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Integral representations for the product of certain polynomials of two variables

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Abstract The main object of this paper is to investigate several integral representations for the product of two polynomials of two variables, e.g. Laguerre, Jacobi, Generalized Bessel, Generalized Rice, Krawtchouk, Meixner, Gottlieb and Poisson–Charlier polynomials of two variables.

1. Introduction
In 1938, Watson [1] gave the integral representation for the product \( L_n^a(x)L_n^b(y) \), which was generalized by Carlitz [2] in the form

\[
L_n^a(x)L_n^b(y) = \frac{2^{a+b+2m+n}}{\pi^2} \frac{\Gamma(z + m + 1)\Gamma(b + n + 1)}{\Gamma(z + b + m + n + 1)} \times \\
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(m+b)\phi + (a-b)\psi} \cos^{m+a} \phi \cos^{b+n} \psi \times \\
L_{m+n}^{a+b} \left( \frac{e^{i(\theta - \phi)} + e^{i(\theta - \psi)}}{\cos \phi} \right) \cos \theta \, d\phi d\theta, \tag{1.1}
\]

where \( L_n^a(x) \) denotes the general Laguerre polynomials of degree \( n \).

The Jacobi polynomials of two variables $L_n^{(a,b)}(x,y)$ are defined as:

$$L_n^{(a,b)}(x,y) = \frac{(1 + 2z)(1 + \beta)_n}{(n!)^2} \sum_{r=0}^{n} \frac{(-n)_r (1 + x)_r (1 + \beta)_r}{(r!)^2} x^r y^r.$$  

(1.2)

The Laguerre polynomials of two variables $L_n^{(a,b)}(x,y)$ are defined as:

$$L_n^{(a,b)}(x,y) = \frac{(1 + 2z)(1 + \beta)_n}{(n!)^2} \sum_{r=0}^{n} \frac{(-n)_r (1 + x)_r (1 + \beta)_r}{(r!)^2} x^r y^r.$$  

(1.3)

The Krawtchouk polynomials of two variables $K_n^{(a,b)}(x,y)$ are defined as:

$$K_n^{(a,b)}(x,y) = \frac{(-x)_n (1 - y)_n}{(x)_n (y)_n} \left( \frac{1}{x} \right)^a \left( \frac{1}{y} \right)^b.$$  

(1.6)

The Meixner polynomials of two variables $M_n^{(a,b)}(x,y)$ are defined as:

$$M_n^{(a,b)}(x,y) = \frac{(-x)_n (1 - y)_n}{(x)_n (y)_n} \left( \frac{1}{x} \right)^a \left( \frac{1}{y} \right)^b.$$  

(1.7)

The Gottlieb polynomials of two variables $G_n^{(a,b)}(x,y)$ are defined as:

$$G_n^{(a,b)}(x,y) = \frac{(-x)_n (1 - y)_n}{(x)_n (y)_n} \left( \frac{1}{x} \right)^a \left( \frac{1}{y} \right)^b.$$  

(1.8)

The Poisson–Charlier polynomials of two variables $C_n^{(a,b)}(x,y)$ are defined as:

$$C_n^{(a,b)}(x,y) = \frac{(-x)_n (1 - y)_n}{(x)_n (y)_n} \left( \frac{1}{x} \right)^a \left( \frac{1}{y} \right)^b.$$  

(1.9)

2. Integral representations for the product of two variables polynomials

For the polynomials $I_n^{(a,b)}(x,y)$ defined by (1.2), we begin by considering the following product:

$$I_n^{(a,b)}(x,y) = \frac{(-x)_n (1 - y)_n}{(x)_n (y)_n} \left( \frac{1}{x} \right)^a \left( \frac{1}{y} \right)^b.$$  

Now we notice the results [4]

$$I_n^{(a,b)}(x,y) = \int \frac{e^{x+y}}{\pi} \int e^{(x+y) \cos \theta} \cos \phi \, d\theta d\phi, \quad (\mu + v > -1)$$  

and

The Bessel polynomials of two variables $Y_n(x, a_1, b_1; y, a_2, b_2)$ are defined as:

$$Y_n(x, a_1, b_1; y, a_2, b_2) = \frac{(-x)_n (1 - y)_n (a_1 - 1 + n)(a_2 - 1 + n)}{n!} \left( \frac{x}{b_1} \right) \left( \frac{y}{b_2} \right)^n.$$  

(1.5)
\[ \Gamma(\mu) \Gamma(\nu) = \int_0^1 t^{\mu-1}(1-t)^{\nu-1} dt, \quad (\mu > 0, \nu > 0). \]  

(2.3)

It therefore follows from (2.1) that

\[ I_{m,n}^{(x,y)}(x,y) = \frac{2^{x+y+2+\delta+\epsilon} m! n!}{\pi^3} \frac{\Gamma(x + m + 1)\Gamma(\beta + m + 1)\Gamma(\gamma + n + 1)\Gamma(\delta + n + 1)}{m!n!(m+n)!\Gamma(x + \gamma + 1)\Gamma(\beta + \delta + 1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(m-n)\phi+i(\beta-\gamma)\psi+i(\delta-\beta)\theta} \]

\[ \times \cos^{m+n}\psi \cos^{x+y}\phi \cos^{\delta+\beta}\theta \times \sum_{\nu=0}^m \sum_{\omega=0} n \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (m-n)_{x+k+z+1} \left( x e^{i(\beta-\phi)\psi} \cos \phi \cos \psi \right)^{x} \left( y e^{i(\gamma-\phi)\psi} \cos \theta \cos \psi \right)^{y} d\phi d\theta d\psi. \]

(2.4)

Since

\[ \sum_{m,n=0}^\infty f(m+n) \frac{y^m}{m!} \frac{x^n}{n!} = \sum_{N=0}^\infty f(N) \frac{(x+y)^N}{N!}, \]

(2.5)

so that

\[ I_{m,n}^{(x,y)}(x,y) = \frac{2^{x+y+2+\delta+\epsilon} m! n!}{\pi^3} \frac{\Gamma(x + m + 1)\Gamma(\beta + m + 1)\Gamma(\gamma + n + 1)\Gamma(\delta + n + 1)}{m!n!(m+n)!\Gamma(x + \gamma + 1)\Gamma(\beta + \delta + 1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(m-n)\phi+i(\beta-\gamma)\psi+i(\delta-\beta)\theta} \]

\[ \times \sum_{\nu=0}^m \sum_{\omega=0} n \frac{(m-n)_{x+k+z+1} M(1+x+\gamma)_{x+k+z+1}(1+\beta+\delta)_{x+k+z+1}}{M!N!(1+x+\gamma)_{x+k+z+1}(1+\beta+\delta)_{x+k+z+1}} \left( x e^{i(\beta-\phi)\psi} \cos \phi \cos \psi \right)^{x} \left( y e^{i(\gamma-\phi)\psi} \cos \theta \cos \psi \right)^{y} d\phi d\theta d\psi. \]

(2.6)

which by virtue of (1.2), yields

\[ I_{m,n}^{(x,y)}(x,y) = \frac{2^{x+y+2+\delta+\epsilon} m! n!}{\pi^3} \frac{\Gamma(x + m + 1)\Gamma(\beta + m + 1)\Gamma(\gamma + n + 1)\Gamma(\delta + n + 1)}{m!n!(m+n)!\Gamma(x + \gamma + 1)\Gamma(\beta + \delta + m + n + 1)} \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{i(m-n)\phi+i(\beta-\gamma)\psi+i(\delta-\beta)\theta} \]

\[ \times \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left( x e^{i(\beta-\phi)\psi} \cos \phi \cos \psi \right)^{x} \left( y e^{i(\gamma-\phi)\psi} \cos \theta \cos \psi \right)^{y} d\phi d\theta d\psi. \]

(2.7)
For $\alpha = \beta = \gamma = \delta = 0$, (2.7) reduces to

$$L_m(x,y)L_n(u,v) = \frac{2^{m+n}}{\pi^3} \frac{m!n!}{(m+n)!} \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(m-n)\phi + (m-n)\psi} \cos^{m+n} \phi \times L_m(x,y) \left( \frac{xe^{(\theta-\phi)} + u}{\cos \phi} \right) \cos \phi, \frac{xe^{(\theta-\phi)} + u}{\cos \phi} \cos \theta \right) d\phi d\psi d\theta. \tag{2.8}$$

In a similar manner, we can derive the following integral representations for the product of two polynomials of two variables defined by (1.3)-(1.9):

$$P_m(x,y)P_n(u,v) = \frac{2^{m+n}}{\pi^3} \frac{m!n!}{(m+n)!} \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(m-n)\phi + (m-n)\psi} \cos^{m+n} \phi \times P_m(x,y) \left( \frac{1 - (1-x)e^{(\theta-\phi)} + (1-u)(1-t)e^{(\phi-\psi)}}{\cos \phi} \right) \cos \phi, \frac{1 - (1-x)e^{(\theta-\phi)} + (1-u)(1-t)e^{(\phi-\psi)}}{\cos \phi} \cos \theta \right) d\phi d\psi d\theta dt, \tag{2.9}$$

As a particular case of (2.9) we note that

$$P_m(x,y)P_n(u,v) = \frac{2^{m+n}}{\pi^3} \frac{m!n!}{(m+n)!} \int_0^1 \int_0^1 \int_0^1 \int_0^1 e^{(m-n)\phi + (m-n)\psi} \cos^{m+n} \phi \times P_m(x,y) \left( \frac{1 - (1-x)e^{(\theta-\phi)} + (1-u)(1-t)e^{(\phi-\psi)}}{\cos \phi} \right) \cos \phi, \frac{1 - (1-x)e^{(\theta-\phi)} + (1-u)(1-t)e^{(\phi-\psi)}}{\cos \phi} \cos \theta \right) d\phi d\psi d\theta dt, \tag{2.10}$$

$$H^{(\alpha+\beta,\gamma+\delta)}_{m,n}(x_1,z_1,p_1,q_1,x_2,z_2,p_2,q_2) = \frac{2^{m+n}a^{m+1}b^{n+1}c^{m+1}d^{n+1}}{(m+n)!} \Gamma(z_1 + m + 1) \Gamma(z_2 + n + 1) \Gamma(x_1 + 1) \Gamma(x_2 + 1) \Gamma(p_1 + 1) \Gamma(q_1 + 1) \Gamma(p_2 + 1) \Gamma(q_2 + 1) \cos^{m+n} \phi \cos^{m+n} \psi \left( 1 - (1-z)e^{(\theta-\phi)} \right) \cos \phi, \frac{1 - (1-z)e^{(\theta-\phi)}}{\cos \phi} \cos \theta \right) d\phi d\psi d\theta dz df, \tag{2.11}$$
As a particular case of (2.11) we note that

\[ H_m(\xi_1, \xi_2, p_1, p_2, x, y) = \frac{2^{n_1+n_2} \pi^{n_1}}{m_{n_1}} (m+n)! \left( \frac{\Gamma(\xi_1+n_1) \Gamma(\xi_2+n_2) \Gamma(p_1) \Gamma(p_2) \Gamma(q_1) \Gamma(q_2)}{\Gamma(\xi_1) \Gamma(\xi_2) \Gamma(p_1+q_1-1) \Gamma(p_2+q_2-1)} \right) \times \frac{1}{2} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} e^{(m-n)(x-y)+(x+y)(p_1-q_1)+(p_2-q_2)} \phi \cos \phi \psi \cos \psi \, d\phi \, d\psi \, dx \, dy \]

\[ \times \left( \xi_1 + \xi_2 + p_1 + q_1 - 1, p_2 + q_2 - 1, \frac{4(4 \pi \sqrt{2} \pi e^{(\theta+\varphi+\lambda)/(\psi+\phi+\mu)+y(1-\tau)} e^{(\theta+\varphi+\lambda)/(\psi+\phi+\mu)+y(1-\tau)}}{\cos \theta \cos \phi} \right) \, d\theta \, d\phi \, d\psi \, d\lambda \, dx \, dy \]

(2.12)

\[ Y_m(x, a_1, b_1; y, a_2, b_2) Y_a(u, c_1, d_1; v, c_2, d_2) = \frac{2^{n_1+n_2} \pi^{n_1}}{m_{n_1} n_{n_2}} \left( \frac{\Gamma(a_1+c_1+m+n-2) \Gamma(a_2+c_2+m+n-2)}{\Gamma(a_1+1) \Gamma(a_2+1) \Gamma(c_1+1) \Gamma(c_2+1)} \right) \times \frac{1}{2} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} e^{(m-n)(x-y)+(x+y)(a_1-c_1)+(a_2-c_2)} \phi \cos \phi \psi \cos \psi \, d\phi \, d\psi \, dx \, dy \]

\[ \times \left( x_1 + a_1, y_1 + b_1 e^{(1-\tau)/(\psi+\phi+\mu)+y(1-\tau)} e^{(1-\tau)/(\psi+\phi+\mu)+y(1-\tau)}}, b_2 y, c_2, d_2 \right) \, d\theta \, d\phi \, d\psi \, d\lambda \, dx \, dy \]

(2.13)

\[ K_m(x; \lambda_1, \lambda_2; M_N; \lambda_1, \lambda_2; M_N) = \frac{2^{n_1+n_2+m_{n_1}} \pi^{n_1+m_{n_1}}}{m_{n_1}} \left( \frac{M_1+1}{M_1} \right)^{m_{n_1}} \Gamma(\lambda_1+1) \Gamma(\lambda_2+1) \Gamma(\lambda_1+n+1) \Gamma(\lambda_2+n+1) \times \frac{1}{2} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} e^{(m-n)(x-y)+(x+y)(\lambda_1-c_1)+(\lambda_2-c_2)} \phi \cos \phi \psi \cos \psi \, d\phi \, d\psi \, dx \, dy \]

\[ \times \left( x+y, \lambda_1, \lambda_2; M_1, M_2 \right) \, d\theta \, d\phi \, d\psi \, d\lambda \, dx \, dy \]

(2.14)

\[ M_m(x; \beta_1, \beta_2; \lambda_1, \lambda_2; M_0) = \frac{2^{n_1+n_2+m_{n_1}} \pi^{n_1+m_{n_1}}}{m_{n_1}} \left( \frac{M_1+1}{M_1} \right)^{m_{n_1}} \Gamma(\beta_1+1) \Gamma(\beta_2+1) \Gamma(\lambda_1+1) \Gamma(\lambda_2+1) \Gamma(\beta_1+\lambda_1+n+1) \Gamma(\beta_2+\lambda_2+n+1) \times \frac{1}{2} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} \int_{-\frac{1}{2}}^{1} e^{(m-n)(x-y)+(x+y)(\beta_1-c_1)+(\beta_2-c_2)} \phi \cos \phi \psi \cos \psi \, d\phi \, d\psi \, dx \, dy \]

\[ \times \left( x+y, \beta_1+\lambda_1-1, \beta_2+\lambda_2-1 \right) \, d\theta \, d\phi \, d\psi \, d\lambda \, dx \, dy \]

(2.15)
\[ I_m(x; \lambda; y; \mu) I_n(u; \nu; \delta) \]
\[ = \frac{2^{2x+y+\mu+\nu+m+n}}{\pi^4} \frac{m!n!x!y!u!v!}{(m+n)!(x+u)!(y+v)!} \times \int \frac{2}{2} \int \frac{2}{2} e^{\lambda e^{y+y} + \mu e^{-y-y} + \nu e^{-u-u} + \delta e^{u+u}} \frac{\cos \psi \cos \phi \psi \cos^{n+1} \theta}{\cos \phi} \]
\[ \times I_{m+n}(x+u, y+v, \log \left( \frac{1 - (1 - e^{y+y})e^\psi \cos \phi}{2 \cos \psi \cos \phi} \right), y+v, \log \left( \frac{1 - (1 - e^{-y-y})e^{-\psi} \cos \phi}{2 \cos \psi \cos \phi} \right)) \, d\psi d\phi d\theta \quad (2.16) \]

and

\[ C_m(x; z_1; y; z_2) C_n(u; \beta_1; v; \beta_2) \]
\[ = \frac{2^{2x+y+z_1+z_2+m+n}}{\pi^3} \frac{m!n!x!y!u!v!}{(m+n)!(x+u)!(y+v)!} \times \int \frac{2}{2} \int \frac{2}{2} e^{z_1 e^{y+y} + \beta_1 e^{-y-y} + z_2 e^{u+u} + \beta_2 e^{-u-u}} \frac{\cos \psi \cos \phi \psi \cos^{n+1} \theta}{\cos \phi} \]
\[ \times C_{m+n}(x+u, z_1 e^{y+y} + \beta_1 e^{-y-y} + z_2 e^{u+u} + \beta_2 e^{-u-u}) \, d\psi d\phi d\theta. \quad (2.17) \]

References


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