Enumeration of the Branched $m\mathbb{Z}_p$-Coverings of Closed Surfaces

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In this paper, we enumerate the equivalence classes of regular branched coverings of surfaces whose covering transformation groups are the direct sum of $m$ copies of $\mathbb{Z}_p$, $p$ prime.

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1. INTRODUCTION

Throughout this paper, a surface $\mathbb{S}$ means a compact connected 2-manifold without boundary. By the classification theorem of surfaces, $\mathbb{S}$ is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} 
\text{the orientable surface with } k \text{ handles} & \text{if } k \geq 0, \\
\text{the nonorientable surface with } -k \text{ crosscaps} & \text{if } k < 0.
\end{cases}$$

A continuous function $p : \tilde{\mathbb{S}} \to \mathbb{S}$ between two surfaces $\tilde{\mathbb{S}}$ and $\mathbb{S}$ is called a branched covering if there exists a finite set $B$ in $\mathbb{S}$ such that the restriction of $p$ to $\tilde{\mathbb{S}} - p^{-1}(B)$, $p | \tilde{\mathbb{S}} - p^{-1}(B) : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$, is a covering projection in the usual sense. The smallest set $B$ of $\mathbb{S}$ which has this property is called the branch set. A branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is regular if there exists a (finite) group $\mathcal{A}$ which acts pseudofreely on $\tilde{\mathbb{S}}$ so that the surface $\mathbb{S}$ is homeomorphic to the quotient space $\tilde{\mathbb{S}}/\mathcal{A}$, say by $h$, and the quotient map $\tilde{\mathbb{S}} \to \tilde{\mathbb{S}}/\mathcal{A}$ is the composition $h \circ p$ of $p$ and $h$. In this case, the group $\mathcal{A}$ is the group of covering transformations of the branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$. We call it a branched $\mathcal{A}$-covering. Two branched coverings $p : \tilde{\mathbb{S}} \to \mathbb{S}$ and $q : \tilde{\mathbb{S}}' \to \mathbb{S}$ are equivalent if there exists a homeomorphism $h : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}'$ such that $p = q \circ h$.

Since A. Hurwitz showed how to classify the branched coverings of a given surface [6], this area has been studied in [1, 2, 5] and their references. Recently, Kwak et al. enumerated the number of equivalence classes of the branched $\mathcal{A}$-coverings of surfaces, when $\mathcal{A}$ is the cyclic group $\mathbb{Z}_p$ or the dihedral group $D_p$ of order $2p$. $p$ prime [7, 9].

In this paper, we enumerate the equivalence classes of branched $\mathcal{A}$-coverings $p : \tilde{\mathbb{S}}_i \to \mathbb{S}$ with branch set $B$, when $\mathcal{A}$ is the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \equiv m\mathbb{Z}_p$, the direct sum of $m$ copies of $\mathbb{Z}_p$.

2. A CLASSIFICATION OF REGULAR BRANCHED COVERINGS

Let $G$ be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. We allow self-loops and multiple edges. Notice regarding the vertices of $G$ as 0-cells and the edges of $G$ as 1-cells, the graph $G$ can be identified with a one-dimensional CW complex in the Euclidean 3-space $\mathbb{R}^3$ so that every graph map is continuous.

A graph map $p : \tilde{G} \to G$ is said to be a regular covering (simply, $\mathcal{A}$-covering) if $p : \tilde{G} \to G$ is a covering projection in a topological sense and there is a subgroup $\mathcal{A}$ of the automorphism group $\text{Aut}(\tilde{G})$ of $\tilde{G}$ acting freely on $\tilde{G}$ so that $G$ is isomorphic to the quotient graph $\tilde{G}/\mathcal{A}$, say by $h$, and the quotient map $\tilde{G} \to \tilde{G}/\mathcal{A}$ is the composition $h \circ p$ of $p$ and $h$.

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Every regular covering of a graph $G$ can be constructed as follows [3]: every edge of a graph $G$ gives rise to a pair of edges in opposite directions. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of $G$ by $D(G)$. An $A$-voltage assignment on $G$ is a function $\phi : D(G) \to A$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of $\phi$ are called voltages. For a graph $G$ and a finite group $A$, let $C^1(G; A)$ denote the set of all $A$-voltage assignments on $G$. The ordinary derived graph $G \times_\phi A$ derived from an $A$-voltage assignment $\phi : D(G) \to A$ has as its vertex set $V(G) \times A$ and as its edge set $E(G) \times A$. For every edge $e$ between $u$ and $v$ in $G$ and $g \in A$, there is an edge from $(u, g)$ to $(v, g\phi(e))$ in the derived graph $G \times_\phi A$. In the ordinary derived graph $G \times_\phi A$, a vertex $(u, g)$ is denoted by $ug$, and an edge $(e, g)$ by $e_g$. The first coordinate projection $p_\phi : G \times_\phi A \to G$, called the natural projection, commutes with the right multiplication action of the $A$-voltage assignments on $A$ on the fibers, which is free and transitive, so that $p_\phi : G \times_\phi A \to G$ is an $A$-covering.

An embedding of a graph $G$ into the surface $S$ is a continuous one-to-one function $i : G \to S$. If every component of $S - i(G)$, called a region, is homeomorphic to an open disk, then $i : G \to S$ is called a 2-cell embedding. An embedding scheme $(\rho, \lambda)$ for a graph $G$ consists of a rotation scheme $\rho$ which assigns a cyclic permutation $\rho_v$ on $N(v) = \{e \in D(G) : v \in V(G)\}$ and a voltage assignment $\lambda$ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{-1, 1\}$ to each $e \in E(G)$.

Stahl [10] showed that every embedding scheme determines a 2-cell embedding of $G$ into an orientable or nonorientable surface $S$, and every 2-cell embedding of $G$ into a surface $S$ is determined by such a scheme.

If an embedding scheme $(\rho, \lambda)$ for a graph $G$ determines a 2-cell embedding of $G$ into a surface $S$, then the orientability of $S$ can be detected by looking at the values of cycles of $G$ under the voltage assignment $\lambda$. In fact, $S$ is orientable if and only if every cycle of $G$ is $\lambda$-trivial, that is, the number of edges $e$ with $\lambda(e) = -1$ is even in every cycle of $G$.

Let $i : G \to S$ be a 2-cell embedding with embedding scheme $(\rho, \lambda)$ and $\phi$ be an $A$-voltage assignment. The ordinary derived graph $G \times_\phi A$ has the derived embedding scheme $(\tilde{\rho}, \tilde{\lambda})$, which is defined by $\tilde{\rho}_e(e_g) = (\rho(e))(ug)$ and $\tilde{\lambda}(e_g) = \lambda(e)$ for each $e_g \in D(G \times_\phi A)$. Then it induces a 2-cell embedding $\tilde{i} : G \times_\phi A \to S^\phi$ such that the following diagram commutes.

$$
\begin{array}{ccc}
G \times_\phi A & \xrightarrow{\tilde{i}} & S^\phi \\
p_\phi \downarrow & & \downarrow \tilde{p}_\phi \\
G & \xrightarrow{i} & S
\end{array}
$$

Moreover, if $G \times_\phi A$ is connected, then $S^\phi$ is connected and $\tilde{p}_\phi : S^\phi \to S$ is a branched $A$-covering. Conversely, let $p : \tilde{S} \to S$ be a branched $A$-covering of a surface $S$. Then there exist a 2-cell embedding $i : G \to S$ such that each face of the embedding has at most one branch point interior to it and an $A$-voltage assignment $\phi : D(G) \to A$ such that the branched $A$-covering $\tilde{p}_\phi : S^\phi \to S$ is equivalent to the given branched $A$-covering $p : \tilde{S} \to S$ [4].

A surface $S_k$ can be represented by a 4k-gon with identification data $\prod_{i=1}^{k} a_i b_i a_i^{-1} b_i^{-1}$ on its boundary if $k > 0$; a bigon with identification data $a a^{-1}$ on its boundary if $k = 0$; and a $-2k$-gon with identification data $\prod_{i=1}^{-k} a_i b_i$ on its boundary if $k < 0$.

Let $B$ be a finite set of points in $S_k$. We note the fact that the fundamental group $\pi_1(S_k - B, *)$ of the punctured surface $S_k - B$ with the base point $* \in S_k - B$ can be represented by

$$\left\{a_1, \ldots, a_s, b_1, \ldots, b_k, c_1, \ldots, c_{|B|} \mid \prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1} \prod_{i=1}^{|B|} c_i = 1 \right\} \text{ if } k > 0;$$
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\[
\begin{align*}
(a_1, \ldots, a_{-k}, c_1, \ldots, c_{|B|}) &: \prod_{s=1}^{k} a_s a_s \prod_{t=1}^{|B|} c_t = 1 \quad \text{if } k < 0; \\
(c_1, \ldots, c_{|B|}) &: \prod_{t=1}^{|B|} c_t = 1 \quad \text{if } k = 0.
\end{align*}
\]

We call this the standard presentation of the fundamental group $\pi_1(S_k - B, \ast)$. For each $t = 1, 2, \ldots, |B|$, we take a simple closed curve based at $\ast$ lying in the face determined by the polygonal representation of the surface $S_k$ so that it represents the homotopy class of the generator $c_t$. Then, it induces a 2-cell embedding of a bouquet of $n$ circles, denoted by $\mathbb{B}_n$, into the surface $S_k$ such that the embedding has $|B|$ 1-sided regions and one $(|B| + 4k)$-sided region if $k > 0$; $|B|$ 1-sided regions and one $(|B| - 2k)$-sided region if $k < 0$; and $|B|$ 1-sided regions and one $|B|$-sided region if $k = 0$, where $n$ is the number of the generators of the corresponding fundamental group. We call this embedding $\iota : \mathbb{B}_n \to S_k$ the standard embedding, simply denoted by $\mathbb{B}_n \hookrightarrow S_k - B$.

Now, we identify each loop $\ell_i$ in $\mathbb{B}_n$ with a generator in $\pi_1(S_k - B)$ and this identification gives a direction on each loop of $\mathbb{B}_n$ so that a positively oriented loop $\ell_i$ represents the corresponding generator of $\pi_1(S_k - B)$.

Let $C^1(\mathbb{B}_n \hookrightarrow S_k - B; A)$ denote the subset of $C^1(\mathbb{B}_n; A)$ consisting of all $A$-voltage assignments $\phi$ of $\mathbb{B}_n$ which satisfy the following two conditions:

1. $\phi(\ell_1), \ldots, \phi(\ell_n)$ generate $A$, and
2. (i) if $k \geq 0$, then $\phi(\ell_i) \neq id_A$ for each $i = 2k + 1, \ldots, 2k + |B| = n$ and
   \[
   \prod_{i=1}^{k} \phi(\ell_i) \phi(\ell_{k+i}) \phi(\ell_i)^{-1} \phi(\ell_{k+i})^{-1} \prod_{i=1}^{|B|} \phi(\ell_{2k+i}) = 1,
   \]
   (ii) if $k < 0$, then $\phi(\ell_i) \neq id_A$ for each $i = -k + 1, \ldots, -k + |B| = n$ and
   \[
   \prod_{i=1}^{-k} \phi(\ell_i) \phi(\ell_i) \prod_{i=1}^{-|B|} \phi(\ell_{-k+i}) = 1.
   \]

Note that the condition (1) guarantees that $S^\phi$ is connected, and the condition (2) guarantees that the set $B$ is the same as the branch set of the branched covering $\tilde{p}_\phi : S^\phi \to S$.

By using these, Kwak et al. obtained the following theorem.

**Theorem 1. (Existence and Classification of Regular Branched Coverings [7].)** Let $n = 2k + |B|$ if $k \geq 0$, and let $n = -k + |B|$ if $k < 0$. Every voltage assignment in $C^1(\mathbb{B}_n \hookrightarrow S_k - B; A)$ induces a connected branched $A$-covering of $S_k$ with branch set $B$. Conversely, every connected branched $A$-covering of $S_k$ with branch set $B$ is induced by a voltage assignment in $C^1(\mathbb{B}_n \hookrightarrow S_k - B; A)$. Moreover, for two given $A$-voltage assignments $\phi, \psi \in C^1(\mathbb{B}_n \hookrightarrow S - B; A)$, two branched $A$-coverings $\tilde{p}_\phi : S^\phi \to S$ and $\tilde{p}_\psi : S^\psi \to S$ are equivalent if and only if there exists a group automorphism $\sigma : A \to A$ such that

\[
\psi(\ell_i) = \sigma(\phi(\ell_i))
\]

for all $\ell_i \in D(\mathbb{B}_n)$.

In fact, the last statement is equivalent to saying that the two graph coverings $p_\phi : \mathbb{B}_n \times_\phi A \to \mathbb{B}_n$ and $p_\psi : \mathbb{B}_n \times_\psi A \to \mathbb{B}_n$ of $\mathbb{B}_n$ are equivalent as graph coverings [7, 8].
To enumerate the equivalence classes of connected branched \( A \)-coverings of a surface \( S \), we define an \( \text{Aut}(A) \)-action on \( C^1(\mathcal{B}_n \hookrightarrow S - B; A) \) as follows:

\[
(\sigma \phi)(\ell_i) = \sigma(\phi(\ell_i)),
\]

for each \( \sigma \in \text{Aut}(A) \), \( \phi \in C^1(\mathcal{B}_n \hookrightarrow S - B; A) \) and all \( \ell_i \in D(\mathcal{B}_n) \). Notice that this \( \text{Aut}(A) \)-action on the set \( C^1(\mathcal{B}_n \hookrightarrow S - B; A) \) is free, because \( \phi(\ell_1), \ldots, \phi(\ell_n) \) generate \( A \).

**Corollary 1.** [7] Let \( A \) be a finite group. Then the number of equivalence classes of connected branched \( A \)-coverings of the surface \( S_k \) with branch set \( B \) is equal to

\[
\frac{|C^1(\mathcal{B}_n \hookrightarrow S_k - B; A)|}{|\text{Aut}(A)|},
\]

where

\[
n = \begin{cases} 2k + |B| & \text{if } k \geq 0, \\ -k + |B| & \text{if } k < 0.
\end{cases}
\]

For convenience, let \( a_k = 2k \) if \( k \geq 0 \), and \( a_k = -k \) if \( k < 0 \). Let \( b \) be the cardinality of a branch set \( B \). Thereby \( n \) in Theorem 1 and Corollary 1 is \( a_k + b \) for each integer \( k \).

3. **Enumeration of Branched \( m\mathbb{Z}_p \)-Coverings of Orientable Surfaces**

In this section, we enumerate the equivalence classes of the branched \( m\mathbb{Z}_p \)-coverings of the orientable surfaces.

Notice that there is a one-to-one correspondence between the set \( C^1(\mathcal{B}_n; m\mathbb{Z}_p) \) of all \( m\mathbb{Z}_p \)-voltage assignments on \( \mathcal{B}_n \) and the set \( M_{m \times n}(\mathbb{Z}_p) \) of all \( m \times n \) matrices over the field \( \mathbb{Z}_p \). Indeed, for each \( M \in M_{m \times n}(\mathbb{Z}_p) \), we define a voltage assignment \( \phi_M \) in \( C^1(\mathcal{B}_n; m\mathbb{Z}_p) \) so that \( \phi_M(\ell_i) \) is the \( i \)-th column of \( M \) for each \( \ell_i \in D(\mathcal{B}_n) \). Conversely, for each \( \phi \in C^1(\mathcal{B}_n; m\mathbb{Z}_p) \), we define \( M_\phi \) as the matrix whose \( i \)-th column is \( \phi(\ell_i) \) for each \( i = 1, \ldots, n \). Under this correspondence, each voltage assignment \( \phi \) in \( C^1(\mathcal{B}_{a_k+b} \hookrightarrow S_k - B; m\mathbb{Z}_p) \) can be identified with the matrix \( M_\phi \in M_{m \times (a_k+b)}(\mathbb{Z}_p) \) which has the following properties;

1. The rank of \( M_\phi \) is \( m \).
2. (i) if \( k \geq 0 \), then \( M_\phi \cdot \mathbf{U}_k = \mathbf{O} \), where the matrix \( \mathbf{U}_k = \begin{bmatrix} a_k &=& 2k & \begin{array}{c} 0 \cdots 0 \ b \end{array} \\ a_k &=& -k & \begin{array}{c} 2 \cdots 2 \ 1 \cdots 1 \ b \end{array} \end{bmatrix} \).

   (ii) if \( k < 0 \), then \( M_\phi \cdot \mathbf{U}_k = \mathbf{O} \), where the matrix \( \mathbf{U}_k = \begin{bmatrix} 0 \cdots 0 & b \\ 2 \cdots 2 & 1 \cdots 1 \end{bmatrix} \),

   where \( ^tA \) is the transpose of a matrix \( A \) and \( \mathbf{O} \) is the zero matrix, and

3. For each \( i = a_k + 1, \ldots, a_k + b \), the \( i \)-th column of \( M_\phi \) is a non-zero vector in \( m\mathbb{Z}_p \).

Consequently, the cardinality of the set \( C^1(\mathcal{B}_{a_k+b} \hookrightarrow S_k - B; m\mathbb{Z}_p) \) is equal to that of the set \( \{ M \in M_{m \times (a_k+b)}(\mathbb{Z}_p) : M \text{ satisfies the conditions (1'),(2') and (3')} \} \). Now, for each integer \( k \) and non-negative integer \( b \), we denote

\[
\zeta(m, k, b) = |\{ M \in M_{m \times (a_k+b)}(\mathbb{Z}_p) : M \text{ satisfies the conditions (1'),(2'), and (3')} \} |.
\]

Notice that \( m\mathbb{Z}_p \) is a vector space over the field \( \mathbb{Z}_p \) and \( \text{Aut}(m\mathbb{Z}_p) \) is the set of all linear isomorphisms on \( m\mathbb{Z}_p \). Hence we have

\[
|\text{Aut}(m\mathbb{Z}_p)| = (p^m - 1)(p^m - p) \cdots (p^m - p^{m-1}) = p^{\frac{m(m-1)}{2}} \prod_{i=1}^{m} (p^i - 1).
\]
Now, Corollary 1 implies that the number of equivalence classes of connected branched $m\mathbb{Z}_p$-coverings of a given surface $\mathcal{S}_k$ with branch set $B$ is equal to

$$\frac{\zeta(m, k, b)}{p^{m(m-1)} \prod_{s=1}^{m} (p^s - 1)}.$$ 

To obtain a formula for computing the number $\zeta(m, k, b)$, we define a number $\eta(m, k, b)$ as follows:

$$\eta(m, k, b) = |\{ M \in M_{m \times (a_k + b)}(\mathbb{Z}_p) : M \text{satisfies the conditions (1') and (2')}\}|.$$

**Lemma 1.** For an integer $k$ and a non-negative integer $b$, we have:

1. $\zeta(m, k, b) \leq \eta(m, k, b)$.
2. If $b = 0$, then $\zeta(m, k, b) = \eta(m, k, b)$. If also $a_k < m$, then these values are zero.
3. If $b \neq 0$ and $a_k + b \leq m$, then $\zeta(m, k, b) = \eta(m, k, b) = 0$.
4. If $b \neq 0$ and $a_k + b \geq m + 1$, then

$$\zeta(m, k, b) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \eta(m, k, b - t).$$

If also $a_k < m$, then

$$\zeta(m, k, b) = \sum_{t=0}^{a_k + b - (m + 1)} (-1)^t \binom{b}{t} \eta(m, k, b - t).$$

**Proof.** First, (1) and (2) are clear by the properties of the matrix $M_\phi$. Next, if $b \neq 0$ and $a_k + b \leq m$, then $\eta(m, k, b)$ must be zero, because the sum of at most $m$ linearly independent vectors in $m\mathbb{Z}_p$ cannot be the zero vector. Hence, (3) comes from (1). Finally, we prove (4). For each $i = a_k + 1, \ldots, a_k + b$, let $\mathcal{P}_i$ be the property that the $i$th column vector is zero. For each subset $S$ of $\{a_k + 1, \ldots, a_k + b\}$, let $N(\mathcal{P}_S)$ be the number of matrices in $M_{m \times (a_k + b)}(\mathbb{Z}_p)$ which satisfy the properties (1'),(2') and $\mathcal{P}_i$ for all $i \in S$. Then

$$\sum_{S \subseteq \{a_k + 1, \ldots, a_k + b\} \mid |S| = m} N(\mathcal{P}_S) = \binom{b}{t} \eta(m, k, b - t).$$

Notice that the number $\zeta(m, k, b)$ is equal to the number of matrices in $M_{m \times (a_k + b)}(\mathbb{Z}_p)$ which have properties (1') and (2'), but does not have any property $\mathcal{P}_i$ for each $i = a_k + 1, \ldots, a_k + b$. Now, it follows from the principle of inclusion and exclusion that

$$\zeta(m, k, b) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \eta(m, k, b - t).$$

Moreover, if $a_k < m$, then it follows from (3) that $\eta(m, k, b - t) = 0$ for each $t \geq a_k + b - m$. That completes the proof.

To complete the computation of the number $\zeta(m, k, b)$, we need to obtain a formula to compute the number $\eta(m, k, b)$. 

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LEMMA 2. For a non-negative integer \( b \), we have:

(a) For each \( k \geq 0 \),

\[
\eta(m, k, b) = \begin{cases} 
  p^{m(m-1)} \prod_{x=0}^{m-1} (p^{2k-x} - 1) & \text{if } 2k \geq m, \ b = 0, \\
  p^{m(m-1)} \prod_{x=0}^{m} (p^{2k+b-x} - 1) & \text{if } 2k + b \geq m + 1, \ b \neq 0, \\
  0 & \text{otherwise.}
\end{cases}
\]

(b) For each \( k < 0 \),

\[
\eta(m, k, b) = \begin{cases} 
  0 & \text{if } -k + b \leq m, \\
  p^{m(m-1)} \prod_{x=0}^{m-1} (p^{-k-x} - 1) & \text{if } -k \geq m + 1, \ b = 0, \ p = 2, \\
  p^{m(m-1)} \prod_{x=1}^{m} (p^{-k+b-x} - 1) & \text{otherwise.}
\end{cases}
\]

PROOF. Obviously, \( \eta(m, k, b) = 0 \) for the case \( a_k + b < m \) or the case \( a_k + b = m, \ b \neq 0 \).

First, we prove the case \( k \geq 0 \).

Case 1. \( 2k \geq m, \ b = 0 \).

Since the matrix \( U_k \) is the zero matrix, \( \eta(m, k, b) \) is equal to the number of matrices in \( M_{m \times (2k+b)}(\mathbb{Z}_p) \) whose rank is \( m \). By looking at the row space, we see that the number of such matrices is equal to that of all sequences of linearly independent vectors of length \( m \) in the \( 2k \)-dimensional vector space over the finite field \( \mathbb{Z}_p \). Hence, \( \eta(m, k, b) = (p^{2k} - 1)(p^{2k} - p) \cdots (p^{2k} - p^{m-1}). \)

Case 2. \( 2k + b \geq m + 1, \ b \neq 0 \).

The condition \( M_\phi \cdot U_k = \mathbf{0} \) means that each row vector of \( M_\phi \) is an element of \( \ker(U_k) \) which is a subspace of \( (2k + b)\mathbb{Z}_p \). Since the rank of \( U_k \) is \( 1 \), the dimension of \( \ker(U_k) \) is \( 2k + b - 1 \). Since the rank of \( M_\phi \) is \( m \), \( \eta(m, k, b) \) is equal to the number of all sequences of linearly independent vectors of length \( m \) in the \( (2k + b - 1) \)-dimensional subspace \( \ker(U_k) \). Hence, \( \eta(m, k, b) = (p^{2k+b-1} - 1)(p^{2k+b-1} - p) \cdots (p^{2k+b-1} - p^{m-1}). \)

Now, we prove the case \( k < 0 \).

Case 3. \( -k \geq m + 1, \ b = 0, \ p = 2 \).

The proof of this case is similar to the proof of Case 1, because the matrix \( U_k \) is also the zero matrix.

Case 4. \( -k + b \geq m + 1, \ b \geq 0, \ p \neq 2; \) or \( -k + b \geq m + 1, \ b \neq 0, \ p = 2 \).

The proof of this case is similar to the proof of Case 2. \( \square \)

From the definition of \( C^1(\mathbb{B}_{a_k+b} \hookrightarrow S_{k}^\phi - B; m\mathbb{Z}_p) \), the derived embedding \( \mathbb{B}_{a_k+b} \times \phi m\mathbb{Z}_p \hookrightarrow S_{k}^\phi - B \) is determined by the lifting of the embedding scheme for the standard embedding \( \mathbb{B}_{a_k+b} \hookrightarrow S_{k}^\phi - B \). Moreover, if \( k \geq 0 \), then \( a_k = 2k \), and the derived embedding has \( p^m \) \((4k+b)\)-sided regions, and \( bp^{m-1} \) \( p \)-sided regions. If \( k < 0 \), then \( a_k = -k \), and the regions of the derived embedding are similarly determined. Thus the Euler characteristic of the branched covering surface \( S_{k}^\phi \) is obtained as follows.
**Lemma 3.** Let $B$ be a finite subset of $\mathbb{S}_k$ and $|B| = b$. Then for each voltage assignment $\phi$ 

in $C^1(\mathcal{W}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$, the Euler characteristic $\chi(\mathbb{S}_k^\phi)$ of the surface $\mathbb{S}_k^\phi$ is

\[
\begin{cases}
p^{m-1}\{b - p (2k + b - 2)\} & \text{if } k \geq 0, \\
p^{m-1}\{b - p (-k + b - 2)\} & \text{if } k < 0.
\end{cases}
\]

Now, we enumerate the equivalence classes of branched $m\mathbb{Z}_p$-coverings of the orientable 

surface $\mathbb{S}_k$ with branch set $B$. Notice that every branched $m\mathbb{Z}_p$-covering surface of the orientable 

surface $\mathbb{S}_k$ is orientable. Hence, from Lemma 3, the genus of $\mathbb{S}_k^\phi$, $k \geq 0$, is

\[
1 - \frac{\chi(\mathbb{S}_k^\phi)}{2} = 1 - \frac{p^{m-1}}{2} \{b - p (2k + b - 2)\}
\]

for each $\phi \in C^1(\mathcal{W}_{2k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$. Thus, the following theorem comes from Corol-

lary 1 and Lemma 2.

**Theorem 2.** Let $\mathbb{S}_k$, $k \geq 0$, be an orientable surface and $B$ a finite set of points in $\mathbb{S}_k$ and let $b = |B|$. The number of equivalence classes of branched $m\mathbb{Z}_p$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ of $\mathbb{S}_k$ with branch set $B$ is equal to

\[
\begin{align*}
&\sum_{i=0}^{m-1} \prod_{x=0}^{m} (p^{2k-x} - 1) \\
&\prod_{x=1}^{b} (p^x - 1) \\
&\sum_{t=0}^{b-1} (-1)^t \begin{pmatrix} b \\ t \end{pmatrix} \prod_{x=1}^{m} (p^{2k+b-t-x} - 1) + (-1)^b \prod_{x=0}^{m-1} (p^{2k-x} - 1) \\
&\prod_{x=1}^{m} (p^x - 1)
\end{align*}
\]

\[
\begin{cases}
m-1 & \text{if } i = 1 + p^m(k - 1) \geq 0, \quad b = 0, \\
\sum_{t=0}^{b-1} (-1)^t \begin{pmatrix} b \\ t \end{pmatrix} \prod_{x=1}^{m} (p^{2k+b-t-x} - 1) + (-1)^b \prod_{x=0}^{m-1} (p^{2k-x} - 1) \\
\prod_{x=1}^{m} (p^x - 1) & \text{if } i = 1 + p^m(k - 1) + \frac{bp^{m-1}}{2}(p - 1) \geq 0, \quad b \neq 0, \\
0 & \text{otherwise}.
\end{cases}
\]

For convenience, we introduce a polynomial $R_{(\mathbb{S}_k, B, A)}(x)$, called the branched covering 

distribution polynomial, which was defined by Kwak et al. [7]. For a finite group $A$, the polynomial $R_{(\mathbb{S}_k, B, A)}(x)$ is defined by

\[
R_{(\mathbb{S}_k, B, A)}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}_k, B, A)x^i,
\]

where $a_i(\mathbb{S}_k, B, A)$ denotes the number of equivalence classes of branched $A$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ with branch set $B$. Notice that

\[
a_i(\mathbb{S}_k, B, m\mathbb{Z}_p) = \frac{\zeta(m, k, b)}{p^{\frac{m(m-1)}{2}} \prod_{x=1}^{m} (p^x - 1)},
\]

where $|B| = b$.

Some polynomials $R_{(\mathbb{S}, B, m\mathbb{Z}_p)}(x)$ are listed in Tables 1 and 2 when $\mathbb{S}$ is the sphere $\mathbb{S}_0$ or the torus $\mathbb{S}_1$. 

**Enumeration of branched $m\mathbb{Z}_p$-coverings**

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Table 1.

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<th>b = 0</th>
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Table 2.

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4. Enumeration of Branched $m\mathbb{Z}_p$-Coverings of Nonorientable Surfaces

In this section, we enumerate the equivalence classes of branched $m\mathbb{Z}_p$-coverings of the nonorientable surface $S_k$ with branch set $B$. In this case, $k < 0$ and $a_k = -k$. We observe that the embedding scheme $(\rho, \lambda)$ for the standard embedding $\mathcal{B}_{-k+b} \hookrightarrow S_k - B$ can be described as follows:

$$\rho_v = (\ell_1 \ell_1^{-1} \ell_2 \ell_2^{-1} \cdots \ell_{-k+b} \ell_{-k+b}^{-1}),$$

$$\lambda(\ell_i) = \begin{cases} 
-1 & \text{if } t = 1, 2, \ldots, -k, \\
1 & \text{if } t = -k + 1, -k + 2, \ldots, -k + b.
\end{cases}$$

Notice that a branched covering surface of a nonorientable surface can be orientable. In order to investigate the orientability of a surface $S_k$, we recall that the surface $S_k^\phi$ is orientable if and only if every cycle of the covering graph $\mathcal{B}_{-k+b} \times_\phi m\mathbb{Z}_p$ is $\lambda$-trivial, i.e., the number of edges $e$ with $\tilde{\lambda}(e) = -1$ is even in every cycle of the covering graph $\mathcal{B}_{-k+b} \times_\phi m\mathbb{Z}_p$.

If $p$ is an odd prime, then every branched $m\mathbb{Z}_p$-covering surface of the nonorientable surface $S_k$ is nonorientable. Indeed, for any loop $\ell_i$, $i = 1, \ldots, -k$, and for any voltage assignment $\phi \in C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_p)$, $p_\phi^{-1}(\ell_i)$ contains either a loop or a cycle of length $p$. Since $\lambda(\ell_i) = -1$ for each $i = 1, \ldots, -k$, the covering graph $\mathcal{B}_{-k+b} \times_\phi m\mathbb{Z}_p$ contains at least one cycle which is not $\lambda$-trivial. Therefore the following is a consequence of Corollary 1 and Lemmas 1, 2, and 3.

**Theorem 3.** Let $S_k$, $k < 0$, be a nonorientable surface and $B$ a finite set of points in $S_k$, let $p$ be an odd prime number, and let $b = |B|$. The number of equivalence classes of branched $m\mathbb{Z}_p$-coverings $p : S_1 \rightarrow S_k$ of the nonorientable surface $S_k$ with branch set $B$ is equal to

$$\prod_{s=1}^m (p^{s-1} - 1)$$

if $i = p^m(k + 2) - 2 < 0$, $b = 0$,

$$\sum_{i=0}^{b-1} (-1)^i \binom{b}{i} \prod_{s=1}^m (p^{s-1} - 1)$$

if $i = p^m(k + 2) - bp^{m-1}(b - 1) - 2 < 0$, $b \neq 0$,

0 otherwise.

However, if $p = 2$, then some branched $m\mathbb{Z}_2$-covering surfaces of a nonorientable surface can be orientable.

**Example.** Let $\mathcal{B}_3 \hookrightarrow S_{-1} - B$ be a standard embedding of the bouquet of three loops $\ell_1$, $\ell_2$, and $\ell_3$. Then the corresponding embedding scheme is $(\rho, \lambda)$, where $\rho_v = (\ell_1 \ell_1^{-1} \ell_2 \ell_2^{-1} \ell_3 \ell_3^{-1})$ and $\lambda(\ell_1) = -1, \lambda(\ell_2) = \lambda(\ell_3) = 1$. Let $\phi$ be an element in $C^1(\mathcal{B}_3 \hookrightarrow S_k - B; \mathbb{Z}_2 \times \mathbb{Z}_2)$ such that $\phi(\ell_1) = (0,1)$, $\phi(\ell_2) = (1,0)$, and $\phi(\ell_3) = (1,0)$. Let $W$ be a closed walk in $\mathcal{B}_3 \times_\phi (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then $p_\phi(W) = e_1 \cdots e_m$ is a closed walk in $\mathcal{B}_3$, with $m$ branches.
and \( \phi(e_1) \cdots \phi(e_m) \) must be the identity \((0,0)\). This implies that the edge \( \ell \) must appear an even number of times in the projection \( p_\phi(W) \) of \( W \), which means \( W \) is \( \lambda \)-trivial. Therefore, the derived surface \( S_k^\phi \) is orientable.

We now aim to classify the orientable branched \( m\mathbb{Z}_2 \)-covering surfaces \( S_k^\phi \) of the nonorientable surface \( S_k \) with branch set \( B \). For each \( \phi \in C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \), we define the net \( \phi \)-voltage of closed walk \( C = e_1^n e_2^n \cdots e_m^n \) in \( \mathcal{B}_{-k+b} \) by the product \( \phi(e_1)^{i_1} \phi(e_2)^{i_2} \cdots \phi(e_m)^{i_m} \). Let \( (m\mathbb{Z}_2)^0_\phi(v) \) be the local group of all net \( \phi \)-voltages occurring on \( v \)-based closed walks. Let \( (m\mathbb{Z}_2)^0_\phi(v) \) be the set of the net \( \phi \)-voltages on all closed walks \( C \) which are \( \lambda \)-trivial in the standard embedding \( \mathcal{B}_{-k+b} \hookrightarrow S_k - B \). Then \( (m\mathbb{Z}_2)^0_\phi(v) \) is a subgroup of the local group \( (m\mathbb{Z}_2)_\phi(v) \). By the definition of \( C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \), the local voltage group \( (m\mathbb{Z}_2)_\phi(v) \) is equal to the group \( m\mathbb{Z}_2 \). It is known [4] that the derived surface \( S_k^\phi \) is orientable if and only if the group \( (m\mathbb{Z}_2)^0_\phi(v) \) is a subgroup of index 2 in the group \( m\mathbb{Z}_2 \). We observe that the net \( \phi \)-voltage of any closed walk \( C \) of \( \mathcal{B}_{-k+b} \) is an element of \( (m\mathbb{Z}_2)^0_\phi(v) \) if and only if the closed walk \( C \) contains an even number of loops in the set \( \{ \ell_i : i = 1, \ldots, k \} \).

Let \( C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) be the set of the voltage assignments \( \phi \in C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) such that the subgroup \( (m\mathbb{Z}_2)^0_\phi(v) \) of index 2 in \( m\mathbb{Z}_2 \). Thus for any \( \phi \in C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \), the derived surface \( S_k^\phi \) is orientable.

Now, let \( C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) be the set of elements in \( C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) such that

\[
\phi(\ell_i) \in (m-1)\mathbb{Z}_2 \oplus \{1\} \text{ for each } i = 1, \ldots, k, \text{ and }
\phi(\ell_j) \in (m-1)\mathbb{Z}_2 \oplus \{0\} \text{ for each } j = -k+1, \ldots, -k.
\]

Then, for each \( \phi \in C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \), the group \( (m\mathbb{Z}_2)^0_\phi(v) \) is clearly isomorphic to the group \( (m-1)\mathbb{Z}_2 \). Notice that the voltage assignment given in our example is of this type. Let \( S \) be a subgroup of index 2 in \( m\mathbb{Z}_2 \). Then, it is not hard to show that there exists an automorphism \( \sigma : m\mathbb{Z}_2 \to m\mathbb{Z}_2 \) such that \( \sigma(S) = (m-1)\mathbb{Z}_2 \oplus \{0\} \). Therefore, the following lemma comes from Theorem 1.

**Lemma 4.** Let \( S_k \) be a nonorientable surface and let \( \phi \) be an \( m\mathbb{Z}_2 \)-voltage assignment in the set \( C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \). If the branched \( m\mathbb{Z}_2 \)-covering \( \tilde{p}_\phi : S_k^\phi \to S_k \) is orientable, then there exists \( \psi \in C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) such that the two branched \( m\mathbb{Z}_2 \)-coverings \( \tilde{p}_\phi : S_k^\phi \to S_k \) and \( \tilde{p}_\psi : S_k^\psi \to S_k \) are equivalent. Moreover, if \( \phi, \psi \in C^1(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \), then the two branched \( m\mathbb{Z}_2 \)-coverings \( \tilde{p}_\phi : S_k^\phi \to S_k \) and \( \tilde{p}_\psi : S_k^\psi \to S_k \) are equivalent if and only if there exists an automorphism \( \sigma : m\mathbb{Z}_2 \to m\mathbb{Z}_2 \) such that \( \sigma((m-1)\mathbb{Z}_2 \oplus \{0\}) = (m-1)\mathbb{Z}_2 \oplus \{0\} \).

Consequently, \( C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) contains representatives of all equivalence classes of orientable branched \( m\mathbb{Z}_2 \)-coverings of the nonorientable surface \( S_k \).

It is not hard to show that there is a one-to-one correspondence between the set \( C^1_\phi(\mathcal{B}_{-k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2) \) and the set \( M^\phi_{m \times (-k+b)} \) of all \( m \times (-k+b) \)-matrices over the field \( \mathbb{Z}_p \) which satisfy the conditions (1'), (2'), (3'), and the following additional condition:

\[
(4') \text{ The } m \text{th row of the matrix in } M^\phi_{m \times (-k+b)} \text{ is } \begin{pmatrix} -k & \cdots & b \\ 1 & \cdots & 0 & \cdots & 0 \end{pmatrix}.
\]

For convenience, let \( \xi_o(m, k, b) \) denote the cardinality of the set \( M^\phi_{m \times (-k+b)} \), and let \( \eta_o(m, k, b) \) denote the cardinality of the set of all elements in \( M^\phi_{m \times (-k+b)} \) which satisfy the conditions (1'), (2'), and (4').
Let $\text{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})$ denote the subgroup of all automorphisms $\sigma$ of $m\mathbb{Z}_2$ such that $\sigma((m-1)\mathbb{Z}_2 \oplus \{0\}) = (m-1)\mathbb{Z}_2 \oplus \{0\}$. Then

$$|\text{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})| = (2^{m-1} - 1)(2^{m-1} - 2) \cdots (2^{m-1} - 2^{m-2}) 2^{m-1}$$

$$= 2^{\frac{m(m-1)}{2}} \prod_{s=1}^{m-1} (2^s - 1).$$

Now, by Lemma 4 and Corollary 1, the number of equivalence classes of orientable branched $m\mathbb{Z}_2$-coverings of the nonorientable surface $S_k$ with branch set $B$ is equal to

$$\frac{|C_1(\mathbb{B}_{k+b} \hookrightarrow S_k - B; m\mathbb{Z}_2)|}{|\text{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})|} = \frac{\zeta_o(m, k, b)}{2^{\frac{m(m-1)}{2}} \prod_{s=1}^{m-1} (2^s - 1)}.$$

Here, by using a method similar to the proof of Lemmas 1 and 2, we can see the following:

**Lemma 5.** For a negative integer $k$ and a non-negative integer $b$, we have:

1. $$\eta_o(m, k, b) = \begin{cases} 1 & \text{if } m = 1, b = 0 \\ 2^{\frac{m(m-1)}{2}} \prod_{s=1}^{m-1} (2^{-k-s} - 1) & \text{if } -k \geq m, m \neq 1, b = 0 \\ 2^{\frac{m(m-1)}{2}} \prod_{s=2}^{m-1} (2^{-k+b-s} - 1) & \text{if } -k + b \geq m + 1, m \neq 1, b \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

2. $$\zeta_o(m, k, b) = \sum_{t=0}^{b} (-1)^t \binom{b}{t} \eta_o(m, k, b - t).$$ In particular, if $-k < m$, then

$$\zeta_o(m, k, b) = \sum_{t=0}^{-k+b-m} (-1)^t \binom{b}{t} \eta_o(m, k, b - t),$$

where $\binom{0}{0}$ is defined to be 1, and the summation over the empty index set is defined to be 0.

Now, by applying Lemmas 3 and 5, we obtain the following theorem.

**Theorem 4.** Let $S_k$, $k < 0$, be a nonorientable surface, $B$ a finite set of points in $S_k$, and $b = |B|$. Then the number of equivalence classes of orientable branched $m\mathbb{Z}_2$-coverings $p : S_i \rightarrow S_k$ of the surface $S_k$ with branch set $B$ is equal to
\[
\begin{align*}
\left\{ \begin{array}{ll}
\frac{m-1}{\prod_{s=1}^{m-1}(2^s - 1)} & \quad \text{if } i = 1 - 2^{m-1}(k + 2) \geq 0, \ b = 0, \\
\frac{b-1}{\prod_{s=1}^{m-1}(2^s - 1)} \sum_{t=0}^{b-1} (-1)^t {\binom{b}{t}} \prod_{s=2}^{m} (2^{-k+b-t-s} - 1) + (-1)^b \prod_{s=1}^{m-1} (2^{-k-s} - 1) & \quad \text{if } i = 1 - 2^{m-1}(k + 2) + 2^{m-2}b \geq 0, \ b \neq 0, \\
0 & \quad \text{otherwise},
\end{array} \right.
\end{align*}
\]

where \( \binom{0}{0} \) is defined to be 1, and the product over the empty index set is defined to be 1.

On the other hand, from Lemmas 1 and 2, we can see that the number of equivalence classes of branched \( m \mathbb{Z}_2 \)-covering surfaces \( S_\phi \) of the nonorientable surface \( S_k \) is equal to

\[
\left\{ \begin{array}{ll}
\frac{m-1}{\prod_{s=0}^{m} (2^s - 1)} & \quad \text{if } b = 0, \\
\prod_{s=1}^{b} (2^s - 1) \sum_{t=0}^{b} (-1)^t {\binom{b}{t}} \eta(m, k, b - t) & \quad \text{if } b \neq 0, \\
2^{\frac{m-1}{2}} \prod_{s=1}^{m} (2^s - 1) & \quad \text{otherwise},
\end{array} \right.
\]

Now, the number of equivalence classes of nonorientable branched \( m \mathbb{Z}_2 \)-coverings \( p : S_i \rightarrow S_k \) of the nonorientable surface \( S_k \) with branch set \( B \) comes from the above discussion and Theorem 4.

**Theorem 5.** Let \( S_k, k < 0 \), be a nonorientable surface and \( B \) a finite set of points in \( S_k \) and \( b = |B| \). Then the number of equivalence classes of nonorientable branched \( m \mathbb{Z}_2 \)-coverings \( p : S_i \rightarrow S_k \) of the surface \( S_k \) with branch set \( B \) is equal to
Table 3 lists the branched \( m\mathbb{Z}_p \)-covering distribution polynomial \( R_{(S, B, m\mathbb{Z}_p)}(x) \), when \( S \) is the projective plane \( S_{-1} \).

Table 3.

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<tr>
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<td>36x^{-13}</td>
<td>216x^{-19}</td>
<td>1296x^{-25}</td>
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\[
\left(2^{-k} - 2^m\right) \prod_{x=1}^{m-1} (2^{-k-x} - 1) \prod_{x=1}^{m} (2^x - 1) \sum_{t=0}^{b-1} (-1)^t \binom{b}{t} m \prod_{x=2}^{m} (2^{-k+b-t-x} - 1)(2^{-k+b-t-1} - 2^m) + (-1)^b (2^{-k} - 2^m) \prod_{x=1}^{m-1} (2^{-k-x} - 1) \prod_{x=1}^{m} (2^x - 1)
\]

\[
\begin{cases}
(2^{-k} - 2^m) \prod_{x=1}^{m-1} (2^{-k-x} - 1) \prod_{x=1}^{m} (2^x - 1) & \text{if } i = 2^m(k + 2) - 2 < 0, \ b = 0, \\
\sum_{t=0}^{b-1} (-1)^t \binom{b}{t} m \prod_{x=2}^{m} (2^{-k+b-t-x} - 1)(2^{-k+b-t-1} - 2^m) + (-1)^b (2^{-k} - 2^m) \prod_{x=1}^{m-1} (2^{-k-x} - 1) \prod_{x=1}^{m} (2^x - 1) & \text{if } i = 2^m(k + 2) - 2(1 + 2^{m-2}b) < 0, \ b \neq 0, \\
0 & \text{otherwise},
\end{cases}
\]

where \( \binom{0}{0} \) is defined to be 1, and the product over the empty index set is defined to be 1.

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REFERENCES


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