$$I_8 \le \eta \int_0^t \|\psi_{xx}(\tau)\|^2 \, d\tau + C_\eta \epsilon^2 \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C_\eta \epsilon, \tag{3.36}$$

$$I_9 \le \eta \int_0^t \|\psi_{xx}(\tau)\|^2 \, d\tau + C_\eta \int_0^t \|(\varphi_x, \psi_x)(\tau)\|^2 \, d\tau + C_\eta \epsilon, \tag{3.37}$$

$$I_{10} \le \eta \int_0^t \|\psi_{xx}(\tau)\|^2 \, d\tau + C_\eta \delta^2 \int_0^t \|\varphi_x(\tau)\|^2 \, d\tau + C_\eta \epsilon, \tag{3.38}$$

and

$$I_{11} \le (\epsilon + \delta) \int_0^t \|(\varphi_{xx}, \psi_{xx})(\tau)\|^2 \, d\tau + C\epsilon. \tag{3.39}$$

Substituting (3.36)–(3.39) into (3.34), then (3.32) follows by Lemma 3.3 and the smallness of η, ϵ and δ. Thus the proof of Lemma 3.4 is completed. □

Now, we estimate $\|\psi_{xx}(\tau)\|$.

Lemma 3.5. *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that for $0 \le t \le T$,*

$$\|(\psi_{xx}, \varphi_{xxx})(t)\|^2 + \int_0^t \|\psi_{xxx}(\tau)\|^2 \, d\tau \le C \left(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \epsilon^{\frac{2}{3}} + (\epsilon + \delta) \int_0^t \|\varphi_{xxx}(\tau)\|^2 \, d\tau \right) \tag{3.40}$$

provided that ϵ and δ are suitably small.

Proof. Differentiating $(3.1)_2$ twice with respect to x, then multiplying the resultant equation by ψ_{xx}, and using the Eq. $(3.1)_1$, we have by a long calculation that

$$\left(\frac{\psi_{xx}^2}{2} + \frac{\kappa\varphi_{xxx}^2}{2v^5} \right)_t + \mu \frac{\psi_{xxx}^2}{v} = R_1 + R_2 + R_3 + \{\cdots\}_x, \tag{3.41}$$

where

$$\begin{aligned} R_1 \ &= \ \big\{p''(\varphi + V)\varphi_x V_x + p''(\varphi + V)V_x^2 - p''(V)V_x^2 + [p'(\varphi + V) - p'(V)]V_{xx} \\ &\quad + p''(\varphi + V)\varphi_x(\varphi_x + V_x) + p'(\varphi + V)\varphi_{xx} + g(V)_{xx}\big\} \, \psi_{xxx}, \\ R_2 \ &= \ -\mu \frac{U_{xxx}\psi_{xxx}}{v} + 2\mu \frac{(U_{xx} + \psi_{xx})(\varphi_x + V_x)\psi_{xxx}}{v^2} \\ &\quad + \mu \frac{(U_x + \psi_x)(\varphi_{xx} + V_{xx})\psi_{xxx}}{v^2} - 2\mu \frac{(V_x + \varphi_x)^2(\psi_x + U_x)\psi_{xxx}}{v^3}, \\ R_3 \ &= \ -\frac{5\kappa\varphi_{xxx}^2(\psi_x + U_x)}{2v^6} + \kappa\psi_{xxx} \left\{ \frac{V_{xxxx}}{v^5} - 5\frac{(V_x + \varphi_x)V_{xxx}}{v^6} - 10\frac{(V_{xx} + \varphi_{xx})^2}{v^6} \right. \\ &\quad \left. - 10\frac{(V_{xxx} + \varphi_{xxx})(\varphi_x + V_x)}{v^6} + 105\frac{(V_{xx} + \varphi_{xx})(\varphi_x + V_x)^2}{v^7} - 105\frac{(\varphi_x + V_x)^4}{v^8} \right\}. \end{aligned} \tag{3.42}$$

Integrating (3.41) with respect to t and x over $[0, t] \times \mathbb{R}$, we have

$$\|(\psi_{xx}, \varphi_{xxx})(t)\|^2 + \int_0^t \|\psi_{xxx}(\tau)\|^2\, d\tau \le C\left(\|(\varphi_{0xxx}, \psi_{0xx})\|^2 + \int_0^t \int_{\mathbb{R}} (|R_1| + |R_2| + |R_3|)\, dx d\tau\right). \tag{3.43}$$

Similar to the estimates in previous lemmas, we obtain

$$\begin{aligned} \int_0^t \int_{\mathbb{R}} |R_1|\, dx d\tau &\le \eta \int_0^t \|\psi_{xxx}(\tau)\|^2\, d\tau + C_\eta \int_0^t \|\varphi_x(\tau)\|_1^2\, d\tau + C_\eta \epsilon, \\ \int_0^t \int_{\mathbb{R}} |R_2|\, dx d\tau &\le \eta \int_0^t \|\psi_{xxx}(\tau)\|^2\, d\tau + C_\eta(\epsilon + \delta) \int_0^t \|(\varphi_x, \psi_x)(\tau)\|_1^2\, d\tau + C_\eta \epsilon, \\ \int_0^t \int_{\mathbb{R}} |R_3|\, dx d\tau &\le \eta \int_0^t \|\psi_{xxx}(\tau)\|^2\, d\tau + C_\eta(\epsilon + \delta) \int_0^t \|\varphi_x(\tau)\|_2^2\, d\tau + C_\eta \epsilon. \end{aligned} \tag{3.44}$$

Combining (3.43) with (3.44), we have (3.40) by the smallness of η and Lemmas 3.3 and 3.4. This completes the proof of Lemma 3.5. □

To close the a priori estimates, we control the term $\int_0^t \|\varphi_{xxx}(\tau)\|^2\, d\tau$ in the following lemma.

Lemma 3.6. *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that for $0 \le t \le T$,*

$$\|\varphi_{xx}(t)\|^2 + \int_0^t \|\varphi_{xxx}(\tau)\|^2\, d\tau \le C\left(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \epsilon^{\frac{2}{3}}\right) \tag{3.45}$$

provided that ϵ and δ are suitably small.

Proof. Differentiating $(3.1)_2$ with respect to x, then multiplying the resultant equation by $\frac{\varphi_{xx}}{v}$, and using the Eq. $(3.1)_1$, we have

$$\left(\frac{\mu \varphi_{xx}^2}{2v^2} - \psi_x \frac{\varphi_{xx}}{v}\right)_t + \kappa \frac{\varphi_{xxx}^2}{v^6} = A_1 + A_2 + A_3 + A_4 + \{\cdots\}_x, \tag{3.46}$$

where

$$\begin{aligned} A_1 &= -\frac{\psi_x \psi_{xxx}}{v} + \frac{(U_x + \psi_x)\psi_x \varphi_{xx}}{v^2}, \\ A_2 &= \left\{[p'(V) - p'(\varphi + V)]V_x - p'(\varphi + V)\varphi_x - g(V)_x\right\}\left(\frac{\varphi_{xxx}}{v} - \frac{\varphi_{xx}(\varphi_x + V_x)}{v^2}\right), \\ A_3 &= \mu \varphi_{xx}\left\{-\frac{U_{xxx}}{v^2} + \frac{V_{xx}(\psi_x + U_x)}{v^3} + 2\frac{(\psi_{xx} + U_{xx})(\varphi_x + V_x)}{v^3} - 2\frac{(\varphi_x + V_x)^2(\psi_x + U_x)}{v^4}\right\}, \\ A_4 &= -\kappa \frac{V_{xxx}\varphi_{xxx}}{v^6} + \kappa \varphi_{xxx}\left\{10\frac{(\varphi_x + V_x)(\varphi_{xx} + V_{xx})}{v^7} - 15\frac{(\varphi_x + V_x)^3}{v^8}\right\} \\ &\quad + \kappa \varphi_{xx}(\varphi_x + V_x)\left\{\frac{\varphi_{xxx} + V_{xxx}}{v^7} - 10\frac{(\varphi_x + V_x)(\varphi_{xx} + V_{xx})}{v^8} + 15\frac{(\varphi_x + V_x)^3}{v^9}\right\}. \end{aligned} \tag{3.47}$$

Integrating (3.46) with respect to t and x over $[0, t] \times \mathbb{R}$, by the similar estimates as Lemma 3.5, using (3.40) and the smallness of ϵ and δ, we can get (3.45). The details are omitted. Thus the proof of Lemma 3.6 is completed. □

Combining Lemmas 3.3–3.6, we have

Lemma 3.7. *Under the assumptions of Proposition 3.2, there exists a positive constant $C_3 > 0$ such that for $0 \le t \le T$,*

$$\|\varphi(t)\|_3^2 + \|\psi(t)\|_2^2 + \int_0^t \left(\|(\sqrt{V_t}\varphi)(\tau)\|^2 + \|(\varphi_x, \psi_x)(\tau)\|_2^2\right) d\tau \le C_3\left(\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 + \epsilon^{\frac{2}{3}}\right) \tag{3.48}$$

provided that ϵ and δ are suitably small.

By the same argument as above, we also obtain

Lemma 3.8. *Under the assumptions of Proposition 3.2, there exists a positive constant $C > 0$ such that for $0 \le t \le T$,*

$$\|\varphi_{xxx}(t)\|_1^2 + \|\psi_{xxx}(t)\|^2 + \int_0^t \|(\varphi_{xxxx}, \psi_{xxxx})(\tau)\|^2 d\tau \le C\left(\|\varphi_0\|_4^2 + \|\psi_0\|_3^2 + \epsilon^{\frac{2}{3}}\right) \tag{3.49}$$

provided that ϵ and δ are suitably small.

Proof of Proposition 3.2. Proposition 3.2 follows immediately from Lemmas 3.7 and 3.8. □

Now, we begin to prove our main theorem.

Proof of Theorem 1.1. Since Proposition 3.2 is proved, we can close the a priori estimates (3.5) by choosing $\|\varphi_0\|_3^2 + \|\psi_0\|_2^2$ and ϵ sufficiently small such that

$$\|\varphi_0\|_3^2 + \|\psi_0\|_2^2 < \frac{\delta^2}{4C_3}, \qquad \epsilon < \min\left\{\frac{\delta^3}{(4C_3)^{\frac{3}{2}}}, \epsilon_0\right\}.$$

Then by the standard continuity argument, one can extend the local solution to be a global one, i.e., $T = +\infty$. Thus, (1.14) follows immediately from (3.6). Moreover, the estimate (3.6) and the system (3.1) imply that

$$\int_0^{+\infty} \left(\|(\varphi_x, \psi_x)(t)\|^2 + \left| \frac{d}{dt} \|(\varphi_x, \psi_x)(t)\|^2 \right| \right) dt < \infty, \tag{3.50}$$

which as well as (3.6) and the Sobolev inequality lead to the following asymptotic behavior of solutions

$$\lim_{t\to+\infty} \sup_{x\in\mathbb{R}} \{|(v(t,x) - V(t,x), u(t,x) - U(t,x))|\} = 0. \tag{3.51}$$

From (3.51) and (iv) of Lemma 2.2, we have (1.15) at once. This completes the proof of Theorem 1.1. □

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