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ANOSOV MAPS, POLYCYCLIC GROUPS AND HOMOLOGY

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§1. INTRODUCTION

LET M DENOTE a compact Riemannian manifold. A C^1 map $f: M \to M$ is called *Anosov* if there exists a continuous splitting $TM = E^u \oplus E^s$ of the tangent bundle of M, and constants $C > 0$, $\lambda > 1$, such that $Tf(E^u) = E^u$, $Tf(E^s) \subset E^s$, and furthermore for all positive integers m and tangent vectors $X \in TM$:

$$|Tf^m(X)| \geq C\lambda^m |X| \qquad \text{if } X \in E^u.$$

$$|Tf^m(x)| \leq C^{-1}\lambda^{-m} |X| \qquad \text{if } X \in E^s.$$

This condition is independent of the Riemannian metric. Anosov maps have been studied in [1, 2, 7, 11], and elsewhere.

Examples of Anosov maps can be obtained from a Lie group G, a discrete subgroup Γ with G/Γ compact, and an endomorphism $\phi: G \to G$ such that $\phi(\Gamma) \subset \Gamma$; see [8]. If the derivative of ϕ at the identity of G has no eigenvalues of absolute value one, then the map $G/\Gamma \to G/\Gamma$ induced by ϕ is an Anosov map. All known Anosov maps are intimately related to maps of this type.

As an example take $G = \mathbf{R}^n$ and $\Gamma = \mathbf{Z}^n$, the integer lattice; then ϕ is defined by an $n \times n$ integer matrix A and G/Γ is the n-torus T^n. In this case we can identify A with the linear transformation

$$f_*: H_1(T^n; \mathrm{R}) \to H_1(T^n; \mathrm{R})$$

induced by f on the first homology group with real coefficients.

John Franks [2] has proved that if $f: T^n \to T^n$ is any Anosov diffeomorphism, then f_* has no root of unity among its eigenvalues.

Theorem 1 extends Franks' result to Anosov maps on a wider class of manifolds which includes all nilmanifolds. As an application, many manifolds that do not admit Anosov diffeomorphisms are constructed. For example: the Cartesian product of the Klein bottle and a torus.

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§2. STATEMENT OF RESULTS

THEOREM 1. *Let $f: M \to M$ be an Anosov map of a compact manifold. Let $u \in H^1(M; \mathbf{Z})$ be a nonzero integral cohomology class of dimension 1 such that $(f^*)^n u = u$, for some $n \in \mathbf{Z}_+$. Then the infinite cyclic covering space corresponding to u has infinite dimensional rational homology.*

The proof is postponed to §3.

COROLLARY 2. *Let M be a compact manifold such that every infinite cyclic covering space has finite dimensional rational homology. Then if $f: M \to M$ is an Anosov map, the induced homomorphism*

$$f_*: H_1(M; \mathbf{R}) \to H_1(M; \mathbf{R})$$

has no root of unity for an eigenvalue.

Proof. The eigenvalues of f_* are the same as those of $f^*: H^1(M; \mathbf{R}) \to H^1(M; \mathbf{R})$, since the Kronecker index

$$H^1(M; \mathbf{R}) \times H_1(M; \mathbf{R}) \to \mathbf{R}$$

is a dual pairing under which f^* and f_* are adjoint homomorphisms. A basis for the finitely generated free abelian group $H^1(M; \mathbf{Z})$ is also a vector space basis for $H^1(M; \mathbf{R})$ under the natural inclusion

$$H^1(M; \mathbf{Z}) \to H^1(M; \mathbf{R})$$

and with this basis, f^* is defined by an integral matrix. Therefore if $f_*: H_1(M; \mathbf{R}) \to H_1(M; \mathbf{R})$ has an eigenvalue ω with $\omega^n = 1$, there exists a nonzero vector $x \in H^1(M; \mathbf{R})$ having rational coordinates in the basis defined above, such that $(f^*)^n(x) = x$. Some nonzero integer multiple u of x lies in $H^1(M; \mathbf{Z})$, and a contradiction is reached by applying Theorem 1.

Polycyclic groups are introduced to ensure the validity of the topological hypothesis on M in Corollary 2.

Definition. A group π is *polycyclic* if it is solvable and every subgroup is finitely generated.

The following lemma collects some facts about polycyclic groups; see [3, 9, 12].

LEMMA 3. *The following conditions are pairwise equivalent:*

(*a*) *π is polycyclic.*

(*b*) *π is solvable and every Abelian subgroup is finitely generated.*

(*c*) *There exists an exact sequence*

$$0 \to N \to \pi \to A \to 0$$

where N is a finitely generated nilpotent group and A is a finitely generated Abelian group.

(*d*) *π is built from cyclic groups by forming a finite number of extensions.*

(*e*) *π is isomorphic to a solvable multiplicative group of integer matrices.*

The following is our main result:

THEOREM 4. *Let M be a compact manifold such that* (*a*) *the fundamental group has a polycyclic subgroup of finite index;* and (*b*) *the universal covering M has finite dimensional rational homology*

$$H_*(\tilde{M}; \mathbf{Q}) = \bigoplus_{i \geq 0} H_i(\tilde{M}; \mathbf{Q}).$$

Then if $f: M \to M$ is an Anosov map, there is no root of unity among the eigenvalues of $f_*: H_1(M; \mathbf{R}) \to H_1(M; \mathbf{R})$.

Proof. Follows from Corollary 2 and Lemma 6, below.

A useful property of polycyclic groups is the following.

LEMMA 5. *Let $\mathbf{Q}^n$ denote a rational vector space of finite dimension n. Given a representation ρ of a polycyclic group π in the group $GL(\mathbf{Q}^n)$ of automorphisms of $\mathbf{Q}^n$, the corresponding homology group*

$$H_*(\pi; \mathbf{Q}^n, \rho)$$

is a finite dimensional vector space over $\mathbf{Q}$.

Proof. Let B_π be a classifying space for π. That is, $B_\pi = E_\pi/\pi$ where E is a contractible space on which π operates freely and B_π is its orbit space; the natural map $E_\pi \to B_\pi$ defines a covering space. Then B_π is connected, $\pi_1(B_\pi) \cong \pi$ and $\pi_i(B_\pi) = 0$ for $i > 1$; these properties characterize B_π up to weak homotopy type.

By definition,

$$H_*(\pi^n j \mathbf{Q}^n \rho) = H_*(B_\pi; \mathbf{Q}^n, \rho),$$

the total homology group of the space B_π with coefficients twisted by the homomorphism

$$\rho: \pi_1(B_\pi) \cong \pi \to \mathrm{Aut}(\mathbf{Q}^n).$$

If it is known that B_π has the homotopy type of a finite simplicial complex, the lemma follows immediately. This is not the case in general but π *has a subgroup* π_0 *of finite index whose classifying space* B_{π_0} *has the homotopy type of a finite complex.* We now show this.

It is proved in Malcev [3] that π has a subgroup π_0 of finite index such that in the exact sequence

$$0 \to N_0 \to \pi_0 \to A_0 \to 0$$

of Lemma 3(c), N_0 and A_0 are torsion free. For any exact sequence of groups $0 \to \alpha \xrightarrow{i} \beta \xrightarrow{j} \gamma \to 0$, the homomorphisms i and j determine maps $B_\alpha \to B_\beta \to B_\gamma$ which, with proper choice of classifying spaces, may be taken to be a fibration. Therefore B_{π_0} is a fibre space over B_{A_0} with fibre B_{N_0}. Now B_{A_0} can be chosen to be an n-torus, and B_{N_0} has the homotopy type of a nilmanifold [4]. It is easily proved by induction on the number of cells in B that if $F \to E \to B$ is a fibration over a finite complex B whose fibre F has the homotopy type of a finite complex, then so has the total space E. Therefore B_{π_0} has the homotopy type of a finite complex.

Now the map $p: B_{\pi_0} \to B_\pi$ corresponding to the inclusion homomorphism $\pi_0 \to \pi$ can be chosen to be a finite covering space. It is well known that the homomorphism induced by p maps $H_*(B_{\pi_0}; \mathbf{Q}^n, \rho_0)$ onto $H_*(B_\pi; Q^n, \rho)$, where ρ_0 denotes the composite homomorphism

$$\pi_0 \to \pi^\rho \to GL(\mathbf{Q}^n).$$

(A right inverse to p_* can be constructed by considering the inverse image by p of a chain in B_π.) Therefore $H_*(\pi; \mathbf{Q}^n, \rho)$ is finite dimensional. This proves Lemma 5.

Remark. It follows immediately from Wang [10; Corollary 2, p. 17] that a polycyclic group π has a normal subgroup π_0 of finite index such that B_{π_0} is a compact smooth manifold.

LEMMA 6. *Let X be a finite simplicial complex. Suppose (a) $\pi_1(M)$ has a polycyclic subgroup of finite index, and (b) $H_*(\tilde{X}; \mathbf{Q})$ is finite dimensional where $\tilde{X}$ is the universal covering of X. Then $H_*(\hat{X}; \mathbf{Q})$ is finite dimensional for any covering $\hat{X}$ of X.*

Proof. By passing to a finite covering of X, we may assume $\pi_1(M)$ is polycyclic. Then so is $\pi_1(\hat{X}) = \pi$. Now π acts freely on $\tilde{X}$ with orbit space $\hat{X}$. In this situation there is a spectral sequence with $E^2 = H_*(\pi; H_*(\tilde{X}; \mathbf{Q}), \rho)$, converging to $H_*(\hat{X}; \mathbf{Q})$; see MacLane [5]. Here $\rho: \pi \to \operatorname{Aut} H_*(\tilde{X}; \mathbf{Q})$ is the homomorphism determined by the action of π on $\tilde{X}$. By assumption $H_*(\tilde{X}; \mathbf{Q})$ is finite dimensional, and then by Lemma 4, E^2 is finite dimensional. Lemma 6 follows by the usual spectral sequence argument: $H_*(\hat{X}; \mathbf{Q})$ is exhibited as being obtained from E^2 by passing alternately to subspaces and quotient spaces a finite number of times. This completes the proof of Lemma 6.

§3. PROOF OF THEOREM 1

It is well known that if $f: M \to M$ is an Anosov map then some iterate f^m of f has a fixed point, $m \geq 1$; compare Proposition 1.7 of Franks [1]. Since all the fixed points of an Anosov map have the same index ± 1, this means that *every iterate f^{mk} of f^m, $k \geq 1$, has nonzero Lefschetz number $L(f^{mk})$.* It suffices to prove Theorem 1 for an iterate of f; therefore, we may suppose $L(f) \neq 0$. Theorem 1 is thus a corollary of

LEMMA 7. *Let X be a finite complex, $f: X \to X$ a continuous map, and $u \in H^1(M; \mathbf{Z})$ a nonzero class such that $f^*u = u$. Let $p: Y \to X$ be the infinite cyclic covering corresponding to u. If $H_*(Y; \mathbf{Q})$ is finite dimensional then $L(f) = 0$.*

Proof. The covering space corresponding to u is the one whose fundamental group is annihilated by the composite homomorphism

$$\pi_1(X) \xrightarrow{h} H_1(X; \mathbf{Z}) \xrightarrow{\phi} \mathbf{Z}$$

where h is the Hurewicz map and ϕ denotes the Kronecker index with u.

Since $f^*u = u$, there is a commutative diagram

$$\begin{array}{ccc} Y & \xrightarrow{g} & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & X \end{array} .$$

Let $t: Y \to Y$ be a generator for the group of deck transformations of Y over X; then $t \circ g = g \circ t$. Imitating Milnor [6] we consider a commutative diagram of chain complexes and chain maps:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C_*(Y;Q) & \xrightarrow{t-1} & C_*(Y;Q) & \xrightarrow{p_*} & C_*(X;Q) & \longrightarrow & 0 \\ & & \downarrow g_* & & \downarrow g_* & & \downarrow f_* & & \\ 0 & \longrightarrow & C_*(Y;Q) & \longrightarrow & C_*(Y;Q) & \longrightarrow & C_*(X;Q) & \longrightarrow & 0 \end{array}$$

The rows are exact. There corresponds a commutative ladder of homomorphisms

$$\begin{array}{ccccccccc} \longrightarrow & H_i(Y;Q) & \longrightarrow & H_i(Y;Q) & \longrightarrow & H_i(X;Q) & \longrightarrow & H_{i-1}(Y;Q) & \longrightarrow \\ & \downarrow g_* & & \downarrow g_* & & \downarrow f_* & & \downarrow & \\ \longrightarrow & H_i(Y;Q) & \longrightarrow & H_i(Y;Q) & \longrightarrow & H_i(X;Q) & \longrightarrow & H_{i-1}(Y;Q) & \longrightarrow \end{array}$$

with exact rows. A well-known computation with traces shows that for such a ladder, the Lefschetz number of the middle endomorphism equals the sum of the Lefschetz numbers of the outer endomorphisms; in this case:

$$L(g) + L(f) = L(g).$$

Hence $L(f) = 0$. This proves Lemma 7 and thereby completes the proofs of Theorems 1, 2 and 4.

§4. APPLICATIONS

The following is an immediate consequence of Theorem 4:

THEOREM 8. *Suppose M is a compact manifold such that*

(a) $\pi_1(M)$ *has a polycyclic subgroup of finite index,*

(b) $H_*(\tilde{M}; \mathbf{Q})$ *is finite dimensional, and*

(c) $H^1(M; \mathbf{Z}) \cong \mathbf{Z}$.

Then M does not admit an Anosov diffeomorphism.

Proof. If $f: M \to M$ is any diffeomorphism, condition (c) forces f^* to have ± 1 for an eigenvalue.

Examples of such manifolds are easy to construct; for example, the Cartesian product of a simply connected manifold with a circle or Klein bottle. Another type comprises the solvmanifolds constructed as follows. Let A be an invertible $n \times n$ integer matrix acting as a diffeomorphism of the n-torus T^n in the usual way. Define $M_A = (T^n \times \mathbf{R})/\mathbf{Z}$ where the generator of $\mathbf{Z}$ acts on $T^n \times \mathbf{R}$ by $(x, y) \mapsto (A(x), y + 1)$. The fundamental group of M_A is the semidirect product $\mathbf{Z} \cdot \mathbf{Z}^n$, the generator of $\mathbf{Z}$ acting on $\mathbf{Z}^n$ by A; its commutator subgroup is the range of $A - I$ in $\mathbf{Z}$. If 1 is not an eigenvalue of A, then

$$H_1(M_A; \mathbf{Z})/\text{torsion} \cong \mathbf{Z} \cong H^1(M_A; \mathbf{Z}).$$

Clearly $\tilde{M}_A \approx \mathbf{R}^{n+1}$, and the exact sequence

$$0 \to \mathbf{Z}^n \to \pi_1(M_A) \to \mathbf{Z} \to 0$$

proves $\pi_1(M_A)$ polycyclic. Therefore by Theorem 4: *M_A does not admit an Anosov diffeomorphism if 1 is not an eigenvalue of A.*

The following theorem gives a different method of exploiting the fundamental group of M_A.

THEOREM 9. *Suppose A is an invertible integer matrix such that $A^k \neq 1$ for all $k \neq 0$. Then*:

(*a*) *$M_A \times S^1$ does not admit an Anosov diffeomorphism*;

(*b*) *If 1 is not an eigenvalue of A then $M_A \times T^m$ does not admit an Anosov diffeomorphism for any $m \geq 0$.*

Proof. The hypothesis on A implies that the center of $\pi_1(M_A) \cong \mathbf{Z} \cdot \mathbf{Z}^n$ is trivial. Therefore the center C of $\pi_1(M_A \times T^m) \cong \pi_1(M_A) \times \mathbf{Z}^m$ is $\mathbf{Z}^m$, and C meets the commutator subgroup of $Z_1(M_A \times T^m)$ only in the identity element. Therefore under the Hurewicz map C maps isomorphically onto a subgroup $C_1 \subset H_1(M_A \times T^m; \mathbf{Z})$ which is invariant under every diffeomorphism of $M_A \times T^m$. Under assumption (a), $C_1 \cong \mathbf{Z}$ and therefore the automorphism f_* of

$$H_1(M_A \times S^1; \mathbf{Z})/\text{torsion}$$

has ± 1 as an eigenvalue. Under assumption (b), $C_1 \cong \mathbf{Z}^m$ and $H_1(M_A \times S^1; \mathbf{Z}) \cong \mathbf{Z}^{m+1}$, and again f_* has an eigenvalue ± 1. Therefore Theorem 4 proves Theorem 9.

Next we consider the Klein bottle K^2. The fundamental group embeds in an exact sequence $0 \to \mathbf{Z}^2 \to \pi_1(K^2) \to \mathbf{Z}_2 \to 0$ and is therefore polycyclic.

THEOREM 10. *Let M be a compact orientable manifold such that*

(*a*) *$\pi_1(M)$ has a polycyclic subgroup of finite index*

(*b*) *$H_*(\tilde{M}; \mathbf{Q})$ is finite dimensional.*

Then $K^2 \times M$ does not admit an Anosov diffeomorphism.

Proof. Let $p : T^2 \to K^2$ be the double covering of the Klein bottle by the torus. The subgroup $p_\#(\pi_1(T^2)) \times \pi_1(M) \subset \pi_1(K^2 \times M)$ is represented by loops in $K^2 \times M$ that preserve local orientation; it is therefore preserved by every diffeomorphism f of $K^2 \times M$. Consequently f is covered by a diffeomorphism g of $T^2 \times M$. Now g^* preserves the subgroup $G \subset H^1(T^2 \times M; \mathbf{Z})$ that is the image of $(p \times I)^* : H^1(K^2 \times M; \mathbf{Z}) \to H^1(T^2 \times M; \mathbf{Z})$. By the Künneth theorem,

$$H^1(K^2 \times M; \mathbf{Z}) \cong \mathbf{Z} \oplus H^1(M; \mathbf{Z})$$

and

$$H^1(T^2 \times M; \mathbf{Z}) \cong \mathbf{Z}^2 \oplus H^1(M; \mathbf{Z}).$$

Thus G has corank 1, which forces $g^* : H^1(T^2 \times M; \mathbf{Z}) \to H^1(T^2 \times M; \mathbf{Z})$ to have ± 1 for an eigenvalue. By Theorem 4, g cannot be an Anosov diffeomorphism. Therefore f cannot be one either.

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