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Maps preserving operator pairs whose products are projections[☆]

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ABSTRACT

Let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on a complex Hilbert space $\mathcal{H}$ with $\dim \mathcal{H} \geqslant 2$. It is proved that a surjective map φ on $\mathcal{B}(\mathcal{H})$ preserves operator pairs whose products are nonzero projections in both directions if and only if there is a unitary or an anti-unitary operator U on $\mathcal{H}$ such that $\varphi(A) = \lambda U^*AU$ for all A in $\mathcal{B}(\mathcal{H})$ for some constants λ with $\lambda^2 = 1$. Related results for surjective maps preserving operator pairs whose triple Jordan products are nonzero projections in both directions are also obtained. These show that the operator pairs whose products or triple Jordan products are nonzero projections are isometric invariants of $\mathcal{B}(\mathcal{H})$.

1. Introduction

In the last few decades, many researchers have studied preserver problems on operator algebras motivated by theory and applications. For example these problems are strongly connected to the Kaplansky's problem concerning the characterization of invertibility preserving linear maps. While a lot of interesting results have been obtained, there are still many open problems. It is well-known that nilpotents, idempotents and projections are all very important subsets in operator algebras and related preserver problems that refer to those subsets have been studied (cf. [1,15–17]. In recent years, several authors have considered preserver problems concerning certain properties of products of operators and some linear and non-linear maps preserving commutativity, spectrum, spectral radius, nilpotency

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or idempotency of products of two operators on operator algebras are extensively studied and some interesting characterizations are given (cf. [3–5,8–12,14] and the references therein). We consider maps preserving operator pairs whose products are nonzero projections on the algebra of all linear bounded operators on a complex Hilbert space. We have considered linear maps with this property in [11]. In this paper, we consider those not necessarily linear maps preserving operator pairs whose products or triple Jordan products are nonzero projections in both directions. We will find that the operator pairs whose products or triple Jordan products are nonzero projections are isometric invariants of $\mathcal{B}(\mathcal{H})$.

Let $\mathcal{H}$ be a complex Hilbert space with $\dim\mathcal{H} \geqslant 2$ and let $\mathcal{B}(\mathcal{H})$ be the algebra of all bounded linear operators on $\mathcal{H}$. $\dim\mathcal{H}$ denotes the dimension of $\mathcal{H}$. For a subset S of $\mathcal{H}$, $[S]$ denotes the closed subspace of $\mathcal{H}$ spanned by S. For every pair of vectors $x, y \in \mathcal{H}$, (x, y) denotes the inner product of x and y. The symbol $x \otimes y$ stands for the rank-1 linear operator on $\mathcal{H}$ defined by $(x \otimes y)z = (z, y)x$ for any $z \in \mathcal{H}$. The rank-1 operator $x \otimes x$ is a projection for any unit vector x. Rank-1 operator $x \otimes y$ is idempotent (resp. nilpotent) if $(x, y) = 1$ (resp. $(x, y) = 0$). For a finite rank operator A we denote by *rankA* the rank of A. Given two projections $P, Q \in \mathcal{B}(\mathcal{H})$, we say $P \leqslant Q$ if $PQ = QP = P$ and we say $P < Q$ if $P \leqslant Q$ and $P \neq Q$. Projections P and Q are orthogonal if $PQ = QP = 0$. We recall that a conjugate linear bijective map U on $\mathcal{H}$ is said to be anti-unitary if $(Ux, Uy) = (y, x)$ for all $x, y \in \mathcal{H}$. Throughout this paper, we will denote by I the identity operator on any Hilbert space without confusion.

In this paper, we consider surjective maps on $\mathcal{B}(\mathcal{H})$ which preserve operator pairs whose products or triple Jordan products are nonzero projections in both directions.

2. Maps preserving operator pairs whose products are nonzero projections

Let φ be a map on $\mathcal{B}(\mathcal{H})$. If for any $A, B \in \mathcal{B}(\mathcal{H})$, $\varphi(A)\varphi(B)$ is a nonzero projection whenever AB is, then we say that φ preserves operator pairs whose products are nonzero projections. If for any $A, B \in \mathcal{B}(\mathcal{H})$, $\varphi(A)\varphi(B)$ is a nonzero projection if and only if AB is, then we say that φ preserves operator pairs whose products are nonzero projections in both directions. Of course φ preserves operator pairs whose products are nonzero projections in both directions if and only if both φ and φ^{-1} preserve operator pairs whose products are nonzero projections if φ is bijective.

Theorem 2.1. *Let $\mathcal{H}$ be a Hilbert space with* $\dim\mathcal{H} \geqslant 2$ *and let φ be a surjective map on $\mathcal{B}(\mathcal{H})$. Then φ preserves operator pairs whose products are nonzero projections in both directions if and only if there exist a unitary or an anti-unitary operator U on $\mathcal{H}$ and a constant λ with $\lambda^2 = 1$ such that $\varphi(A) = \lambda U^*AU$ for all $A \in \mathcal{B}(\mathcal{H})$.*

We note that the sufficiency is clear. To prove the necessity of this theorem, we need some lemmas. We next assume that φ is a surjective map on $\mathcal{B}(\mathcal{H})$ preserving operator pairs whose products are nonzero projections in both directions.

Lemma 2.2. *Let $A, B \in \mathcal{B}(\mathcal{H})$ be nonzero operators. Then the following assertions are equivalent*:

(*i*) *$A = B$.*
(*ii*) *For every $T \in \mathcal{B}(\mathcal{H})$, BT is a nonzero projection whenever AT is.*
(*iii*) *For every $T \in \mathcal{B}(\mathcal{H})$, TBT is a nonzero projection whenever TAT is.*

Proof. The implications from (*i*) to both (*ii*) and (*iii*) are obvious.

(*ii*) $\Rightarrow$ (*i*) Take any $x \in \mathcal{H}$. Suppose that $Ax \neq 0$. Put $T = \frac{x\otimes Ax}{\|Ax\|^2}$, then AT is a nonzero projection. This implies that BT is. Note that $BT = \frac{Bx\otimes Ax}{\|Ax\|^2}$ so that $Ax = Bx$. If $Ax = 0$, then there is a vector $y \in \mathcal{H}$ such that $Ay \neq 0$ and $A(x + y) \neq 0$ since $A \neq 0$. Then by the proof above we have $Ay = By$ and $A(x + y) = B(x + y)$, which means that $Bx = 0$. Thus, $A = B$.

$(iii) \Longrightarrow (i)$ Let $x \in \mathcal{H}$ be a unit vector. If $(Ax, x) \neq 0$, then $\frac{x\otimes x}{\sqrt{(Ax,x)}} A \frac{x\otimes x}{\sqrt{(Ax,x)}} = x \otimes x$ is a nonzero projection, where $\sqrt{(Ax,x)}$ is a square root of (Ax, x). It follows that $\frac{x\otimes x}{\sqrt{(Ax,x)}} B \frac{x\otimes x}{\sqrt{(Ax,x)}} = x \otimes x$ and then $(Ax, x) = (Bx, x)$. Suppose that $(Ax, x) = 0$. We may take a unit vector sequence $\{x_n\}$ in $\mathcal{H}$ such that $\lim_{n\to\infty} x_n = x$ and $(Ax_n, x_n) \neq 0$. Thus, we also have $(Bx, x) = 0 = (Ax, x)$. Hence, $A = B$. The proof is complete. □

Corollary 2.3. *φ is bijective.*

Proof. This follows from Lemma 2.2. □

Lemma 2.4. *$\varphi(0) = 0$ and $\varphi(I) = I$ or $\varphi(I) = -I$.*

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ such that $\varphi(A) = 0$. If $A \neq 0$, then there exists a vector $x \in \mathcal{H}$ such that $Ax \neq 0$. Let $B = \frac{x\otimes Ax}{\|Ax\|^2}$. Then AB is a nonzero projection, which implies that $\varphi(A)\varphi(B)$ also is. But $\varphi(A)\varphi(B) = 0$, a contradiction. Thus, $A = 0$.

Suppose $\varphi(I) = A$. If $A \notin \mathbb{C}I$, then there exists a nonzero vector $x \in \mathcal{H}$ such that x and Ax are linearly independent. Put $B = \frac{x\otimes Ax}{\|Ax\|^2}$, then AB is a nonzero projection. So is $\varphi^{-1}(A)\varphi^{-1}(B) = \varphi^{-1}(B)$. Moreover, $\varphi^{-1}(B)\varphi^{-1}(B) = \varphi^{-1}(B)$ is a nonzero projection, too. Thus, B^2 is a nonzero projection. Now $B^2 = \frac{(x,Ax)}{\|Ax\|^4} x \otimes Ax$. Then x and Ax are linearly dependent. This contradiction shows that $A = aI$ for some constant a. $A^2 = a^2 I$ must be a nonzero projection. Then $a^2 = 1$ which completes the proof. □

We may replace φ by $-\varphi$ if $\varphi(I) = -I$. Without loss of generality we may assume that $\varphi(I) = I$. Then φ preserves nonzero projections in both directions. We observe that if projections P, Q satisfy that both PQ and QP are projections, then they commute. Moreover, if $rankP = 1$ and PQ, QP are both nonzero projections, then $P \leqslant Q$.

Lemma 2.5. *Let $\varphi(I) = I$. Then φ preserves rank-n projections in both directions for any $n \geqslant 1$.*

Proof. Let E be a nonzero projection. Then $\varphi(E) = P$ is a nonzero projection, too. For any $z \in \mathbb{C}\backslash\{0, 1\}$, put $A(z) = E + z(I - E)$ and $B(z) = \varphi(A(z))$. We will complete the proof by three steps.

Step 1. φ preserves rank-1 projections in both directions.

Let $E = e \otimes e$ for some unit vector $e \in \mathcal{H}$. Since $A(z)E = EA(z) = E$ is a nonzero projection, both $B(z)P$ and $PB(z)$ are, too. Under the direct sum decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we have $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $B(z) = \begin{pmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{pmatrix}$. Then $B(z)P = \begin{pmatrix} B_{11}(z) & 0 \\ B_{21}(z) & 0 \end{pmatrix}$ and $PB(z) = \begin{pmatrix} B_{11}(z) & B_{12}(z) \\ 0 & 0 \end{pmatrix}$. It follows that both $B_{12}(z)$ and $B_{21}(z)$ are 0. It is also known that $B_{11}(z)$ is a nonzero projection.

Put $P(z) = PB(z) = B(z)P$. Then $P(z)$ is a nonzero projection with $P(z) \leqslant P$ and $P(z)P = PP(z) = P(z)$. Let $E(z) = \varphi^{-1}(P(z))$. It is known that $E(z)$ is a nonzero projection by assumption. Then $E(z)E = \varphi^{-1}(P(z))\varphi^{-1}(P)$ and $EE(z) = \varphi^{-1}(P)\varphi^{-1}(P(z))$ are nonzero projections. Hence, $E(z)E = EE(z) = E$. Since $P(z)B(z) = B(z)P(z) = P(z)$, both $E(z)A(z)$ and $A(z)E(z)$ are nonzero projections, too. However, $E(z)A(z) = E(z)E + z(E(z) - E(z)E)$ for any $z \in \mathbb{C}\backslash\{0, 1\}$. Then $E(z) - E(z)E = 0$, that is $E(z) = E(z)E = E$ and therefore $P(z) = P$ for any $z \in \mathbb{C}\backslash\{0, 1\}$. This implies that $B(z) = \begin{pmatrix} I & 0 \\ 0 & B_{22}(z) \end{pmatrix}$. If $rankP \geqslant 2$, then for any rank-1 projection Q with $0 < Q < P$, we similarly have that both $\varphi^{-1}(Q)\varphi^{-1}(P) = \varphi^{-1}(Q)E$ and $\varphi^{-1}(P)\varphi^{-1}(Q) = E\varphi^{-1}(Q)$ are equal to E. On the other hand, $QB(z) = B(z)Q = Q$. So for any $z \in \mathbb{C}\backslash\{0, 1\}$, we have $\varphi^{-1}(Q)A(z) = \varphi^{-1}(Q)E + z(\varphi^{-1}(Q) - \varphi^{-1}(Q)E)$ is also a nonzero projection. Thus, $\varphi^{-1}(Q) = \varphi^{-1}(Q)E = E$ and $Q = \varphi(E) = P$, a contradiction. Therefore, $rankP = 1$.

Step 2. $B(z) = \varphi(A(z)) = P + (I - P)B(z)(I - P)$ for for any nonzero projection E.

Since $A(z)E = EA(z) = E$ are nonzero projections, both $B(z)P$ and $PB(z)$ are nonzero projections, too. It easily follows that $B(z) = PB(z)P + (I - P)B(z)(I - P)$ such that $P(z) = PB(z) = PB(z)P$ is a nonzero projection. Since $P(z)B(z) = P(z)$ and $P(z)P = P(z)$ are nonzero projections, both $\varphi^{-1}(P(z))A(z)$ and $\varphi^{-1}(P(z))E$ are also nonzero projections. However,

$$\varphi^{-1}((z))A(z) = \varphi^{-1}(P(z))E + z(\varphi^{-1}(P(z)) - \varphi^{-1}(P(z))E),$$

which implies that $\varphi^{-1}(P(z)) = \varphi^{-1}(P(z))E$ for any $z \in \mathbb{C}\backslash\{0, 1\}$. Therefore, $\varphi^{-1}(P(z)) \leqslant E$.

For any rank-1 projection $E_1 \leqslant E$, Step 1 shows that $\varphi(E_1)$ is a rank-1 projection. Since $E_1E = EE_1 = E_1$, $\varphi(E_1)\varphi(E)$ and $\varphi(E)\varphi(E_1)$ are nonzero projections and $rank(\varphi(E_1)\varphi(E)) \leqslant rank\varphi(E_1) = 1$. We thus have that $rank(\varphi(E_1)\varphi(E)) = 1$, that is $\varphi(E_1)\varphi(E) = \varphi(E_1)$. It follows that $\varphi(E_1) \leqslant \varphi(E) = P$. Note that $E_1A(z) = E_1 = A(z)E_1$, so $\varphi(E_1)B(z)$ is a rank-1 projection. $\varphi(E_1)P(z) = \varphi(E_1)B(z)$ is a rank-1 projection. Thus, $\varphi(E_1) \leqslant P(z)$ and $E_1\varphi^{-1}(P(z))$ is also a nonzero projection. Since E_1 is of rank-1, $E_1 \leqslant \varphi^{-1}(P(z))$. We can get $E \leqslant \varphi^{-1}(P(z))$ from the arbitrariness of E_1. Thus, we have $E \leqslant \varphi^{-1}(P(z)) \leqslant E$, that is $E = \varphi^{-1}(P(z))$. Thus, $P = \varphi(E) = P(z)$, which means $P = PB(z)$. Hence, $B(z) = P + (I - P)B(z)(I - P)$.

Step 3. φ preserves rank-n projections in both directions.

Step 1 shows that this assertion holds when $n = 1$. If $\dim \mathcal{H} = 2$, then the proof is complete. We assume that $\dim \mathcal{H} \geqslant 3$.

Now we assume that φ preserves rank-k projections in both directions for any $k \leqslant n$ and we prove that φ preserves rank-$(n + 1)$ projections in both directions. Let E be a rank-$(n + 1)$ projection, then $rankP \geqslant n + 1$. $\varphi(A(z)) = B(z) = P + (I - P)B(z)(I - P)$ from Step 2. Take any projection $Q \leqslant P$ with $rankQ = n + 1$. Then $rank\varphi^{-1}(Q) \geqslant n + 1$. Since $QP = PQ = Q$, both $\varphi^{-1}(P)\varphi^{-1}(Q) = E\varphi^{-1}(Q)$ and $\varphi^{-1}(Q)\varphi^{-1}(P) = \varphi^{-1}(Q)E$ are nonzero projections. We also have that $QB(z) = B(z)Q = Q$, which implies that $\varphi^{-1}(Q)A(z)$ is a nonzero projection, too. However, $\varphi^{-1}(Q)A(z) = \varphi^{-1}(Q)E + z(\varphi^{-1}(Q) - \varphi^{-1}(Q)E)$, so $\varphi^{-1}(Q) = \varphi^{-1}(Q)E \leqslant E$. By the inductive assumption, rank $E = n + 1$, we have $\varphi^{-1}(Q) = E$ and $Q = \varphi(E) = P$. Hence, $rankP = n + 1$. Thus, φ preserves rank-$(n + 1)$ projections and so does φ^{-1}. The proof is complete. □

Lemma 2.6. *Let $\varphi(I) = I$. Then φ preserves the order as well as the orthogonality of projections in both directions.*

Proof. We firstly show that φ preserves the order of projections in both directions. Let E be a nonzero projection and $\varphi(E) = P$. Take any nonzero projection $F \leqslant E$ and set $Q = \varphi(F)$. Then $FE = EF = F$. So $\varphi(F)\varphi(E) = QP$ and $\varphi(E)\varphi(F) = PQ$ are nonzero projections. Hence, $PQ = QP$.

Take any rank-1 projection $Q_1 \leqslant Q$. Then $\varphi^{-1}(Q_1)F$ is a rank-1 projection by Lemma 2.5. Thus, $\varphi^{-1}(Q_1) \leqslant F \leqslant E$, which means that $\varphi^{-1}(Q_1)E$ is a rank-1 projection. It follows that Q_1P is also a rank-1 projection. Thus, $Q_1 \leqslant P$. By the arbitrariness of Q_1, we have $Q \leqslant P$.

Next we prove that φ preserves the orthogonality of rank-1 projections in both directions. Suppose $\dim \mathcal{H} \geqslant 3$ and there are orthogonal unit vectors $e_1, e_2 \in \mathcal{H}$ such that $\varphi(e_1 \otimes e_1)$ and $\varphi(e_2 \otimes e_2)$ are not orthogonal. Take any unit vector $e_3 \in \mathcal{H} \ominus \{e_1, e_2\}$. We know that there are unit vectors $\{f_i : i = 1, 2, 3\}$ such that $\varphi(e_i \otimes e_i) = f_i \otimes f_i$ for $i = 1, 2, 3$ by Lemma 2.5. It is easy to see that f_1, f_2 and f_3 are linearly independent. Let P_{ij} (resp. Q_{ij}) be the projection from $\mathcal{H}$ onto $[e_i, e_j]$ (resp. $[f_i, f_j]$) $(1 \leqslant i < j \leqslant 3)$. Then we have that $Q_{ij} = \varphi(P_{ij})$ for all $1 \leqslant i < j \leqslant 3$ from Lemma 2.5. It follows that $Q_{12}Q_{13} = Q_{13}Q_{12} = f_1 \otimes f_1$ since $P_{12}P_{13} = P_{13}P_{12} = e_1 \otimes e_1$. Put $f_2 = \alpha f_1 + y$ and $f_3 = \beta f_1 \oplus z$ for some $y, z \in \{f_1\}^{\perp}$. It is clear that $Q_{12}y = y$ and $Q_{13}z = z$. We then have that $\alpha \neq 0$ by the assumption and Thus, $f_1 = \alpha^{-1}(f_2 - y)$. Note that $\beta f_1 = Q_{12}Q_{13}f_3 = Q_{12}(\beta f_1 + z) = \beta f_1 + Q_{12}z$. Then $Q_{12}z = 0$, which implies that $y \perp z$. Hence, $f_3 = \beta\alpha^{-1}f_2 - \beta\alpha^{-1}y + z$. We similarly have that $Q_{12}Q_{23} = Q_{23}Q_{12} = f_2 \otimes f_2$. Thus,

$$\begin{aligned} Q_{23}Q_{12}f_3 &= Q_{23}Q_{12}(\beta\alpha^{-1}f_2 - \beta\alpha^{-1}y + z) \\ &= Q_{23}(\beta\alpha^{-1}f_2 - \beta\alpha^{-1}y) \\ &= \beta\alpha^{-1}f_2 - \beta\alpha^{-1}Q_{23}y \\ &= Q_{12}Q_{23}f_3 = Q_{12}f_3 \end{aligned}$$

$$\begin{aligned}&=Q_{12}(\beta\alpha^{-1}f_2-\beta\alpha^{-1}y+z)\\&=\beta\alpha^{-1}f_2-\beta\alpha^{-1}y.\end{aligned}$$

It follows that $\beta\alpha^{-1}Q_{23}y=\beta\alpha^{-1}y$. Note that $f_1\notin Q_{23}\mathcal{H}=[f_2,f_3]$. Thus, we have that $\beta=0$, that is f_1 and $f_3=z$ are orthogonal. Note that $y\perp z$. We then have that f_2 and f_3 are orthogonal.

On the other hand, we also have that $Q_{13}Q_{23}=Q_{23}Q_{13}=f_3\otimes f_3$. Then $Q_{13}Q_{23}f_1=0$. However,

$$\begin{aligned}Q_{13}Q_{23}f_1&=Q_{13}Q_{23}(\alpha^{-1}(f_2-y))\\&=\alpha^{-1}Q_{13}Q_{23}f_2=\alpha^{-1}Q_{13}f_2\\&=\alpha^{-1}Q_{13}(\alpha f_1+y)\\&=f_1+\alpha^{-1}Q_{13}y=f_1\end{aligned}$$

since $y\in\{f_1,z\}^{\perp}=\{f_1,f_3\}^{\perp}$. This is a contradiction. Thus, f_1 and f_2 are orthogonal.

Suppose $\dim\mathcal{H}=2$. It is known that φ preserves rank-1 projections in both directions from Lemma 2.5. Let E be a rank-1 projection and P a rank-1 idempotent such that $PE=E$ (resp. $EP=E$). We claim that $\varphi(P)$ is a rank-1 idempotent such that $\varphi(P)\varphi(E)=\varphi(E)$ (resp. $\varphi(E)\varphi(P)=\varphi(E)$). In fact, we may assume that $E\neq P$, $E=e\otimes e$ and $P=e\otimes x$ with $(e,x)=1$. Note that E and $\varphi(E)$ are unitarily similar. Without loss of generality, we may assume that $\varphi(E)=E$. Take a unit vector $f\in\mathcal{H}$ such that $(e,f)=0$. Then $P=\begin{pmatrix}1&\xi\\0&0\end{pmatrix}$ ($\xi\neq0$). Put $A=\varphi(P)=\begin{pmatrix}a_{11}&a_{12}\\a_{21}&a_{22}\end{pmatrix}$. It is known that $a_{11}=1$ and $a_{21}=0$ since $PE=E$. It is enough to show that $a_{22}=0$. Otherwise, if $a_{22}\neq0$, then A is invertible with inverse $A^{-1}=\begin{pmatrix}1&-a_{12}a_{22}^{-1}\\0&a_{22}^{-1}\end{pmatrix}$. We also have that $\varphi^{-1}(A^{-1})=\begin{pmatrix}1&x\\0&y\end{pmatrix}$ since $A^{-1}E=E$. However, $\varphi^{-1}(A^{-1})P=P$ is not a projection. This is a contradiction since $A^{-1}A=I$. Hence, $a_{22}=0$. Note that φ^{-1} has the same property.

Suppose that F is a rank-1 projection such that $EF=0$. As above we may assume that $\varphi(E)=E$. If $\varphi(F)\neq F$, then $\varphi(F)=\frac{1}{1+|z|^2}\begin{pmatrix}1&z\\\bar z&|z|^2\end{pmatrix}$ for some nonzero constant z. Put $Q=\begin{pmatrix}1&0\\\bar z&0\end{pmatrix}$. Then $Q\varphi(F)=\varphi(F)$, $EQ=E$ and $P=\varphi^{-1}(Q)$ is a rank-1 idempotent just as we have proved such that $PF=F$ and $EP=E$. This is a contradiction. Therefore $\varphi(F)=F$. Consequently, $\varphi(E)$ and $\varphi(F)$ are orthogonal whenever E and F are.

Now let E and F be two orthogonal projections and set $P=\varphi(E)$ and $Q=\varphi(F)$. If P and Q are not orthogonal, then there are two rank-1 projections $P_1\leqslant P$ and $Q_1\leqslant Q$ such that P_1 and Q_1 are not orthogonal. However, $E_1=\varphi^{-1}(P_1)\leqslant E$ and $F_1=\varphi^{-1}(Q_1)\leqslant F$. This is a contradiction. Thus, P and Q are orthogonal. By considering φ^{-1}, we know that φ preserves orthogonality of projections in both directions. The proof is complete. □

If $\dim\mathcal{H}\geqslant3$, then φ is a bijection on the set of all projections of $\mathcal{B}(\mathcal{H})$ preserving orthogonality in both directions. It follows from the Uhlhorn's theorem in [18] that there is a unitary or an anti-unitary operator U on $\mathcal{H}$ such that

$$\varphi(E)=UEU^*$$

for any projection $E\in\mathcal{B}(\mathcal{H})$. Next lemma shows that this form holds when $\dim\mathcal{H}=2$ by use of the Wigner's fundamental theorem (cf. [13,19]).

Lemma 2.7. *Suppose that* $\dim\mathcal{H}=2$ *and* $\varphi(I)=I$. *Then there is a unitary or an anti-unitary operator* U *on* $\mathcal{H}$ *such that* $\varphi(E)=UEU^*$ *for any projection* $E\in\mathcal{B}(\mathcal{H})$.

Proof. We firstly claim that there is a function on $\mathbb{C}$ such that $\varphi(zP)=f(z)\varphi(P)$ for all $z\in\mathbb{C}$ and any idempotent P.

Let $E=e\otimes e$ and $F=f\otimes f$ be arbitrary two rank-1 projections such that $EF=0$. By Lemma 2.6, $\varphi(E)$ and $\varphi(F)$ are orthogonal such that $\varphi(E)+\varphi(F)=I$. We may assume that $\varphi(E)=E$ and

$\varphi(F) = F$. There is a function f_e on $\mathbb{C} \setminus \{0, 1\}$ such that $\varphi(zE + F) = f_e(z)E + F$ from Step 2 in the proof of Lemma 2.5 for any $z \in \mathbb{C} \setminus \{0, 1\}$. It is clear that $f_e\left(\frac{1}{z}\right) = \frac{1}{f_e(z)}$. We show that $\varphi(zE) = f_e(z)E$. In fact, $\varphi(zE)\varphi\left(\frac{1}{z}E + F\right) = \varphi(zE)\left(f_e\left(\frac{1}{z}\right)E + F\right)$ is a rank-1 projection. Put $\varphi(zE) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. Then $\varphi(zE)\left(f_e\left(\frac{1}{z}\right)E + F\right) = \begin{pmatrix} f_e\left(\frac{1}{z}\right)x_{11} & x_{12} \\ f_e\left(\frac{1}{z}\right)x_{21} & x_{22} \end{pmatrix}$ is a rank-1 projection and thus, $f_e\left(\frac{1}{z}\right)x_{21} = \overline{x_{12}}$. Suppose that $x_{22} \neq 0$. Put $Q = \begin{pmatrix} 0 & 0 \\ \frac{f_e\left(\frac{1}{z}\right)x_{21}}{x_{22}} & 1 \end{pmatrix}$, then $\varphi(zE)Q = \varphi(zE)\left(f_e\left(\frac{1}{z}\right)E + F\right)Q = \varphi(zE)\left(f_e\left(\frac{1}{z}\right)E + F\right)$, that is, $\varphi(zE)Q$ is a rank-1 projection. However, $\varphi^{-1}(Q) = \begin{pmatrix} 0 & 0 \\ y & 1 \end{pmatrix}$ for some $y \in \mathbb{C}$ from the proof of Lemma 2.6 and $zE\varphi^{-1}(Q) = 0$. Thus, $x_{22} = 0$ and $\varphi(zE) = f_e(z)E = f_e(z)\varphi(E)$. Then $\varphi(zI)f_e\left(\frac{1}{z}\right)\varphi(E)$ is rank-1 projection for any rank-1 projection E since $(zI)\left(\frac{1}{z}E\right) = E$. This means that $\varphi(zI)\varphi(E) = f_e(z)\varphi(E)$ for any rank-1 projection E. Thus, $\varphi(zI)$ is a multiple of the identity I and $f_e(z)$ is a constant $f(z)$ independent on e such that $\varphi(zI) = f(z)I$. It is also shown that $\varphi(zE) = f(z)\varphi(E)$ for any rank-1 projection E by the arbitrariness of E.

On the other hand, if $\varphi(E) = E$ and $P(z) = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}$, then $\varphi(P(z)) = P(h(z))$ for some $h(z) \in \mathbb{C}$ from the proof of Lemma 2.6 again. Note that a projection which is not orthogonal to E has the form $E(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & z \\ \bar{z} & |z|^2 \end{pmatrix}$ for some $z \in \mathbb{C}\setminus\{0, 1\}$. It follows that $\varphi(E(z)) = E(g(z))$for some $g(z) \in \mathcal{C}\setminus\{0, 1\}$. We have that $h(z) = g(z)$ for all $z \in \mathbb{C}\setminus\{0, 1\}$ since $E(z)P(z) = E(z)$. Similarly we have that $\varphi(P(z)^*) = (\varphi(P(z)))^*$for any z. Put $\varphi(wP(z)) = \begin{pmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{pmatrix}$. It is known that $y_{11} = f(w)$ and $y_{21} = y_{22} = 0$. On the other hand, $(\frac{1}{w}E(z))(wP) = E(z)$ for any $w \in \mathbb{C}\setminus\{0\}$. This means that $\frac{1}{f(w)}\varphi(E(z))\varphi(wP(z))$ is a nonzero projection. It follows that $y_{12} = f(w)g(z)$. Hence, $\varphi(wP(z)) = f(w)\varphi(P(z))$. Put $f(0) = 0$ and $f(1) = 1$. Then $\varphi(wP) = f(w)\varphi(P)$ for any idempotent P and any $z \in \mathbb{C}$.

Next we assume that $\varphi(E) = E$ and $\varphi(F) = F$ again. Put $N(w) = \begin{pmatrix} 0 & 0 \\ w & 0 \end{pmatrix}$for any nonzero $w \in \mathbb{C}$. We show that $\varphi(N(w)) = N\left(\left(g\left(\frac{1}{w}\right)\right)^{-1}\right)$. In fact, it is easily known that $\varphi(N(w))$ is of rank-1. Otherwise, if $\varphi(N(w))$ is invertible, then there is a rank-1 projection Q and a constant α such that $\varphi(N(w))(\alpha Q)$ is a rank-1 projection. Since $N(w)F = 0, \varphi^{-1}(Q) = E(z)$ for some $z \in \mathbb{C}$. It follows that $N(w)\varphi^{-1}(\alpha Q) = N(w)(\beta E(z))$ is a rank-1 projection for some constant β. This is a contradiction. Note that $P\left(\frac{1}{w}\right)N(w) = E$. This implies that $\varphi(N(w)) = N\left(\left(g\left(\frac{1}{w}\right)\right)^{-1}\right)$. Similarly, $\varphi(N(w)^*) = (\varphi(N(w)))^*$. Thus, it is easily follows that $|g(1)| = 1$ since $(N(1))^*N(1) = E$. Note that $\frac{1}{z}P(z)N(1) = E$. Then $f(z) = g(1)g(z)$ for all z. In particular, $|f(z)| = |g(z)|$. For any $a > 0$, $b > 0$, $\frac{1}{ab}P(a)N(b) = E$. Then $\frac{1}{f(ab)}\varphi(P(a))\varphi(N(b))$ is a rank-1 projection. Thus, $|f(ab)| = |g(a)||g\left(\frac{1}{b}\right)^{-1}| = |f(a)||f(b)|$, that is $|f|$ is multiplicative on $[0, +\infty)$. Again $P(z)P(z)^* = (1 + |z|^2)E$. We easily have that $f(1 + |z|^2) = 1 + |g(z)|^2 = 1 + |f(z)|^2$ and $|f(a + b)| = |f\left(b\left(1 + \frac{a}{b}\right)\right)| = |f(b)f\left(1 + \frac{a}{b}\right)| = |f(b)|\left(1 + |f\left(\sqrt{\frac{a}{b}}\right)|^2\right) = |f(a)| + |f(b)|$. Thus, $|f|$ is additive as well as multiplicative on $[0, \infty)$. Note That $f(1) = 1$. Thus, we have $|f(a)| = a$ for all $a \in [0, \infty)$. It follows that $|g(z)| = f(|z|) = |z|$ from the equality $1 + (f(|z|))^2 = f(1 + |z|^2) = 1 + |g(z)|^2$ for all $z \in \mathbb{C}$. Now for any rank-1 projection $E(z)$, we have $\mathrm{tr}(EE(z)) = \mathrm{tr}(\varphi(E)\varphi(E(z))) = \frac{1}{1+|z|^2}$. By the Wigner's fundamental theorem(cf.[13,19]), there is unitary or an anti-unitary operator U on $\mathcal{H}$ such that $\varphi(E) = UEU^*$ for all rank-1 projection E. The proof is complete. $\square$

Lemma 2.8. *Suppose that $\varphi(E) = E$ for any projection $E \in \mathcal{B}(\mathcal{H})$. Then there is a complex function $f(z)$ on $\mathbb{C}$ such that $\varphi(zE) = f(z)E$ for any projection $E \in \mathcal{B}(\mathcal{H})$.*

Proof. If $\dim \mathcal{H} = 2$, then this is easy from Lemma 2.7. Next we assume that $\dim \mathcal{H} \geqslant 3$. First let $e \in \mathcal{H}$ be a unit vector and $E = e \otimes e$. Take any nonzero projections P and Q such that $I = P \oplus E \oplus Q$. Let $\mathcal{H} = P(\mathcal{H}) \oplus E(\mathcal{H}) \oplus Q(\mathcal{H})$. We claim that there is a complex function $f_e(z)$ on $\mathbb{C} \setminus \{0, 1\}$ such that $\varphi(P + zE) = P + f_e(z)E$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. Take any $z \in \mathbb{C} \setminus \{0, 1\}$. It is elementary that

$$\varphi(P + zE) = \begin{pmatrix} I & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & B_{33} \end{pmatrix}$$

since $P_0(P + zE) = (P + zE)P_0 = P_0$ is a nonzero projection for any nonzero projection $P_0 \leqslant P$. On the other hand, we have $(P + zE)(I - E) = (I - E)(P + zE) = P$. It follows that $B_{23} = 0, B_{32} = 0$ and B_{33} is a projection. If $B_{33} \neq 0$ then $\varphi(P + zE)(I - (P + E)) = 0 \oplus 0 \oplus B_{33}$ is a nonzero projection. However, $(P + zE)(I - (P + E)) = 0$. This is a contradiction. Hence, $B_{33} = 0$. It is trivial that B_{22} is a constant dependent on z. Set $f_e(z) = B_{22}$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. Thus, $\varphi(P + zE) = P + f_e(z)E$. Note that $\varphi\left(P + \frac{1}{z}E\right) = P + f_e(\frac{1}{z})E$ and $(P + zE)\left(P + \frac{1}{z}E\right) = P + E$. It follows that $f_e\left(\frac{1}{z}\right) = \frac{1}{f_e(z)}$. A similar way shows that $\varphi(Q + zE) = Q \oplus f_e(z)E$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. Put

$$\varphi(zE) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{22} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

Note that $\left(P + \frac{1}{z}E\right)(zE) = (zE)\left(P + \frac{1}{z}E\right) = E$, which implies that $\left(P + f_e\left(\frac{1}{z}\right)E\right)\varphi(zE)$ is a nonzero projection. A simple calculation shows that $C_{ij} = 0$ for any $i = 1$ or $j = 1$. Replacing P by Q, it is known that $C_{ij} = 0$ for all i, j except $i = j = 2$. It is clear that $C_{22} = f_e(z)$. Thus, $\varphi(zE) = f_e(z)E$.

For any $z \in \mathbb{C} \setminus \{0, 1\}$, it is known that $\varphi(zI)\left(\frac{1}{f_e(z)}E\right)$ is a nonzero projection since $zI\left(\frac{1}{z}E\right) = E$. It follows that $\varphi(zI)e = f_e(z)e$ for any unit vector $e \in \mathcal{H}$. Therefore $\varphi(zI)$ is a multiple of I and $f_e(z)$ is a constant $f(z)$ for any $e \in \mathcal{H}$. In particular, $\varphi(zI) = f(z)I$ and $\varphi(E) = f(z)E$ for any $z \in \mathbb{C} \setminus \{0, 1\}$ and any rank-1 projection E.

Let F be any nonzero projection and set $\varphi(zF) = F_z$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. It follows that $F_z = f(z)P_z$ for some nonzero projection P_z from the fact that $\left(\frac{1}{z}I\right)(zF) = F$. Take any rank-1 projection $E \leqslant F$. We have that $\left(\frac{1}{z}E\right)zF = E$. Then $\varphi\left(\frac{1}{z}E\right)\varphi(zF) = \left(\frac{1}{f(z)}E\right)(f(z)P_z) = EP_z$ is a nonzero projection. Hence, $E \leqslant P_z$. Thus, we have $F \leqslant P_z$. In fact we must have $F = P_z$. Otherwise, there is a rank-1 projection $E \leqslant P_z$ such that $EF = FE = 0$. We then have that $(f(z)P_z)\left(\frac{1}{f(z)}E\right) = E$. However, $\varphi^{-1}(f(z)P_z)\varphi^{-1}\left(\frac{1}{f(z)}E\right) = (zF)\left(\frac{1}{z}E\right) = 0$. This is a contradiction. Hence, $P_z = F$ and $\varphi(zF) = f(z)F$ for any projection F and any $z \in \mathbb{C} \setminus \{0, 1\}$.

If we define $f(0) = 0$ and $f(1) = 1$, then f is a complex function on $\mathbb{C}$ such that $\varphi(zE) = f(z)E$ for any projection E and $z \in \mathbb{C}$. The proof is complete. □

Lemma 2.9. *Suppose that $\varphi(E) = E$ for any projection $E \in \mathcal{B}(\mathcal{H})$ and f is the function defined in Lemma 2.8. Then $\varphi(zP) = f(z)P$ for any $z \in \mathbb{C}$ and any rank-1 idempotent P.*

Proof. Take any rank-1 idempotent P. We may assume that $P = e \otimes (e + \xi x)$ for some unit vectors $e, x \in \mathcal{H}$ and nonzero constant ξ such that $e \perp x$. Let $\mathcal{H} = \{e\} \oplus \{x\} \oplus \{e, x\}^{\perp}$. Next we can assume that $z \neq 0$. Then $f(z) \neq 0$. We firstly claim that if

$$T = \begin{pmatrix} z & T_{12} & 0 \\ 0 & T_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then

$$\varphi(T)=\begin{pmatrix} f(z) & X_{12} & 0\\ 0 & X_{22} & 0\\ 0 & X_{32} & 0\end{pmatrix}.$$

In fact, put $\varphi(T)=\begin{pmatrix} X_{11} & X_{12} & X_{13}\\ X_{21} & X_{22} & X_{23}\\ X_{31} & X_{32} & X_{33}\end{pmatrix}$. Since $T\left(\frac{1}{z}E\right)=E$, $\varphi(T)\varphi\left(\frac{1}{z}E\right)=\varphi(T)\left(\frac{1}{f(z)}E\right)$ is a nonzero projection, which implies that $X_{11}=f(z)$, $X_{21}=0$ and $X_{31}=0$. Now it is trivial that $T\left(\frac{1}{z}(I-x\otimes x)\right)=E$. We then have that $\varphi(T)\varphi\left(\frac{1}{z}(I-x\otimes x)\right.=\varphi(T)\left(\frac{1}{f(z)}(I-x\otimes x)\right)$ is also a nonzero projection. Note that

$$\begin{aligned}\varphi(T)\left(\frac{1}{f(z)}(I-x\otimes x)\right) &= \begin{pmatrix} f(z) & X_{12} & X_{13}\\ 0 & X_{22} & X_{23}\\ 0 & X_{32} & X_{33}\end{pmatrix}\begin{pmatrix} \frac{1}{f(z)} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \frac{1}{f(z)}\end{pmatrix}\\ &= \begin{pmatrix} 1 & 0 & \frac{1}{f(z)}X_{13}\\ 0 & 0 & \frac{1}{f(z)}X_{23}\\ 0 & 0 & \frac{1}{f(z)}X_{33}\end{pmatrix},\end{aligned}$$

so $X_{13}=0, X_{23}=0$ and $\frac{1}{f(z)}X_{33}$ is a projection. If $X_{33}\neq 0$, then we know that $\varphi(T)\left(\frac{1}{f(z)}\right.(I-E-x\otimes x)=0\oplus 0\oplus\frac{1}{f(z)}X_{33}$ is a nonzero projection. However, we have $T\left(\frac{1}{z}(I-E-x\otimes x)\right)=0$. This is a contradiction. Hence, $X_{33}=0$ and the claim holds.

Now If $\mathcal{H}=[e,x]\oplus[e,x]^{\perp}$, then $P=P(\xi)\oplus 0$, where $P(\xi)$ is defined in the proof of Lemma 2.7. By the claim we have shown,

$$\varphi(zP)=\begin{pmatrix} f(z) & A_{12} & 0\\ 0 & A_{22} & 0\\ 0 & A_{32} & 0\end{pmatrix}.$$

Put $F=E(\xi)\oplus 0$, then F is a rank-1 projection and a simple calculation shows that $\left(\frac{1}{z}\right.(F+(I-E-x\otimes x))(zP)=F$. It now follows that $\varphi\left(\frac{1}{z}\right.(F+(I-E-x\otimes x))\varphi(zP)=\left(\frac{1}{f(z)}\right.(F+(I-E-x\otimes x))\varphi(zP)$ is a nonzero projection. Note that

$$\begin{aligned}&\left(\frac{1}{f(z)}\right)(F+(I-E-x\otimes x)))\varphi(zP)\\ &\quad=\frac{1}{f(z)}\begin{pmatrix} \frac{1}{1+|\xi|^2} & \frac{\xi}{1+|\xi|^2} & 0\\ \frac{\bar{\xi}}{1+|\xi|^2} & \frac{|\xi|^2}{1+|\xi|^2} & 0\\ 0 & 0 & I\end{pmatrix}\begin{pmatrix} f(z) & A_{12} & 0\\ 0 & A_{22} & 0\\ 0 & A_{32} & 0\end{pmatrix}\\ &\quad=\begin{pmatrix} * & * & 0\\ * & * & 0\\ 0 & \frac{1}{f(z)}A_{32} & 0\end{pmatrix},\end{aligned}$$

which implies that $A_{32}=0$.

We next show that $A_{22}=0$. Suppose that $A_{22}\neq 0$. Let $B=\begin{pmatrix} \frac{1}{f(z)} & \frac{-A_{22}^{-1}}{f(z)} & 0\\ 0 & A_{22}^{-1} & 0\\ 0 & 0 & 0\end{pmatrix}$. Then $B\varphi(zP)=E+x\otimes x$. Thus, $\varphi^{-1}(B)(zP)$ is also a nonzero projection. By use of the claim that we just have proved to φ^{-1}, we have that $\varphi^{-1}(B)=\begin{pmatrix} \frac{1}{z} & C_{12} & 0\\ 0 & C_{22} & 0\\ 0 & C_{32} & 0\end{pmatrix}$. However,

$$\varphi^{-1}(B)(zP) = \begin{pmatrix} \frac{1}{z} & C_{12} & 0 \\ 0 & C_{22} & 0 \\ 0 & C_{32} & 0 \end{pmatrix} \begin{pmatrix} z & z\xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & \xi & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is not a projection. This contradiction shows that $A_{22} = 0$. Again note that $\left(\frac{1}{z}F\right)(zP) = F$. It follows that $A_{12} = f(z)\xi$ and the proof is complete. □

Proof of Theorem 2.1. We can assume that $\varphi(E) = E$ for any projection $E \in \mathcal{B}(\mathcal{H})$ by the Uhlhorn's theorem in [18] and Lemma 2.7. We first show that for any nonzero $A \in \mathcal{B}(\mathcal{H})$, $\varphi(A) = \lambda(A)A$ for some nonzero constant $\lambda(A) \in \mathbb{C}$. Take any nonzero $x \in \mathcal{H}$. If $Ax = 0$, then Ax and $\varphi(A)x$ are linearly dependent. We may assume that $Ax \neq 0$. If $(x, Ax) \neq 0$, then $\frac{x \otimes Ax}{\|Ax\|^2} = \alpha P$ for some nonzero constant $\alpha \in \mathbb{C}$ and a rank-1 idempotent P and Hence, $\varphi(\alpha P) = f(\alpha)P$ from Lemma 2.9. It is trivial that $A(\alpha P)$ is a rank-1 projection. Then $\varphi(A)(f(\alpha)P)$ is a nonzero projection. That is, $\varphi(A)(x \otimes Ax) = \varphi(A)x \otimes Ax$ is a multiple of a nonzero projection. It follows that $\varphi(A)x$ and Ax are linearly dependent. If $(x, Ax) = 0$, then there is a nonzero $x_0 \in \mathcal{H}$ such that $(x_0, Ax_0) \neq 0$. We may choose a positive sequence $\{\alpha_n : n = 1, 2, \ldots\}$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $(x + \alpha_n x_0, A(x + \alpha_n x_0)) \neq 0$. In fact, if there is a positive constant a such that

$$(x + \alpha x_0, A(x + \alpha x_0)) = (x, Ax) + ((x, Ax_0) + (x_0, Ax))\alpha + (x_0, Ax_0)\alpha^2 = 0$$

for all $\alpha \in [0, a]$, then $(x_0, Ax_0) = 0$, a contradiction. Then just as we have shown above, $\varphi(A)(x + \alpha_n x_0) = \xi_n A(x + \alpha_n x_0)$ for a complex sequence $\{\xi_n : n = 1, 2, \ldots\}$. Note that $Ax \neq 0$ and $\lim_{n\to\infty} \alpha_n = 0$. We may assume that $\lim_{n\to\infty} \xi_n = \xi$. Thus, $\varphi(A)x = \xi Ax$, which means that Ax and $\varphi(A)x$ are linearly dependent. Consequently, we have shown that for all $x \in \mathcal{H}$, Ax and $\varphi(A)x$ are linearly dependent. It follows that one of the following assertions holds from Theorem 2.3 in [2].

(i) A and $\varphi(A)$ are linearly dependent.
(ii) There are vectors $x_0, y_1, y_2 \in \mathcal{H}$ such that $A = x_0 \otimes y_1$ and $\varphi(A) = x_0 \otimes y_2$.

If (i) holds, then $\varphi(A) = \lambda(A)A$ for some $\lambda(A) \in \mathbb{C}$.

Next we assume that (ii) holds. If $(x_0, y_1) \neq 0$, then A is a nonzero multiple of a rank-1 idempotent. Then $\varphi(A)$ and A are linearly dependent from Lemma 2.9. Moreover, this assertion shows that $\varphi(A)$ is a rank-1 nilpotent operator if and only if A is and both A and $\varphi(A)$ have the same ranges. Assume that $(x_0, y_1) = 0$. Put $A_1 = \frac{y_1 \otimes x_0}{\|x_0\|^2 \|y_1\|^2}$. Then A_1 is nilpotent. Hence, we have $\varphi(A_1) = y_1 \otimes x_1$ for some $x_1 \in \mathcal{H}$ such that $(y_1, x_1) = 0$. Note that $A_1 A$ is a rank-1 projection. Then $\varphi(A_1)\varphi(A) = (y_1 \otimes x_1)(x_0 \otimes y_1) = (x_0, x_1) y_1 \otimes y_2$ is a nonzero projection. This means that y_1 and y_2 are linearly dependent. Thus, $\varphi(A) = \lambda(A)A$ for some constant $\lambda(A) \in \mathbb{C}$.

At last, we prove that $f(z) = z$ for any $z \in \mathbb{C}$ and $\lambda(A) = 1$ for all $A \in \mathcal{B}(\mathcal{H})$. Let E and F be any rank-1 projections such that $EF = 0$. Then we have $\varphi(E + zF) = E + f(z)F = \lambda(E + zF)(E + zF)$. It follows that $\lambda(E + zF) = 1$ and $f(z) = z$ for any $z \in \mathbb{C}$. This also implies that $\varphi(zP) = zP$ for every rank-1 idempotent P by Lemma 2.9. For any nonzero operator $A \in \mathcal{B}(\mathcal{H})$, there is a vector $x_0 \in \mathcal{H}$ such that $(x_0, Ax_0) \neq 0$. Put $P = \frac{x_0 \otimes Ax_0}{(x_0, Ax_0)}$ and $A_1 = \frac{(x_0, Ax_0)}{\|Ax_0\|^2} P$. Then P is a rank-1 idempotent and AA_1 is a rank-1 projection. Thus, $\varphi(A)\varphi(A_1) = \lambda(A) AA_1$ is a nonzero projection. This implies that $\lambda(A) = 1$. Hence, $\varphi(A) = A, \forall A \in \mathcal{B}(\mathcal{H})$. The proof is complete. □

3. Maps preserving operator pairs whose triple Jordan products are nonzero projections

Let $A, B \in \mathcal{B}(\mathcal{H})$. The triple Jordan product of A and B is defined to be ABA. Very recently, some preserver problems on triple Jordan products of operators are considered by several authors (cf. [3,4,9–11]) We say that a map φ on $\mathcal{B}(\mathcal{H})$ preserves operator pairs whose triple Jordan products are nonzero projections in both directions if $\varphi(A)\varphi(B)\varphi(A)$ is a nonzero projection if and only if ABA is for any

$A, B \in \mathcal{B}(\mathcal{H})$. We consider those maps in this section. It is noted that the treatment of triple Jordan products is little different from that of products of two operators.

Theorem 3.1. *Let φ be a surjective map on $\mathcal{B}(\mathcal{H})$. Then φ preserves operator pairs whose triple Jordan products are nonzero projections in both directions if and only if there exist a unitary or an anti-unitary operator U on $\mathcal{H}$ and a constant α with $\alpha^3 = 1$ such that one of the following forms holds.*

(1) $\varphi(A) = \alpha UAU^*, \forall A \in \mathcal{B}(\mathcal{H})$;
(2) $\varphi(A) = \alpha UA^*U^*, \forall A \in \mathcal{B}(\mathcal{H})$.

It is sufficient to consider the necessity. Let φ be a surjective map on $\mathcal{B}(\mathcal{H})$ preserving operator pairs whose triple Jordan products are nonzero projections in both directions. Then φ is also injective from Lemma 2.1.

Lemma 3.2. *$\varphi(0) = 0$ and $\varphi(I) = \alpha I$ for some $\alpha \in \mathbb{C}$ such that $\alpha^3 = 1$.*

Proof. Let $A \in \mathcal{B}(\mathcal{H})$ such that $\varphi(A) = 0$. If $A \neq 0$, then there exists a unit vector $x \in \mathcal{H}$ such that $(Ax, x) \neq 0$. Let $B = \frac{x \otimes x}{\sqrt{(Ax,x)}}$. It is clear that $BAB = x \otimes x$. Then $\varphi(B)\varphi(A)\varphi(B)$ is a nonzero projection. However, $\varphi(B)\varphi(A)\varphi(B) = 0$, a contradiction. Thus, $\varphi(0) = 0$.

On the other hand, suppose $\varphi(I) = A$. We claim that $A \in \mathbb{C}I$. In fact we have $A^3 = E$ is a nonzero projection. If $E \neq I$, then $AE = EA$. Let $\mathcal{H} = E\mathcal{H} \oplus (I - E)\mathcal{H}$. Then $A = \begin{pmatrix} A_{11} & 0 \\ 0 & A_{22} \end{pmatrix}$ with $A_{11}^3 = I$ and $A_{22}^3 = 0$. Let P be the projection from $(I - E)\mathcal{H}$ onto the kernel of A_{22}. Then $P \neq 0$ and $A_{22}PA_{22} = 0$. Let $B_z = A_{11} \oplus zP$ for all $z \in \mathbb{C}$. Note that $AB_zA = E \neq 0$. Then $\varphi^{-1}(A)\varphi^{-1}(B_z)\varphi^{-1}(A) = \varphi^{-1}(B_z)$ is a nonzero projection. Thus, $B_z^3 = A_{11}^3 \oplus z^3P$ is a nonzero projection. This is a contradiction. It follows that $E = I$ and A is invertible with $A^3 = I$.

We show that $A = \alpha I$ for some constants $\alpha \in \mathbb{C}$, $\alpha^3 = 1$. For any unit vector $x \in \mathcal{H}$, there is a nonzero vector $y \in \mathcal{H}$ such that $Ax = A^*y$ since A is invertible. Put $B = \frac{x \otimes y}{\|Ax\|}$. Then $ABA = \frac{Ax \otimes Ax}{\|Ax\|^2}$ is a rank-1 projection, which implies that $\varphi^{-1}(A)\varphi^{-1}(B)\varphi^{-1}(A) = \varphi^{-1}(B)$ is a nonzero projection. Therefore $B^3 = \frac{\|Ax\|^3}{(x,y)} x \otimes y$ is a nonzero projection. It follows that $y = \alpha_x x$ for some nonzero constant $\alpha_x \in \mathbb{C}$. Hence, $A^*y = \alpha_x A^*x = Ax$. Note that A is invertible. Then it follows that $A^* = \alpha A$ for some constant $\alpha \in \mathbb{C}$ from Theorem 2.3 in [2]. Thus, A is normal such that $\sigma(A) \subseteq \{z : z^3 = 1\}$, that is A is unitary. Therefore $A^*A = \alpha A^2 = I$, which implies that $A = \alpha A^3 = \alpha I$. The proof is complete. $\square$

If $\alpha^3 = 1$ and $\alpha \neq 1$, then $\bar{\alpha}\varphi$ preserves operator pairs whose triple Jordan product are nonzero projections in both directions such that $\bar{\alpha}\varphi(I) = I$. Without loss of generality, we may assume that $\varphi(I) = I$. Then φ preserves nonzero projections in both directions.

Lemma 3.3. *Suppose $\varphi(I) = I$. Then φ preserves rank-n projections in both directions.*

Proof. Just as in the proof of lemma 2.5, we show that φ preserves rank-1 projections in both directions.

Let $E = e \otimes e$ for some unit vector $e \in \mathcal{H}$, then $P = \varphi(E)$ is a nonzero projection. Put $A(z) = E + z(I - E)$ for any $z \in \mathbb{C}\backslash\{0, 1\}$. Let $B(z) = \varphi(A(z))$. Since $EA(z)E = E$ is a nonzero projection, $PB(z)P$ is also a nonzero projection. Under the direct sum decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we have $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$ and $B(z) = \begin{pmatrix} B_{11}(z) & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{pmatrix}$. Then $PB(z)P = P(z) = \begin{pmatrix} B_{11}(z) & 0 \\ 0 & 0 \end{pmatrix}$ is a nonzero projection. It follows that $E(z) = \varphi^{-1}(P(z))$ is a nonzero projection, too. Note that $EE(z)E$ is a nonzero projection since $PP(z)P$ is. Then $E \leqslant E(z)$. On the other hand, $P(z)B(z)P(z) = P(z)$. Then $E(z)A(z)E(z) = E + z(E(z) - E)$ is a nonzero projection. We now have $E(z) = E$ and therefore $P(z) = P$. That is $B_{11}(z) = I$. Again, $A(z)EA(z) = E$. Then

$$\begin{aligned} B(z)PB(z) &= \begin{pmatrix} I & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} I & B_{12}(z) \\ B_{21}(z) & B_{22}(z) \end{pmatrix} \\ &= \begin{pmatrix} I & B_{12}(z) \\ B_{21}(z) & B_{21}(z)B_{12}(z) \end{pmatrix} \end{aligned}$$

is a nonzero projection. This shows that $B_{12}(z) = 0$ and $B_{21}(z) = 0$. Take any rank-1 projection Q with $Q \leqslant P$. Then $QPQ = Q$. It follows that $\varphi^{-1}(Q)E\varphi^{-1}(Q)$ is a nonzero projection, which implies that $\varphi^{-1}(Q) \geqslant E$. Again it follows that $\varphi^{-1}(Q)A(z)\varphi^{-1}(Q) = E + z(\varphi^{-1}(Q) - E)$ is also a nonzero projection from the fact that $QB(z)Q = Q$. Hence, $\varphi^{-1}(Q) = E$ and $P = Q$ is of rank-1.

By a similar method (Steps 2 and 3) used in the proof of Lemma 2.5, we may show φ preserves rank-n projections in both directions by induction. The proof is complete. □

Lemma 3.4. *Suppose $\varphi(I) = I$. Then φ preserves the order as well as the orthogonality of projections in both directions.*

Proof. Let E is a nonzero projection and $\varphi(E) = P$. For any nonzero projection $F \leqslant E$, Put $Q = \varphi(F)$. Note that $FEF = EFE = F$. Then both $\varphi(F)\varphi(E)\varphi(F) = \varphi(F)P\varphi(F)$ and $\varphi(E)\varphi(F)\varphi(E) = P\varphi(F)P$ are nonzero projections. Under the direct sum decomposition $\mathcal{H} = P\mathcal{H} \oplus P^{\perp}\mathcal{H}$, we have

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ and } \varphi(F) = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix}.$$

It follows that Q_{11} is a nonzero projection. We easily have $Q_{12} = 0$ since Q is a projection. Thus, Q_{22} is also a projection. We next claim that $Q_{22} = 0$. If $Q_{22} \neq 0$, then there is a rank-1 projection $Q_0 \leqslant Q_{22}$ such that $QQ_0Q = Q_0$. We then have that $\varphi^{-1}(Q_0)$ is a rank-1 projection such that $F\varphi^{-1}(Q_0)F = E\varphi^{-1}(Q_0)E = \varphi^{-1}(Q_0) \neq 0$. However, $PQ_0P = 0$. This is a contraction. Hence, $Q \leqslant P$. That is, φ preserves the order of projections. We note that φ^{-1} has the same properties. Thus, φ preserves the order of projections in both directions.

To prove the second assertion, we claim a fact: If P and Q are projections such that both PQP and QPQ are projections, then $QP = PQ$. In fact, put $\mathcal{H} = P\mathcal{H} \oplus (I - P)\mathcal{H}$ and $P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}$. Let $Q = \begin{pmatrix} Q_{11} & Q_{12} \\ Q_{12}^* & Q_{22} \end{pmatrix}$. Then Q_{11} is a projection. Note that $QPQ = \begin{pmatrix} Q_{11} & Q_{11}Q_{12} \\ Q_{12}^*Q_{11} & Q_{12}^*Q_{12} \end{pmatrix}$. It follows that $Q_{11}Q_{12} = 0$ from the fact that QPQ is a projection. This means that $PQP \leqslant QPQ$. By the symmetry, we have that $PQP = QPQ$. Moreover, $P - QPQ = P - PQP \geqslant 0$. Then $Q(P - QPQ)Q = 0$. It follows that $(P - QPQ)Q = 0$, that is $PQ = QPQ$. Thus, $PQ = QP$. Now we may use the proof of Lemma 2.6 to complete the proof if $\dim \mathcal{H} \geqslant 3$.

Suppose $\dim \mathcal{H} = 2$. It is known that φ preserves rank-1 projections in both directions from Lemma 3.3. Let $E = e \otimes e$ and $F = f \otimes f$ be two rank-1 projections such that $EF = 0$ and $E + F = I$. Without loss of generality, we may assume that $\varphi(E) = E$. We claim that $\varphi(T)^2 = I$ if $T^2 = I$. If $\varphi(T)^2 = Q$ is a rank-1 projection, then $\varphi(T) = \alpha Q$ for some α with $\alpha^2 = 1$ and $T\varphi^{-1}(Q)T$ is also a rank-1 projection. Thus, so is $I - T\varphi^{-1}(Q)T = T(I - \varphi^{-1}(Q))T$. It follows that both $\varphi(I - \varphi^{-1}(Q))$ and $\varphi(T)\varphi(I - \varphi^{-1}(Q))\varphi(T) = Q\varphi(I - \varphi^{-1}(Q))Q$ must be rank-1 projections. Hence, $\varphi(I - \varphi^{-1}(Q)) = Q$. This is a contradiction. Thus, $\varphi(T)^2 = I$.

If $\varphi(F) \neq F$, then there is a nonzero constant z such that $\varphi(E(z)) = F$, where $E(z) = \frac{1}{1+|z|^2}\begin{pmatrix} 1 & z \\ \bar{z} & |z|^2 \end{pmatrix}$ is defined in the proof of Lemma 2.7. Put $P(z) = \begin{pmatrix} 1 & z \\ 0 & 0 \end{pmatrix}$ and $T = \begin{pmatrix} 1 & z \\ 0 & -1 \end{pmatrix}$. Then $T^2 = I$ such that $TPT = E$. It is known that $\varphi(P) = \begin{pmatrix} 1 & a_{12} \\ a_{21} & 1 \end{pmatrix}$ by considering $EPE = E$ and $E(z)PE(z) = E(z)$. On the other hand, let $\varphi(T) = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}$. Then $b_{11} = 1$, $b_{12}b_{21} = 0$ and $b_{22}^2 = 1$ since $ETE = E$ and $\varphi(T)^2 = I$.

Suppose $b_{21} = 0$. Then $b_{22} = -1$. Note that

$$\varphi(T)\varphi(P)\varphi(T) = \begin{pmatrix} 1 + b_{12}a_{21} & b_{12}^2 a_{21} - a_{12} \\ -a_{21} & 1 - b_{12}a_{21} \end{pmatrix}$$

is a projection. Then we have that $b_{12}a_{21}$ is real such that both $1 + b_{12}a_{21}$ and $1 - b_{12}a_{21}$ are in $[0, 1]$. This is impossible unless $b_{12}a_{21} = 0$. Whenever either $a_{21} = 0$ or $b_{12} = 0$, it does follows that $\varphi(P) = I$. This is a contradiction. If $b_{12} = 0$, we can induce a contradiction, too. Hence, $\varphi(F) = F$. Therefore φ preserves orthogonality of projections in both directions. The proof is complete. $\square$

We again have that φ is a bijection on the set of all projections of $\mathcal{B}(\mathcal{H})$ preserving orthogonality in both directions and there is a unitary or an anti-unitary operator U on $\mathcal{H}$ such that

$$\varphi(E) = UEU^*$$

for any projection $E \in \mathcal{B}(\mathcal{H})$ from the Uhlhorn's theorem in [18] if $\dim \mathcal{H} \geqslant 3$.

Lemma 3.5. *Suppose that* $\dim \mathcal{H} \geqslant 3$ *and* $\varphi(E) = E$ *for any projection* $E \in \mathcal{B}(\mathcal{H})$. *Then there is a complex function* $f(z)$ *on* $\mathbb{C}$ *such that* $\varphi(zE) = f(z)E$ *for any projection* $E \in \mathcal{B}(\mathcal{H})$.

Proof. First let $e \in \mathcal{H}$ be any unit vector. Let $E = e \otimes e$ be a rank-1 projection and take any nonzero projections P and Q such that $I = P \oplus E \oplus Q$. Let $\mathcal{H} = P(\mathcal{H}) \oplus E(\mathcal{H}) \oplus Q(\mathcal{H})$. We claim that there is a complex function $f_e(z)$ on $\mathbb{C} \setminus \{0, 1\}$ such that $\varphi(P + zE) = P + f_e(z)E$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. Let

$$\varphi(P + zE) = \begin{pmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \\ B_{31} & B_{32} & B_{33} \end{pmatrix}.$$

As in the proof of Lemma 2.7, we know that for any nonzero projection $P_0 \leqslant P$, we have $P_0(P + zE)P_0 = P_0$, which implies that $P_0 B_{11} P_0$ is a nonzero projection. It easily follows that $B_{11} = I$. Again we have $(P + zE)P(P + zE) = P$. An elementary calculation shows that the nonzero projection $\varphi(P + zE)P\varphi(P + zE)$ is of the form

$$\begin{pmatrix} I & B_{12} & B_{13} \\ B_{21} & * & * \\ B_{31} & * & * \end{pmatrix}.$$

Then $B_{12} = 0$, $B_{13} = 0$, $B_{21} = 0$ and $B_{31} = 0$. On the other hand, we have $(P + Q)(P + zE)(P + Q) = P$. It follows that B_{33} is a projection. If $B_{33} \neq 0$ then $Q\varphi(P + zE)Q = 0 \oplus 0 \oplus B_{33}$ is a nonzero projection. However, $Q(P + zE)Q = 0$. This is a contradiction. Hence, $B_{33} = 0$. It is trivial that B_{22} is a scalar dependent on z. Thus,

$$\varphi(P + zE) = \begin{pmatrix} I & 0 & 0 \\ 0 & B_{22} & B_{23} \\ 0 & B_{32} & 0 \end{pmatrix}.$$

Similarly we have for any $w \in \mathbb{C} \setminus \{0, 1\}$,

$$\varphi(Q + wE) = \begin{pmatrix} 0 & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I \end{pmatrix}$$

with $A_{22} \neq 0$. For any $z \neq 0$, we have that $\left(\frac{1}{\sqrt{z}}E + Q\right)(P + zE)\left(\frac{1}{\sqrt{z}}E + Q\right) = E$. Again by an elementary calculation, we have

$$\varphi\left(\frac{1}{\sqrt{z}}E + Q\right)\varphi(P + zE)\varphi\left(\frac{1}{\sqrt{z}}E + Q\right) = \begin{pmatrix} * & * & * \\ * & * & A_{22}B_{23} \\ * & B_{32}A_{22} & 0 \end{pmatrix}.$$

Note that E is of rank-1 and $A_{22} \neq 0$. Then $B_{32} = 0$ and $B_{23} = 0$. Set $f_e(z) = B_{22}$. Thus, $\varphi(P + zE) = P + f_e(z)E$. We also easily get

$$f_e(z)\left(f_e\left(\frac{1}{\sqrt{z}}\right)\right)^2 = 1 \tag{3.1}$$

for all $z \in \mathbb{C} \setminus \{0, 1\}$. Similarly we have that $\varphi(zE + Q) = f_e(z)E + Q$.

On the other hand, we can similarly show that $\varphi(P + zE + Q) = P + g_e(z)E + Q$ for some constant $g_e(z) \in \mathcal{C} \setminus \{0, 1\}$ by use of the facts that $F(P + zE + Q)F = F$ for any nonzero projection $F \leqslant P + Q$ and $(P + zE + Q)(P + Q)(P + zE + Q) = P + Q$. Note that $\left(P + \frac{1}{\sqrt{z}}E\right)(P + zE + Q)\left(P + \frac{1}{\sqrt{z}}E\right) = P + E$. It easily follows that $f_e(z) = g_e(z)$.

Put

$$\varphi(zE) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}$$

and $w = f_e\left(\frac{1}{\sqrt{z}}\right)$. Note that $\left(P + \frac{1}{\sqrt{z}}E + Q\right)(zE)\left(P + \frac{1}{\sqrt{z}}E + Q\right) = \left(P + \frac{1}{\sqrt{z}}E\right)(zE)\left(P + \frac{1}{\sqrt{z}}E\right) = E$, which implies that both $(P + wE + Q)\varphi(zE)(P + wE + Q)$ and $(P + wE)\varphi(zE)(P + wE)$ are nonzero projections. Now

$$(P + wE + Q)\varphi(zE)(P + wE + Q) = \begin{pmatrix} C_{11} & wC_{12} & C_{13} \\ wC_{21} & w^2C_{22} & wC_{23} \\ C_{31} & wC_{32} & C_{33} \end{pmatrix} \tag{3.2}$$

and

$$(P + wE)\varphi(zE)(P + wE) = \begin{pmatrix} C_{11} & wC_{12} & 0 \\ wC_{21} & w^2C_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{3.3}$$

Then $C_{11} \geqslant 0$, $wC_{21} = (wC_{12})^*$, $C_{31} = C_{13}^*$ and $wC_{32} = (wC_{23})^*$ by (3.2) and (3.3). It follows that both C_{13} and C_{23} are 0 from the facts that $C_{11} = C_{11}^2 + |w|^2C_{12}C_{12}^* + C_{13}C_{13}^* = C_{11}^2 + |w|^2C_{12}C_{12}^*$ and $w^2C_{22} = |w|^2C_{12}C_{12}^* + |w|^4C_{22}C_{22}^* + |w|^2C_{23}C_{23}^*$. We also have C_{33} is a projection from (3.2). Thus, $C_{33} = 0$ since $Q(zE)Q = 0$. We symmetrically have $C_{11} = 0, C_{12} = 0, C_{21} = 0$ and $w^2C_{22} = 1$, that is, $f_e(\frac{1}{\sqrt{z}})^2C_{22} = 1$ and $\varphi(zE) = C_{22}E = f_e(z)E$ from (3.1).

Put $\varphi(zI) = A_z$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. It is known that $A_z\left(f_e\left(\frac{1}{z^2}\right)E\right)A_z$ is a nonzero projection since $(zI)\left(\frac{1}{z^2}E\right)(zI) = E$. Note that

$$A_z\left(f_e\left(\frac{1}{z^2}\right)E\right)A_z = f_e\left(\frac{1}{z^2}\right)(A_ze \otimes A_z^*e).$$

It follows that $\left|f_e\left(\frac{1}{z^2}\right)\right| \|A_ze\|\|A_z^*e\| = 1$ and

$$A_z^*e = \left(\frac{1}{\overline{f_e\left(\frac{1}{z^2}\right)}\|A_ze\|^2}\right)A_ze \neq 0 \tag{3.4}$$

for any unit vector $e \in \mathcal{H}$. On the other hand, $\left(\frac{1}{\sqrt{z}}E\right)(zI)\left(\frac{1}{\sqrt{z}}E\right) = E$. It follows that $\left(f_e\left(\frac{1}{\sqrt{z}}\right)E\right)A_z\left(f_e\left(\frac{1}{\sqrt{z}}\right)E\right)$ is a nonzero projection. That is, $\left(f_e\left(\frac{1}{\sqrt{z}}\right)\right)^2(A_ze, e) = 1$. Thus,

$$(A_ze, e) = \frac{1}{\left(f_e\left(\frac{1}{\sqrt{z}}\right)\right)^2} = f_e(z) \neq 0 \tag{3.5}$$

for any unit vector $e \in \mathcal{H}$ form (3.1). We recall that the numerical range of an operator $A \in \mathcal{B}(\mathcal{H})$ is the set $W(A) = \{(Ax, x) : x \in \mathcal{H},\ \|x\| = 1\}$(cf. [7]). It follows that $0 \notin W(A_z) = \{f_e(z) : e \in \mathcal{H},\ \|e\| = 1\}$

and therefore A_z is injective. Moreover, $A_{\frac{1}{\sqrt{z}}} A_z A_{\frac{1}{\sqrt{z}}}$ is a nonzero projection since $\left(\frac{1}{\sqrt{z}}I\right)(zI)\left(\frac{1}{\sqrt{z}}I\right) = I$. It immediately follows that $A_{\frac{1}{\sqrt{z}}} A_z A_{\frac{1}{\sqrt{z}}} = I$ and A_z is invertible. Thus,

$$A_z = \left(A_{\frac{1}{\sqrt{z}}}\right)^{-2}. \tag{3.6}$$

From (3.4) we know that $A_z^* = \lambda_z A_z$ for some constant $\lambda_z \in \mathbb{C}$ by Theorem 2.3 in [2], that is, $\lambda_z = \frac{1}{f_e\left(\frac{1}{z^2}\right)\|A_z e\|^2}$ is independent on e. Thus, for any $z \in \mathbb{C} \setminus \{0, 1\}$,

$$\overline{\lambda_{\frac{1}{\sqrt{z}}}} f_e(z) = \lambda_{\frac{1}{\sqrt{z}}} \overline{f_e(z)} = \frac{1}{\|A_{\frac{1}{\sqrt{z}}} e\|^2} \geqslant 0.$$

Therefore $\overline{\lambda_{\frac{1}{\sqrt{z}}}} A_z$ is a positive operator for any $z \in \mathbb{C} \setminus \{0, 1\}$. Let β be the constant such that $D = \beta A_{\frac{1}{\sqrt{z}}} \geqslant 0$. We know that

$$\left(f_e\left(\frac{1}{\sqrt{z}}\right)\right)^2 f_e(z) = \left(A_{\frac{1}{\sqrt{z}}} e, e\right)^2 (A_z e, e) = \left(A_{\frac{1}{\sqrt{z}}} e, e\right)^2 \left(\left(A_{\frac{1}{\sqrt{z}}}\right)^{-2} e, e\right) = 1 \tag{3.7}$$

for any unit vector $e \in \mathcal{H}$ from (3.1). Then D is an invertible positive operator such that $(De, e)^2 (D^{-2} e, e) = 1$ for any unit vector $e \in \mathcal{H}$ from (3.7). We next claim that $D = dI$ for some positive constant $d \in \mathbb{C}$. In fact, for any unit vector $e \in \mathcal{H}$, $(De, e)^2 (D^{-2} e, e) = (De, e)^2 \|D^{-1} e\|^2 = 1$. Then $(De, e)\|D^{-1} e\| = 1$. Thus, $(De, e)(D^{-1} e, e) \leqslant (De, e)\|D^{-1} e\| = 1$. On the other hand, $1 = (e, e)^2 = \left(D^{\frac{1}{2}} e, D^{-\frac{1}{2}} e\right)^2 \leqslant \left\|D^{\frac{1}{2}} e\right\|^2 \left\|D^{-\frac{1}{2}} e\right\|^2 = (De, e)(D^{-1} e, e) \leqslant 1$. It follows that

$$(De, e)(D^{-1} e, e) = (e, e) = 1$$

for any unit vector $e \in \mathcal{H}$. Thus, $D = dI$ for some positive constant d from Theorem 3.4 in [6].

Therefore A_z is a multiple of I and $f_e(z)$ is a constant $f(z)$ independent on e for any $z \in \mathbb{C} \setminus \{0, 1\}$. In particular, $\varphi(zI) = f(z)I$ and $\varphi(E) = f(z)E$ for any $z \in \mathbb{C} \setminus \{0, 1\}$ and any rank-1 projection E.

Let F be any nonzero projection and set $\varphi(zF) = F_z$ for any $z \in \mathbb{C} \setminus \{0, 1\}$. It follows that $F_z = f(z)P_z$ for some nonzero projection P_z from the fact that $\left(\frac{1}{\sqrt{z}}I\right)(zF)\left(\frac{1}{\sqrt{z}}I\right) = F$ and (3.1). Take any rank-1 projection $E \leqslant F$. We have that $\left(\frac{1}{\sqrt{z}}E\right)(zF)\left(\frac{1}{\sqrt{z}}E\right) = E$. Then

$$\varphi\left(\left(\frac{1}{\sqrt{z}}E\right)\right)\varphi(zF)\varphi\left(\frac{1}{\sqrt{z}}E\right) = \left(f\left(\frac{1}{\sqrt{z}}\right)E\right)(f(z)P_z)\left(f\left(\frac{1}{\sqrt{z}}\right)E\right) = EP_zE$$

is a nonzero projection. Hence, $E \leqslant P_z$. Thus, we have $F \leqslant P_z$. In fact we must have $F = P_z$. Otherwise, if there is a rank-1 projection $E \leqslant P_z$ such that $EF = FE = 0$, then $\varphi^{-1}\left(f\left(\frac{1}{\sqrt{z}}\right)E\right)\varphi^{-1}(f(z)P_z)\varphi^{-1}\left(f\left(\frac{1}{\sqrt{z}}\right)E\right) = \left(\frac{1}{\sqrt{z}}E\right)(zF)\left(\frac{1}{\sqrt{z}}E\right) = 0$. However, $\left(f\left(\frac{1}{\sqrt{z}}\right)E\right)(f(z)P_z)\left(f\left(\frac{1}{\sqrt{z}}\right)E\right) = E$ is a rank-1 projection. This is a contradiction. Hence, $P_z = F$ and $\varphi(zF) = f(z)F$ for any projection F and any $z \in \mathbb{C} \setminus \{0, 1\}$.

We define $f(0) = 0$ and $f(1) = 1$ again, then f is a complex function on $\mathbb{C}$ such that $\varphi(zE) = f(z)E$ for any projection E and $z \in \mathbb{C}$. The proof is complete. ☐

Proof of Theorem 3.1. We firstly suppose that $\dim \mathcal{H} \geqslant 3$. By use of the Uhlhorn's theorem in [18], we may assume that $\varphi(E) = E$ for any projection. Then $\varphi(zE) = f(z)E$ for any projection from Lemma 3.5. We claim that $\varphi(zE + wF) = f(z)E + f(w)F$ for any nonzero projections E and F with $EF = 0$ and any

$z, w \in \mathbb{C}$. It is trivial that the claim holds if either $zw = 0$ or $z = w$. Thus, we may assume that $z \neq w$ and $zw \neq 0$ next. Put $P = I - (E + F)$, $\mathcal{H} = E\mathcal{H} \oplus F\mathcal{H} \oplus P\mathcal{H}$ and

$$\varphi(zE + wF) = \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}.$$

It is easily shown that $C_{11} = f(z)I$ and $C_{22} = f(w)I$ by the facts $\left(\frac{1}{\sqrt{z}}E\right)(zE + wF)\left(\frac{1}{\sqrt{z}}E\right) = E$ and $\left(\frac{1}{\sqrt{w}}F\right)(zE + wF)\left(\frac{1}{\sqrt{w}}F\right) = F$. Again we note that $\left(\frac{1}{\sqrt{z}}(E + P)\right)(zE + wF)\left(\frac{1}{\sqrt{z}}(E + P)\right) = E$ and

$$\begin{aligned} &\varphi\left(\tfrac{1}{\sqrt{z}}(E + P)\right)\varphi(zE + wF)\varphi\left(\tfrac{1}{\sqrt{z}}(E + P)\right) \\ &\quad = f\left(\tfrac{1}{\sqrt{z}}\right)(E + P)\varphi(zE + wF)f\left(\tfrac{1}{\sqrt{z}}\right)(E + P) \\ &\quad = f\left(\tfrac{1}{\sqrt{z}}\right)^2 \begin{pmatrix} C_{11} & 0 & C_{13} \\ 0 & 0 & 0 \\ C_{31} & 0 & C_{33} \end{pmatrix} = \begin{pmatrix} 1 & 0 & f\left(\frac{1}{\sqrt{z}}\right)^2 C_{13} \\ 0 & 0 & 0 \\ f\left(\frac{1}{\sqrt{z}}\right)^2 C_{31} & 0 & f\left(\frac{1}{\sqrt{z}}\right)^2 C_{33} \end{pmatrix} \end{aligned}$$

by (3.1). Thus, $C_{13} = 0$, $C_{31} = 0$ and $f\left(\frac{1}{\sqrt{z}}\right)^2 C_{33}$ is a projection. It is known that $C_{33} = 0$ by a simple calculation. Then we also have $C_{23} = 0$ and $C_{32} = 0$. Moreover, $(zE + wF)\left(\frac{1}{z^2}E\right)(zE + wF) = E$ and $f(z)^2 f\left(\frac{1}{z^2}\right) = 1$ by (3.1). Note that

$$\varphi(zE + wF)\left(f\left(\frac{1}{z^2}\right)E\right)\varphi(zE + wF) = \begin{pmatrix} 1 & f(z)f\left(\frac{1}{z^2}\right)C_{12} & 0 \\ f(z)f\left(\frac{1}{z^2}\right)C_{21} & f\left(\frac{1}{z^2}\right)C_{21}C_{12} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Then $C_{12} = 0$ and $C_{21} = 0$. That is $\varphi(zE + wF) = f(z)E + f(w)F$.

Let $A \in \mathcal{B}(\mathcal{H})$. For any unit vector $x \in \mathcal{H}$, if $(Ax, x) \neq 0$, then $\frac{x\otimes x}{\sqrt{(Ax,x)}}A\frac{x\otimes x}{\sqrt{(Ax,x)}} = x \otimes x$. This implies that $\varphi\left(\frac{x\otimes x}{\sqrt{(Ax,x)}}\right)\varphi(A)\varphi\left(\frac{x\otimes x}{\sqrt{(Ax,x)}}\right) = \left(f\left(\frac{1}{\sqrt{(Ax,x)}}\right)\right)^2(\varphi(A)x, x)(x \otimes x)$ is a nonzero projection. It follows that $\left(f\left(\frac{1}{\sqrt{(Ax,x)}}\right)\right)^2(\varphi(A)x, x) = 1$ and Thus, $f((Ax, x)) = (\varphi(A)x, x)$ from (3.1). If $(Ax, x) = 0$, then we know $(\varphi(A)x, x) = 0$ by considering φ^{-1}. Therefore,

$$f((Ax, x)) = (\varphi(A)x, x) \tag{3.8}$$

for any unit vector $x \in \mathcal{H}$. In particular, $f((Ex, x)) = (Ex, x)$ for any projection E, which implies that $f(\mu) = \mu$ for all $0 \leqslant \mu \leqslant 1$. Take any unit vectors $e, f \in \mathcal{H}$ such that $(e, f) = 0$. Set $E = e \otimes e$, $F = f \otimes f$ and $A = zE + wF$ for all $z, w \in \mathbb{C}$. For any $0 < \lambda < 1$, put $x = \sqrt{\lambda}e + \sqrt{1 - \lambda}f$. Then $\|x\| = 1$ and $(Ax, x) = \lambda z + (1 - \lambda)w$ and $(\varphi(A)x, x) = \lambda f(z) + (1 - \lambda)f(w)$. Then we have

$$f(\lambda z + (1 - \lambda)w) = \lambda f(z) + (1 - \lambda)f(w) \tag{3.9}$$

for all $z, w \in \mathbb{C}$ by (3.8). For any real number $a > 1$, we have that $\left(1 - \frac{1}{a}\right)0 + \frac{1}{a}a = 1$. It now easily follows that $f(a) = a$ from (3.9). Similarly, $f(a) = a$ and $f(ai) = af(i)$ for any real number a. On the other hand, we have $(f(i))^2 = -1$ from (3.1). Then $f(i) = i$ or $f(i) = -i$.

Case 1. $f(i) = i$. Then we easily have $f(z) = z$ for all $z \in \mathbb{C}$ from (3.9). It follows that $(Ax, x) = (\varphi(A)x, x)$ for any unit vector $x \in \mathcal{H}$. Thus, $\varphi(A) = A$ for all $A \in \mathcal{B}(\mathcal{H})$.

Case 2. $f(i) = -i$. Then we easily have $f(z) = \bar{z}$ for all $z \in \mathbb{C}$ from (3.9). It follows that $(\varphi(A)x, x) = \overline{(Ax, x)} = (A^*x, x)$ for any unit vector $x \in \mathcal{H}$. Thus, $\varphi(A) = A^*$ for all $A \in \mathcal{B}(\mathcal{H})$.

If $\dim \mathcal{H} = 2$, then by an elementary treatment, we can show that φ satisfies the condition of the Wigner's fundamental theorem (cf. [13,19]). For completeness, we give the elementary proof. In fact, we claim that there is a function on $\mathbb{C}$ such that $\varphi(zE) = f(z)\varphi(E)$ for any projection E. Let $E = e \otimes e$ and $F = f \otimes f$ be arbitrary two rank-1 projections such that $EF = 0$. By Lemma 3.4, $\varphi(E)$ and $\varphi(F)$ are orthogonal such that $\varphi(E) + \varphi(F) = I$. We may assume that $\varphi(E) = E$ and $\varphi(F) = F$. There is a function f_e on $\mathbb{C} \setminus \{0, 1\}$ satisfying (3.1) such that $\varphi(zE + F) = f_e(z)E + F$ for any $z \in \mathbb{C} \setminus \{0, 1\}$ by considering $F(zE + F)F = (zE + F)F(zE + F) = F$. Put $\varphi(zE) = \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix}$. It is known that $\varphi(zE)$ is of rank-1 and $x_{22} \geqslant 0$ from the fact that $\left(\frac{1}{\sqrt{z}}E + F\right)(zE)\left(\frac{1}{\sqrt{z}}E + F\right) = E$. If $x_{22} > 0$, then $\left(\frac{1}{\sqrt{x_{22}}}F\right)\varphi(zE)\left(\frac{1}{\sqrt{x_{22}}}F\right) = \left(E + (x_{22})^{\frac{1}{4}}F\right)\left(\frac{1}{\sqrt{x_{22}}}F\right)\left(E + (x_{22})^{\frac{1}{4}}F\right) = F$. Put $A = \varphi^{-1}\left(\frac{1}{\sqrt{x_{22}}}F\right) = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ and $B = \varphi^{-1}\left(E + (x_{22})^{\frac{1}{4}}F\right)$. Then $B = E + bF$ for some nonzero constant b just as we have proved by considering φ^{-1}. Thus, both $A(zE)A = \begin{pmatrix} (a_{11})^2 z & a_{11}a_{12}z \\ a_{11}a_{21}z & a_{21}a_{12}z \end{pmatrix}$ and $BAB = \begin{pmatrix} a_{11} & a_{12}b \\ a_{21}b & a_{22}b^2 \end{pmatrix}$ are nonzero projections, which implies that $a_{11} > 0$ and therefore $z > 0$. This means that there is a constant $f_e(z)$ such that $\varphi(zE) = f_e(z)E$ for any non-positive complex number z. By considering φ^{-1} and replacing E by F, we also have that $\varphi^{-1}(wF) = g_f(w)F$ for any non-positive complex number w. In particular, $\varphi^{-1}\left(-\frac{1}{\sqrt{x_{22}}}F\right) = g_f\left(-\frac{1}{\sqrt{x_{22}}}\right)F$. However, $\left(-\frac{1}{\sqrt{x_{22}}}F\right)\varphi(zE)\left(-\frac{1}{\sqrt{x_{22}}}F\right) = F$ while $\varphi^{-1}\left(-\frac{1}{\sqrt{x_{22}}}F\right)(zE)\varphi^{-1}\left(-\frac{1}{\sqrt{x_{22}}}F\right) = 0$. This contradiction shows that there is a constant $f_e(z)$ such that $\varphi(zE) = f_e(z)E$ for any nonzero complex number z. As in the proof of Lemma 3.5, we know that $f_e(z)$ is a constant $f(z)$ independent on e and $\varphi(zE) = f(z)\varphi(E)$ for any $z \in \mathbb{C}$. Again by considering $\varphi(zE + wF) = f(z)\varphi(E) + f(w)\varphi(F)$, it follows that $f(a) = a$ for all $a \in (0, +\infty)$ as above. As in the proof of Lemma 2.7, put $\varphi(E(z)) = E(g(z))$ for any z. Since $E((1 + |z|^2)E(z))E = E$, $Ef(1 + |z|^2)\varphi(E(z))E = \frac{1+|z|^2}{1+|g(z)|^2}E$ is a nonzero projection, which implies that $|g(z)| = |z|$. Thus, $\operatorname{tr}(EE(z)) = \operatorname{tr}(\varphi(E)\varphi(E(z)))$. By the Wigner's fundamental theorem (cf. [13,19]), there is a unitary or an anti-unitary operator U on $\mathcal{H}$ such that $\varphi(E) = UEU^*$ for all rank-1 projection E. By repeating the preceding treatment when $\dim \mathcal{H} \geqslant 3$, we have the desired result. The proof is complete. $\square$

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