Europ. J. Combinatorics (2001) **22**, 1125–1138
doi:10.1006/eujc.2001.0530
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Enumeration of the Branched $m\mathbb{Z}_p$-Coverings of Closed Surfaces

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In this paper, we enumerate the equivalence classes of regular branched coverings of surfaces whose covering transformation groups are the direct sum of m copies of $\mathbb{Z}_p$, p prime.

1. INTRODUCTION

Throughout this paper, a surface $\mathbb{S}$ means a compact connected 2-manifold without boundary. By the classification theorem of surfaces, $\mathbb{S}$ is homeomorphic to one of the following:

$$\mathbb{S}_k = \begin{cases} \text{the orientable surface with } k \text{ handles} & \text{if } k \geq 0, \\ \text{the nonorientable surface with } -k \text{ crosscaps} & \text{if } k < 0. \end{cases}$$

A continuous function $p : \tilde{\mathbb{S}} \to \mathbb{S}$ between two surfaces $\tilde{\mathbb{S}}$ and $\mathbb{S}$ is called a *branched covering* if there exists a finite set B in $\mathbb{S}$ such that the restriction of p to $\tilde{\mathbb{S}} - p^{-1}(B)$, $p_{\mid \tilde{\mathbb{S}} - p^{-1}(B)} : \tilde{\mathbb{S}} - p^{-1}(B) \to \mathbb{S} - B$, is a covering projection in the usual sense. The smallest set B of $\mathbb{S}$ which has this property is called the *branch set*. A branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ is *regular* if there exists a (finite) group $\mathcal{A}$ which acts pseudofreely on $\tilde{\mathbb{S}}$ so that the surface $\mathbb{S}$ is homeomorphic to the quotient space $\tilde{\mathbb{S}}/\mathcal{A}$, say by h, and the quotient map $\tilde{\mathbb{S}} \to \tilde{\mathbb{S}}/\mathcal{A}$ is the composition $h \circ p$ of p and h. In this case, the group $\mathcal{A}$ is the group of covering transformations of the branched covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$. We call it a *branched $\mathcal{A}$-covering*. Two branched coverings $p : \tilde{\mathbb{S}} \to \mathbb{S}$ and $q : \tilde{\mathbb{S}}' \to \mathbb{S}$ are *equivalent* if there exists a homeomorphism $h : \tilde{\mathbb{S}} \to \tilde{\mathbb{S}}'$ such that $p = q \circ h$.

Since A. Hurwitz showed how to classify the branched coverings of a given surface [6], this area has been studied in [1, 2, 5] and their references. Recently, Kwak *et al*. enumerated the number of equivalence classes of the branched $\mathcal{A}$-coverings of surfaces, when $\mathcal{A}$ is the cyclic group $\mathbb{Z}_p$ or the dihedral group $\mathbb{D}_p$ of order $2p$, p prime [7, 9].

In this paper, we enumerate the equivalence classes of branched $\mathcal{A}$-coverings $p : \mathbb{S}_i \to \mathbb{S}$ with branch set B, when $\mathcal{A}$ is the group $\mathbb{Z}_p \oplus \mathbb{Z}_p \oplus \cdots \oplus \mathbb{Z}_p \equiv m\mathbb{Z}_p$, the direct sum of m copies of $\mathbb{Z}_p$.

2. A CLASSIFICATION OF REGULAR BRANCHED COVERINGS

Let G be a finite connected graph with vertex set $V(G)$ and edge set $E(G)$. We allow self-loops and multiple edges. Notice by regarding the vertices of G as 0-cells and the edges of G as 1-cells, the graph G can be identified with a one-dimensional CW complex in the Euclidean 3-space $\mathbb{R}^3$ so that every graph map is continuous.

A graph map $p : \tilde{G} \to G$ is said to be a *regular covering* (simply, $\mathcal{A}$-covering) if $p : \tilde{G} \to G$ is a covering projection in a topological sense and there is a subgroup $\mathcal{A}$ of the automorphism group $\mathrm{Aut}(\tilde{G})$ of $\tilde{G}$ acting freely on $\tilde{G}$ so that G is isomorphic to the quotient graph $\tilde{G}/\mathcal{A}$, say by h, and the quotient map $\tilde{G} \to \tilde{G}/\mathcal{A}$ is the composition $h \circ p$ of p and h.

†Supported by Com²MaC-KOSEF (program year: 1999).
‡Supported by KRF (program year: 1998).

0195–6698/01/081125 + 14 $35.00/0

Every regular covering of a graph G can be constructed as follows [3]: every edge of a graph G gives rise to a pair of edges in opposite directions. By $e^{-1} = vu$, we mean the reverse edge to a directed edge $e = uv$. We denote the set of directed edges of G by $D(G)$. An $\mathcal{A}$-*voltage assignment* on G is a function $\phi : D(G) \to \mathcal{A}$ with the property that $\phi(e^{-1}) = \phi(e)^{-1}$ for each $e \in D(G)$. The values of ϕ are called *voltages*. For a graph G and a finite group $\mathcal{A}$, let $C^1(G; \mathcal{A})$ denote the set of all $\mathcal{A}$-voltage assignments on G. The *ordinary derived graph* $G \times_\phi \mathcal{A}$ derived from an $\mathcal{A}$-voltage assignment $\phi : D(G) \to \mathcal{A}$ has as its vertex set $V(G) \times \mathcal{A}$ and as its edge set $E(G) \times \mathcal{A}$. For every edge e between u and v in G and $g \in \mathcal{A}$, there is an edge from (u, g) to $(v, g\phi(e))$ in the derived graph $G \times_\phi \mathcal{A}$. In the ordinary derived graph $G \times_\phi \mathcal{A}$, a vertex (u, g) is denoted by u_g, and an edge (e, g) by e_g. The first coordinate projection $p_\phi : G \times_\phi \mathcal{A} \to G$, called the *natural projection*, commutes with the right multiplication action of the $\phi(e)$ and left action of $\mathcal{A}$ on the fibers, which is free and transitive, so that $p_\phi : G \times_\phi \mathcal{A} \to G$ is an $\mathcal{A}$-covering.

An *embedding* of a graph G into the surface $\mathbb{S}$ is a continuous one-to-one function $\imath : G \to \mathbb{S}$. If every component of $\mathbb{S} - \imath(G)$, called a *region*, is homeomorphic to an open disk, then $\imath : G \to \mathbb{S}$ is called a 2-*cell* embedding. An *embedding scheme* (ρ, λ) for a graph G consists of a rotation scheme ρ which assigns a cyclic permutation ρ_v on $N(v) = \{e \in D(G) :$ the initial vertex of e is $v\}$ to each $v \in V(G)$ and a voltage assignment λ which assigns a value $\lambda(e)$ in $\mathbb{Z}_2 = \{-1, 1\}$ to each $e \in E(G)$.

Stahl [10] showed that every embedding scheme determines a 2-cell embedding of G into an orientable or nonorientable surface $\mathbb{S}$, and every 2-cell embedding of G into a surface $\mathbb{S}$ is determined by such a scheme.

If an embedding scheme (ρ, λ) for a graph G determines a 2-cell embedding of G into a surface $\mathbb{S}$, then the orientability of $\mathbb{S}$ can be detected by looking at the values of cycles of G under the voltage assignment λ. In fact, $\mathbb{S}$ is orientable if and only if every cycle of G is λ-*trivial*, that is, the number of edges e with $\lambda(e) = -1$ is even in every cycle of G.

Let $\imath : G \to \mathbb{S}$ be a 2-cell embedding with embedding scheme (ρ, λ) and ϕ be an $\mathcal{A}$-voltage assignment. The ordinary derived graph $G \times_\phi \mathcal{A}$ has the *derived embedding scheme* $(\tilde{\rho}, \tilde{\lambda})$, which is defined by $\tilde{\rho}_{v_g}(e_g) = (\rho_v(e))_g$ and $\tilde{\lambda}(e_g) = \lambda(e)$ for each $e_g \in D(G \times_\phi \mathcal{A})$. Then it induces a 2-cell embedding $\tilde{\imath} : G \times_\phi \mathcal{A} \to \mathbb{S}^\phi$ such that the following diagram commutes.

$$\begin{array}{ccc} G \times_\phi \mathcal{A} & \xrightarrow{\ \tilde{\imath}\ } & \mathbb{S}^\phi \\ {\scriptstyle p_\phi}\downarrow & & \downarrow{\scriptstyle \tilde{p}_\phi} \\ G & \xrightarrow{\ \imath\ } & \mathbb{S} \end{array}$$

Moreover, if $G \times_\phi \mathcal{A}$ is connected, then $\mathbb{S}^\phi$ is connected and $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ is a branched $\mathcal{A}$-covering. Conversely, let $p : \tilde{\mathbb{S}} \to \mathbb{S}$ be a branched $\mathcal{A}$-covering of a surface $\mathbb{S}$. Then there exist a 2-cell embedding $\imath : G \to \mathbb{S}$ such that each face of the embedding has at most one branch point interior to it and an $\mathcal{A}$-voltage assignment $\phi : D(G) \to \mathcal{A}$ such that the branched $\mathcal{A}$-covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ is equivalent to the given branched $\mathcal{A}$-covering $p : \tilde{\mathbb{S}} \to \mathbb{S}$ [4].

A surface $\mathbb{S}_k$ can be represented by a $4k$-gon with identification data $\prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1}$ on its boundary if $k > 0$; a bigon with identification data aa^{-1} on its boundary if $k = 0$; and a $-2k$-gon with identification data $\prod_{s=1}^{-k} a_s a_s$ on its boundary if $k < 0$.

Let B be a finite set of points in $\mathbb{S}_k$. We note the fact that the fundamental group $\pi_1(\mathbb{S}_k - B, *)$ of the punctured surface $\mathbb{S}_k - B$ with the base point $* \in \mathbb{S}_k - B$ can be represented by

$$\left\langle a_1, \ldots, a_k, b_1, \ldots, b_k, c_1, \ldots, c_{|B|} \,;\, \prod_{s=1}^{k} a_s b_s a_s^{-1} b_s^{-1} \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k > 0;$$

$$\left\langle a_1, \ldots, a_{-k}, c_1, \ldots, c_{|B|} \; ; \; \prod_{s=1}^{-k} a_s a_s \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k < 0;$$
$$\left\langle c_1, \ldots, c_{|B|} \; ; \; \prod_{t=1}^{|B|} c_t = 1 \right\rangle \quad \text{if } k = 0.$$

We call this the *standard presentation* of the fundamental group $\pi_1(\mathbb{S}_k - B, *)$. For each $t = 1, 2, \ldots, |B|$, we take a simple closed curve based at $*$ lying in the face determined by the polygonal representation of the surface $\mathbb{S}_k$ so that it represents the homotopy class of the generator c_t. Then, it induces a 2-cell embedding of a bouquet of n circles, denoted by $\mathfrak{B}_n$, into the surface $\mathbb{S}_k$ such that the embedding has $|B|$ 1-sided regions and one $(|B| + 4k)$-sided region if $k > 0$; $|B|$ 1-sided regions and one $(|B| - 2k)$-sided region if $k < 0$; and $|B|$ 1-sided regions and one $|B|$-sided region if $k = 0$, where n is the number of the generators of the corresponding fundamental group. We call this embedding $\imath : \mathfrak{B}_n \to \mathbb{S}_k$ the *standard embedding*, simply denoted by $\mathfrak{B}_n \hookrightarrow \mathbb{S}_k - B$.

Now, we identify each loop ℓ_i in $\mathfrak{B}_n$ with a generator in $\pi_1(\mathbb{S}_k - B)$ and this identification gives a direction on each loop of $\mathfrak{B}_n$ so that a positively oriented loop ℓ_i represents the corresponding generator of $\pi_1(\mathbb{S}_k - B)$.

Let $C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ denote the subset of $C^1(\mathfrak{B}_n; \mathcal{A})$ consisting of all $\mathcal{A}$-voltage assignments ϕ of $\mathfrak{B}_n$ which satisfy the following two conditions:

(1) $\phi(\ell_1), \ldots, \phi(\ell_n)$ generate $\mathcal{A}$, and

(2) (i) if $k \geq 0$, then $\phi(\ell_i) \neq id_{\mathcal{A}}$ for each $i = 2k+1, \ldots, 2k+|B| = n$ and

$$\prod_{i=1}^{k} \phi(\ell_i)\phi(\ell_{k+i})\phi(\ell_i)^{-1}\phi(\ell_{k+i})^{-1} \prod_{i=1}^{|B|} \phi(\ell_{2k+i}) = 1,$$

(ii) if $k < 0$, then $\phi(\ell_i) \neq id_{\mathcal{A}}$ for each $i = -k+1, \ldots, -k+|B| = n$ and

$$\prod_{i=1}^{-k} \phi(\ell_i)\phi(\ell_i) \prod_{i=1}^{|B|} \phi(\ell_{-k+i}) = 1.$$

Note that the condition (1) guarantees that $\mathbb{S}^\phi$ is connected, and the condition (2) guarantees that the set B is the same as the branch set of the branched covering $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$.

By using these, Kwak *et al*. obtained the following theorem.

THEOREM 1. (EXISTENCE AND CLASSIFICATION OF REGULAR BRANCHED COVERINGS [7]). *Let $n = 2k + |B|$ if $k \geq 0$, and let $n = -k + |B|$ if $k < 0$. Every voltage assignment in $C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$ induces a connected branched $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B. Conversely, every connected branched $\mathcal{A}$-covering of $\mathbb{S}_k$ with branch set B is induced by a voltage assignment in $C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S}_k - B; \mathcal{A})$. Moreover, for two given $\mathcal{A}$-voltage assignments $\phi, \psi \in C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S} - B; \mathcal{A})$, two branched $\mathcal{A}$-coverings $\tilde{p}_\phi : \mathbb{S}^\phi \to \mathbb{S}$ and $\tilde{p}_\psi : \mathbb{S}^\psi \to \mathbb{S}$ are equivalent if and only if there exists a group automorphism $\sigma : \mathcal{A} \to \mathcal{A}$ such that*

$$\psi(\ell_i) = \sigma(\phi(\ell_i))$$

for all $\ell_i \in D(\mathfrak{B}_n)$.

In fact, the last statement is equivalent to saying that the two graph coverings $p_\phi : \mathfrak{B}_n \times_\phi \mathcal{A} \to \mathfrak{B}_n$ and $p_\psi : \mathfrak{B}_n \times_\psi \mathcal{A} \to \mathfrak{B}_n$ of $\mathfrak{B}_n$ are equivalent as graph coverings [7, 8].

To enumerate the equivalence classes of connected branched $\mathcal{A}$-coverings of a surface $\mathbb{S}$, we define an Aut($\mathcal{A}$)-action on $C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S} - B; \mathcal{A})$ as follows:

$$(\sigma\phi)(\ell_i) = \sigma(\phi(\ell_i)),$$

for each $\sigma \in \mathrm{Aut}(\mathcal{A})$, $\phi \in C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S} - B; \mathcal{A})$ and all $\ell_i \in D(\mathfrak{B}_n)$. Notice that this Aut($\mathcal{A}$)-action on the set $C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S} - B; \mathcal{A})$ is free, because $\phi(\ell_1), \ldots, \phi(\ell_n)$ generate $\mathcal{A}$.

COROLLARY 1. [7] *Let $\mathcal{A}$ be a finite group. Then the number of equivalence classes of connected branched $\mathcal{A}$-coverings of the surface $\mathbb{S}_k$ with branch set B is equal to*

$$\frac{|C^1(\mathfrak{B}_n \hookrightarrow \mathbb{S}_k - B; \mathcal{A})|}{|\mathrm{Aut}(\mathcal{A})|},$$

where

$$n = \begin{cases} 2k + |B| & \text{if } k \geq 0, \\ -k + |B| & \text{if } k < 0. \end{cases}$$

For convenience, let $a_k = 2k$ if $k \geq 0$, and $a_k = -k$ if $k < 0$. Let b be the cardinality of a branch set B. Thereby n in Theorem 1 and Corollary 1 is $a_k + b$ for each integer k.

3. ENUMERATION OF BRANCHED $m\mathbb{Z}_p$-COVERINGS OF ORIENTABLE SURFACES

In this section, we enumerate the equivalence classes of the branched $m\mathbb{Z}_p$-coverings of the orientable surfaces.

Notice that there is a one-to-one correspondence between the set $C^1(\mathfrak{B}_n; m\mathbb{Z}_p)$ of all $m\mathbb{Z}_p$-voltage assignments on $\mathfrak{B}_n$ and the set $M_{m\times n}(\mathbb{Z}_p)$ of all $m \times n$ matrices over the field $\mathbb{Z}_p$. Indeed, for each $M \in M_{m\times n}(\mathbb{Z}_p)$, we define a voltage assignment ϕ_M in $C^1(\mathfrak{B}_n; m\mathbb{Z}_p)$ so that $\phi_M(\ell_i)$ is the ith column of M for each $\ell_i \in D(\mathfrak{B}_n)$. Conversely, for each $\phi \in C^1(\mathfrak{B}_n; m\mathbb{Z}_p)$, we define M_ϕ as the matrix whose ith column is $\phi(\ell_i)$ for each $i = 1, \ldots, n$. Under this correspondence, each voltage assignment ϕ in $C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$ can be identified with the matrix M_ϕ in $M_{m\times(a_k+b)}(\mathbb{Z}_p)$ which has the following properties;

(1′) The rank of M_ϕ is m,

(2′) (i) if $k \geq 0$, then $M_\phi\, {}^tU_k = \mathbf{O}$, where the matrix $U_k = [\, \overbrace{0 \cdots 0}^{a_k = 2k}\ \overbrace{1 \cdots 1}^{b}\,]$,

(ii) if $k < 0$, then $M_\phi\, {}^tU_k = \mathbf{O}$, where the matrix $U_k = [\, \overbrace{2 \cdots 2}^{a_k = -k}\ \overbrace{1 \cdots 1}^{b}\,]$,
where ${}^tA$ is the transpose of a matrix A and $\mathbf{O}$ is the zero matrix, and

(3′) For each $i = a_k + 1, \ldots, a_k + b$, the ith column of M_ϕ is a non-zero vector in $m\mathbb{Z}_p$.

Consequently, the cardinality of the set $C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$ is equal to that of the set $\{M \in M_{m\times(a_k+b)}(\mathbb{Z}_p) : M$ satisfies the conditions (1′),(2′) and (3′)$\}$. Now, for each integer k and non-negative integer b, we denote

$$\zeta(m, k, b) = |\{M \in M_{m\times(a_k+b)}(\mathbb{Z}_p) : M \text{ satisfies the conditions (1′),(2′), and (3′)}\}|.$$

Notice that $m\mathbb{Z}_p$ is a vector space over the field $\mathbb{Z}_p$ and Aut($m\mathbb{Z}_p$) is the set of all linear isomorphisms on $m\mathbb{Z}_p$. Hence we have

$$|\mathrm{Aut}(m\mathbb{Z}_p)| = (p^m - 1)(p^m - p)\cdots(p^m - p^{m-1}) = p^{\frac{m(m-1)}{2}} \prod_{s=1}^{m} (p^s - 1).$$

Now, Corollary 1 implies that the number of equivalence classes of connected branched $m\mathbb{Z}_p$-coverings of a given surface $\mathbb{S}_k$ with branch set B is equal to

$$\frac{\zeta(m,k,b)}{p^{\frac{m(m-1)}{2}}\prod_{s=1}^{m}(p^s-1)}.$$

To obtain a formula for computing the number $\zeta(m,k,b)$, we define a number $\eta(m,k,b)$ as follows:

$$\eta(m,k,b) = |\{M \in M_{m\times(a_k+b)}(\mathbb{Z}_p) : M \text{ satisfies the conditions } (1') \text{ and } (2')\}|.$$

LEMMA 1. *For an integer k and a non-negative integer b, we have:*

(1) $\zeta(m,k,b) \le \eta(m,k,b)$.
(2) *If $b = 0$, then $\zeta(m,k,b) = \eta(m,k,b)$. If also $a_k < m$, then these values are zero.*
(3) *If $b \neq 0$ and $a_k + b \le m$, then $\zeta(m,k,b) = \eta(m,k,b) = 0$.*
(4) *If $b \neq 0$ and $a_k + b \ge m+1$, then*

$$\zeta(m,k,b) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}\eta(m,k,b-t).$$

If also $a_k < m$, then

$$\zeta(m,k,b) = \sum_{t=0}^{a_k+b-(m+1)}(-1)^t\binom{b}{t}\eta(m,k,b-t).$$

PROOF. First, (1) and (2) are clear by the properties of the matrix M_ϕ. Next, if $b \neq 0$ and $a_k + b \le m$, then $\eta(m,k,b)$ must be zero, because the sum of at most m linearly independent vectors in $m\mathbb{Z}_p$ cannot be the zero vector. Hence, (3) comes from (1). Finally, we prove (4). For each $i = a_k+1,\ldots,a_k+b$, let $\mathcal{P}_i$ be the property that the ith column vector is zero. For each subset S of $\{a_k+1,\ldots,a_k+b\}$, let $N(\mathcal{P}_S)$ be the number of matrices in $M_{m\times(a_k+b)}(\mathbb{Z}_p)$ which satisfy the properties $(1')$,$(2')$ and $\mathcal{P}_i$ for all $i \in S$. Then

$$\sum_{\substack{S\subset\{a_k+1,\ldots,a_k+b\}\\ |S|=t}} N(\mathcal{P}_S) = \binom{b}{t}\eta(m,k,b-t).$$

Notice that the number $\zeta(m,k,b)$ is equal to the number of matrices in $M_{m\times(a_k+b)}(\mathbb{Z}_p)$ which have properties $(1')$ and $(2')$, but does not have any property $\mathcal{P}_i$ for each $i = a_k+1,\ldots,a_k+b$. Now, it follows from the principle of inclusion and exclusion that

$$\zeta(m,k,b) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}\eta(m,k,b-t).$$

Moreover, if $a_k < m$, then it follows from (3) that $\eta(m,k,b-t) = 0$ for each $t \ge a_k+b-m$. That completes the proof. □

To complete the computation of the number $\zeta(m,k,b)$, we need to obtain a formula to compute the number $\eta(m,k,b)$.

LEMMA 2. *For a non-negative integer b, we have:*

(a) *For each* $k \geq 0$,

$$\eta(m,k,b) = \begin{cases} p^{\frac{m(m-1)}{2}} \prod_{s=0}^{m-1} (p^{2k-s} - 1) & \text{if } 2k \geq m,\ b = 0, \\ p^{\frac{m(m-1)}{2}} \prod_{s=1}^{m} (p^{2k+b-s} - 1) & \text{if } 2k + b \geq m + 1,\ b \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

(b) *For each* $k < 0$,

$$\eta(m,k,b) = \begin{cases} 0 & \text{if } -k + b \leq m, \\ p^{\frac{m(m-1)}{2}} \prod_{s=0}^{m-1} (p^{-k-s} - 1) & \text{if } -k \geq m + 1,\ b = 0,\ p = 2, \\ p^{\frac{m(m-1)}{2}} \prod_{s=1}^{m} (p^{-k+b-s} - 1) & \text{otherwise.} \end{cases}$$

PROOF. Obviously, $\eta(m,k,b) = 0$ for the case $a_k + b < m$ or the case $a_k + b = m,\ b \neq 0$. First, we prove the case $k \geq 0$.

Case 1. $2k \geq m,\ b = 0$.
Since the matrix U_k is the zero matrix, $\eta(m,k,b)$ is equal to the number of matrices in $M_{m\times(2k+b)}(\mathbb{Z}_p)$ whose rank is m. By looking at the row space, we see that the number of such matrices is equal to that of all sequences of linearly independent vectors of length m in the $2k$-dimensional vector space over the finite field $\mathbb{Z}_p$. Hence, $\eta(m,k,b) = (p^{2k} - 1)(p^{2k} - p)\cdots(p^{2k} - p^{m-1})$.

Case 2. $2k + b \geq m + 1,\ b \neq 0$.
The condition $M_\phi\, {}^tU_k = \mathbf{O}$ means that each row vector of M_ϕ is an element of $ker({}^tU_k)$ which is a subspace of $(2k + b)\mathbb{Z}_p$. Since the rank of ${}^tU_k$ is 1, the dimension of $ker({}^tU_k)$ is $2k + b - 1$. Since the rank of M_ϕ is m, $\eta(m,k,b)$ is equal to the number of all sequences of linearly independent vectors of length m in the $(2k + b - 1)$-dimensional subspace $ker({}^tU_k)$. Hence, $\eta(m,k,b) = (p^{2k+b-1} - 1)(p^{2k+b-1} - p)\cdots(p^{2k+b-1} - p^{m-1})$.

Now, we prove the case $k < 0$.

Case 3. $-k \geq m + 1,\ b = 0,\ p = 2$.
The proof of this case is similar to the proof of Case 1, because the matrix U_k is also the zero matrix.

Case 4. $-k + b \geq m + 1,\ b \geq 0,\ p \neq 2$; or $-k + b \geq m + 1, b \neq 0, p = 2$.
The proof of this case is similar to the proof of Case 2. □

From the definition of $C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$, the derived embedding $\mathfrak{B}_{a_k+b} \times_\phi m\mathbb{Z}_p \hookrightarrow \mathbb{S}_k^\phi$ is determined by the lifting of the embedding scheme for the standard embedding $\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B$. Moreover, if $k \geq 0$, then $a_k = 2k$, and the derived embedding has p^m $(4k+b)$-sided regions, and bp^{m-1} p-sided regions. If $k < 0$, then $a_k = -k$, and the regions of the derived embedding are similarly determined. Thus the Euler characteristic of the branched covering surface $\mathbb{S}_k^\phi$ is obtained as follows.

LEMMA 3. *Let B be a finite subset of $\mathbb{S}_k$ and $|B| = b$. Then for each voltage assignment ϕ in $C^1(\mathfrak{B}_{a_k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$, the Euler characteristic $\chi(\mathbb{S}_k^{\phi})$ of the surface $\mathbb{S}_k^{\phi}$ is*

$$\begin{cases} p^{m-1}\{b - p\,(2k + b - 2)\} & \text{if } k \geq 0, \\ p^{m-1}\{b - p\,(-k + b - 2)\} & \text{if } k < 0. \end{cases}$$

Now, we enumerate the equivalence classes of branched $m\mathbb{Z}_p$-coverings of the orientable surface $\mathbb{S}_k$ with branch set B. Notice that every branched $m\mathbb{Z}_p$-covering surface of the orientable surface $\mathbb{S}_k$ is orientable. Hence, from Lemma 3, the genus of $\mathbb{S}_k^{\phi}$, $k \geq 0$, is

$$1 - \frac{\chi(\mathbb{S}_k^{\phi})}{2} = 1 - \frac{p^{m-1}}{2}\,(b - p\,(2k + b - 2))$$

for each $\phi \in C^1(\mathfrak{B}_{2k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$. Thus, the following theorem comes from Corollary 1 and Lemma 2.

THEOREM 2. *Let $\mathbb{S}_k$, $k \geq 0$, be an orientable surface and B a finite set of points in $\mathbb{S}_k$ and let $b = |B|$. The number of equivalence classes of branched $m\mathbb{Z}_p$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ of $\mathbb{S}_k$ with branch set B is equal to*

$$\begin{cases} \dfrac{\prod_{s=0}^{m-1}(p^{2k-s} - 1)}{\prod_{s=1}^{m}(p^s - 1)} & \textit{if } i = 1 + p^m(k-1) \geq 0,\ b = 0, \\[2ex] \dfrac{\sum_{t=0}^{b-1}(-1)^t\binom{b}{t}\prod_{s=1}^{m}(p^{2k+b-t-s} - 1) + (-1)^b\prod_{s=0}^{m-1}(p^{2k-s} - 1)}{\prod_{s=1}^{m}(p^s - 1)} & \\ & \textit{if } i = 1 + p^m(k-1) + \dfrac{bp^{m-1}}{2}(p-1) \geq 0,\ b \neq 0, \\[2ex] 0 & \textit{otherwise}. \end{cases}$$

For convenience, we introduce a polynomial $R_{(\mathbb{S}_k,B,\mathcal{A})}(x)$, called the branched covering distribution polynomial, which was defined by Kwak *et al*. [7]. For a finite group $\mathcal{A}$, the polynomial $R_{(\mathbb{S}_k,B,\mathcal{A})}(x)$ is defined by

$$R_{(\mathbb{S}_k,B,\mathcal{A})}(x) = \sum_{i=-\infty}^{\infty} a_i(\mathbb{S}_k, B, \mathcal{A})x^i,$$

where $a_i(\mathbb{S}_k, B, \mathcal{A})$ denotes the number of equivalence classes of branched $\mathcal{A}$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ with branch set B. Notice that

$$a_i(\mathbb{S}_k, B, m\mathbb{Z}_p) = \frac{\zeta(m, k, b)}{p^{\frac{m(m-1)}{2}}\prod_{s=1}^{m}(p^s - 1)},$$

where $|B| = b$.

Some polynomials $R_{(\mathbb{S},B,m\mathbb{Z}_p)}(x)$ are listed in Tables 1 and 2 when $\mathbb{S}$ is the sphere $\mathbb{S}_0$ or the torus $\mathbb{S}_1$.

TABLE 1.
$R_{(\mathbb{S}_0, B, m\mathbb{Z}_p)}(x)$.

(m, p)	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$\cdots$
(1, 2)	0	0	1	0	x	$\cdots$
(1, 3)	0	0	1	x	$3x^2$	$\cdots$
(1, 5)	0	0	1	$3x^2$	$13x^4$	$\cdots$
(1, 7)	0	0	1	$5x^3$	$31x^6$	$\cdots$
$\vdots$			$\vdots$			
(2, 2)	0	0	0	1	$3x$	$\cdots$
(2, 3)	0	0	0	x	$9x^4$	$\cdots$
(2, 5)	0	0	0	x^6	$27x^{16}$	$\cdots$
(2, 7)	0	0	0	x^{15}	$53x^{36}$	$\cdots$
$\vdots$			$\vdots$			
(3, 2)	0	0	0	0	x	$\cdots$
(3, 3)	0	0	0	0	x^{10}	$\cdots$
(3, 5)	0	0	0	0	x^{76}	$\cdots$
(3, 7)	0	0	0	0	x^{246}	$\cdots$
$\vdots$			$\vdots$			

TABLE 2.
$R_{(\mathbb{S}_1, B, m\mathbb{Z}_p)}(x)$.

(m, p)	$b = 0$	$b = 1$	$b = 2$	$b = 3$	$b = 4$	$\cdots$
(1, 2)	$3x$	0	$4x^2$	0	$4x^3$	$\cdots$
(1, 3)	$4x$	0	$9x^3$	$9x^4$	$27x^5$	$\cdots$
(1, 5)	$6x$	0	$25x^5$	$75x^7$	$325x^9$	$\cdots$
(1, 7)	$8x$	0	$49x^7$	$245x^{10}$	$1519x^{13}$	$\cdots$
$\vdots$			$\vdots$			
(2, 2)	x	0	$6x^3$	$16x^4$	$54x^5$	$\cdots$
(2, 3)	x	0	$12x^7$	$93x^{10}$	$765x^{13}$	$\cdots$
(2, 5)	x	0	$30x^{21}$	$715x^{31}$	$17265x^{41}$	$\cdots$
(2, 7)	x	0	$56x^{43}$	$2681x^{64}$	$128989x^{85}$	$\cdots$
$\vdots$			$\vdots$			
(3, 2)	0	0	x^5	$12x^7$	$101x^9$	$\cdots$
(3, 3)	0	0	x^{19}	$37x^{28}$	$1056x^{37}$	$\cdots$
(3, 5)	0	0	x^{101}	$153x^{151}$	$19688x^{201}$	$\cdots$
(3, 7)	0	0	x^{295}	$399x^{442}$	$138456x^{589}$	$\cdots$
$\vdots$			$\vdots$			

4. ENUMERATION OF BRANCHED $m\mathbb{Z}_p$-COVERINGS OF NONORIENTABLE SURFACES

In this section, we enumerate the equivalence classes of branched $m\mathbb{Z}_p$-coverings of the nonorientable surface $\mathbb{S}_k$ with branch set B. In this case, $k < 0$ and $a_k = -k$. We observe that the embedding scheme (ρ, λ) for the standard embedding $\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B$ can be described as follows:

$$\rho_v = (\ell_1\ell_1^{-1}\ell_2\ell_2^{-1}\cdots\ell_{-k+b}\ell_{-k+b}^{-1}),$$
$$\lambda(\ell_t) = \begin{cases} -1 & \text{if } t = 1, 2, \ldots, -k, \\ 1 & \text{if } t = -k+1, -k+2, \ldots, -k+b. \end{cases}$$

Notice that a branched covering surface of a nonorientable surface can be orientable. In order to investigate the orientability of a surface $\mathbb{S}_k^{\phi}$, we recall that the surface $\mathbb{S}_k^{\phi}$ is orientable if and only if every cycle of the covering graph $\mathfrak{B}_{-k+b} \times_{\phi} m\mathbb{Z}_p$ is $\tilde{\lambda}$-trivial, i.e., the number of edges e with $\tilde{\lambda}(e) = -1$ is even in every cycle of the covering graph $\mathfrak{B}_{-k+b} \times_{\phi} m\mathbb{Z}_p$.

If p is an odd prime, then every branched $m\mathbb{Z}_p$-covering surface of the nonorientable surface $\mathbb{S}_k$ is nonorientable. Indeed, for any loop ℓ_i, $i = 1, \ldots, -k$, and for any voltage assignment $\phi \in C^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_p)$, $p_{\phi}^{-1}(\ell_i)$ contains either a loop or a cycle of length p. Since $\lambda(\ell_i) = -1$ for each $i = 1, \ldots, -k$, the covering graph $\mathfrak{B}_{-k+b} \times_{\phi} m\mathbb{Z}_p$ contains at least one cycle which is not $\tilde{\lambda}$-trivial. Therefore the following is a consequence of Corollary 1 and Lemmas 1, 2, and 3.

THEOREM 3. *Let $\mathbb{S}_k$, $k < 0$, be a nonorientable surface and B a finite set of points in $\mathbb{S}_k$, let p be an odd prime number, and let $b = |B|$. The number of equivalence classes of branched $m\mathbb{Z}_p$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ of the nonorientable surface $\mathbb{S}_k$ with branch set B is equal to*

$$\begin{cases} \dfrac{\prod_{s=1}^{m}(p^{-k-s}-1)}{\prod_{s=1}^{m}(p^{s}-1)} & \text{if } i = p^m(k+2) - 2 < 0,\ b = 0, \\[2ex] \dfrac{\sum_{t=0}^{b-1}(-1)^t\binom{b}{t}\prod_{s=1}^{m}(p^{-k+b-t-s}-1) + (-1)^b\prod_{s=1}^{m}(p^{-k-s}-1)}{\prod_{s=1}^{m}(p^{s}-1)} & \\ \qquad \text{if } i = p^m(k+2) - bp^{m-1}(b-1) - 2 < 0,\ b \neq 0, \\[2ex] 0 & \text{otherwise}. \end{cases}$$

However, if $p = 2$, then some branched $m\mathbb{Z}_2$-covering surfaces of a nonorientable surface can be orientable.

EXAMPLE. Let $\mathfrak{B}_3 \hookrightarrow \mathbb{S}_{-1} - B$ be a standard embedding of the bouquet of three loops ℓ_1, ℓ_2, and ℓ_3. Then the corresponding embedding scheme is (ρ, λ), where $\rho_v = (\ell_1\ell_1^{-1}\ell_2\ell_2^{-1}\ell_3\ell_3^{-1})$ and $\lambda(\ell_1) = -1$, $\lambda(\ell_2) = \lambda(\ell_3) = 1$. Let ϕ be an element in $C^1(\mathfrak{B}_3 \hookrightarrow \mathbb{S}_k - B; \mathbb{Z}_2 \times \mathbb{Z}_2)$ such that $\phi(\ell_1) = (0, 1)$, $\phi(\ell_2) = (1, 0)$, and $\phi(\ell_3) = (1, 0)$. Let $\widetilde{W}$ be a closed walk in $\mathfrak{B}_3 \times_{\phi} (\mathbb{Z}_2 \times \mathbb{Z}_2)$. Then $p_{\phi}(\widetilde{W}) = e_1 \cdots e_m$ is a closed walk in $\mathfrak{B}_3$, and $\phi(e_1)\cdots\phi(e_m)$ must be the identity $(0,0)$. This implies that the edge ℓ_1 must appear an even number of times in the projection $p_\phi(\widetilde{W})$ of $\widetilde{W}$, which means $\widetilde{W}$ is $\tilde{\lambda}$-trivial. Therefore, the derived surface $\mathbb{S}_k^\phi$ is orientable.

We now aim to classify the orientable branched $m\mathbb{Z}_2$-covering surfaces $\mathbb{S}_k^\phi$ of the nonorientable surface $\mathbb{S}_k$ with branch set B. For each $\phi \in C^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$, we define the *net ϕ-voltage* of closed walk $C = e_1^{\varepsilon_1}e_2^{\varepsilon_2}\cdots e_n^{\varepsilon_n}$ in $\mathfrak{B}_{-k+b}$ by the product $\phi(e_1)^{\varepsilon_1}\phi(e_2)^{\varepsilon_2}\cdots\phi(e_n)^{\varepsilon_n}$. Let $(m\mathbb{Z}_2)_\phi(v)$ be the local group of all net ϕ-voltages occurring on v-based closed walks. Let $(m\mathbb{Z}_2)_\phi^0(v)$ be the set of the net ϕ-voltages on all closed walks C which are λ-trivial in the standard embedding $\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B$. Then $(m\mathbb{Z}_2)_\phi^0(v)$ is a subgroup of the local group $(m\mathbb{Z}_2)_\phi(v)$. By the definition of $C^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$, the local voltage group $(m\mathbb{Z}_2)_\phi(v)$ is equal to the group $m\mathbb{Z}_2$. It is known [4] that the derived surface $\mathbb{S}_k^\phi$ is orientable if and only if the group $(m\mathbb{Z}_2)_\phi^0(v)$ is a subgroup of index 2 in the group $m\mathbb{Z}_2$. We observe that the net ϕ-voltage of any closed walk C of $\mathfrak{B}_{-k+b}$ is an element of $(m\mathbb{Z}_2)_\phi^0(v)$ if and only if the closed walk C contains an even number of loops in the set $\{\ell_i : i = 1, \ldots, -k\}$.

Let $C_o^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ be the set of the voltage assignments $\phi \in C^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ such that the subgroup $(m\mathbb{Z}_2)_\phi^0(v)$ is of index 2 in the group $m\mathbb{Z}_2$. Thus for any ϕ in $C_o^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$, the derived surface $\mathbb{S}_k^\phi$ is orientable.

Now, let $C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ be the set of all elements in $C_o^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ such that

$$\phi(\ell_i) \in (m-1)\mathbb{Z}_2 \oplus \{1\} \text{ for each } i = 1, \ldots, -k, \text{ and}$$
$$\phi(\ell_j) \in (m-1)\mathbb{Z}_2 \oplus \{0\} \text{ for each } j = -k+1, \ldots, -k+b.$$

Then, for each $\phi \in C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$, the group $(m\mathbb{Z}_2)_\phi^0(v)$ is clearly isomorphic to the group $(m-1)\mathbb{Z}_2$. Notice that the voltage assignment given in our example is of this type. Let $\mathcal{S}$ be a subgroup of index 2 in $m\mathbb{Z}_2$. Then, it is not hard to show that there exists an automorphism $\sigma : m\mathbb{Z}_2 \to m\mathbb{Z}_2$ such that $\sigma(\mathcal{S}) = (m-1)\mathbb{Z}_2 \oplus \{0\}$. Therefore, the following lemma comes from Theorem 1.

LEMMA 4. *Let $\mathbb{S}_k$ be a nonorientable surface and let ϕ be an $m\mathbb{Z}_2$-voltage assignment in the set $C_o^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$. If the branched $m\mathbb{Z}_2$-covering $\tilde{p}_\phi : \mathbb{S}_k^\phi \to \mathbb{S}_k$ is orientable, then there exists $\psi \in C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ such that the two branched $m\mathbb{Z}_2$-coverings $\tilde{p}_\phi : \mathbb{S}_k^\phi \to \mathbb{S}_k$ and $\tilde{p}_\psi : \mathbb{S}_k^\psi \to \mathbb{S}_k$ are equivalent. Moreover, if $\phi, \psi \in C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$, then the two branched $m\mathbb{Z}_2$-coverings $\tilde{p}_\phi : \mathbb{S}_k^\phi \to \mathbb{S}_k$ and $\tilde{p}_\psi : \mathbb{S}_k^\psi \to \mathbb{S}_k$ are equivalent if and only if there exists an automorphism $\sigma : m\mathbb{Z}_2 \to m\mathbb{Z}_2$ such that $\sigma((m-1)\mathbb{Z}_2 \oplus \{0\}) = (m-1)\mathbb{Z}_2 \oplus \{0\}$.*

Consequently, $C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ contains representatives of all equivalence classes of orientable branched $m\mathbb{Z}_2$-coverings of the nonorientable surface $\mathbb{S}_k$.

It is not hard to show that there is a one-to-one correspondence between the set $C_{so}^1(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)$ and the set $M_{m\times(-k+b)}^o$ of all $m \times (-k+b)$-matrices over the field $\mathbb{Z}_p$ which satisfy the conditions $(1')$, $(2')$, $(3')$, and the following additional condition:

$$(4')\ \text{The } m\text{th row of the matrix in } M_{m\times(-k+b)}^o \text{ is } \big(\overbrace{1\ \cdots\ 1}^{-k}\ \overbrace{0\ \cdots\ 0}^{b}\big).$$

For convenience, let $\zeta_o(m,k,b)$ denote the cardinality of the set $M_{m\times(-k+b)}^o$, and let $\eta_o(m,k,b)$ denote the cardinality of the set of all elements in $M_{m\times(-k+b)}^o$ which satisfy the conditions $(1')$, $(2')$, and $(4')$.

Let $\mathrm{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})$ denote the subgroup of all automorphisms σ of $m\mathbb{Z}_2$ such that $\sigma((m-1)\mathbb{Z}_2 \oplus \{0\}) = (m-1)\mathbb{Z}_2 \oplus \{0\}$. Then

$$\begin{aligned}|\mathrm{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})| &= (2^{m-1}-1)(2^{m-1}-2)\cdots(2^{m-1}-2^{m-2})\,2^{m-1}\\ &= 2^{\frac{m(m-1)}{2}}\prod_{s=1}^{m-1}(2^s-1).\end{aligned}$$

Now, by Lemma 4 and Corollary 1, the number of equivalence classes of the orientable branched $m\mathbb{Z}_2$-coverings of the nonorientable surface $\mathbb{S}_k$ with branch set B is equal to

$$\frac{|C^1_{so}(\mathfrak{B}_{-k+b} \hookrightarrow \mathbb{S}_k - B; m\mathbb{Z}_2)|}{|\mathrm{Aut}(m\mathbb{Z}_2, (m-1)\mathbb{Z}_2 \oplus \{0\})|} = \frac{\zeta_o(m,k,b)}{2^{\frac{m(m-1)}{2}}\prod_{s=1}^{m-1}(2^s-1)}.$$

Here, by using a method similar to the proof of Lemmas 1 and 2, we can see the following:

LEMMA 5. *For a negative integer k and a non-negative integer b, we have:*

(1)

$$\eta_o(m,k,b) = \begin{cases} 1 & \text{if } m=1, b=0\\ 2^{\frac{m(m-1)}{2}}\prod_{s=1}^{m-1}(2^{-k-s}-1) & \text{if } -k \geq m, m \neq 1, b=0,\\ 2^{\frac{m(m-1)}{2}}\prod_{s=2}^{m}(2^{-k+b-s}-1) & \text{if } -k+b \geq m+1, m \neq 1, b \neq 0.\\ 0 & \text{otherwise.}\end{cases}$$

(2) $\zeta_o(m,k,b) = \sum_{t=0}^{b}(-1)^t\binom{b}{t}\eta_o(m,k,b-t)$. *In particular, if $-k < m$, then*

$$\zeta_o(m,k,b) = \sum_{t=0}^{-k+b-m}(-1)^t\binom{b}{t}\eta_o(m,k,b-t),$$

where $\binom{0}{0}$ is defined to be 1, and the summation over the empty index set is defined to be 0.

Now, by applying Lemmas 3 and 5, we obtain the following theorem.

THEOREM 4. *Let $\mathbb{S}_k$, $k < 0$, be a nonorientable surface, B a finite set of points in $\mathbb{S}_k$, and $b = |B|$. Then the number of equivalence classes of orientable branched $m\mathbb{Z}_2$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ of the surface $\mathbb{S}_k$ with branch set B is equal to*

$$\begin{cases} \dfrac{\prod_{s=1}^{m-1}(2^{-k-s}-1)}{\prod_{s=1}^{m-1}(2^{s}-1)} & \textit{if } i=1-2^{m-1}(k+2)\geq 0,\ b=0,\\ \dfrac{\sum_{t=0}^{b-1}(-1)^t\binom{b}{t}\prod_{s=2}^{m}(2^{-k+b-t-s}-1)+(-1)^b\prod_{s=1}^{m-1}(2^{-k-s}-1)}{\prod_{s=1}^{m-1}(2^{s}-1)} & \\ & \textit{if } i=1-2^{m-1}(k+2)+2^{m-2}b\geq 0,\ b\neq 0,\\ 0 & \textit{otherwise}, \end{cases}$$

where $\binom{0}{0}$ *is defined to be* 1*, and the product over the empty index set is defined to be* 1*.*

On the other hand, from Lemmas 1 and 2, we can see that the number of equivalence classes of branched $m\mathbb{Z}_2$-covering surfaces $\mathbb{S}_k^{\phi}$ of the nonorientable surface $\mathbb{S}_k$ with branch set B is equal to

$$\begin{cases} \dfrac{\prod_{s=0}^{m-1}(2^{-k-s}-1)}{\prod_{s=1}^{m}(2^{s}-1)} & \text{if} \quad b=0,\\ \dfrac{\sum_{t=0}^{b}(-1)^t\binom{b}{t}\eta(m,k,b-t)}{2^{\frac{m(m-1)}{2}}\prod_{s=1}^{m}(2^{s}-1)} & \text{if} \quad b\neq 0. \end{cases}$$

Now, the number of equivalence classes of nonorientable branched $m\mathbb{Z}_2$-coverings $p : \mathbb{S}_i \to \mathbb{S}_k$ of the nonorientable surface $\mathbb{S}_k$ with branch set B comes from the above discussion and Theorem 4.

THEOREM 5. *Let* $\mathbb{S}_k$*,* $k < 0$*, be a nonorientable surface and* B *a finite set of points in* $\mathbb{S}_k$ *and* $b = |B|$*. Then the number of equivalence classes of nonorientable branched* $m\mathbb{Z}_2$*-coverings* $p : \mathbb{S}_i \to \mathbb{S}_k$ *of the surface* $\mathbb{S}_k$ *with branch set* B *is equal to*

TABLE 3.
$R_{(\mathbb{S}_{-1},B,m\mathbb{Z}_p)}(x)$.

$m, p)$	$b=0$	$b=1$	$b=2$	$b=3$	$b=4$	$b=5$	$\cdots$
(1, 2)	1	0	$2x^{-2}$	0	$2x^{-4}$	0	$\cdots$
(1, 3)	0	x^{-1}	$2x^{-3}$	$4x^{-5}$	$8x^{-7}$	$16x^{-9}$	$\cdots$
(1, 5)	0	x^{-1}	$4x^{-5}$	$16x^{-9}$	$64x^{-13}$	$256x^{-17}$	$\cdots$
(1, 7)	0	x^{-1}	$6x^{-7}$	$36x^{-13}$	$216x^{-19}$	$1296x^{-25}$	$\cdots$
$\vdots$				$\vdots$			
(2, 2)	0	0	1	$4x^{-4}$	x^3+12x^{-6}	$40x^{-8}$	$\cdots$
(2, 3)	0	0	x^{-5}	$10x^{-11}$	$84x^{-17}$	$680x^{-23}$	$\cdots$
(2, 5)	0	0	x^{-17}	$28x^{-37}$	$688x^{-57}$	$16576x^{-77}$	$\cdots$
(2, 7)	0	0	x^{-37}	$54x^{-79}$	$2628x^{-121}$	$126360x^{-163}$	$\cdots$
$\vdots$				$\vdots$			
(3, 2)	0	0	0	x^3	$3x^5+8x^{-10}$	$10x^7+80x^{-14}$	$\cdots$
(3, 3)	0	0	0	x^{-29}	$36x^{-47}$	$1020x^{-65}$	$\cdots$
(3, 5)	0	0	0	x^{-177}	$152x^{-277}$	$19536x^{-337}$	$\cdots$
(3, 7)	0	0	0	x^{-541}	$396x^{-835}$	$138060x^{-1129}$	$\cdots$
$\vdots$				$\vdots$			

$$\begin{cases} \dfrac{(2^{-k}-2^m)\prod_{s=1}^{m-1}(2^{-k-s}-1)}{\prod_{s=1}^{m}(2^s-1)} & \textit{if } i = 2^m(k+2)-2<0,\ b=0, \\[2ex] \dfrac{\sum_{t=0}^{b-1}(-1)^t\binom{b}{t}\prod_{s=2}^{m}(2^{-k+b-t-s}-1)(2^{-k+b-t-1}-2^m)+(-1)^b(2^{-k}-2^m)\prod_{s=1}^{m-1}(2^{-k-s}-1)}{\prod_{s=1}^{m}(2^s-1)} & \\ \qquad \textit{if } i = 2^m(k+2)-2(1+2^{m-2}b)<0,\ b\neq 0, & \\[2ex] 0 & \textit{otherwise}, \end{cases}$$

where $\binom{0}{0}$ *is defined to be* 1*, and the product over the empty index set is defined to be* 1*.*

Table 3 lists the branched $m\mathbb{Z}_p$-covering distribution polynomial $R_{(\mathbb{S},B,m\mathbb{Z}_p)}(x)$, when $\mathbb{S}$ is the projective plane $\mathbb{S}_{-1}$.

ACKNOWLEDGEMENT

The authors would like to express their gratitude to the referee for valuable comments.

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Received 15 February 2000 and accepted 3 July 2001

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