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Attractors and Inertial Manifolds for the Dynamics of a Closed Thermosyphon

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1. INTRODUCTION

In this paper we are concerned with the dynamics of a fluid inside a closed pipe, whose motion is driven by natural convection, i.e., due to the exchange of heat between the fluid and the ambient environment. The differences of temperature in different parts of the circuit and gravity forces initiate the motion of the fluid. Several models of such devices, usually called thermosyphons, have been derived [9, 5, 6]. Here we analyze a very general one, though very accurate from the physical point of view, that has been derived in [9], to which we refer for physical considerations; see also the references therein.

The equations governing the motion of fluid and the temperature distribution inside the circuit are given, in dimensionless and normalized form, by

$$\varepsilon \frac{dv}{dt} + G(v)v = \oint Tf, \qquad v(0) = v_0$$

$$\frac{\partial T}{\partial t} + v\frac{\partial T}{\partial x} = \frac{G(v)}{\varepsilon}(T_a - T), \qquad T(0,x) = T_0(x),$$

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where $\varepsilon > 0$, $x \in (0, 1)$ is the arc length, and $\oint = \int_0^1 dx$ means integration along the closed path of the circuit. The function $f = dz/dx$ represents the variation of height along the circuit, and so f introduces the geometry of the problem and the distribution of gravity forces. Note that $\oint f = 0$. The unknowns v and T represent the velocity and the temperature of the fluid, and $T_a(x)$ is the (given) ambient temperature distribution. The function G is given by $G(v) = g(Re|v|)|v|$, where Re is a Reynolds-like number that is assumed to be large, and g is a regular strictly positive function on $(0, \infty)$ such that $g(s) \approx A/s$ as $s \approx 0$ and $g(s) \approx 1$ as $s \approx \infty$.

Some simplifying assumptions have been made from the full 3-D problem. For example, the diameter of the pipe's cross section is assumed to be small compared to the dimension of the physical device, so the geometry has been reduced to a one dimensional one; i.e., the circuit has been reduced to a closed curve in the 3-D space. In fact the parameter ε is proportional to the diameter/length ratio and is supposed to be small. On the other hand, the motion of the fluid is assumed to be of turbulent type, i.e., the Reynolds number is assumed to be large, so the velocity of the fluid is assumed to be the same at each point of the circuit. Therefore, the unknown velocity is a simple space-independent scalar quantity $v(t)$. However, in our analysis the sizes of ε and Re are unimportant.

We will show below that (1.1) defines a unique solution in several spaces for any initial data for the velocity and temperature. Note that all space-dependent functions defined on the circuit must be periodic of period one. The choice of the functional space to work in is given by the properties of the data f and T_a.

We also exhibit several special solutions that include, besides the stationary ones, solutions in which the fluid is not moving but the temperature changes with time and is approaching equilibrium. For this, as well as for most of the relevant features of the model, special, although not restrictive, conditions are required for the given functions f and T_a.

More surprising is the fact that the model, although containing a transport equation, presents a subtle asymptotic smoothing effect (again depending on f and T_a) that allows us to prove the existence of a global compact attractor. Some uniform bounds on $\varepsilon \in (0, \varepsilon_0)$ and $Re \in (0, Re_0)$ will be given.

Even more, depending again on f and T_a, we can show the existence of an inertial manifold, not necessarily of finite dimension, i.e., an invariant exponentially attracting manifold for the flow defined by (1.1). It is worth noting that the existence of the inertial manifold does not rely on the existence of big gaps in the spectrum of a linear operator. For a similar result see [1]. We refer the reader to [4, 8, 2, 7] for general discussions about attractors and inertial manifolds.

Finally, we will show a reduced explicit set of ordinary differential equations that captures all the asymptotic behavior of the system. By properly choosing f and T_a we can have any prescribed odd number of equations in that system. This result may have important consequences from the numerical point of view.

2. Existence and Uniqueness Results

We will introduce some function spaces that will be used in the study of the existence of solutions of (1.1). Let $\Omega = (0, 1)$ and consider the spaces

$$\begin{aligned} L^p_{\text{per}}(\Omega) &= \{u \in L^p_{\text{loc}}(\mathbb{R}),\ u(x+1) = u(x), \text{a.e. } x \in \mathbb{R}\} \\ W^{m,p}_{\text{per}}(\Omega) &= W^{m,p}_{\text{loc}}(\mathbb{R}) \cap L^p_{\text{per}}(\Omega) \\ C^k_{\text{per}}(\Omega) &= \{u \in C^k(\mathbb{R}),\ u(x+1) = u(x), \text{for all } x \in \mathbb{R}\} \\ C^{k,\alpha}_{\text{per}}(\Omega) &= \{u \in C^{k,\alpha}(\mathbb{R}),\ u(x+1) = u(x), \text{for all } x \in \mathbb{R}\}, \end{aligned} \tag{2.1}$$

where $1 \leq p < \infty$, m, $k \in \mathbb{N} \cup \{0\}$, $\alpha \in (0, 1]$, and $u \in L^p_{\text{loc}}(\mathbb{R})$ (or $W^{m,p}_{\text{loc}}(\mathbb{R})$) iff for every open set $\omega \subset\subset \mathbb{R}$ one has $u \in L^p(\omega)$ (or $W^{m,p}(\omega)$, respectively). We will refer to these spaces as *admissible spaces*, and for convenience we will eliminate the reference to Ω.

Since (1.1) has two components, we will consider a phase space of the form $Y = \mathbb{R} \times X$, where X is an admissible space, i.e., $X = L^p_{\text{per}}$, $W^{m,p}_{\text{per}}$, C^k_{per}, or $C^{k,\alpha}_{\text{per}}$. Any particular choice of such a space will depend upon the regularity properties of the given functions T_a and f, as stated below. Our main result in this section is the following.

THEOREM 2.1. *Assume either*

$$\begin{aligned} &T_a \in X = L^p_{\text{per}},\ 1 \leq p < \infty, && \textit{and} && f \in W^{1,q}_{\text{per}},\ \textit{where } 1/p + 1/q = 1, \\ &T_a \in X = W^{1,p}_{\text{per}},\ 1 \leq p < \infty, && \textit{and} && f \in L^q_{\text{per}},\ \textit{where } 1/p + 1/q = 1, \\ &T_a \in X = C_{\text{per}}, && \textit{and} && f \in W^{1,1}_{\text{per}}, \\ &T_a \in X = C^{0,1}_{\text{per}}, && \textit{and} && f \in L^1_{\text{per}} \end{aligned}$$

and denote $Y = \mathbb{R} \times X$. *Then* (1.1) *has a unique solution* $(v, T) \in C([0, \infty), Y)$. *Moreover, the mapping* $S(t)(v_0, T_0) = (v(t), T(t))$, $t \geq 0$, *defines a semigroup of class* C^0 *in* Y.

Even more, assume $T_a, T_0 \in X_1 \subset X$, *where* X_1 *is as in* (2.1). *Then* $T \in C([0, \infty), X_1)$.

Proof. The proof will be based in a fixed point argument as follows.

(i) If $v \in C[0, \tau]$ is given, then the second equation in (1.1) yields, through integration along characteristics, to a function $T = T(v)$ given by

$$T(t, x) = T_0\left(x - \int_0^t v(r)\, dr\right) e^{-\int_0^t G_\varepsilon(r)dr} + \int_0^t \left[G_\varepsilon(s) e^{-\int_s^t G_\varepsilon(r)dr}\, T_a\left(x - \int_s^t v(r)\, dr\right)\right] ds, \tag{2.2}$$

where $G_\varepsilon(s) = G(s)/\varepsilon$ and we have, temporarily, dropped the dependence of v in $G_\varepsilon(v(s))$. With this explicit form we get, from the first equation in (1.1),

$$v(t) = v_0 + \int_0^t \left[-H_\varepsilon(v(s)) + \frac{1}{\varepsilon}\left(\oint T(v)(s) f\right)\right] ds, \tag{2.3}$$

where $H_\varepsilon(s) = G_\varepsilon(s)s$. Therefore, the fixed point argument will be worked out on (2.3).

First, observe that if X is any admissible space and $g \in X$ then $\|g(\cdot + h)\|_x = \|g\|_x$ for every $h \in \mathbb{R}$; i.e., for fixed h, the translation is a linear isometric isomorphism of X. Moreover, for fixed g the translation map $h \mapsto g(\cdot + h)$ is continuous in X. Therefore, if $v \in C[0, \tau]$ and $T_0 \in X$, $T_0(x - \int_0^t v) e^{-\int_0^t G_\varepsilon(r)dr}$ is continuous from $[0, \tau]$ to X, and for fixed $t \in [0, \tau]$ has norm $\|T_0\|_x e^{-\int_0^t G_\varepsilon(r)dr}$. On the other hand, observe that if $T_a \in X$, then for fixed $t \in [0, \tau]$ the X-valued function $s \mapsto G_\varepsilon(s) e^{-\int_s^t G_\varepsilon(r)dr} T_a(x - \int_s^t v)$ is in $L^1(0, t, X)$ (it is in fact continuous) since its norm is in $L^1(0, t)$ and $\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon(r)dr} \|T_a(x - \int_s^t v)\|_x\, ds = \|T_a\|_x(1 - e^{-\int_0^t G_\varepsilon(r)dr})$. Consequently, (2.2) defines, for each $t \in [0, \tau]$, an element in X. Moreover, the term $\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon(r)dr} T_a(x - \int_s^t v)\, ds$ is also continuous from $[0, \tau]$ to X. To prove this it suffices to prove the continuity of $g(t) = \int_0^t G_\varepsilon(s) e^{\int_0^s G_\varepsilon(r)dr} T_a(x - \int_s^t v)\, ds$. To this end, take $t_1 < t_2$ and then

$$\|g(t_2) - g(t_1)\|_X \le \|T_a\|_X \int_{t_1}^{t_2} G_\varepsilon(s) e^{\int_0^s G_\varepsilon}\, ds + \int_0^{t_1} G_\varepsilon(s) e^{\int_0^s G_\varepsilon} \left\| T_a\left(\cdot - \int_s^{t_2} v\right) - T_a\left(\cdot - \int_s^{t_1} v\right) \right\|_X ds.$$

But $\|T_a(\cdot - \int_s^{t_2} v) - T_a(\cdot - \int_s^{t_1} v)\|_X = \|T_a(\cdot - \int_{t_1}^{t_2} v) - T_a(\cdot)\|_X$ and then, since v is bounded on $[0, \tau]$, and again using the fact that the translation map is continuous, we see that for every $\eta > 0$, if $t_2 - t_1$ is sufficiently small then, $\int_{t_1}^{t_2} v$ can be made arbitarily small and therefore $\int_0^{t_1} G_\varepsilon(s) e^{\int_0^s G_\varepsilon(r)dr}\|T_a(\cdot - \int_s^{t_2} v) - T_a(\cdot - \int_s^{t_1} v)\|_X \, ds \le \eta$. Thus, g is continuous. Consequently, (2.2) defines a function $T(v) \in C([0, \tau], X)$ and it just remains to prove that (2.3) has a unique globally defined solution.

(ii) Now, denote by $\mathcal{F}(v)$ the right hand side of (2.3), i.e.,

$$\mathcal{F}(v) = v_0 + \int_0^t \left[-H_\varepsilon(v(s)) + \frac{1}{\varepsilon}\left(\oint T(v)(s) f \right) \right] ds$$

for $v \in C[0, \tau]$, and look for a fixed point of $\mathcal{F}$. To accomplish this, we fix a positive constant M and denote $W = \{v \in C[0, \tau], v(0) = v_0, |v(t) - v_0| \le M\}$, endowed with the sup norm, denoted $|||\cdot|||$. We will prove below that under certain assumptions on M, τ, T_a, and f, $\mathcal{F}$ is a contraction on W. First we show that $\mathcal{F}$ maps W into itself. As proved above, if $v \in W$ and $T_0, T_a \in X$ then $T(v) \in C[0, \tau], X)$, and since by hypothesis $f \in X'$, then $\oint T(v) f \in C[0, \tau]$. Since H_ε is also continuous, we get $\mathcal{F}(v) \in C[0, \tau]$. Moreover, $\mathcal{F}(v)(0) = v_0$ and

$$|\mathcal{F}(v)(t) - v_0| \le \int_0^t \left[|H_\varepsilon(v(s))| + \frac{1}{\varepsilon}\|T(v)(s)\|_X \|f\|_{X'} \right] ds.$$

But from (2.2), we get

$$\|T(v)(t)\|_X \le \|T_0\|_X e^{-\int_0^t G_\varepsilon(s)ds} + \|T_a\|_X (1 - e^{-\int_0^t G_\varepsilon(s)ds}); \tag{2.4}$$

hence, we have $\|T(v)(t)\|_X \le \max\{\|T_0\|_X, \|T_a\|_X\}$ for every $t \in [0, \tau]$. On the other hand, since $v \in W$ then $|v(t)| \le |v_0| + M$ and denoting $K_0 = \sup_{|s| \le |v_0| + M} H_\varepsilon'(s)$, we have $|\mathcal{F}(v)(t) - v_0| \le \tau(K_0 + (1/\varepsilon) \max\{\|T_0\|_X, \|T_a\|_X\}\|f\|_{X'})$; therefore, with fixed M, we can choose τ small, such that the right hand side above is less than M. Consequently, $\mathcal{F}(W) \subset W$.

Now we prove that $\mathcal{F}$ is Lipschitz on W. First, we start by proving the lipchitzness of $\oint T(v) f$ in two different cases.

(a) Assume either $T_a, T_0 \in W^{1,p}_{\text{per}} \subset X = L^p_{\text{per}}$ with $1 \le p < \infty$ or $T_a, T_0 \in C^{0,1}_{\text{per}} \subset X = C_{\text{per}}$. Then $W \ni v \mapsto T(v) \in C([0, \tau], X)$ is Lipschitz with constant $L = L(\tau, M, T_a, T_0)$, such that $L \overset{\tau \to 0}{\to} 0$.

If $v_i \in W$, $i = 1, 2$, and denoting $T_i = T(v_i)$ and again $G_{\varepsilon,i}(t) = G(v_i(t))/\varepsilon$, we have for every $t \in [0, \tau]$

$$\|T_1 - T_2\|_X \le e^{-\int_0^t G_{\varepsilon,1}} \left\| T_0\left(\cdot - \int_0^t v_1\right) - T_0\left(\cdot - \int_0^t v_2\right) \right\|_X$$
$$+ \|T_0\|_X |e^{-\int_0^t G_{\varepsilon,1}} - e^{-\int_0^t G_{\varepsilon,2}}|$$
$$+ \int_0^t \left[G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} \left\| T_a\left(\cdot - \int_s^t v_1\right) - T_a\left(\cdot - \int_s^t v_2\right) \right\|_X \right.$$
$$\left. + \|T_a\|_X |G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} - G_{\varepsilon,2}(s) e^{-\int_s^t G_{\varepsilon,2}}| \right] ds.$$

Since T_a, $T_0 \in W^{1,p}_{\text{per}} \subset X = L^p_{\text{per}}$ or T_a, $T_0 \in C^{0,1}_{\text{per}} \subset X = C_{\text{per}}$, then there exist constants c_a and c_0, respectively, such that $\|T(\cdot - h) - T(\cdot)\|_X \le c|h|$ for $h \in \mathbb{R}$, and then

$$\|T_1 - T_2\|_X \le c_0 e^{-\int_0^t G_{\varepsilon,1}} \left| \int_0^t v_1 - \int_0^t v_2 \right| + \|T_0\|_X |e^{-\int_0^t G_{\varepsilon,1}} - e^{-\int_0^t G_{\varepsilon,2}}|$$
$$+ \int_0^t \left[c_a G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} \left| \int_s^t v_1 - \int_s^t v_2 \right| \right.$$
$$\left. + \|T_a\|_X |G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} - G_{\varepsilon,2}(s) e^{-\int_s^t G_{\varepsilon,2}}| \right] ds.$$

Introducing the quantities $K_1 = \sup_{|s| \le |v_0| + M} G_\varepsilon(s)$, $K_2 = \sup_{|s| \le |v_0| + M} G'_\varepsilon(s)$, we have $c_0 e^{-\int_0^t G_{\varepsilon,1}} |\int_0^t v_1 - \int_0^t v_2| \le c_0 \tau \|\|v_1 - v_2\|\|$, $\|T_0\|_X |e^{-\int_0^t G_{\varepsilon,1}} - e^{-\int_0^t G_{\varepsilon,2}}| \le K_2 \tau \|T_0\|_X \|\|v_1 - v_2\|\|$, and also $\int_0^t c_a G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} |\int_s^t v_1 - \int_s^t v_2| \, ds \le K_1 c_a \tau^2 \|\|v_1 - v_2\|\|$ and

$$\|T_a\|_X \int_0^t |G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} - G_{\varepsilon,2}(s) e^{-\int_s^t G_{\varepsilon,2}}| \, ds \le K_2 \tau \|T_a\|_X (K_1 \tau + 1) \|\|v_1 - v_2\|\|.$$

Putting all these together, we get $\sup_{t \in [0,\tau]} \|T_1 - T_2\|_X \le L \|\|v_1 - v_2\|\|$, where

$$L = \tau(c_0 + K_2 \|T_0\|_X + K_1 c_a \tau + \|T_a\|_X (K_1 K_2 \tau + K_2)).$$

Thus, the statement is proved.

In particular, if $f \in L^q_{\text{per}}$, where $1/p + 1/q = 1$, or if $f \in L^1_{\text{per}}$, respectively, then $W \ni v \mapsto \oint T(v) f \in C[0, \tau]$ is Lipschitz with constant $L\|f\|_{X'} \overset{\tau \to 0}{\to} 0$.

(b) Assume either T_a, $T_0 \in X = L^p_{\text{per}}$ with $1 \le p < \infty$ and $f \in W^{1,q}_{\text{per}}$, where $1/p + 1/q = 1$, or T_a, $T_0 \in X = C_{\text{per}}$ and $f \in W^{1,1}_{\text{per}}$. Then

$W \ni v \mapsto \oint T(v)f \in C[0,\tau]$ is Lipschitz with constant $L = L(\tau, M, T_a, T_0, f)$, such that $L \overset{\tau\to 0}{\rightrightarrows} 0$.

As above, if $v_i \in W$, $i = 1, 2$, $T_i = T(v_i)$, and $G_{\varepsilon,i}(t) = G(v_i(t))/\varepsilon$, then, for every $t \in [0,\tau]$, using $\oint g(x-h)f(x) = \oint g(x)f(x+h)$, we get

$$\begin{aligned}
\left|\oint (T_1 - T_2)f\right| \le{}& e^{-\int_0^t G_{\varepsilon,1}} \left|\oint T_0(x)\left(f\left(x + \int_0^t v_1\right) - f\left(x + \int_0^t v_2\right)\right) dx\right| \\
&+ \left|e^{-\int_0^t G_{\varepsilon,1}} - e^{-\int_0^t G_{\varepsilon,2}}\right| \left|\oint T_0\left(x - \int_0^t v_2\right) f(x)\,dx\right| \\
&+ \int_0^t \left[G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} \left|\oint T_a(x)\left(f\left(x + \int_s^t v_1\right)\right.\right.\right. \\
&\left.\left. - f\left(x + \int_s^t v_2\right)\right) dx\right| \\
&+ \left|\oint T_a\left(x - \int_s^t v_2\right) f(x)\,dx\right| \left|G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}}\right. \\
&\left.\left. - G_{\varepsilon,2}(s) e^{-\int_s^t G_{\varepsilon,2}}\right| \right] ds.
\end{aligned}$$

From here we get the bound

$$\begin{aligned}
\left|\oint (T_1 - T_2)f\right| \le{}& e^{-\int_0^t G_{\varepsilon,1}} \|T_0\|_X \left\| f\left(\cdot + \int_0^t v_1\right) - f\left(\cdot + \int_0^t v_2\right)\right\|_{X'} \\
&+ \left|e^{-\int_0^t G_{\varepsilon,1}} - e^{-\int_0^t G_{\varepsilon,2}}\right| \|T_0\|_X \|f\|_{X'} \\
&+ \int_0^t \left[G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} \|T_a\|_X \left\| f\left(\cdot + \int_s^t v_1\right)\right.\right. \\
&\left. - f\left(\cdot + \int_s^t v_2\right)\right\|_{X'} \\
&\left. + \|T_a\|_X \|f\|_{X'} \left|G_{\varepsilon,1}(s) e^{-\int_s^t G_{\varepsilon,1}} - G_{\varepsilon,2}(s) e^{-\int_s^t G_{\varepsilon,2}}\right| \right] ds.
\end{aligned}$$

Now, since there exists a constant c_f such that $\|f(\cdot + h) - f(\cdot)\|_{X'} \le c_f|h|$ for $h \in \mathbb{R}$, the rest follows as above and we get $\sup_{t\in[0,\tau]} \left|\oint (T_1 - T_2)f\right| \le L|||v_1 - v_2|||$, with $L \overset{\tau\to 0}{\rightrightarrows} 0$.

(iii) Now, with $K_3 = \sup_{|s|\le|v_0|+M} H'_\varepsilon(s)$, if T_a and f are as in case (a) or case (b) above and from (2.3), we get for $v_i \in W$, $i = 1, 2$,

$$|\mathcal{F}(v_1) - \mathcal{F}(v_2)| \le \int_0^t \left[|H_\varepsilon(v_1) - H_\varepsilon(v_2)| + \left| \oint (T(v_1) - T(v_2)) f \right| \right] ds.$$

Therefore, $\mathcal{F}$ is Lipschitz on W, with Lipschitz constant $\tau(K_3 + L)$. Hence, if τ is sufficiently small, $\mathcal{F}$ is a contraction on W and thus it has a unique fixed point.

(iv) Now we prove that the local solutions found above are actually defined on $[0, \infty)$. To see this we follow a classical prolongation argument. Assume the solution has been prolonged to a maximal interval of time $[0, \tau_{\max})$. Then, we have the alternative $\tau_{\max} = \infty$ or $(v(t), T(t))$ becomes unbounded as $T \to \tau_{\max}$, and from (2.4) this is equivalent to having $v(t)$ become unbounded. In fact, if $\tau_{\max} < \infty$ but $(v(t), T(t))$ remains bounded as $t \to \tau_{\max}$, we prove below that $(v(t), T(t))$ is Cauchy as $t \to \tau_{\max}$, reaching then a contradiction with the maximality of $\tau_{\max}$.

We first prove that $v(t)$ is Cauchy as $t \to \tau_{\max}$ and the limit $\lim_{t\to\tau_{\max}} v(t) = v_1$ exists.With $t > s$, from the equation for v, integrated in $[s, t]$, we get

$$v(t) - v(s) = \int_s^t \left[-H_\varepsilon(v(r)) + \frac{1}{\varepsilon}\left(\oint T(r) f \right) \right] dr$$

and then, since $(v(t), T(t))$ remains bounded $|v(t) - v(s)| \le c|t - s|$ for some positive c. Now recalling (2.2), and arguing as when proving the continuity of $T(v)$, we can prove that the limit $\lim_{t\to\tau_{\max}} T(t, x) = T_1(x)$ exists in X.

It remains to prove that $v(t)$ remains bounded for a finite time. From the first equation in (1.1), we get

$$v(t) = v_0 e^{-\int_0^t G_\varepsilon(s)ds} + \int_0^t \frac{1}{\varepsilon}\left(\oint Tf \right) e^{-\int_s^t G_\varepsilon(r)dr}\, ds \tag{2.5}$$

and the right hand side is finite on finite intervals, since G_ε is strictly positive. The proof is now complete. ▮

Remark 2.1. (i) Observe that the spaces in (2.1) are taken for convenience, but are not the only possible choices. In fact, we can reproduce the proof above assuming that X is a Banach space of 1-periodic functions on the line, such that, for every $h \in \mathbb{R}$ and $g \in X$, $\|g(\cdot + h)\|_X = \|g\|_X$ and, for fixed g, the map $h \mapsto g(\cdot + h)$ is continuous in X. Also, we can assume that $f \in X'$, the dual space of X. In that case, the term $\oint Tf$ should be understood as a duality product.

(ii) Note that depending on the regularity of T_a and T_0, the temperature, given by (2.2), verifies the partial differential equation a.e. (x, t) or is a generalized or integral solution.

3. Special Solutions

Below we give several types of special solutions that may occur for (1.1).

Proposition 3.1. *The set of stationary solutions of (1.1), with $v \neq 0$, is in a one-to-one correspondence with the solutions of the scalar equation*

$$\frac{u^2}{Re^2} = J_\varepsilon(u) = \sum_{k\in\mathbb{Z}} \frac{b_k \bar{c}_k}{2\pi i k\varepsilon + R(u)}, \tag{3.1}$$

where $R(u) = g(|u|)\mathrm{sig}(u)$, and $\{b_k\}$, $\{c_k\}$ denote, respectively the complex Fourier coefficients of T_a and f. Moreover, if $\oint T_a f = 0$ then $(0, T_a)$ is a stationary solution.

Proof. From (1.1) the equation of the stationary solution is

$$G(v)v = \oint Tf \tag{3.2}$$

$$v\frac{\partial T}{\partial x} = \frac{G(v)}{\varepsilon}(T_a - T).$$

Thus, $v = 0$ is only possible if $T = T_a$ and verifies $\oint T_a f = 0$, and in such a case $(0, T_a)$ is a stationary solution. If $v \neq 0$ and assuming the expansion $T(x) = \sum_{k\in\mathbb{Z}} a_k e^{2\pi kix}$, $T_a(x) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi kix}$, and $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi kix}$, then (3.2) reduces to $G(v)v = \sum_{k\in\mathbb{Z}} a_k c_{-k}$ and $(2\pi ikv + G(v)/\varepsilon)a_k = (G(v)/\varepsilon)b_k$, $k \in \mathbb{Z}$. Since f is real valued then $c_{-k} = \bar{c}_k$ and plugging the second equation into the first one yields $v = \sum_{k\in\mathbb{Z}}(b_k\bar{c}_k/(2\pi i\varepsilon kv + G(v))$. Multiplying both sides by v and recalling that $G(v) = g(Re|v|)|v|$, and so writing $G(v)/v = g(Re|v|)\mathrm{sig}(v)$, and denoting $u = Rev$, we get (3.1). ∎

See [9] for a detailed analysis of the stationary solutions and their linear stability.

Proposition 3.2. *Assume X is an admissible space such that Theorem 2.1 holds true. Moreover, assume $\oint T_a f = 0$. Then, for any $T_0 \in X$ such that $\oint T_0 f = 0$, the solution of (1.1) with initial data $v_0 = 0$, T_0, is such that for every $t \geq 0$*

$$v(t) = 0$$

and T is given by

$$T(t, x) = T_a(x) + (T_0(x) - T_a(x))e^{-(A/\varepsilon Re)t}.$$

In particular, these solutions converge in Y, at the uniform exponential rate $A/\varepsilon Re$, to the stationary solution $(0, T_a)$ as t goes to ∞.

Proof. Just note that from (1.1), $v(t) = 0$ implies $\oint T(t)f = 0$. But at the same time, since $G(0) = A/Re$, one must have $\partial T/\partial t = (A/\varepsilon Re)(T_a - T)$. Hence, if one multiplies by f and integrates, $a(t) = \oint T(t)f$ must verify $\dot{a}(t) = -(A/\varepsilon Re)a(t)$. Therefore, if $a(0) = \oint T_0 f = 0$ then $a(t) = \oint T(t)f = 0$ for every t, and $(0, T(t, x))$ is actually the solution of (1.1). ∎

PROPOSITION 3.3. *Assume T_a is constant. Then, there exist solutions of the form $(v(t), T(t))$, where $T(t)$ is constant in x. Furthermore, any solution (v, T) of* (1.1) *converges exponentially to* $(0, T_a)$.

Proof. Just note that since $\oint f = 0$, then the equations for constants in x solutions reduce to $\varepsilon(dv/dt) + G(v)v = 0$ and $dT/dt = (G(v)/\varepsilon)(T_a - T)$. Hence, $v = v_0 e^{-\int_0^t G_\varepsilon}$ and $T = T_a + (T_0 - T_a)e^{-\int_0^t G_\varepsilon}$ and they converge exponentially to $(0, T_a)$ since G_ε is strictly away from zero.

Now, if $(v(t), T(t, x))$ is an arbitrary solution, note that by integrating in (1.1), $\overline{T}(t) = \oint T(t)$ is a solution of the second ODE above, since $\oint T_a = T_a$. Therefore, it goes exponentially to T_a. Now, if we denote $\theta = T - \overline{T}$, and again use $\oint T_a = T_a$, θ verifies $\partial\theta/\partial t + v(\partial\theta/\partial x) = -(G(v)/\varepsilon)\theta$ and therefore, integrating along characteristics as in (2.2), we have $\theta(t, x) = \theta_0(x - \int_0^t v)e^{-\int_0^t G_\varepsilon}$ so it goes exponentially to zero in X. Consequently, T and $\oint Tf = \oint \theta f = e^{-\int_0^t G_\varepsilon} \oint \theta_0(x - \int_0^t v)f(x)\,dx$ go exponentially to T_a and 0, respectively. Therefore, v verifies $\varepsilon(dv/dt) + G(v)v = \oint \theta f$ and then, from the expressions above and (2.5), v goes to zero exponentially and the result is proved. ∎

Remark 3.1. In the next section we will give a proof of this result in a much more general case for T_a than the one above. We have included this proof because of its simplicity.

4. ATTRACTORS AND INERTIAL MANIFOLDS

In this section we are concerned with the asymptotic behavior of the solutions of Eq. (1.1). Since we have two parameters (ε, Re), we denote the semigroup generated by (1.1) by $S_\lambda(t)$, where $\lambda = (\varepsilon, Re)$. That explicit dependence will be dropped unless some confusion may arise. Thus, we can prove

THEOREM 4.1. *Assume X is an admissible space and T_a, f are such that Theorem 2.1 holds true, and denote $Y = \mathbb{R} \times X$.*

(i) *For every $\lambda = (\varepsilon, Re)$ denote $K(\lambda) = \|T_a\|_X \|f\|_{X'} Re/h_*$, where $h_* = \inf_{s>0}\{g(s)s\} > 0$. Then the set*

$$[-K(\lambda), K(\lambda)] \times \overline{B}_X(0, \|T_a\|_X)$$

is invariant for S_λ in Y.

(ii) *For every $\lambda_0 = (\varepsilon_0, Re_0)$ and $\eta > 0$ the set*

$$(-K(\lambda_0) - \eta, K(\lambda_0) + \eta) \times B_X(0, \|T_a\|_X + \eta)$$

is absorbing for S_λ in Y, for every $0 < \lambda \le \lambda_0$, that is, for $0 < \varepsilon \le \varepsilon_0$ and $0 < Re \le Re_0$.

(iii) *Moreover, for every λ, there exists a compact connected attractor, $\mathcal{A}_\lambda$, in Y. Even more, for every λ_0, $\bigcup_{0<\lambda\le\lambda_0} \mathcal{A}_\lambda$ is a bounded set of Y.*

Proof. Again recalling (2.4), and denoting, for fixed $\lambda = (\varepsilon, Re)$, $M = M(\lambda) = h_*/\varepsilon Re$, we have $e^{-\int_0^t G_\varepsilon(s)ds} \le e^{-Mt}$, and then we get

$$\|T(t)\|_X \le \|T_a\|_X + (\|T_0\|_X - \|T_a\|_X)_+ e^{-Mt}, \tag{4.1}$$

where $(x)_+$ denotes the positive part of x. In particular, if $\|T_0\|_X \le \|T_a\|_X$ then $\|T(t)\|_X \le \|T_a\|_X$ for every $t > 0$, while if $\|T_0\|_X \ge \|T_a\|_X$, then $\|T(t)\|_X \le \|T_0\|_X$ for $t > 0$.

For convenience, we will write hereafter $\|\cdot\|$ instead of $\|\cdot\|_X$ unless we work in a different space. Analogously, the norm of f will be written $\|f\|$ instead of $\|f\|_{X'}$.

Now, from (4.1) and (2.5) we get

$$|v(t)| \le K(\lambda) + e^{-Mt}(|v_0| - K(\lambda) + (\|T_0\| - \|T_a\|)_+ t)_+. \tag{4.2}$$

From (4.1), (4.2), the first statement is straightforward. If we now fix $\lambda_0 = (\varepsilon_0, Re_0)$, and a positive constant R, then for any initial data such that $|v_0| + \|T_0\| \le R$ and any $0 < \lambda \le \lambda_0$, we get, in a way analogous to (4.1), (4.2),

$$\|T(t)\| \le \|T_a\| + (R - \|T_a\|)_+ e^{-M(\lambda_0)t} \tag{4.3}$$

and

$$|v(t)| \le K(\lambda_0) + e^{-M(\lambda_0)t}(R + (R - \|T_a\|)_+ t). \tag{4.4}$$

Now it is clear that, for every $\eta > 0$, there exists a $t_0 = t_0(\eta, R, \lambda_0) > 0$ such that $S_\lambda(v_0, T_0) \in (-K(\lambda_0) - \eta, K(\lambda_0) + \eta) \times B_X(0, \|T_a\|_X + \eta)$ for $t \geq t_0$ and $0 < \lambda \leq \lambda_0$. Therefore, this set absorbs bounded sets of Y and (ii) is proved.

Finally, from (2.2), (2.5), $S_\lambda(t)(v_0, T_0) = (v(t), T(t,x))$ can be decomposed as $S_\lambda(t)(v_0, T_0) = S_1(t)(v_0, T_0) + S_2(t)(v_0, T_0)$ with

$$S_1(t)(v_0, T_0) = (v_0 e^{-\int_0^t G_\varepsilon}, T_0(x - \int_0^t v) e^{-\int_0^t G_\varepsilon})$$

$$S_2(t)(v_0, T_0) = \left(\int_0^t \frac{1}{\varepsilon} \left(\oint Tf \right) e^{-\int_s^t G_\varepsilon} \, ds, \int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} T_a \left(x - \int_s^t v \right) ds \right).$$

From the estimates above, we have $\|S_1(t)(v_0, T_0)\| \leq (|v_0| + \|T_0\|) e^{-M(\lambda)t}$. Now we prove that $S_2(t)$ is compact. Clearly, if $T_a \in X_1$, where X_1 is an admissible space such that the inclusion $X_1 \hookrightarrow X$ is compact, then $S_2(t)$ is completely continuous in Y, since $\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} \|T_a\|_{X_1} \, ds \leq \|T_a\|_{X_1}$ so, $S_2(t)$ maps bounded sets of Y into bounded sets of $Y_1 = \mathbb{R} \times X_1$. Note that, in that case, the range of $S_2(t)$ is in a fixed compact set of X independent of t.

A closer look at $S_2(t)$ will reveal that, even without this compactness assumption on T_a, it is a completely continuous operator. For this, we introduce the following notation. Let $s(T_a) = \{T_a(\cdot + h), h \in \mathbb{R}\}$ be the set of all translates of T_a. Then, $s(T_a)$ is a compact set in X and it is in fact homeomorphic to the sphere $\mathbb{S}^1$. Also, for an interval $I \subset \mathbb{R}$, we denote $I \cdot s(T_a) = \{rT, r \in I, T \in s(T_a)\}$, which is clearly a compact set in X if I is compact.

Now we take a bounded set B in Y. From (4.3) and (4.4) we know that $S(t)B$ remains in a bounded set for all $t \geq 0$. Therefore, there exists $G_* = G_*(B)$ such that $0 \leq G_\varepsilon(v(s)) \leq G_*$ for any initial data in B and any $s \geq 0$. Hence, for fixed t, the integrand $G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} T_a(x - \int_s^t v)$ is continuous as a function of s and takes values in the compact set $[0, G_*] \cdot s(T_a)$. Therefore, using the mean value theorem, we get

$$\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} T_a(x - \int_s^t v) \, ds \in t \cdot \overline{\text{co}}([0, G_*] \cdot s(T_a)),$$

where $\overline{\text{co}}([0, G_*] \cdot s(T_a))$ denotes the closed convex hull of the compact set $[0, G_*] \cdot s(T_a)$, which thanks to Mazur's theorem, is also compact in X. That proves that $S_2(t)$ is completely continuous. Moreover, we can still prove that $S_2(t)B$ remains in a fixed compact set independent of t since $\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} \|T_a\|_X \, ds \leq \|T_a\|_X$ and then

$$\int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} T_a\left(x - \int_s^t v\right) ds \in t \cdot \overline{\mathrm{co}}([0, G_*] \cdot s(T_a)) \cap \overline{B}_X(0, \|T_a\|_X)$$

and the right hand side above is a compact set of X. But since $[0, G_*] \cdot s(T_a)$ is a star shaped domain, with respect to the origin, so is $\overline{\mathrm{co}}(0, G_*] \cdot s(T_a))$ and therefore, for sufficiently large t, the set $t \cdot \overline{\mathrm{co}}(0, G_*] \cdot s(T_a)) \cap \overline{B}_X(0, \|T_a\|_X)$ does not depend on t. Since this family of sets is increasing with t, we conclude that the integral above takes values in a fixed compact set of X independent of t.

Using this splitting, the existence of A_λ follows from [4]. Just note that if $\mathcal{B} = \mathcal{B}_\lambda$ denotes either the invariant set in (i) or the absorbing set in (ii) then

$$\mathcal{A}_\lambda = \omega(\mathcal{B}_\lambda) = \bigcap_{t>0} \overline{\bigcup_{s>t} S(s)\mathcal{B}_\lambda}^X; \tag{4.5}$$

see [4, 8]. The property that $\bigcup_{0<\lambda\leq\lambda 0} \mathcal{A}_\lambda$ is a bounded set of Y follows from the uniform absorbing set $(-K(\lambda_0) - \eta, K(\lambda_0) + \eta) \times B_X(0, \|T_a\|_X + \eta)$. ∎

Remark 4.1. The idea of splitting the semigroup as $S = S_1 + S_2$ is by now a well known trick to handle damped hyperbolic equations; see [4, 10] and references therein, for example.

In what follows we will make use of Fourier expansions for the functions involved. So, we restrict ourselves to work on admissible spaces $X \hookrightarrow L^2_{\mathrm{per}}$. In particular we will consider the cases $X = H^m_{\mathrm{per}}$, where $m \geq 0$, and, for an arbitrary function, we write its Fourier expansion in complex form as $g = \sum_{k\in\mathbb{Z}} a_k e^{2\pi kix}$. We will find below the case in which functions have Fourier expansions $g = \sum_{k\in K} a_k e^{2\pi kix}$, with $K \subset \mathbb{Z}$. Note that all the functions involved are real, and then $a_{-k} = \overline{a}_k$, and the sets $K \subset \mathbb{Z}$ are symmetric, i.e., $-K = K$. With these notations we have

Proposition 4.1. *Let $A \subset \mathbb{Z}$ be any nonempty set. Then, if $\pi = \pi_A$ denotes the projection onto this set of Fourier modes, i.e., for $g = \sum_{k\in\mathbb{Z}} a_k e^{2\pi kix}$, then $\pi(g) = \sum_{k\in A} a_k e^{2\pi kix}$, and if (v, T) is a solution of* (1.1) *in $Y = \mathbb{R} \times X$ with $X \hookrightarrow L^2_{\mathrm{per}}$, then $\pi(T)$ verifies*

$$\pi(T) = \pi(T_0)\left(x - \int_0^t v\right) e^{-\int_0^t G_\varepsilon} + \int_0^t \left[G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} \pi(T_a)\left(x - \int_0^t v\right) \right] ds, \tag{4.6}$$

where, again, $G_\varepsilon(s) = G(v(s))/\varepsilon$. *That means* $\pi(T)$ *is an integral solution of*

$$\frac{\partial\pi(T)}{\partial t} + v\frac{\partial\pi(T)}{\partial x} = \frac{G(v)}{\varepsilon}(\pi(T_a) - \pi(T)), \qquad \pi(T)(0) = \pi(T_0).$$

In particular, using the expansions $T(t, x) = \sum_{k\in\mathbb{Z}} a_k(t)e^{2\pi kix}$, $T_a(x) = \sum_{k\in\mathbb{Z}} b_k e^{2\pi kix}$, *and* $f(x) = \sum_{k\in\mathbb{Z}} c_k e^{2\pi kix}$, *we have for* $k \in \mathbb{Z}$

$$a_k(t) = a_k(0)e^{-\int_0^t 2\pi ikv+G_\varepsilon} + b_k \int_0^t G_\varepsilon(s)e^{-\int_s^t 2\pi ikv+G_\varepsilon}\,ds; \tag{4.7}$$

i.e., $a_k(t)$ *is a solution of* $\dot a_k(t) + (2\pi ikv(t) + G(v(t))/\varepsilon)a_k(t) = (G(v(t))/\varepsilon)b_k$.

Proof. The proof is based in the following property of the projection π: for every g and $h \in \mathbb{R}$, we have $\pi(g(\cdot + h)) = \pi(g)(\cdot + h)$. The rest is obvious, by projecting (2.2). ∎

COROLLARY 4.1. *Assume* T_a *and* f *are such that Theorems* 2.1 *and* 4.1 *hold true in* $Y = \mathbb{R} \times X$ *with* $X \hookrightarrow L^2_{\text{per}}$. *Then, if* $T_a \in H^m_{\text{per}}$

$$\mathcal{A} \subset \mathbb{R} \times H^m_{\text{per}}$$

and is a compact set in that space.

Proof. From (4.7), we get

$$|a_k(t)| \le |a_k(0)|e^{-\int_0^t G_\varepsilon(s)ds} + |b_k|(1 - e^{-\int_0^t G_\varepsilon(s)ds}). \tag{4.8}$$

Let $(v_0, T_0) \in \mathcal{A}$; then, from (4.5), there exist sequences (v_n, T_n), bounded in Y (in fact lying in a bounded invariant set), and t_n going to ∞, such that $S(t_n)(v_n, T_n) \to (v_0, T_0)$ in Y as $n \to \infty$. But then, for each $k \in \mathbb{Z}$, we have

$$|a_k^n(t_n)| \le |a_k^n(0)|e^{-\int_0^{t_n} G_\varepsilon^n(s)ds} + |b_k|(1 - e^{-\int_0^{t_n} G_\varepsilon^n(s)ds})$$

and $|a_k^n(0)| \le M_k < \infty$ and $M(\lambda) \le G_\varepsilon^n(s) \le M_1 < \infty$ for every n, s. Therefore, passing to the limit, we get $|a_k| \le |b_k|$. That means $\mathcal{A} \subset [-K(\lambda), K(\lambda)] \times \mathcal{B}$, where $K(\lambda)$ is as in Theorem 4.1 and $\mathcal{B} = \{T \in H^m_{\text{per}}, |a_k| \le |b_k|\}$, since $\|T_a\|_{H^m} = (2\pi)^m (\sum_{k\in\mathbb{Z}} k^{2m}|b_k|^2)^{1/2} < \infty$. But the set $\mathcal{B}$ is compact in H^m_{per} and, therefore, the result is proved. ∎

Remark 4.2. Observe that the previous result shows the asymptotic smoothing effect of (1.1), since the attractor is as smooth as T_a. Also, note that the result is optimal in the sense that when $(0, T_a)$ is a stationary solution, then the attractor cannot be more regular than H^m_{per}. Also the same argument holds if we take the ω-limit sets in the weak topology of a single solution or the weak attractor, given by (4.5), were $\mathcal{B}$ denotes again a bounded absorbing set for (1.1) and the closures are taken in the weak sense.

Assume now that the ambient temperature is given by

$$T_a(x) = \sum_{k \in K} b_k e^{2\pi kix} \in H^m_{\text{per}}, \qquad m \geq 0,$$

where $K \subset \mathbb{Z}$. Then, denote by V_m the closed linear subspace of H^m_{per} spanned by $\{e^{2\pi kix}, k \in K\}$, i.e., $V_m = \overline{\text{span}\{e^{2\pi kix}, k \in K\}}$, and consider the following spectral decomposition in H^m_{per}: $T = T_1 + T_2$, where T_1 denotes the projection of T onto V_m and T_2 the projection onto the space generated by $\{e^{2\pi kix}, k \notin K\}$, i.e., $T_1 = \pi_k(T)$ and $T_2 = \pi_{\mathbb{Z}\setminus K}(T) = T - T_1$. Note that (1.1) is formally equivalent to

$$\begin{aligned} &\varepsilon \frac{dv}{dt} + G(v)v = \oint (T_1 + T_2)f \\ &\frac{\partial T_1}{\partial t} + v\frac{\partial T_1}{\partial x} = \frac{G(v)}{\varepsilon}(T_a - T_1) \\ &\frac{\partial T_2}{\partial t} + v\frac{\partial T_2}{\partial x} = -\frac{G(v)}{\varepsilon}T_2 \end{aligned} \tag{4.9}$$

and that integral equations for T_1 and T_2 are obtained by projecting (2.2) as in Proposition 4.1.

The following result shows the existence of inertial manifolds for (1.1) using in an essential way the fact that the equation for the temperature leaves invariant the Fourier spectrum of the function T_a. Therefore, an inertial manifold is obtained without requiring large gaps in the spectrum of a linear operator. A similar idea has been used in [1].

THEOREM 4.2. *With the notations above, assume T_a and f are such that Theorem* 2.1 *holds true. Then $Z_m = \mathbb{R} \times V_m$ is an inertial manifold for the flow of* (1.1) *in $Y_m = \mathbb{R} \times H^m_{\text{per}}$. Moreover, the attraction rate is uniform for* $0 < \lambda \leq \lambda_0$.

Even more, if $f \in V_m$ the inertial manifold Z_m has the exponential tracking property; that is, for every $(v_0, T_0) \in Y_m$ there exists $(v_1, T_1) \in Z_m$ such that if $(v_i(t), T_i(t))$, $i = 0, 1$, are the corresponding solutions of (1.1), *then*

$$(v_0(t), T_0(t)) - (v_1(t), T_1(t)) \to 0$$

in Y_m, at a fixed exponential rate, as $t \to \infty$.

Proof. From Proposition 4.1, taking $\pi = \pi_{Z\setminus K}$ and using $\pi_K(T_a) = T_a$, i.e., $T_{a_1} = T_a$, $T_{a_2} = 0$, we get $T_2(t, x) = T_{0_2}(x - \int_0^t v)e^{-\int_0^t G_\varepsilon}$. Therefore, if $T_{0_2} = 0$ then $T_2(t, x) = 0$ for every t, i.e., Z_m is an invariant manifold. Now, observe that $\operatorname{dist}_{H_m}(T, V_m) = \|T_2\|_m$. Hence, if $(v_0, T_0) \in Y_m$, then

$$\operatorname{dist}_{Y_m}((v(t), T(t)), Z_m) = \operatorname{dist}_{H_m}(T(t), V_m) = \|T_2(t)\|_m \leq \|T_{0_2}\|_m e^{-M(\lambda)t},$$

where, as above, $M(\lambda) = h_*/\varepsilon Re$. That proves the uniform exponential attraction rate for $0 < \lambda \leq \lambda_0$, since $M(\lambda) \geq M(\lambda_0)$.

To prove the exponential tracking property just note that the flow inside Y_m is given by

$$\varepsilon \frac{dv}{dt} + G(v)v = \oint T_1 f$$

$$T_1 = T_{0_1}\left(x - \int_0^t v\right) e^{-\int_0^t G_\varepsilon} + \int_0^t G_\varepsilon(s) e^{-\int_s^t G_\varepsilon} T_a(x - \int_0^t v)\, ds \qquad (4.10)$$

$$T_2 = 0.$$

Therefore, if $f \in V_m$ then $\oint Tf = \oint T_1 f$ and then if $(v(t), T(t)) \in Y_m$ is a solution of (1.1), then $(v(t), T_1(t)) \in Z_m$ and it is still a solution of (4.10). Hence, $(v(t), T(t)) - (v(t), T_1(t)) = (0, T_2(t))$ and the right hand side is of order $e^{-M(\lambda)t}$. Thus, the theorem is proved. ∎

Remark 4.3. Note that in the proof above, if T_0 is more regular, namely, if $T_0 \in H^j_{\text{per}}$ for $j > m$, then the norm of $T_2(t, x)$ can be computed in H^j_{per}, since $\|T_2(t)\|_j \leq \|T_{0_2}\|_j e^{-M(\lambda)t}$ and it converges exponentially to zero. Therefore, the manifold attracts in a stronger topology.

Assume, as above, that the ambient temperature is given by $T_a(x) = \sum_{k\in K} b_k e^{2\pi kix}$ and the geometry of the loop is given by

$$f(x) = \sum_{k\in J} c_k e^{2\pi kix}.$$

Moreover, assume $T_a \in H^m_{\text{per}}$, $m \geq 0$. Consider the flow on $Y_m = \mathbb{R} \times H^m_{\text{per}}$ and recall the spectral decomposition in H^m_{per}, $T = T_1 + T_2$, as defined above. We further decompose T_1 as

$$T_1 = \tau + \theta,$$

where τ is the projection onto the space generated by $\{e^{2\pi kix}, k \in K \cap J\}$ and θ is the projection onto the space generated by $\{e^{2\pi kix}, k \in K\setminus J\}$. Finally, denote by P the projection $P(v, T) = (v, \tau, 0, 0)$ and $Q = I - P$.

With these notations, and decomposing T_a, so $T_a = T_{a_1} = \tau_a + \theta_a$, $T_{a_2} = 0$, (1.1) and (4.9) can be formally decomposed as a system

$$\begin{aligned} \varepsilon \frac{dv}{dt} + G(v)v &= \oint (\tau + T_2) f \\ \frac{\partial \tau}{\partial t} + v \frac{\partial \tau}{\partial x} &= \frac{G(v)}{\varepsilon}(\tau_a - \tau) \\ \frac{\partial \theta}{\partial t} + v \frac{\partial \theta}{\partial x} &= \frac{G(v)}{\varepsilon}(\theta_a - \theta) \\ \frac{\partial T_2}{\partial t} + v \frac{\partial T_2}{\partial x} &= -\frac{G(v)}{\varepsilon} T_2 \end{aligned} \tag{4.11}$$

since $\oint \theta f = 0$. Note that, when one sets $T_2 = 0$, the first three equations give the flow inside the inertial manifold Y_m, i.e., they are equivalent to (4.10), while the first two are the only nonlinearly coupled ones. Therefore, once this subsystem is solved the other unknowns are determined through linear nonhomogeneous equations.

Before working out the general case we go to a special, but important, case of the above.

PROPOSITION 4.2. *Assume $K \cap J = \emptyset$; then for every $(v_0, T_0) \in Y_m$, the solution of* (1.1) *verifies*

$$(v(t), T(t, x)) \to (0, T_a(x))$$

as $t \to \infty$. In other words, $(0, T_a(x))$ is the global attractor of (1.1).

Proof. From the previous theorem it is enough to prove the result for the flow inside the inertial manifold Z_m. Since $K \cap J = \emptyset$, then the equation for v in (4.11), when $T_2 = 0$, i.e., on the inertial manifold Z_m, reduces to $\varepsilon(dv/dt) + G(v)v = 0$ so $v(t) = v_0 e^{-\int_0^t G_\varepsilon}$ and it goes exponentially to zero. On the other hand, taking $s < t$ as initial time for $T = T_1$, and denoting $T_s = T(s, \cdot)$, we have

$$T(t, x) = T_s\left(x - \int_s^t v\right) e^{-\int_s^t G_\varepsilon} + \int_s^t G_\varepsilon(z) e^{-\int_s^t G_\varepsilon} T_a\left(x - \int_z^t v\right) dz.$$

We prove below that $\int_z^t v$ is uniformly small, so the first term above decays exponentially, while the second is close to T_a. From the expression for v, we have $|\int_z^t v| \le \int_z^\infty |v| \le (|v_0|/M) e^{-Mz}$, where $M = M(\lambda)$ as in Theorem 4.1. Consequently, for every $\delta > 0$ there exists s_0 such that, for every $s_0 \le z \le t < \infty$, we have $|\int_z^t v| \le \delta$, and then, again using the fact that the translation map is continuous, for any $\eta > 0$, there exists s_0 such that, for every $s_0 \leq z \leq t$, $\|T_a(\cdot - \int_z^t v) - T_a(\cdot)\|_m \leq \eta$. Therefore, taking the H^m_{per}-norm in the expression

$$T(t,x) - \left(1 - e^{-\int_s^t G_\varepsilon}\right) T_a(x) = T_s\left(x - \int_s^t v\right) e^{-\int_s^t G_\varepsilon}$$
$$+ \int_s^t G_\varepsilon(z) e^{-\int_z^t G_\varepsilon}\left(T_a\left(x - \int_z^t v\right) - T_a(x)\right) dz$$

we see that the first term in the right hand side above decays exponentially to zero, at an e^{-Mt} rate, while the second can be made as small as needed. We have therefore proved that the flow on the inertial manifold converges to $(0, T_a)$. ∎

COROLLARY 4.2. *Assume that either*

the circuit lies horizontally, i.e., $f = 0$, or T_a is constant;

then all solutions of (1.1) *converge to* $(0, T_a(x))$.

Now, for the general case, we have the following notation. We denote by $S_I(t)$ the semigroup generated by (1.1) on Y_m restricted to the inertial manifold Z_m. We will find below a reduced semigroup denoted $S_R(t)$ that, in a sense, determines the asymptotic behavior of $S_I(t)$ and therefore that of $S(t)$. Note that $S(t)$ and $S_I(t)$ have the same attractor.

THEOREM 4.3. *Assume that T_a and f verify the assumptions in Theorems* 4.1 *and* 4.2. *Then*

(i) *The system of equations*

$$\begin{aligned} \varepsilon \frac{dv}{dt} + G(v)v &= \oint \tau f \\ \frac{\partial \tau}{\partial t} + v\frac{\partial \tau}{\partial x} &= \frac{G(v)}{\varepsilon}(\tau_a - \tau) \end{aligned} \tag{4.12}$$

defines a nonlinear semigroup, denoted $S_R(t)$, on $Y_R = P(Y_m)$, that can be identified with $PS_I(t)P = PS_I(t)$, restricted to Y_R.

(ii) *If $\mathcal{A}$ denotes the maximal attractor of $S_I(t)$, then $\mathcal{A}_R = P(\mathcal{A})$ is the maximal attractor of $S_R(t)$. Moreover*

$$\mathcal{A} = G(\mathcal{A}_R),$$

where $G: \mathcal{A}_R \to \mathcal{A}$ *is continuous. Even more, if* $a_0 = (v_0, \tau_0, \theta_0, 0) \in \mathcal{A}$, *then*

$$S(t)a_0 = G(t)(v_0, \tau_0) = G(t)P(a_0),$$

where $G(t)$ *is continuous.*

(iii) *If the set* $K \cap J$ *is finite,* (4.12) *is equivalent to a system of complex ODEs of the form*

$$\begin{aligned} &\varepsilon \frac{dv}{dt} + G(v)v = \sum_{k \in K \cap J} a_k(t)\bar{c}_k \\ &\dot{a}_k(t) + \left(2\pi i k v(t) + \frac{G(v(t))}{\varepsilon}\right) a_k(t) = \frac{G(v(t))}{\varepsilon} b_k, \qquad k \in K \cap J, \end{aligned} \tag{4.13}$$

where $a_{-k} = \bar{a}_k$, b_{-k}, *and* $c_{-k} = \bar{c}_k$. *Consequently, the asymptotic behavior of* (1.1) *is described by an explicit system of ODEs in* $\mathbb{R}^N$ *with* $N = |K \cap J| + 1$ *an odd number.*

Proof. (i) The proof of the existence and uniqueness of globally defined solutions for (4.12) follows the same lines as those for (1.1), i.e., as in Theorem 2.1. Therefore, the semigroup $S_R(t)$ is well defined. On the other hand, from (4.11) it is clear that the projection onto Y_R of any solution of (4.11) lying in the inertial manifold Z_m, i.e., with $T_2 = 0$, is a solution of (4.12), that is, $S_R(t) = PS_I(t)$. At the same time, it is also clear that $PS_I(t) = PS_I(t)P$. Note, however, that Y_R is not invariant for $S_I(t)$.

(ii) The existence of the attractor for $S_R(t)$ is the same as that in Theorem 4.1. Therefore, $\mathcal{A}_R$ is well defined. Moreover, since $S_R(t) = PS_I(t)P = PS_I(t)$ on Y_R, and $S_I(t)\mathcal{A} = \mathcal{A}$ then $P(\mathcal{A})$ is compact and invariant for $S_R(t)$, i.e., $S_R(t)P(\mathcal{A}) = P(\mathcal{A})$. To see this, just note that

$$S_R(t)P(\mathcal{A}) = PS_I(t)P(\mathcal{A}) = PS_I(t)(\mathcal{A}) = P(\mathcal{A}).$$

Hence, $P(\mathcal{A}) \subset \mathcal{A}_R$, since $\mathcal{A}_R$ is the maximal bounded invariant set for $S_R(t)$ [4, 8].

Conversely, if $(v_0, \tau_0) \in \mathcal{A}_R$, we will show below that there exists a unique θ_0 such that $(v_0, \tau_0, \theta_0, 0) \in \mathcal{A}$. That would prove that $\mathcal{A}_R \subset P(\mathcal{A})$. While proving this, we will actually show that $\theta_0 = F(v_0, \tau_0)$, where F is a continuous function. With this, we will have proved that $\mathcal{A} = G(\mathcal{A}_R)$ with $G = (I_d, F, 0)$ and G is continuous.

So, if θ_0 is such that $(v_0, \tau_0, \theta_0, 0) \in \mathcal{A}$ then, the solution of (1.1) is defined on $\mathbb{R}$ and then, from the equation for θ in (4.11) and for $t > s$ we have

$$\theta(t, x) = \theta_s\left(x - \int_s^t v\right) e^{-\int_s^t G_\varepsilon} + \int_s^t G_\varepsilon(z) e^{-\int_z^t G_\varepsilon} \theta_a\left(x - \int_z^t v\right) dz, \tag{4.14}$$

where we have denoted $\theta_s(\cdot) = \theta(s, \cdot)$. But, since we are on the attractor, θ_s is bounded in $H^m_{\rm per}$, so taking $t = 0$ and letting s go to $-\infty$, we get

$$\theta_0(x) = F(v_0, \tau_0)(x) = \int_{-\infty}^{0} G_\varepsilon(z) e^{-\int_z^0 G_\varepsilon} \theta_a\left(x - \int_z^0 v\right) dz. \qquad (4.15)$$

Note that this actually defines an element $H^m_{\rm per}$, since the integral of the norm verifies

$$\int_{-\infty}^{0} G_\varepsilon(z) e^{-\int_z^0 G_\varepsilon} \|\theta_a\|_m \, dz \leq \|\theta_a\|_m < \infty.$$

Conversely, it is easy to check that the solution of (4.11) starting at $(v_0, \tau_0, \theta_0, 0)$ actually exists globally and is bounded in $H^m_{\rm per}$. Hence, it lies on the attractor $\mathcal{A}$. Moreover, for this solution, it holds that

$$\theta(t, x) = \int_{-\infty}^{t} G_\varepsilon(z) e^{-\int_z^t G_\varepsilon} \theta_a\left(x - \int_z^t v\right) dz, \qquad t \in \mathbb{R}. \qquad (4.16)$$

From this, we see that if $a_0 = (v_0, \tau_0, \theta_0, 0) \in \mathcal{A}$, then $S(t)a_0 = G(t)(v_0, \tau_0) = G(t)P(a_0)$.

It just remains to prove that F is continuous. To this end, we denote, for $M > 0$,

$$F_M(v_0, \tau_0) = \int_{-M}^{0} \left[G_\varepsilon(z) e^{-\int_z^0 G_\varepsilon} \theta_a\left(x - \int_z^0 v\right)\right] dz.$$

We will prove in the lemmas below that F_M is continuous. Assuming this, for a moment, we can prove that $F_M \to F$ uniformly on $\mathcal{A}_{\rm R}$ and this proves that F is continuous. In fact, from (4.14), with $t = 0$ and $s = -M$, we have $F(v_0, \tau_0) - F_M(v_0, \tau_0) = \theta_{-M}(x - \int_{-M}^0 v) e^{-\int_{-M}^0 G_\varepsilon}$, so the right hand side goes to zero at a uniform exponential rate, as M goes to ∞, since $\|\theta_{-M}\|$ is uniformly bounded.

(iii) The proof is obvious. Just note that $K \cap J$ is a symmetric set and $0 \notin K \cap J$ since $\oint f = 0$, so $|K \cap J|$ is even. Also, the equations for a_k and $a_{-k} = \bar{a}_k$ are conjugate, so when going to real variables, $a_k = x_k + iy_k$, we have a system in $\mathbb{R}^N$ with $N = |K \cap J| + 1$ odd. ∎

Now we prove that F_M is continuous. For this, we first need the following.

LEMMA 4.1. *For every $M > 0$ there exists a constant $C_M > 0$ such that for every (v_1, τ_1) and (v_2, τ_2) lying in $\mathscr{A}_R$, it holds that*

$$|v_1(t) - v_2(t)| \le C_M(|v_1 - v_2| + \|\tau_1 - \tau_2\|), \qquad \text{for every } t \in [-M, M].$$

Proof. Note that by changing t to $-t$ in (4.12), and integrating along characteristics, we find that the solution starting at $(v_0, \tau_0) \in \mathscr{A}_R$ is given, in $[-M, M]$, by

$$\tau(t, x) = \tau_0\left(x \pm \int_0^t v\right) e^{\pm\int_0^t G_\varepsilon(v)} \mp \int_0^t G_\varepsilon(v) e^{\pm\int_s^t G_\varepsilon(v)} \tau_a\left(x \pm \int_s^t v\right) ds$$

and $v(t) = v_0 + \int_0^t [\pm H_\varepsilon(v(s)) \mp (1/\varepsilon)(\oint \tau f)]\, ds$ for $t \ge 0$, where the upper symbol corresponds to backward integration and the lower to forward integration. Let (v_1, τ_1) and (v_2, τ_2) be on $\mathscr{A}_R$; then

$$\begin{aligned}\tau_1(t, x) - \tau_2(t, x) &= \tau_1\left(x \pm \int_0^t v_1\right) e^{\pm\int_0^t G_{\varepsilon,1}} - \tau_2\left(x \pm \int_0^t v_2\right) e^{\pm\int_0^t G_{\varepsilon,2}} \\ &\mp \int_0^t G_{\varepsilon,1} e^{-\int_s^t G_{\varepsilon,1}} \tau_a\left(x \pm \int_s^t v_1\right) ds \\ &\pm \int_0^t G_{\varepsilon,2} e^{-\int_s^t G_{\varepsilon,2}} \tau_a\left(x \int_s^t v_2\right) ds.\end{aligned}$$

Now, we proceed as in Theorem 2.1, using the fact that $v_i(t)$ are uniformly bounded, but instead of making the bounds dependent on $\sup_{[0,M]} |v_1 - v_2|(s)$ we let $\int_0^t |v_1 - v_2|$ appear, and then we get $|\oint (\tau_1 - \tau_2) f| \le c_1(M)(\|\tau_1 - \tau_2\| + \int_0^t |v_1 - v_2|)$. Then, from the expression for the velocity, we get $|v_1(t) - v_2(t)| \le |v_1 - v_2| + c_2(M)\, \|\tau_1 - \tau_2\| + c_3(M) \int_0^t |v_1 - v_2|$. We conclude by using Gronwall's lemma. ∎

We finally have

LEMMA 4.2. *The function F_M, defined above, is continuous. Moreover, if θ_a is such that $\|\theta_a(\cdot + h) - \theta_a(\cdot)\| \le c_a\, |h|$ for every $h \in \mathbb{R}$, then F_M is a Lipschitz function.*

Proof. Let (v_1, τ_1) and (v_2, τ_2) be on $\mathscr{A}_R$; then denoting $\theta_{Mi} = F_M(v_i, \tau_i)$, we have

$$\begin{aligned}\|\theta_{M1} - \theta_{M2}\| \le \int_{-M}^0 \Bigg[& G_{\varepsilon,1}(s) e^{-\int_s^0 G_{\varepsilon,1}} \left\|\theta_a\left(\cdot - \int_s^0 v_1\right) - \theta_a\left(\cdot - \int_s^0 v_2\right)\right\| \\ &+ \|\theta_a\|\, \left|G_{\varepsilon,1}(s) e^{-\int_s^0 G_{\varepsilon,1}} - G_{\varepsilon,2}(s) e^{-\int_s^0 G_{\varepsilon,1}}\right| \Bigg] ds\end{aligned}$$

and note that $\|\theta_a(\cdot - \int_s^0 v_1) - \theta_a(\cdot - \int_s^0 v_2)\| = \|\theta_a(\cdot - \int_s^0 (v_1 - v_2)) - \theta_a(\cdot)\|$. Since the solutions lie on the attractor, there exist constants such that $0 < G_{\varepsilon,i} \leq c_1$ and $|G_{\varepsilon,1} - G_{\varepsilon,2}| \leq c_2 |v_1 - v_2|$, for $-M \leq t \leq 0$. Therefore, we get

$$\|\theta_{M1} - \theta_{M2}\| \leq c_3(M) \sup_{[-M,0]} \left\| \theta_a \left(\cdot - \int_s^0 (v_1 - v_2) \right) - \theta_a(\cdot) \right\|$$

$$+ \|\theta_a\| \, c_4(M) \, \|v_1 - v_2\|_\infty,$$

where $\|v_1 - v_2\|_\infty = \sup_{[-M,0]} |v_1 - v_2|(s)$. On the other hand, $|\int_s^0 (v_1 - v_2| \leq M \|v_1 - v_2\|_\infty$ and, from the previous lemma, this quantity is small if $|v_1 - v_2| + \|\tau_1 - \tau_2\|$ is small. Since the translation map is continuous, we see that $\|\theta_a(\cdot - \int_s^0 (v_1 - v_2)) - \theta_a(\cdot)\|$ is also small. Moreover, if θ_a is such that $\|\theta_a(\cdot + h) - \theta_a(\cdot)\| \leq c_a |h|$ for every $h \in \mathbb{R}$, then $\|\theta_a(\cdot - \int_s^0 (v_1 - v_2)) - \theta_a(\cdot)\| \leq c_a M \|v_1 - v_2\|_\infty$ and then, again using the previous lemma, we get $\|\theta_{M1} - \theta_{M2}\| \leq c_5(M)(|v_1 - v_2| + \|\tau_1 - \tau_2\|)$ and the lemma is proved. ∎

Remark 4.4. Note that when F_M is Lipschitz, the Lipschitz constant may blow-up with M; therefore we are not able to prove that F itself is Lipschitz. If this last situation were true, then the statement of Theorem 4.3 would be improved to saying that $\mathcal{A}$ and $\mathcal{A}_R$ have the same dimension. Finally, note that the proofs above hold in any admissible space.

5. Final Remarks

Although we have worked out all the results on (1.1) and we have certainly used some of the particularities of that system, the existence of attractors holds for similar but more general systems of transport equations under periodic boundary conditions. To be more precise, consider the system

$$\begin{aligned} &\dot{v} = F(v, T), && v(0) = v_0 \\ &\frac{\partial T}{\partial t} + \mathbf{a}(v) \nabla T = G(v)(T_a - T), && T(0, x) = T_0(x), \end{aligned} \tag{5.1}$$

where T, T_0, T_a are periodic functions in $\mathbb{R}^N$; i.e., they satisfy $T(x + L_i e_i) = T(x)$ for every $i = 1, \ldots, N$, where $\{e_i\}_{i=1}^N$ is a basis in $\mathbb{R}^N$. Moreover, in (5.1) we assume that $v \in \mathbb{R}^m$ and

$$\mathbf{a}: \mathbb{R}^m \to \mathbb{R}^N, \qquad G: \mathbb{R}^m \to \mathbb{R}$$

are locally Lipschitz and $G(v) \geq G_0 > 0$.

If we denote $\Omega = \{x \in \mathbb{R}^N, x = \sum_i x_i e_i, 0 \le x_i \le L_i\}$, an admissible function space for (5.1) will be a Banach space X of Ω-periodic functions defined on $\mathbb{R}^N$, i.e., satisfying $T(x + L_i e_i) = T(x)$ for every $i = 1, \ldots, N$, such that for every $h \in \mathbb{R}^N$ and $g \in X$, $\|g(\cdot + h)\|_x = \|g\|_x$ and, for fixed g, the map $h \mapsto g(\cdot + h)$ is continuous in X. For example,

$$L^p_{\text{per}}(\Omega) = \{u \in L^p_{\text{loc}}(\mathbb{R}^N), u(x + L_i e_i) = u(x), \text{a.e. } x \in \mathbb{R}^N, \forall\, i = 1, \ldots, N\},$$

endowed with the norm of $L^p(\Omega)$ and $W^{m,p}_{\text{per}}(\Omega) = W^{m,p}_{\text{loc}}(\mathbb{R}^N) \cap L^p_{\text{per}}(\Omega)$ with the norm of $W^{m,p}(\Omega)$, verify those assumptions. Analogously, we can consider the spaces $C^{k,\alpha}_{\text{per}}(\Omega)$ with the obvious definitions.

Now we assume that

$$F\colon \mathbb{R}^m \times X \to \mathbb{R}^m$$

is locally Lipschitz and moreover we assume that (5.1) has a unique, globally defined solution for any initial data in $Y = \mathbb{R}^m \times X$ and T is given by integration along characteristics, i.e., $T(t, x) = T_0(x - \int_0^t \mathbf{a}(v))e^{-\int_0^t G(v)} + \int_0^t G(v)e^{-\int_s^t G(v)}T_a(x - \int_s^t \mathbf{a}(v))\, ds$. Note that again we have $\|T(t)\|_x \le \|T_0\|_x e^{-\int_0^t G(v)} + \|T_a\|_x (1 - e^{-\int_0^t G(v)}) \le \|T_a\|_x + (\|T_0\|_x - \|T_a\|_x)_+ e^{-G_0 t}$. So, we just need to control the velocity components to have dissipativity. Therefore, we can prove

THEOREM 5.1. *With the above notations assume*

(i) *There exists a constant $K \ge 0$ such that for every $(v_0, T_0) \in Y$ there exists $t_0 \ge 0$ such that $|v(t)| \le K$ for every $t \ge t_0$.*

(ii) *For every $(v_0, T_0) \in B$, and B any bounded set in Y, the set $\{|v(t)|, t \ge 0\}$ is bounded.*

Then, (5.1) *has a compact attractor $\mathcal{A}$ in $Y = \mathbb{R}^m \times X$.*

Proof. Note that (i) implies that the solution map for (5.1), $S(t)(v, T)$, is point dissipative, i.e., there exists a bounded set $\mathcal{B} \subset Y$, such that for every $(v, T) \in Y$, there exists $t_0 \ge 0$ such that $S(t)(v, T) \in \mathcal{B}$ for every $t \ge t_0$. At the same time, (ii) implies that for every bounded set $B \subset Y$, the trajectory $\{S(t)B, t \ge 0\}$ is bounded in Y. Finally, following the same lines as those in Theorem 4.1, we again decompose $S(t)(v_0, T_0) = S_1(t)(v_0, T_0) + S_2(t)(v_0, T_0)$, where the part of the temperature in S_1 is given by $T_0(x - \int_0^t \mathbf{a}(v))e^{-\int_0^t G(v)}$ and the part in S_2 by $\int_0^t G(v)e^{-\int_s^t G(v)}T_a(x - \int_s^t \mathbf{a}(v))\, ds$. Using now the fact that $s(T_a) = \{T_a(\cdot + h), h \in \mathbb{R}^N\}$ is compact, since it is homeomorphic to the torus T^N, we conclude that S_2 is completely continuous. Again using the results in [4], we complete the proof. ∎

Finally, note that the above can be immediately extended to systems of the form

$$\dot{v} = F(v, T^1, \ldots, T^n), \qquad v(0) = v_0$$
$$\frac{\partial T^j}{\partial t} + \mathbf{a}_j(v)\,\nabla T^j = G_j(v)(T^j_a - T^j) \qquad T^j(0,x) = T^j_0(x) \tag{5.2}$$

with $j = 1, \ldots, n$ and periodic boundary conditions. Now the phase space would be of the form $Y = \mathbb{R}^m \times X_1 \times \cdots \times X_n$, where X_j are admissible spaces as above. We can also consider different dimensions and different periods for the space variables in the X_j spaces.

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