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LINEAR ALGEBRA AND ITS APPLICATIONS

# Limit cycles for successive projections onto hyperplanes in $\mathbb{R}^n$

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**Abstract**

In this paper we consider successive orthogonal projections onto $m$ hyperplanes in $\mathbb{R}^n$, where $m \geqslant 2$ and $n \geqslant 2$. A limit cycle is defined to be a sequence of points formed by projecting onto each of the hyperplanes once in a prescribed order, with the last projection giving the starting point. Several examples, including triangles, quadrilaterals, regular polygons, and arbitrary collections of lines in $\mathbb{R}^2$, are solved for the limit cycle. Limit cycles are found in various ways, including by a limiting process and by solving an $mn \times mn$ linear system of equations. The latter approach will produce all the limit cycles for an arbitrary ordered set of $m$ hyperplanes in $\mathbb{R}^n$. 

## 1. Introduction

In this paper we consider successive orthogonal projections onto $m$ subsets of $\mathbb{R}^n$, where $m$ and $n$ are integers which are at least 2. These subsets take the form of translated $(n-1)$ dimensional subspaces of $\mathbb{R}^n$. We denote orthogonal projections onto hyperplanes by $Q$ or $Q_i$, $i = 1, 2, \ldots, m$; if the hyperplanes are subspaces (i.e. if they contain the origin) we will often denote the projections by

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$P$ or $P_i$ instead. A limit cycle is defined to be a sequence of points formed by projecting a point of one of the hyperplanes onto the next hyperplane (in a given ordering of the hyperplanes), then projecting that point onto the next hyperplane, and continuing in this way until each of the hyperplanes has been projected onto once, with the final projection giving the starting point. We will show that if one starts with an arbitrary point in $\mathbb{R}^n$ and projects successively onto the hyperplanes in a specified order, the points of projection will converge to the points of a limit cycle; further, this limit cycle will be unique, independent of the starting point, if and only if there is no line which is parallel to all the hyperplanes.

One can determine limit cycles iteratively by choosing any starting point in $\mathbb{R}^n$ and computing projections in the prescribed order until convergence occurs, as nearly as it can occur in the presence of roundoff error. One can also compute limit cycles noniteratively by solving the system of linear equations developed in Section 5. If $n = 2$, one can also use the formulas in Theorem 3.1, or its corollaries in Theorems 2.1 and 2.2. If $n = 2$, the hyperplanes define a regular polygon, and the projections are done on adjacent (extended) sides, then Corollary 4.1 can also be used. In this paper we have tried to select examples which (a) illustrate or delimit the results in the paper, (b) have hyperplanes which define common figures from geometry, namely triangles, trapezoids, pentagons, regular polygons, and a tetrahedron, (c) have limit cycles which are relatively easy to determine exactly by noniterative means, and (d) produce interesting pictures. As a generalized example of a slightly different type, we note that if the hyperplanes have a unique common point of intersection, then by the result of Ref. [1], all points of the unique limit cycle will be just this point.

We need to recall the formulas for projecting a point $\mathbf{x}_0 \in \mathbb{R}^n$ onto a hyperplane $H = \{\mathbf{x}: \mathbf{a}^\mathrm{T}\mathbf{x} = d\}$. Since the projection is to be orthogonal, and $\mathbf{a}$ is orthogonal to the hyperplane, the projection $Q\mathbf{x}_0$ of $\mathbf{x}_0$ onto $H$ must have the form $Q\mathbf{x}_0 = \mathbf{x}_0 + \lambda\mathbf{a}$ for some real number $\lambda$. Substituting this into the equation $\mathbf{a}^\mathrm{T}\mathbf{x} = d$ to determine $\lambda$ gives

$$Q\mathbf{x}_0 = \mathbf{x}_0 + \left(\frac{d - \mathbf{a}^\mathrm{T}\mathbf{x}_0}{\|\mathbf{a}\|^2}\right)\mathbf{a}, \tag{1}$$

where $\|\mathbf{a}\|$ denotes the Euclidean norm of $\mathbf{a}$. Note that we can break this into

$$Q\mathbf{x}_0 = P\mathbf{x}_0 + \zeta,$$

where $P\mathbf{x}_0$ is the projection of $\mathbf{x}_0$ onto the subspace $\{\mathbf{x} : \mathbf{a}^\mathrm{T}\mathbf{x} = 0\}$ which is parallel to the hyperplane $H$, that is

$$P\mathbf{x}_0 = \mathbf{x}_0 - \left(\frac{\mathbf{a}^\mathrm{T}\mathbf{x}_0}{\|\mathbf{a}\|^2}\right)\mathbf{a} \quad \text{and} \quad \zeta = \left(\frac{d}{\|\mathbf{a}\|^2}\right)\mathbf{a}.$$

We will also denote by the letter $P$ the matrix representation of the projection, i.e.,

$$P = I - \frac{\mathbf{a}\mathbf{a}^{\mathrm{T}}}{\|\mathbf{a}\|^2}.$$

The case $n = 2$ is of special interest; using $(x, y)$ to represent a vector in $\mathbb{R}^2$, we have that if $H$ is not vertical we can write $H$ in the form $\{(x,y) : y = mx + b\}$ for real numbers $m$ and $b$. Thus $\mathbf{a} = [-m, 1]^{\mathrm{T}}$ and we get

$$Q(x_0, y_0) = \frac{1}{1+m^2}(x_0 + my_0 - mb, mx_0 + m^2y_0 + b). \tag{2}$$

In matrix form, we have

$$Q\left(\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}\right) = \frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{1}{1+m^2}\begin{bmatrix} -mb \\ b \end{bmatrix}, \tag{3}$$

where

$$\frac{1}{1+m^2}\begin{bmatrix} 1 & m \\ m & m^2 \end{bmatrix}$$

is the matrix representation of $P$.

**Example 1.1.** Consider the straight lines $y = x$ and $y = -2x, -\infty < x < \infty$. There are the graphs of the subspaces $H_1 = \{(x,y) : y = x\}$ and $H_2 = \{(x,y) : y = -2x\}$, respectively. Then for arbitrary $(x_0, y_0) \in \mathbb{R}^2$ we have

$$P_1(x_0, y_0) = \tfrac{1}{2}(x_0 + y_0, x_0 + y_0), \tag{4}$$

$$P_2(x_0, y_0) = \tfrac{1}{5}(x_0 - 2y_0, -2(x_0 - 2y_0)), \tag{5}$$

For all $(x_0, y_0) \in \mathbb{R}^2$ we have $\lim_{k\to\infty}(P_2P_1)^k(x_0, y_0) = (0, 0)$ since

$$(P_2P_1)^k = \frac{1}{10^k}\begin{bmatrix} -1 & -1 \\ 2 & 2 \end{bmatrix} \to \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

This agrees with non Neumann's theorem [2], which states that alternating projections onto two closed subspaces of a Hilbert space $X$ converge in norm to the projection onto the intersection of the subspaces, for any initial point $x \in X$. This result was later generalized to $m$ closed subspaces by Halperin [1]. The projection method of Kaczmarz [3] is described in the next example, in which the solution of a linear system of equations is determined by an iterative procedure using alternating projections. Tanabe [4] has shown that the method converges for singular or inconsistent linear systems of equations. We note that it follows from Theorem 1.1 that the limit cycle is unique if the linear equations describe lines in $\mathbb{R}^2$ which are not all parallel, but in general the limit cycle could depend on the initial guess. An example of nonuniqueness is given by the system of equations $x = 1, x = -1, y = 1$, and $y = -1$ in $\mathbb{R}^3$, where the limit cycle depends on the choice of $z_0$ in the initial point $(x_0, y_0, z_0)$.

**Example 1.2**. Consider the $3 \times 2$ linear system

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}. \tag{6}$$

One can construct an iterative procedure to solve (6) by sequential projections on the hyperplanes $H_1 = \{(x,y) : y = 1\}, H_2 = \{(x,y) : x + y = 1\}$, and $H_3 = \{(x,y) : x - y = 1\}$. The procedure of Tanabe [4] defines the function

$$F(\mathbf{x}) = Q_m(Q_{m-1}(\ldots(Q_1(\mathbf{x}))\ldots). \tag{7}$$

The iteration procedure of Kaczmarz [3] is given by

$$\mathbf{x}^{(k+1)} = F(\mathbf{x}^{(k)}), \quad k = 0, 1, 2, \ldots, \tag{8}$$

where $\mathbf{x}^{(0)}$ is an arbitrary initial guess. This is an example of fixed-point iteration; the objective is to find a solution of the equation $\mathbf{x} = F(\mathbf{x})$ (i.e. a "fixed point" of $F$). In this example we have $n = 2, m = 3, \mathbf{a}_1 = [0, 1]^{\mathrm{T}}$, $\mathbf{a}_2 = [1, 1]^{\mathrm{T}}, \mathbf{a}_3 = [1, -1]^{\mathrm{T}}, d_1 = d_2 = d_3 = 1$, so

$$Q_1\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$Q_2\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

$$Q_3\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix},$$

$H_2$ and $H_3$ are orthogonal, and so one can see geometrically that the iteration 8 converges to the point (1, 0) after at most two projections onto $H_3$, regardless of the initial guess. Performing two more projections yields the other two points in the limit cycle (1, 1), and $(1/2, 1/2)$. Due to the numbering of the hyperplanes we have done the projections in a counterclockwise manner; if we instead do them in a clockwise manner, the limit cycle is $\{(1,0), (1,1), (3/2, 1/2)\}$, which illustrates the fact that the limit cycle is dependent on the order in which the projections are done. Fig. 1 depicts the situation.

We now prove a result which allows one to compute one of the points of a limit cycle; once this has been done, the remaining points can be determined by doing one projection on each of the remaining hyperplanes. This is similar to the computation giving Eq. (8) in Ref. [4].

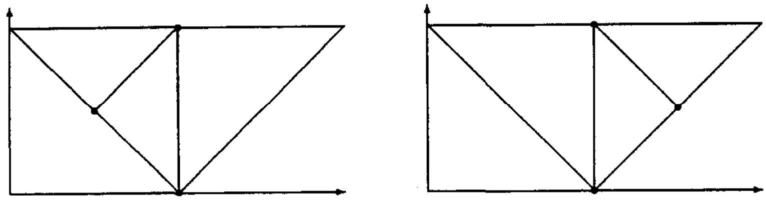


Fig. 1. The two limit cycles.

**Lemma 1.1.** *Let hyperplanes $H_1, H_2, \ldots, H_m \subset \mathbb{R}^n$ be given and let the projections $Q_i$ of a point onto $H_i$ be given by $P_i\mathbf{x} + \zeta_i$ for $i = 1, 2, \ldots, m$, where $P_i$ is the projection onto the subspace parallel to $H_i$. Let $Q = Q_m Q_{m-1} \ldots Q_1$, $P = P_m P_{m-1} \ldots P_1$, and $\boldsymbol{\beta} = \zeta_m + P_m\zeta_{m-1} + P_m P_{m-1}\zeta_{m-2} + \cdots + P_m P_{m-1} \cdots P_2\zeta_1$. Then for every positive integer k and every $\mathbf{x} \in \mathbb{R}^n$*

$$Q^k\mathbf{x} = P^k\mathbf{x} + \sum_{i=1}^{k-1} P^i\boldsymbol{\beta} + \boldsymbol{\beta}. \tag{9}$$

**Proof.** We first prove by induction on $m$ that $Q\mathbf{x} = P\mathbf{x} + \boldsymbol{\beta}$. For $m = 1$, this statement just says $Q_1\mathbf{x} = P_1\mathbf{x} + \zeta_1$, which is true by the definition of $Q_1$. Now suppose that for some $j \geqslant 1$, $Q_j \cdots Q_1\mathbf{x} = P_j \cdots P_1\mathbf{x} + \zeta_j + P_j\zeta_{j-1} + \cdots + P_j P_{j-1} \cdots P_2\zeta_1$. Then $Q_{j+1}Q_j \cdots Q_1\mathbf{x} = P_{j+1}Q_j \cdots Q_1\mathbf{x} + \zeta_{j+1} = P_{j+1}P_j \cdots P_1\mathbf{x} + P_{j+1}\zeta_j + P_{j+1}P_j\zeta_{j-1} + \cdots + P_{j+1}P_j \cdots P_2\zeta_1 + \zeta_{j+1}$ as desired, completing the induction. Now note that $Q^2\mathbf{x} = P(P\mathbf{x} + \boldsymbol{\beta}) + \boldsymbol{\beta} = P^2\mathbf{x} + P\boldsymbol{\beta} + \boldsymbol{\beta}$, and one can show that Eq. (9) is true by a second induction argument on $k$. □

The following theorem guarantees the existence of a limit cycle and gives the conditions under which it is unique.

**Theorem 1.1.** *Consider the ordered collection of m hyperplanes in $\mathbb{R}^n$ of the form $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_j^{\mathrm{T}}\mathbf{x} = d_j\}$, where $\mathbf{a}_j \neq \mathbf{0}$ for all j. Then successive projections onto the hyperplanes will converge to a limit cycle. This limit cycle will be unique, independent of the starting point, if and only if there is no line which is parallel to all the hyperplanes.*

**Proof.** First consider the case where there is no line which is parallel to all the hyperplanes. This is equivalent to each of the following three assertions: There is no $\mathbf{a} \in \mathbb{R}^n, \mathbf{a} \neq \mathbf{0}$, with $\mathbf{a}^{\mathrm{T}}\mathbf{a}_j = 0$ for $j = 1, \ldots, m$; $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m\}$ spans $\mathbb{R}^n$; and the matrix $\mathbf{A}$ whose $j$th row is $\mathbf{a}_j^{\mathrm{T}}$ has rank $n$. The result in Ref. [4] state that this last condition implies that $\|\mathbf{P}\| < 1$, where $\mathbf{P}$ is the matrix representation of $P$ and $\|\ \|$ represents the $l_2$ matrix norm. This condition implies that $\mathbf{I} - \mathbf{P}$ is non singular (where $\mathbf{I}$ is the $n \times n$ identity matrix), $\lim_{k\to\infty} \mathbf{P}^k = \mathbf{0}$ and

$\lim_{k\to\infty}\sum_{i=0}^{k-1}\mathbf{P}^i=(\mathbf{I}-\mathbf{P})^{-1}$. From this and Eq. (9) it follows that $\lim_{k\to\infty}Q^k\mathbf{x}=(\mathbf{I}-\mathbf{P})^{-1}\boldsymbol{\beta}$ for every $\mathbf{x}\in\mathbb{R}^n$. It now follows that $Q(\mathbf{I}-\mathbf{P})^{-1}\boldsymbol{\beta}=(\mathbf{I}-\mathbf{P})^{-1}\boldsymbol{\beta}$, so $(\mathbf{I}-\mathbf{P})^{-1}\boldsymbol{\beta}$ is the point on the $m$th hyperplane of a limit cycle, and the projections onto the other hyperplanes also converge to the points of this limit cycle which lie on those hyperplanes. Since $(\mathbf{I}-\mathbf{P})^{-1}\boldsymbol{\beta}$ is independent of $\mathbf{x}$, this limit cycle is unique.

Now consider the case where there is a line which is parallel to all the hyperplanes. Let $\mathbf{x}$ be any point in $\mathbb{R}^n$, and let $H$ be the unique $(n-1)$-dimensional hyperplane which is perpendicular to the line and contains $\mathbf{x}$. Since $H$ is perpendicular to all the given hyperplanes, all the projections starting with $\mathbf{x}$ must lie in $H$. Consider the $m(n-2)$-dimensional hyperplanes formed by intersecting the original hyperplanes with $H$, and by translating and rotating the axes (if necessary) and deleting an axis, consider $H$ to be $\mathbb{R}^{n-1}$. If $n>2$, and there is no line $\mathbb{R}^{n-1}$ which is parallel to all the new hyperplanes, then by the earlier part of this proof, successive projections starting with $\mathbf{x}$ will converge to a limit cycle. If on the other hand there is a line in $\mathbb{R}^{n-1}$ which is parallel to all the new hyperplanes, then repeat the process to reduce $\mathbb{R}^{n-1}$ to $\mathbb{R}^{n-2}$. Continuing this way, we will eventually show that successive iteration starting with $\mathbf{x}$ converges to a limit cycle, or else we will reduce the space all the way to $\mathbb{R}^1$, but in this latter case it follows geometrically that the first $m$ projections will produce the $m$ points of a limit cycle, so again we will have convergence to a limit cycle.

It only remains to show that if there is a line which is parallel to all the original $(n-1)$-dimensional hyperplanes, then the limit cycle is not unique. To see this, defining $H$ as before, choose a second starting point $\mathbf{x}'$ which is not in $H$, and let $H'$ be the unique $(n-1)$-dimensional hyperplane which is perpendicular to the line and contains $\mathbf{x}'$. Then $H$ and $H'$ have empty intersection, so that limit cycles starting from $\mathbf{x}$ and $\mathbf{x}'$ are not the same, so there is no unique limit cycle independent of the starting point. □

**Example 1.3.** Again consider Example 1.2. Here we have

$$P=P_3P_2P_1=\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&0\\0&0\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$$

and

$$\begin{aligned}\boldsymbol{\beta}&=\zeta_3+P_3\zeta_2+P_3P_2\zeta_1\\&=\frac{1}{2}\begin{bmatrix}1\\-1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1\\1\end{bmatrix}+\frac{1}{2}\begin{bmatrix}1&1\\1&1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}1\\0\end{bmatrix},\end{aligned}$$

so by Lemma 1.1. we have $Q^k[x,y]^{\mathrm{T}}=[1,0]^{\mathrm{T}}$ for all positive integers $k$ and all $(x,y)\in\mathbb{R}^2$, verifying our earlier geometrical observation that the iteration converges after at most two projections onto $H_3$.

In the next two sections we derive formulas for general triangles, quadrilaterals, and collections of lines in the plane. In Section 4 we consider regular polygons, and derive a simple real-value function $g$ such that the limit of the iteration

$$\mathbf{x}^{(k+1)} = g\left(\mathbf{x}^{(k)}\right), \quad k = 0, 1, 2 \ldots \tag{10}$$

determines the location of the points of the limit cycle. In that section we also verify geometrically for convex figures in the plane with all interior angles at least 90° that the figure formed by the limit cycle has the same interior angles as the original figure, but the new figure is rotated 90°; however the side lengths of the new figure are not necessarily proportional to the side lengths of the original figure. In Section 5 we formulate a linear system of equations whose solutions give the points of all limit cycles for $m$ hyperplanes in $n$-dimensional space.

## 2. Triangles and Quadrilaterals

In this section we give formulas for one of the points of a limit cycle for two situations. The first one is the triangle with vertices $(0, y_1), (\alpha, 0)$, and $(y_1/m_2 + \alpha, y_1)$ where $y_1 = -m_1\alpha, y_1 > 0, \alpha > 0, y_1/m_2 + \alpha > 0$, so the triangle is determined by the lines $y = m_1 x + y_1, y = m_2(x - \alpha), y = y_1$ (see Fig. 2).

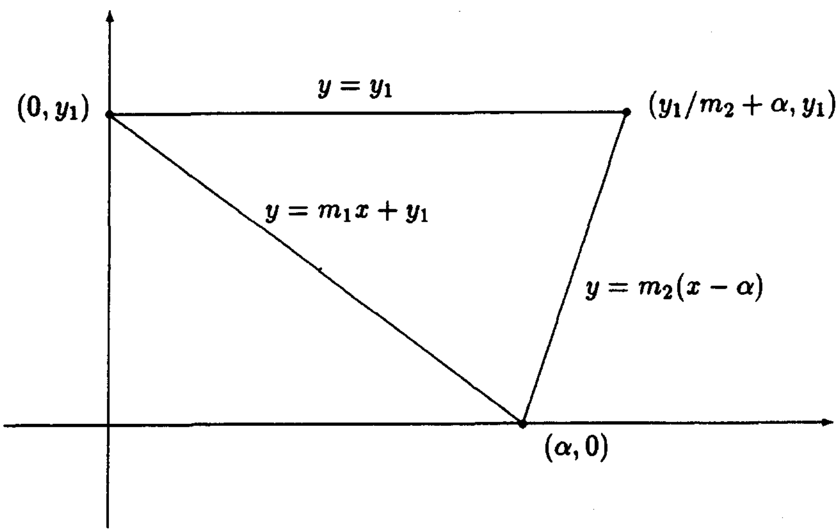



Fig. 2.

Any nondegenerate triangle can be put into this position by rotating and translating axes, if necessary. We note that the location of the limit cycle points is independent of the placement of the axes since the points are determined by orthogonal projections, which are geometrically independent of the placement of the axes; thus one could rotate/translate the axes to get the triangle in the desired position, compute the limit cycle points using the formula given in this section, then rotate/translate the axes back to their original position. Similar comments apply to the other figures considered in this paper.

The proof of the following theorem will be given later in Example 3.1 as a corollary of Theorem 3.1.

**Theorem 2.1.** *For the triangle described above, regardless of the starting point* $(x_0, y_0)$, *counterclockwise iteration converges to a unique limit cycle; the point of projection on the side* $y = y_1$ *is given by*

$$\left(\frac{m_2(1+m_1^2)(y_1+m_2\alpha)}{(1+m_1^2)(1+m_2^2)-(1+m_1m_2)}, y_1\right). \tag{11}$$

**Remark 2.1.** If the triangle is a right triangle, then it follows from Remark 3.1 in Section 3 that the point in Eq. (11) will be achieved after at most two projections on the line $y = y_1$.

**Example 2.1.** Consider the triangle formed by the lines, $y = -x + 2$, $y = \frac{1}{3}(x-2)$, and $y = 2$ (see Fig. 3). We have $m_1 = -1, y_2 = 2, m_2 = 1/3, \alpha = 2$, so substituting into Eq. (11) gives the point of the limit cycle on $y = 2$ as

$$\left(\frac{\frac{1}{3}(1+1)(2+\frac{1}{3}(2))}{(1+1)(1+\frac{1}{9})-(1-\frac{1}{3})}, 2\right) = \left(\frac{\frac{16}{9}}{\frac{20}{9}-\frac{2}{3}}, 2\right) = \left(\frac{8}{7}, 2\right).$$

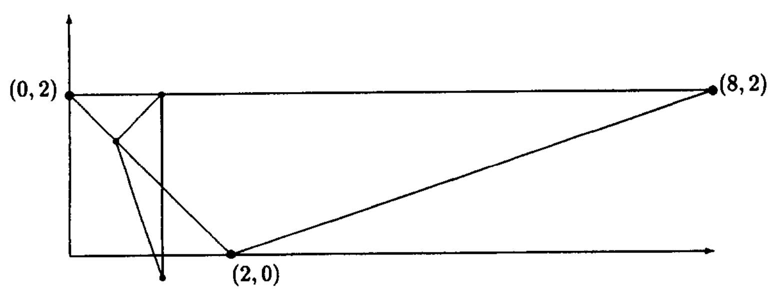



Fig. 3. Original triangle with limit cycle.

The remaining two points of the limit cycle are computed as follows using Eq. (2):

$$Q_1\left(\frac{8}{7},2\right)=\frac{1}{2}\left(\frac{8}{7}-2+2,-\left(\frac{8}{7}-2\right)+2\right)=\left(\frac{4}{7},\frac{10}{7}\right),$$
$$Q_2\left(\frac{4}{7},\frac{10}{7}\right)=\frac{9}{10}\left(\frac{4}{7}+\frac{1}{3}\cdot\frac{10}{7}+\frac{2}{9},\frac{1}{3}\left(\frac{4}{7}+\frac{1}{3}\cdot\frac{10}{7}\right)-\frac{2}{3}\right)=\left(\frac{8}{7},-\frac{2}{7}\right).$$

As a check, we have $Q_3(8/7,-2/7)=(8/7,2)$, the first point of the limit cycle. Note that the limit cycle forms a triangle which is similar to the original triangle, but is rotated $90^0$ clockwise.

We now suppose that a nondegenerate quadrilateral is formed by the lines $y=m_1x+y_1$, $y=m_2(x-\alpha)$, $y=m_3(x-\alpha)$, and $y=y_2$, so the vertices are $((y_2-y_1)/(m_1,\ y_2)$, $((y_1+m_2\alpha)/(m_2-m_1)$, $m_2(y_1+m_1\alpha)/(m_2-m_1))$, $(\alpha,0)$, and $(y_2/m_3+\alpha,y_2)$ (see Fig. 4).

Any nondegenerate quadrilateral which contains a side which is not perpendicular to any other (extended) side can be put into this form by selecting such a side and rotating the axes to make this side horizontal, then translating the axes so that the right-most vertex not on this side is on the $x$-axis. We note that there are quadrilaterals which are not covered by Theorem 2.2, e.g. rectangles and the quadrilateral with vertices (0, 0), (1, 1), (1, 2), and (−2, 2). Such quadrilaterals can be dealt with in other ways, e.g., by inspection or by using Theorem 3.1. One can prove the following theorem directly by the techniques used to prove Theorem 3.1, but instead we will give the proof later in Example 3.2 as a corollary to Theorem 3.1.

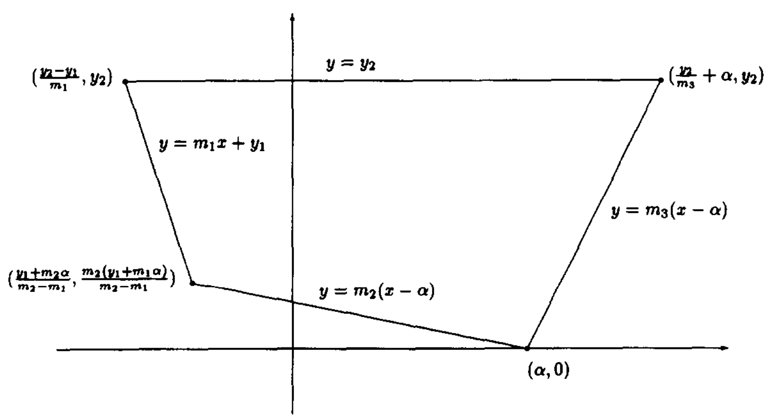



Fig. 4.

**Theorem 2.2.** *For the quadrilateral described above, regardless of the starting point* $(x_0, y_0)$, *counterclockwise iteration converges to a unique limit cycle; the point of projection on the side* $y = y_2$ *is given by*

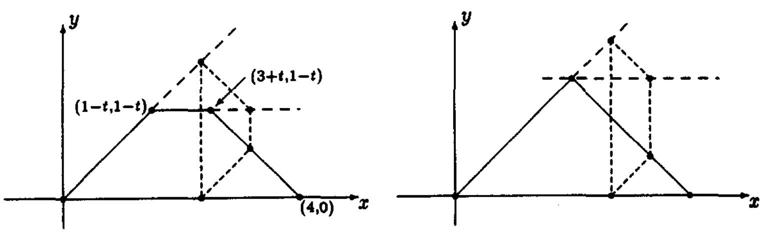
$$\begin{aligned}&\big([\alpha m_3^2(1+m_1^2)(1+m_2^2)+\alpha m_2(m_2-m_3)(1+m_1^2)\\&\quad+y_1(m_2-m_1)(1+m_2m_3)+m_1y_2(1+m_1m_2)(1+m_2m_3)]\\&\quad/[(1+m_1^2)(1+m_2^2)(1+m_3^2)-(1+m_1m_2)(1+m_2m_3)],y_2\big).\end{aligned}\tag{12}$$

**Remark 2.2.** If the quadrilateral contains a right angle, then it follows from Remark 3.1 in Section 3 that the point in Eq. (12) will be achieved after at most two projections on the line $y = y_2$.

**Example 2.2.** Consider the trapezoid formed by the lines $y = x, y = 0$, $y = 4 - x$, and $y = 1 - t$, where $-1 < t < 1$ (see Fig. 5(a)). We have $m_1 = 1$, $y_1 = 0$, $m_2 = 0$, $\alpha = 4$, $m_2 = -1$, $y_2 = 1 - t$, so the point of the limit cycle which is a projection onto the line $y = 1 - t$ is given by

$$\begin{aligned}&\left(\frac{4(-1)^2(1+1^2)(1+0^2)+0+0+1(1-t)\cdot 1\cdot 1}{(1+1^2)(1+0^2)(1+(-1)^2)-1\cdot 1},1-t\right)\\&=\left(\frac{9-t}{3},1-t\right).\end{aligned}\tag{13}$$

The remaining three points of the limit cycle are computed as follows, using Eq. (2):




(a) $(t = -\frac{1}{2})$ (b) $(t = -1)$

Fig. 5.

$$Q_1\left(\frac{9-t}{3},1-t\right)$$
$$=\frac{1}{1+1^2}\left(\frac{9-t}{3}+1\cdot(1-t)-1\cdot 0,1\cdot\left(\frac{9-t}{3}+1\cdot(1-t)\right)+0\right)$$
$$=\left(\frac{6-2t}{3},\frac{6-2t}{3}\right),$$
$$Q_2\left(\frac{6-2t}{3},\frac{6-2t}{3}\right)=\frac{1}{1+0^2}\left(\frac{6-2t}{3}+0-0,0\left(\frac{6-2t}{3}+0\right)+0\right)$$
$$=\left(\frac{6-2t}{3},0\right),$$
$$Q_3\left(\frac{6-2t}{3},0\right)$$
$$=\frac{1}{1+(-1)^2}\left(\frac{6-2t}{3}+(-1)(0)-(-1)(4),(-1)\left(\frac{6-2t}{3}+(-1)(0)\right)+4\right)$$
$$=\left(\frac{9-t}{3},\frac{3+t}{3}\right).$$

Now as $t\to -1$, the points of the limit cycle approach $(10/3,2),(8/3,8/3),(8/3,0)$, and $(10/3,2/3)$, respectively. At first glance this seems to be a discontinuity in the limit cycle as a function of $t$ as $t$ approaches $-1$, since when $t=-1$ the original trapezoid collapses to the right triangle with vertices (0, 0), (4, 0), and (2, 2), and the limit cycle for this triangle (traversed counterclockwise) consists of the points (2, 2), (2, 0), and (3, 1). The discontinuity disappears, however, if we consider the original figure to be a degenerate trapezoid with a horizontal side of length 0; in this case one can verify geometrically from Fig. 5(b) that the four-point limit cycle given above is correct. Note that although the trapezoid formed by the limit cycle has the same angles as the original trapezoid and is rotated $90^0$ clockwise from it, it is not similar to the original trapezoid since the lengths are not proportional.

## 3. Arbitrary collections of lines in $\mathbb{R}^2$

In this section we consider an arbitrary ordered collection of lines in $\mathbb{R}^2$. The only restrictions we impose in part (i) of Theorem 3.1 below are that the lines are not all parallel, and none of them are vertical. If the lines are parallel, then one can see geometrically that the limit cycle depends on the initial point of the iteration. It consists of the points on the intersection of the given lines (in the given order) with the line which contains the initial point and is perpendicular to the given lines. If one or more of the lines are vertical, one could rotate the axes so the lines become nonvertical, compute the unique limit cycle, then rotate the axes back to their original position; an explicit formula in the case of exactly one vertical line is given in part (ii) of Theorem 3.1 below. The method of proof in part (ii) could also be used to derive formulas when more than one line is vertical.

**Theorem 3.1.** *Consider an ordered collection of m lines in* $\mathbb{R}^2$, *where* $m \geqslant 2$ *and the lines are not all parallel to one another. Then regardless of the starting point* $(x_0, y_0)$, *the iteration converges to the unique limit cycle. Furthermore,*

(i) *if none of the lines are vertical, then the lines can be expressed by the equations* $y = m_i x + b_i$ $(m_i, b_i$ *real) for* $i = 1, 2, \ldots, m$. *In this case the point of projection on the line* $y = m_m x + b_m$ *is given by*

$$\frac{1}{\prod_{i=1}^{m}(1+m_i^2) - (1+m_1 m_m)\prod_{j=1}^{m-1}(1+m_j m_{j+1})} \times \left( \sum_{l=2}^{m} b_{l-1}(m_1 - m_{l-1}) \prod_{i=1}^{l-2}(1+m_i^2) \prod_{j=l}^{m-1}(1+m_j m_{j+1}) + b_m \left( m_1 \prod_{j=1}^{m-1}(1+m_j m_{j+1}) - m_m \prod_{i=1}^{m-1}(1+m_i^2) \right), \right.$$
$$\left. m_m \sum_{l=2}^{m} b_{l-1}(m_l - m_{l-1}) \prod_{i=1}^{l-2}(1+m_i^2) \prod_{j=1}^{m-1}(1+m_j m_{j+1}) + b_m \left( \prod_{i=1}^{m-1}(1+m_i^2) - \prod_{j=1}^{m-1}(1+m_j m_{j+1}) \right) \right), \tag{14}$$

*where as usual if* $i_1 > i_2$ *we define* $\sum_{j=i_1}^{i_2}(\cdot) = 0$ *and* $\prod_{j=i_1}^{i_2}(\cdot) = 1$.

(ii) *If exactly one of the lines is vertical, we can assume that it is the mth line, and that the lines are given by* $y = m_i x + b_i (m_i,\ b_i$ *real) for* $i = 1, 2, \ldots, m-1$ *and* $x = x'$ $(x'$ *real). In this case the point of projection of the limit cycle on the mth line has first coordinate* $x'$ *and second coordinate given by*

$$\frac{1}{\prod_{i=1}^{m-1}(1+m_i^2) - m_1 m_{m-1}\prod_{j=1}^{m-2}(1+m_j m_{j+1})} \times \left( m_{m-1} \sum_{l=2}^{m-1} b_{l-1}(m_l - m_{l-1}) \prod_{i=1}^{l-2}(1+m_i^2) \prod_{j=l}^{m-2}(1+m_j m_{j+1}) + b_{m-1} \prod_{i=1}^{m-2}(1+m_i^2) + x' m_{m-1} \prod_{j=1}^{m-2}(1+m_j m_{j+1}) \right). \tag{15}$$

**Proof.** (i) By Eq. (3) the projection of an arbitrary point $(x_0, y_0)$ onto the line $y = m_i x + b_i$ is given by

$$Q_i\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \frac{1}{1+m_i^2}\begin{bmatrix} 1 & m_i \\ m_i & m_i^2 \end{bmatrix}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \frac{b_i}{1+m_i^2}\begin{bmatrix} -m_i \\ 1 \end{bmatrix} \equiv P_i\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \zeta_i.$$

We compute

$$\begin{aligned} P = P_m P_{m-1}\cdots P_1 &= \left(\prod_{i=1}^{m}\frac{1}{1+m_i^2}\right)\begin{bmatrix} 1 & m_m \\ m_m & m_m^2 \end{bmatrix}\cdots\begin{bmatrix} 1 & m_2 \\ m_2 & m_2^2 \end{bmatrix}\begin{bmatrix} 1 & m_1 \\ m_1 & m_1^2 \end{bmatrix} \\ &= \left(\prod_{i=1}^{m}\frac{1}{1+m_i^2}\right)\begin{bmatrix} 1 & m_m \\ m_m & m_m^2 \end{bmatrix}\cdots\begin{bmatrix} 1 & m_3 \\ m_3 & m_3^2 \end{bmatrix}(1+m_1m_2)\begin{bmatrix} 1 & m_1 \\ m_2 & m_1m_2 \end{bmatrix} \\ &= \left(\prod_{i=1}^{m}\frac{1}{1+m_i^2}\right)\begin{bmatrix} 1 & m_m \\ m_m & m_m^2 \end{bmatrix}\cdots\begin{bmatrix} 1 & m_4 \\ m_4 & m_4^2 \end{bmatrix}(1+m_1m_2)\ (1+m_2m_3)\times \\ &\quad \begin{bmatrix} 1 & m_1 \\ m_3 & m_1m_3 \end{bmatrix} = \cdots = \left(\prod_{i=1}^{m}\frac{1}{1+m_i^2}\prod_{j=1}^{m-1}(1+m_jm_{j+1})\right)\begin{bmatrix} 1 & m_1 \\ m_m & m_1m_m \end{bmatrix}. \end{aligned}$$

So defining

$$\gamma = \prod_{i=1}^{m}\frac{1}{1+m_i^2}\prod_{j=1}^{m-1}(1+m_jm_{j+1}) \tag{16}$$

we have

$$P = \gamma\begin{bmatrix} 1 & m_1 \\ m_m & m_1m_m \end{bmatrix}.$$

Thus

$$\begin{aligned} P^2 &= \gamma^2(1+m_1m_m)\begin{bmatrix} 1 & m_1 \\ m_m & m_1m_m \end{bmatrix} = \gamma(1+m_1m_m)P \\ P^3 &= \gamma^2(1+m_1m_m)^2P = \gamma^3(1+m_1m_m)^2\begin{bmatrix} 1 & m_1 \\ m_m & m_1m_m \end{bmatrix} \\ &\cdots \\ P^k &= \gamma[\gamma(1+m_1m_m)]^{k-1}\begin{bmatrix} 1 & m_1 \\ m_m & m_1m_m \end{bmatrix}. \end{aligned} \tag{17}$$

Now we wish to show that the scalar quantity in square brackets in Eq. (17) is less than one in absolute value. Letting $\mathbf{a}_i = [-m_i, 1]^{\mathrm{T}}$ for $i = 1, 2, \ldots, m$ we have by the Cauchy–Schwarz inequality

$$|\gamma(1+m_1m_m)| = \left|\frac{(1+m_1m_2)(1+m_2m_3)\cdots(1+m_{m-1}m_m)(1+m_1m_m)}{(1+m_1^2)(1+m_2)^2\cdots(1+m_m^2)}\right|$$

$$= \frac{|\mathbf{a}_1\cdot\mathbf{a}_m|}{\|\mathbf{a}_1\|\|\mathbf{a}_m\|}\prod_{j=1}^{m-1}\frac{|\mathbf{a}_j\cdot\mathbf{a}_{j+1}|}{\|\mathbf{a}_j\|\|\mathbf{a}_{j+1}\|} \leqslant 1\cdot\prod_{j=1}^{m-1}1 = 1, \tag{18}$$

with equality if and only if $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_m$ are all parallel, i.e., if and only if $m_1 = m_2 = \cdots = m_m$. Since we have assumed the lines are not parallel, we have

$$|\gamma(1+m_1m_m)| < 1. \tag{19}$$

From this and Eq. (17) it follows that

$$\lim_{k\to\infty} P^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Now in order to use Lemma 1.1 we need to compute the quantity

$$\boldsymbol{\beta} = \zeta_m + P_m\zeta_{m-1} + P_mP_{m-1}\zeta_{m-2} + \cdots + P_mP_{m-1}\cdots P_2\zeta_1.$$

Arguing as above, for $2 \leqslant l \leqslant m$ we have

$$P_mP_{m-1}\cdots P_l\zeta_{l-1} = \prod_{i=l}^{m}\frac{1}{1+m_i^2}\prod_{j=1}^{m-1}(1+m_jm_{j+1})\begin{bmatrix} 1 & m_l \\ m_m & m_1m_m \end{bmatrix}\frac{b_{l-1}}{1+m_{l-1}^2}$$

$$\times\begin{bmatrix} -m_{l-1} \\ 1 \end{bmatrix} = b_{l-1}(m_l - m_{l-1})\prod_{i=l-1}^{m}\frac{1}{1+m_i^2}\prod_{j=l}^{m-1}(1+m_jm_{j+1})\begin{bmatrix} 1 \\ m_m \end{bmatrix}.$$

Thus

$$\boldsymbol{\beta} = \frac{b_m}{1+m_m^2}\begin{bmatrix} -m_m \\ 1 \end{bmatrix} + \left[\sum_{l=2}^{m} b_{l-1}(m_l - m_{l-1})\prod_{i=l-1}^{m}\frac{1}{1+m_i^2}\prod_{j=1}^{m-1}(1+m_jm_{j+1})\right]$$

$$\times\begin{bmatrix} 1 \\ m_m \end{bmatrix}.$$

Letting

$$\sigma = \sum_{l=2}^{m} b_{l-1}(m_l - m_{l-1})\prod_{i=l-1}^{m}\frac{1}{1+m_i^2}\prod_{j=1}^{m-1}(1+m_jm_{j+1}), \tag{20a}$$

$$\rho = \frac{b_m}{1+m_m^2}, \tag{20b}$$

we have

$$\boldsymbol{\beta} = \begin{bmatrix} \sigma - m_m\rho \\ m_m\sigma + \rho \end{bmatrix}.$$

Thus, from Lemma 1.1, and also using Eqs. (17) and (19) we have that the point of projection of the limit cycle on the line $y = m_m x + b_m$ is given by

$$\lim_{k\to\infty} Q^k \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \lim_{k\to\infty} P^k \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \sum_{k=1}^{\infty} P^k \boldsymbol{\beta} + \boldsymbol{\beta} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$$+ \sum_{k=1}^{\infty} \left( \gamma[\gamma(1 + m_1 m_m)]^{k-1} \begin{bmatrix} 1 & m_1 \\ m_m & m_1 m_m \end{bmatrix} \begin{bmatrix} \sigma - m_m\rho \\ m_m\sigma + \rho \end{bmatrix} \right) + \begin{bmatrix} \sigma - m_m\rho \\ m_m\sigma + \rho \end{bmatrix}$$

$$= \frac{\gamma}{1 - \gamma(1 + m_1 m_m)} \begin{bmatrix} \sigma - m_m\rho + m_1(m_m\sigma + \rho) \\ m_m(\sigma - m_m\rho) + m_1 m_m(m_m\sigma + \rho) \end{bmatrix} + \begin{bmatrix} \sigma - m_m\rho \\ m_m\sigma + \rho \end{bmatrix}$$

$$= \frac{1}{1 - \gamma(1 + m_1 m_m)} \begin{bmatrix} \gamma m_1\rho(1 + m_m^2) - m_m\rho + \sigma \\ m_m\sigma + \rho - \gamma\rho(1 + m_m^2) \end{bmatrix}$$

$$= \frac{\prod_{i=1}^{m}(1 + m_i^2)}{\prod_{i=1}^{m}(1 + m_i^2) - (1 + m_1 m_m)\prod_{j=1}^{m-1}(1 + m_j m_{j+1})} \begin{bmatrix} \sigma + b_m\left(\gamma m_1 - \frac{m_m}{1+m_m^2}\right) \\ m_m\sigma + b_m\left(\frac{1}{1+m_m^2} - \gamma\right) \end{bmatrix}$$

and now using Eqs. (16) and (20a) leads to Eq. (14).

(ii) If we replace the $m$th line $x = x'$ by the line $y = m_m(x - x')(m_m$ real) we can apply part (i) with $b_m = -m_m x'$, and then divide the numerator and the denominator by $m_m^2$ and let $m_m \to \infty$; the result is the point

$$\frac{1}{\prod_{i=1}^{m-1}(1 + m_i^2) - m_1 m_{m-1}\prod_{j=1}^{m-2}(1 + m_j m_{j+1})}$$
$$\Bigg(0 - x'\left(m_1 m_{m-1}\prod_{j=1}^{m-2}(1 + m_j m_{j+1}) - \prod_{i=1}^{m-1}(1 + m_i^2)\right),$$
$$m_{m-1}\sum_{l=2}^{m-1} b_{l-1}(m_l - m_{l-1})\prod_{i=1}^{l-2}(1 + m_i^2)\prod_{j=l}^{m-2}(1 + m_j m_{j+1})$$
$$+ b_{m-1}\prod_{i=1}^{m-2}(1 + m_i^2) - x'\left(0 - m_{m-1}\prod_{j=1}^{m-2}(1 + m_j m_{j+1})\right)\Bigg),$$

and this leads directly to Eq. (15). To validate this computation we need to verify that the denominator is nonzero, but this follows from the fact that

$$\left|\frac{m_1 m_{m-1}\prod_{j=1}^{m-2}(1 + m_j m_{j+1})}{\prod_{i=1}^{m-1}(1 + m_i^2)}\right| = \frac{|m_1|}{\sqrt{1 + m_1^2}} \left(\prod_{j=1}^{m-2} \frac{|\mathbf{a}_j \cdot \mathbf{a}_{j+1}|}{\|\mathbf{a}_j\|\,\|\mathbf{a}_{j+1}\|}\right) \frac{|m_{m-1}|}{\sqrt{1 + m_{m-1}^2}}$$
$$< 1 \cdot \left(\prod_{j=1}^{m-2} 1\right) \cdot 1 = 1. \qquad (21)$$

Now we have shown that the point $(x'_m, y'_m)$ of the limit cycle for $\{y_i = m_i x + b_i\colon\ i = 1, 2, \ldots, m\}$ which is a projection on the line $y = m_m x + b_m$ approaches the point $(x', y')$ given in Eq. (15) on the line $x = x'$ as $m_m \to \infty$; thus by continuity of the projections, we have $Q_1(x'_m, y'_m) \to Q_1(x', y')$, $Q_2Q_1(x'_m, y'_m) \to Q_2Q_1(x', y'), \ldots, Q_mQ_{m-1}\cdots Q_1(x'_m, y'_m) \to Q_mQ_{m-1}\cdots Q_1(x', y')$, where in the last step we also used the fact that the line $y = m_m x + b_m$ approaches the line $x = x'$ in the region of interest. Since $Q_mQ_{m-1}\cdots Q_1(x'_m, y'_m) = (x'_m, y'_m)$, we also have $Q_mQ_{m-1}\cdots Q_1(x', y') = (x', y')$, so we do indeed have a limit cycle. This limit cycle must be unique, for if there were another limit cycle one could rotate the axes slightly and get a contradiction to the uniqueness in part (i). Note that uniqueness also follows from Theorem 1.1. □

**Remark 3.1.** From Eq. (18) we see that

$$\gamma(1 + m_1 m_m) = \cos\theta_{12}\cos\theta_{23}\cdots\cos\theta_{m-1,m}\cos\theta_{m1},$$

where $\theta_{ij}$ is the angle between $\mathbf{a}_i$ and $\mathbf{a}_j$, and is thus also the angle between the lines $y = m_i x + b_i$ and $y = m_j x + b_j$. Thus if any two adjacent lines (in the given ordering) are perpendicular, we have $\gamma(1 + m_1 m_m) = 0$, so

$$P^k = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

for $k \geqslant 2$ by Eq. (17), so the point in Eq. (14) or Eq. (15) will be achieved after at most two projections onto the $m$th line. Even if no two adjacent lines are perpendicular, one would except the convergence to be faster if two adjacent lines are nearly perpendicular.

**Example 3.1.** We can now prove Theorem 2.1 as a corollary to Theorem 3.1 (i). Thus we take $m = 3, m_3 = 0, b_1 = y_1, b_2 = -m_2\alpha, b_3 = y_1$ and substitute into Eq. (14), getting

$$\begin{aligned} &\frac{1}{(1+m_1^2)(1+m_2^2) - (1+m_1m_2)} \times (y_1(m_2 - m_1)(1 + m_2\cdot 0) \\ &\quad -m_2\alpha(0 - m_2)(1 + m_1^2) + y_1 m_1(1 + m_1m_2)(1 + m_2\cdot 0), \\ &\quad y_1\left((1+m_1^2)(1+m_2^2) - (1m_1m_2)(1 + m_2\cdot 0)\right)) \end{aligned} \tag{22}$$

which reduces to Eq. (11).

**Example 3.2.** We can now prove Theorem 2.2 as a corollary to Theorem 3.1(i). Thus we take $m = 4$, $m_4 = 0$, $b_1 = y_1$, $b_2 = -m_2\alpha$, $b_3 = -m_3\alpha$, $b_4 = y_2$ and substitute into Eq. (14), getting

$$\frac{1}{(1+m_1^2)(1+m_2)^2(1+m_3^2)-(1+m_1m_2)(1+m_1m_3)}$$
$$\times\, (y_1(m_2-m_1)(1+m_2m_3)+(-m_2\alpha)(m_3-m_2)(1+m_1^2)$$
$$+(-m_3\alpha)(-m_3)(1+m_1^2)\ (1+m_2^2)+y_2m_1(1+m_1m_2)(1+m_2m_3),$$
$$y_2((1+m_1^2)(1+m_2^2)(1+m_3^2)\ -(1+m_1m_2)(1+m_2m_3))) \tag{23}$$

which reduces to Eq. (12).

Note that once the first component $x^*$ of Eq. (14) has been computed, the second component $y^*$ can be computed more simply as $y^* = m_m x^* + b_m$ since $(x^*, y^*)$ must be on the line $y = m_m x + b_m$.

## 4. Regular polygons

In this section we consider the case of a regular polygon with $m$ sides, where $m \geqslant 3$. We will assume the projections are done successively on adjacent sides. The next theorem gives a simple relationship between the projection on one (extended) side and the projection on the next (extended) side which holds even if the projections are not part of the limit cycle.

**Theorem 4.1.** *Let a denote the position of a point on an extended side of an m-sided regular polygon relative to the midpoint of that side, that is,* $|a| =$ *distance of the point from the midpoint,* $a \geqslant 0$ *if the points is at the midpoint or on the side of the midpoint away from the next side, and* $a < 0$ *otherwise. let b be the position of the projection of the first point on the next side relative to the midpoint of the next side. Let c be the distance from the center of the polygon to the midpoint of any side. Then*

$$\frac{b}{c} = \frac{a}{c}\cos\left(\frac{2\pi}{m}\right) + \sin\left(\frac{2\pi}{m}\right). \tag{24}$$

**Proof.** Consider Fig. 6, which is drawn for the case $a \geqslant 0$ and $m > 4$; for the other cases the proof is essentially the same, although the pictures look a little different. Without loss of generality, we can assume that the origin is at the center of the polygon and the top side is horizontal. Let $A$ be the first point, let $B$ be its projection onto the (extended) second side, let $C$ be the midpoint of the second side, let $E$ be the intersection of the two sides considered, and let $O$ be the origin (see Fig. 6). Let $D$ be the intersection of the line $AB$ with the line through the origin which is parallel to the line $BC$. Now the measure of $\angle AEC$ is $(m-2)\pi/m$, so the measure of $\angle AEB$ is $2\pi/m$, and thus the line $OD$ makes an angle of $2\pi/m$ with the positive $x$-axis. Therefore the point $(\cos 2\pi/m, \sin 2\pi/m)$ is on the (extended) line $OD$. Now since $OD$ and $CB$

are parallel by construction, and $OC$ and $DB$ are perpendicular to $CB$, it follows that $ODBC$ is a rectangle, and $DB$ (and thus $AD$) are perpendicular to $OD$, so $|OD|$ is the component of vector $OA$ in the direction of the vector $[\cos 2\pi/m, \sin 2\pi/m]^{\mathrm{T}}$. Therefore

$$|OD| = [a, c]\begin{bmatrix} \cos\frac{2\pi}{m} \\ \sin\frac{2\pi}{m} \end{bmatrix} = a\cos\frac{2\pi}{m} + c\sin\frac{2\pi}{m}.$$

Thus

$$b = |CB| = |OD| = a\cos\frac{2\pi}{m} + c\sin\frac{2\pi}{m}$$

and dividing by $c$ completes the proof. □

Theorem 4.1 suggests that we consider the function $g$ defined by

$$g(x) = x\cos\frac{2\pi}{m} + \sin\frac{2\pi}{m} \tag{25}$$

and the iteration

$$x^{(k+1)} = g\left(x^{(k)}\right), \quad k = 0, 1, \ldots \tag{26}$$

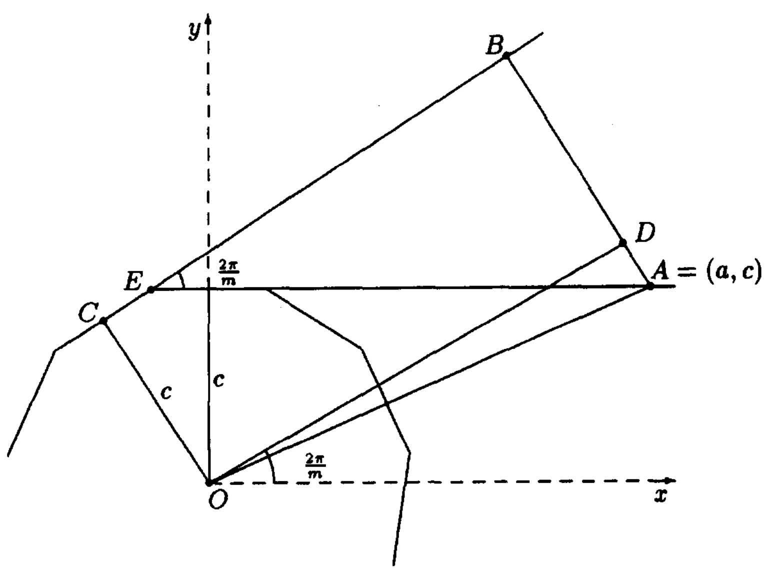



Fig. 6.

since if $x^{(k)}$ gives the position of a point in a sequence of projected points relative to the midpoint of the (extended) side onto which it was projected, then $x^{(k+1)}$ will give the position of the next point relative to the midpoint of the (extended) next side. Thus if $x^*$ is a fixed point if $g$ (i.e. $x^* = g(x^*)$), then if we start at a position $x^*$ relative to the midpoint of a side and do $m$ successive projections onto successive sides, we will return to the starting point, so we will have a limit cycle. Thus the following theorem is of interest.

**Theorem 4.2.** *The function g given by* (25) *has a unique fixed point* $x^* = \cot \pi/m$. *The iteration* (26) *will converge to* $\cot \pi/m$ *for any starting approximation* $x_0$, *with rate of convergence given by the equations*

$$x^{(k+1)} - \cot\frac{\pi}{m} = \left(x^{(k)} - \cot\frac{\pi}{m}\right)\cos\frac{2\pi}{m}, \quad k = 0, 1, 2, \ldots \tag{27}$$

$$x^{(k+m)} - \cot\frac{\pi}{m} = \left(x^{(k)} - \cot\frac{\pi}{m}\right)\cos^m\frac{2\pi}{m}, \quad k = 0, 1, 2, \ldots \tag{28}$$

**Proof.** Solving the linear equation $x^* = g(x^*)$ gives

$$x^*\left(1 - \cos\frac{2\pi}{m}\right) = \sin\frac{2\pi}{m},$$

$$x^* = \frac{2\sin\frac{\pi}{m}\cos\frac{\pi}{m}}{2\sin^2\frac{\pi}{m}} = \cot\frac{\pi}{m}.$$

Now for $k = 0, 1, 2, \ldots$ we have

$$x^{(k+1)} - \cot\frac{\pi}{m} = g\left(x^{(k)}\right) - g\left(\cot\frac{\pi}{m}\right) = \left(x^{(k)} - \cot\frac{\pi}{m}\right)\cot\frac{2\pi}{m}$$

so Eq. (27) holds. Eq. (28) now follows by induction on $m$. □

We next prove result relating the limit cycle of certain polygons to the original polygon.

**Theorem 4.3.** *Let S be a convex polygon in which every angle has measure at least* 90 *degrees, and let S′ be a limit cycle for S created by doing projections on adjacent sides. Then S′ preserves the angles of S, and S′ is rotated* 90° *from S in the direction opposite the direction in which projections are done; that is, each angle of S′ has the same measure as the corresponding angle of S, but the rays forming the sides of the angles are rotated* 90° *clockwise* (*counterclockwise*) *if the projections are done counterclockwise* (*clockwise*).

**Proof.** Without loss of generality, assume the projections are done counterclockwise. We first claim that at least one of the vertices of $S'$ is either on the (unextended) side of $S$ onto which it is the projection, or else is located clockwise from that side. Suppose this were not the case, so that every vertex of $S'$ be located counterclockwise from the side onto whose extension it is a projection. For such a point $\mathbf{x}^{(k)}$, let $d(\mathbf{x}^{(k)})$ be the distance of $\mathbf{x}^{(k)}$ from the counterclockwise endpoint of the side onto whose extension $\mathbf{x}^{(k)}$ is a projection (see Fig. 7(a)). Then the points $\mathbf{x}^{(k)}$ and $\mathbf{x}^{(k+1)}$, together with the counterclockwise endpoint of the side onto whose extension $\mathbf{x}^{(k)}$ is a projection, are vertices of a right triangle with hypotenuse of length $(d\mathbf{x}^{(k)})$. Now $d(\mathbf{x}^{(k+1)})$ is the length of part of one leg of this right triangle, so $d(\mathbf{x}^{(k+1)}) < d(\mathbf{x}^{(k)})$. Now letting $m$ be the number of sides of $S$, we have by induction that $d(\mathbf{x}^{(k+m)}) < d(\mathbf{x}^{(k)})$, but the assumption that $S'$ is a limit cycle implies $\mathbf{x}^{(k+m)} = \mathbf{x}^{(k)}$, so we have a contradiction, which establishes the claim.

Now let $s'$ be a vertex of $S'$ which is on or clockwise from the (unextended) side of $S$ onto whose extension it is a projection (see Fig. 7(b)). Since the angle between this side of $S$ and the next side of $S$ counterclockwise from it has measure between 90° and 180°, the next vertex of $S'$ will be at or clockwise from the (unextended) side of $S$ onto whose extension it is a projection, and by continuing the projections one can see that all vertices of $S'$ must satisfy this condition. Each side of $S'$ is formed by projecting onto (the extension of) a side of $S$, and we consider these two sides to be corresponding. Each vertex of $S'$ is located at the intersection of two sides of $S'$, and we consider this vertex to correspond to the vertex formed by the intersection of the two corresponding sides of $S$. Thus each vertex of $S'$ will correspond to the counterclockwise endpoint of the side onto whose extension it is a projection.

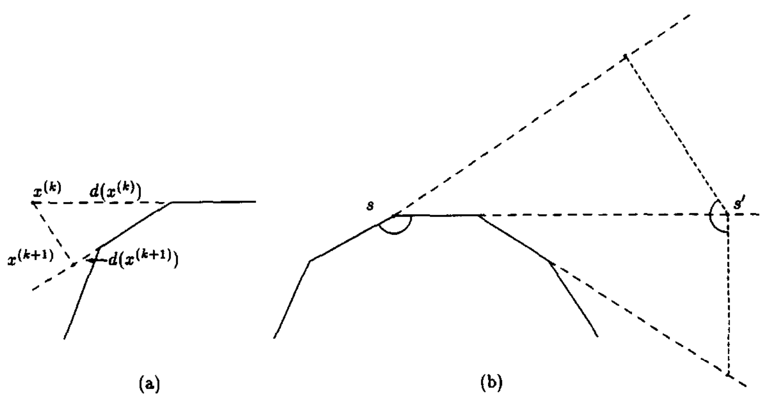



Fig. 7.

Now consider a vertex $s$ of $S$ and the corresponding vertex $s'$ of $S'$. Then $s$ and $s'$ lie on the same (extended) side of $S$, with $s'$ clockwise from $s$. If we were to translate $s$ along this side to $s'$, then rotate the angle at $s$ clockwise 90°, then the rays which form the sides of $s$ will coincide with the rays which form the sides of $s'$. This is because the rays which form the sides of $s'$ are (when extended) perpendicular to the corresponding (extended) rays for $s$, and the fact that $s'$ is clockwise from $s$ guarantees that the corresponding rays will point in the same (not opposite) directions. Thus the angles of $S$ and $S'$ at $s$ and $s'$ respectively have the same measure, but the angle at $s'$ is rotated 90° clockwise from the corresponding angle at $s$. □

**Remark 4.1.** Under the hypotheses of Theorem 4.3, one can see by drawing a picture that $S'$ will surround $S$, but the conclusions of Theorem 4.2 may hold even if $S$ has acute angles, and in this case $S'$ might not surround $S$ (e.g. see Example 2.1). If $S$ has acute angles, however, then the theorem may fail; for example, if in Example 2.2, $t$ is chosen sufficiently close to 1 but less than 1 then, although $S$ is a trapezoid, one can see by using the results of Example 2.2 or by sketching projections until the approximate location of the limit cycle becomes apparent that $S'$ is a quadrilateral which intersects itself, so angles are not preserved (see Fig. 8). The problem in this example is that the vertex of $S'$ which lies on the extended side of $S$ which has negative slope is located counterclockwise from the corresponding vertex of $S$.

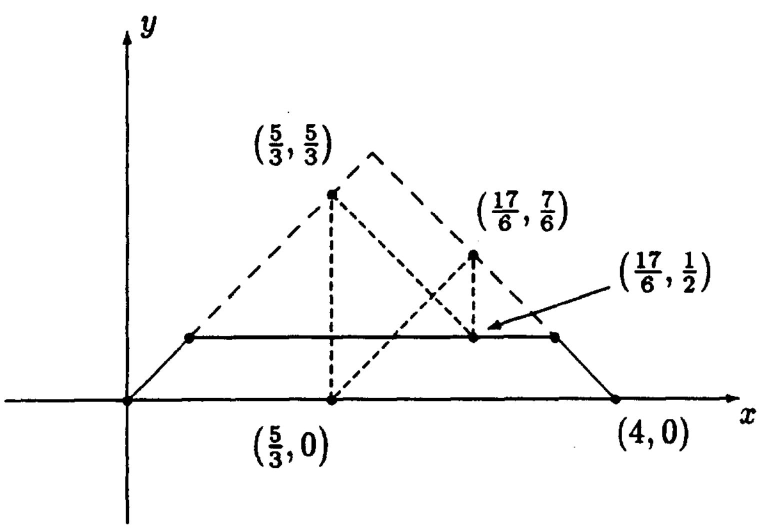



Fig. 8. ($t = \frac{1}{2}$).

**Remark 4.2.** Even under the hypotheses of Theorem 4.3, the limit cycle need not be similar to the original polygon. For example, if $S$ is the pentagon with vertices $(-1,0)$, $(1,0)$, $(4,3)$, $(0,7)$, and $(-4,3)$, then one can verify geometrically that $S'$ is the pentagon with vertices $(-4,3)$, $(-4,0)$, $(-3/2,5/2)$, $(4,3)$, and $(0,7)$. Since the side lengths of $S$ are 2, $3\sqrt{2}$, $4\sqrt{2}$, $4\sqrt{2}$, and $3\sqrt{2}$, and the corresponding side lengths of $S'$ are 3, $(5/2)\sqrt{2}$, $(11/2)\sqrt{2}$, $4\sqrt{2}$, and $4\sqrt{2}$, the side lengths of $S'$ are not proportional to those of $S$, so the polygons are not similar (see Fig. 9).

We now use the results of this section to give a complete description of the limit cycle for a regular polygon.

**Corollary 4.1.** *The unique limit cycle for a regular polygon, assuming the projections are done on adjacent sides, is a similar regular polygon with the same center but rotated* 90° *in the direction opposite that in which the projections were done. If these polygons have m sides, then the length of a side of the limit cycle is* $\cot(\pi/m)$ *times the length of a side of the original polygon, and the distance from*

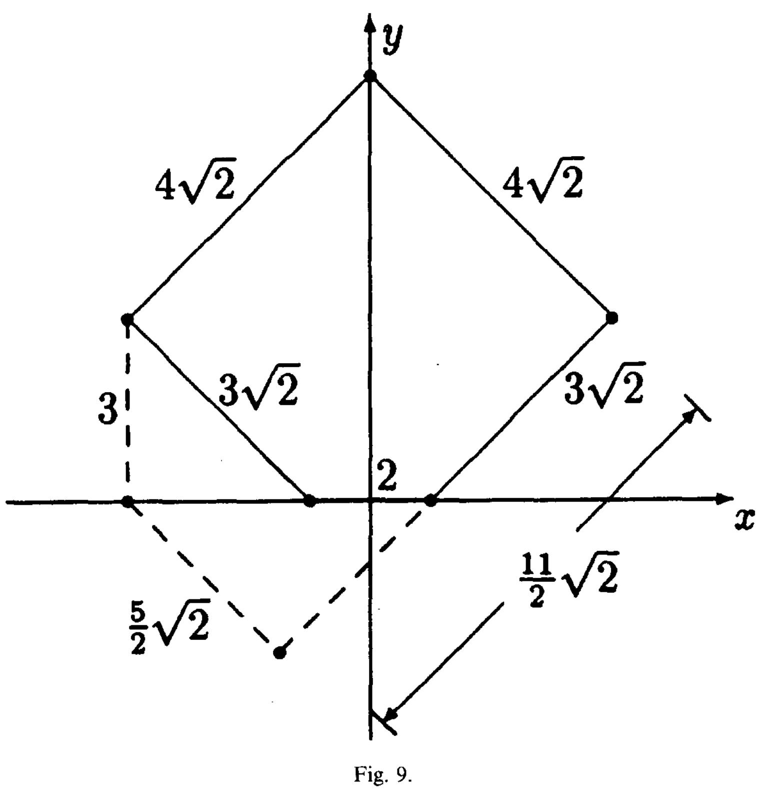



Fig. 9.

*the center to a vertex of the limit cycle is also* $\cot(\pi/m)$ *times the distance from the center to a vertex of the original polygon.*

**Proof.** From Theorems 4.1 and 4.2 we have that if the location of a point on an (extended) side is given by $a/c = \cot(\pi/m)$, then $b/c = \cot(\pi/m)$ also and successive projections will bring us back to the starting point, so we have a limit cycle. Preservation of angles and the 90° rotation follow from Theorem 4.3 when $m \geqslant 4$; for $m = 3$ we have $a/c = \cot\ \pi/3 = 1/\sqrt{3}$, and a picture shows that again preservation of angles and rotation hold. Now from Fig. 6 if $m > 4$ or from a similar picture if $m = 3$ or $m = 4$ we see that the distance from the origin to each vertex of the limit cycle is $|OA| = \sqrt{a^2 + c^2} = c\sqrt{\cot^2(\pi/m) + 1} = c\csc(\pi/m)$. Now triangle $AOB$ is isoceles with vertex angle $2\pi/m$ (since the limit cycle is a regular polygon with $m$ sides), so the length of a side of the limit cycle is $|AB| = 2(|OA|\ \sin(\pi/m)) = 2c\csc(\pi/m)\sin(\pi/m) = 2c$. But from triangle $EOC$ in Fig. 6 (or a similar figure if $m = 3$ or 4), the length of a side of the original polygon is $2|CE| = 2c\ \tan(\pi/m)$, so the ratio of the side length of the limit cycle to the side length of the original polygon is $2c/(2c\ \tan(\pi/m)) = \cot(\pi/m)$, as claimed. Also, from triangle $EOC$ the distance from the origin to a vertex of the original polygon is $c\ \sec(\pi/m)$, so the ratio of the corresponding distance in the limit cycle to this is $c\csc(\pi/m/c)\ \sec(\pi/m) = \cot(\pi/m)$. Finally, uniqueness of the limit cycle follows from Theorem 4.2, since if there were a different limit cycle then it would have a vertex whose location relative to the midpoint of the side on whose extension it was a projection would be some number $x$ other than $\cot(\pi/m)$, but by Eq. (28), after $m$ projections we would arrive at a point with location given by $\cot(\pi/m) + (x - \cot(\pi/m))\cos^m(2\pi/m) \neq x$, so we would not have a limit cycle. Note that uniqueness also follows from Theorems 1.1 and 3.1. □

**Remark 4.3.** This proof contains the striking fact that the side length of the limit cycle is the constant $2c$, independent of $m$.

**Remark 4.4.** For $z > 0$, $z$ small, we have, by using Taylor's Theorem with $\cos z/\sin z$, $\cot z = 1/z + O(z)$ so $\cot(\pi/m) = m/\pi + O(\pi/m)$, so the distance from the center to a vertex of the limit cycle grows like $m$ as $m$ increases.

Fig. 10 shows the original figure and limit cycle for two regular polygons.

**Remark 4.5.** We note that if projections are done on other than adjacent sides the limit cycles can look quite different from the original polygons. For example, if we project onto every $k$th side where $k < m/2$ and $k$ is relatively prime to $m$, then a similar function $g$ to that in (25) can be constructed. In this case we obtain

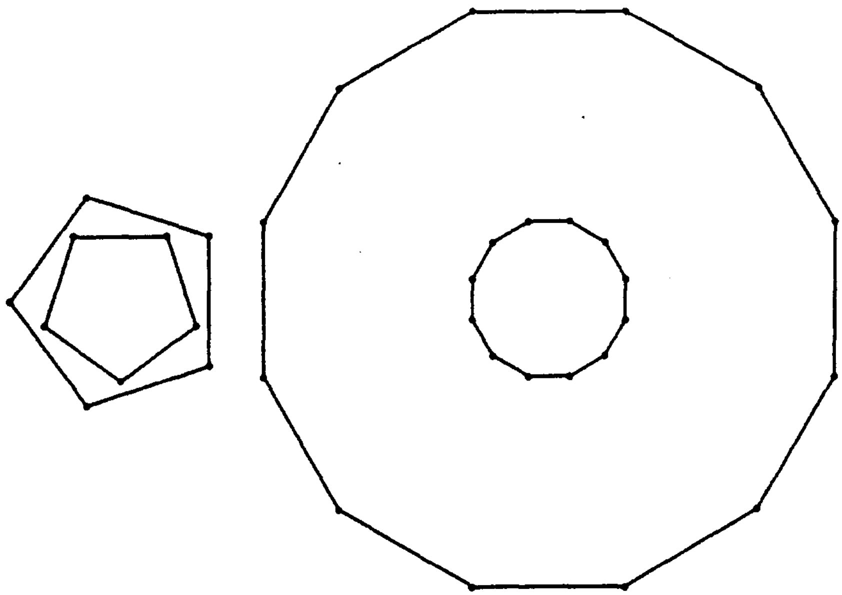


Fig. 10. Pentagon and dodecagon with limit cycle.

$$g(x) = x\cos\frac{2k\pi}{m} + \sin\frac{2k\pi}{m}. \tag{29}$$

A similar analysis as in Corollary 4.1 shows that the distance from the origin to each vertex of the limit cycle is $c\csc k\pi/m$. One can show that the limit cycle will be contained within the original polygon if and only if $\cot k\pi/m \leqslant \tan\pi/\mathrm{m}$, and its convex hull will contain the original polygon if and only if $k \leqslant m/4$. Fig. 11 shows several of these limit cycles.

## 5. General solution to the limit cycle problem

This section contains one of the main results of this paper: The solution of the limit cycle problem through the solution of a linear system of equations. Although Lemma 1.1 allows solution of the problem by a limiting process, here we present an exact solution which can be obtained in finitely many steps (e.g., by Gaussian elimination).

**Theorem 5.1.** *Let $n \geqslant 2$ and $m \geqslant 2$ integers, and suppose $\mathbf{a}_i \in \mathbb{R}^n, \mathbf{a}_i \neq \mathbf{0}, d_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$. Then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ forms a limit cycle for the hyperplanes $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i \cdot \mathbf{x} = d_i\}$ (in this order) if an only if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ satisfies the following $mn \times mn$ system of linear equations:*

$$\mathbf{x}_i - \mathbf{x}_{i-1} + \frac{\mathbf{a}_i \cdot \mathbf{x}_{i-1}}{\|\mathbf{a}_i\|^2}\mathbf{a}_i = \frac{d_i}{\|\mathbf{a}_i\|^2}\mathbf{a}_i, \quad 2 \leqslant i \leqslant m,$$

$$\mathbf{x}_1 - \mathbf{x}_m + \frac{\mathbf{a}_1 \cdot \mathbf{x}_m}{\|\mathbf{a}_m\|^2}\mathbf{a}_1 = \frac{d_1}{\|\mathbf{a}_1\|^2}\mathbf{a}_1. \tag{30}$$

**Proof.** First we show that $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$, forms a limit cycle if and only if there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that the following $m(n+1) \times m(n+1)$ linear system is satisfied.

$$\mathbf{a}_i \cdot \mathbf{x}_i = d_i, \quad 1 \leqslant i \leqslant m, \tag{31a}$$

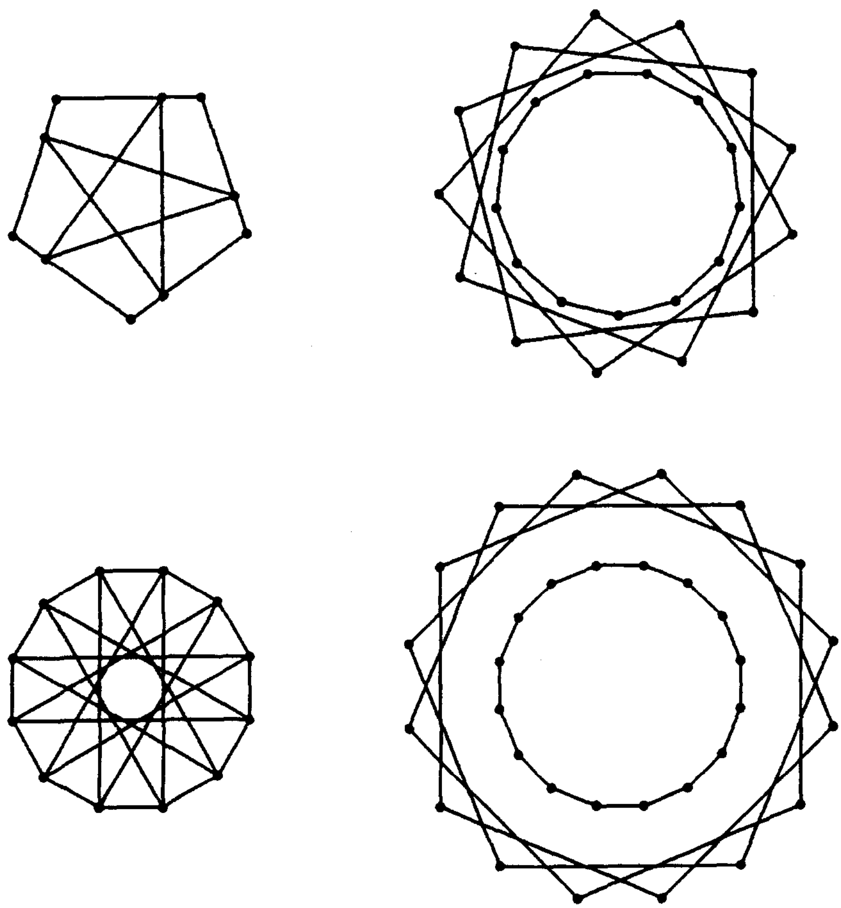


Fig. 11. Pentagon ($k = 2$), 13-gon ($k = 3$), dodecagon ($k = 5$), and 16-gon ($k = 3$) with limit cycles.

$$\mathbf{x}_i - \mathbf{x}_{i-1} = \alpha_i \mathbf{a}_i, \quad 2 \leqslant i \leqslant m, \tag{31b}$$

$$\mathbf{x}_1 - \mathbf{x}_m = \alpha_1 \mathbf{a}_1. \tag{31c}$$

This is true because $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ will be a limit cycle if and only if each $\mathbf{x}_i$ is on the $i$th hyperplane, and the vector from one of the $\mathbf{x}_i$'s to the next (or form $\mathbf{x}_m$ to $\mathbf{x}_1$) is orthogonal to the hyperplane containing the next point. The condition for $\mathbf{x}_i$ being on the $i$th hyperplane is just (31a), while $\mathbf{x}_i - \mathbf{x}_{i-1}$ being orthogonal to the $i$th hyperplane (respectively $\mathbf{x}_1 - \mathbf{x}_m$ being orthogonal to the first hyperplane) is equivalent to $\mathbf{x}_i - \mathbf{x}_{i-1}$ being parallel to $\mathbf{a}_i$ (respectively $\mathbf{x}_1 - \mathbf{x}_m$ being parallel to $\mathbf{a}_1$), and these conditions are equivalent to Eqs. (31b) and (31c).

Now suppose there exist real numbers $\alpha_1, \alpha_2, \ldots, \alpha_m$ such that Eqs. (31a)–(31c) are satisfied. Taking the dot product of each side of Eqs. (31b) and (31c) with the $\mathbf{a}_i$ vector which appears there, using Eq. (31a), and solving for $\alpha_i$ gives

$$\begin{aligned} \alpha_i &= \frac{\mathbf{a}_i \cdot \mathbf{x}_i - \mathbf{a}_i \cdot \mathbf{x}_{i-1}}{\|\mathbf{a}_i\|^2} = \frac{d_i - \mathbf{a}_i \cdot \mathbf{x}_{i-1}}{\|\mathbf{a}_i\|^2}, \quad 2 \leqslant i \leqslant m, \\ \alpha_1 &= \frac{\mathbf{a}_1 \cdot \mathbf{x}_1 - \mathbf{a}_1 \cdot \mathbf{x}_m}{\|\mathbf{a}_1\|^2} = \frac{d_1 - \mathbf{a}_1 \cdot \mathbf{x}_m}{\|\mathbf{a}_1\|^2}, \end{aligned} \tag{32}$$

and substituting back into Eqs. (31b) and (31c) gives Eq. (30). Finally if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ satisfy Eq. (30), then Eq. (31b) and Eq. (31c) are satisfied with $\alpha_i, i = 1, 2, \ldots, m$ as in Eq. (32), and taking the dot product of each side of Eq. (30) with the appropriate $\mathbf{a}_i$ yields Eq. (31a). □

We can also prove a characterization of limit cycles consisting of only $2m$ equations in $mn$ unknowns in which half of the equations are quadratic instead of linear. The proof will be omitted as it is identical to the proof for Eqs. (31a)–(31c), except that the criterion used for two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ being parallel is related to the Cauchy–Schwarz inequality, i.e. $\mathbf{v}_1$ and $\mathbf{v}_2$ are parallel if and only if $|\mathbf{v}_1 \cdot \mathbf{v}_2| = \|\mathbf{v}_1\|\|\mathbf{v}_2\|$.

**Theorem 5.2.** *Let $n \geqslant 2$ and $m \geqslant 2$ be integers, and suppose $\mathbf{a}_i \in \mathbb{R}^n$, $\mathbf{a}_i \neq 0, d_i \in \mathbb{R}$ for $i = 1, 2, \ldots, m$. Then $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m \in \mathbb{R}^n$ forms a limit cycle for the hyperplanes $\{\mathbf{x} \in \mathbb{R}^n : \mathbf{a}_i \cdot \mathbf{x} = d_i\}$ (in this order) if and only if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_m$ satisfies the following $2m \times mn$ system of equations:*

$$\mathbf{a}_i \cdot \mathbf{x}_i = d_i, \quad 1 \leqslant i \leqslant m \tag{33a}$$

$$(\mathbf{a}_i \cdot (\mathbf{x}_i - \mathbf{x}_{i-1}))^2 = \|\mathbf{a}_i\|^2 \|\mathbf{x}_i - \mathbf{x}_{i-1}\|^2, \quad 2 \leqslant i \leqslant m, \tag{33b}$$

$$(\mathbf{a}_1 \cdot (\mathbf{x}_1 - \mathbf{x}_m))^2 = \|\mathbf{a}_1\|^2 \|\mathbf{x}_1 - \mathbf{x}_m\|^2. \tag{33c}$$

**Example 5.1.** In this example we compute the limit cycle of the equations that form the faces of a regular tetrahedron. The equations are given by $F_1$: $-x+y+z=a$, $F_2$: $x-y+z=a$, $F_3$: $x+y-z=a$, and $F_4$: $-x-y-z=a$, where $a>0$. These are the faces of the regular tetrahedron with vertices $(a,a,a),(-a,-a,a),(-a,a,-a)$, and $(a,-a,-a)$. Let $\mathbf{x}_i=[x_i,y_i,z_i]^{\mathrm{T}}$, $i=1,2,3,4$ be the limit cycle, where $\mathbf{x}_i$ is on the (extended) face $F_i$. If we let $\mathbf{x}=[\mathbf{x}_1^{\mathrm{T}},\mathbf{x}_2^{\mathrm{T}},\mathbf{x}_3^{\mathrm{T}},\mathbf{x}_4^{\mathrm{T}},\alpha_1,\alpha_2,\alpha_3,\alpha_4]^{\mathrm{T}}$, then the solution of Eq. (30) is given by

$$\mathbf{x}_1=[-2a/3,a/3,0]^{\mathrm{T}},\qquad \mathbf{x}_2=[0,-a/3,2a/3]^{\mathrm{T}},$$
$$\mathbf{x}_3=[2a/3,a/3,0]^{\mathrm{T}},\qquad \mathbf{x}_4=[0,-a/3,-2a/3]^{\mathrm{T}},$$

and

$$\alpha_1=\alpha_2=\alpha_3=\alpha_4=\frac{2a}{3}.$$

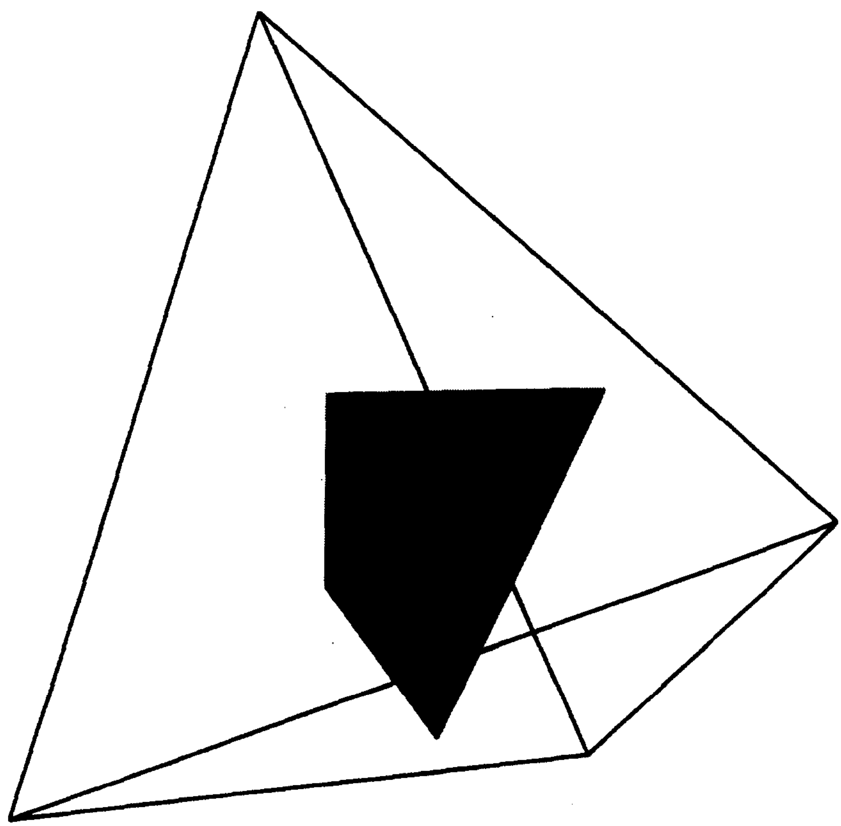


Fig. 12. Tetrahedron with limit cycle (convex hull).

Note that the convex hull of the limit cycle is not a regular tetrahedron. Four of the edges have length $2\sqrt{3}a/3$ and two of the edges have length $4a/3$. Fig. 12 depicts the situation.

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