NORTH-HOLLAND

Permanental Mates and Hwang's Conjecture

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ABSTRACT

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. Two unequal matrices $A, B \in \Omega_n$ are said to form a permanental pair if $\text{per}[tA + (1 - t)B]$ is constant for all $t \in [0, 1]$, in which case A and B are called permanental mates of each other. We characterize the sets of all matrices in Ω_3 and Ω_4 having their transpose as permanental mates. Using a generalization of the results, we disprove Hwang's conjecture which states that for $n \geq 4$, any matrix in the interior of Ω_n has no mate.

1. INTRODUCTION

Let Ω_n denote the set of all $n \times n$ doubly stochastic matrices. Let J_n denote the $n \times n$ matrix all whose entries equal $1/n$, and let I_n be the $n \times n$ identity matrix. According to Wang [5], two matrices $A, B \in \Omega_n$ where $A \neq B$ are said to form a permanental pair if $\text{per}[tA + (1 - t)B]$ is constant for all $t \in [0, 1]$, in which case A and B are called permanental mates or simply mates of each other. In the above definition, if $A = B$, Hwang [3] called A and B trivial mates, and otherwise nontrivial mates. Throughout this paper, we follow the definition of Wang [5] for mates, and by a mate we mean a nontrivial mate according to Hwang [3]. From the definition of mates it follows that matrices in Ω_2 have no mates. Wang [5]

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showed: (1) for every $n \geq 3$, there exist infinitely many permanental pairs, and (2) permutation matrices have no mates. Later, Brenner and Wang [1] proved that if $A \in \Omega_n$ has the form $P(J_{n_1} \oplus \cdots \oplus J_{n_k})Q$, where $\oplus$ denotes the direct sum and P and Q are permutation matrices, $k \geq 1$, $1 \leq n_i \leq n$, $n_1 + \cdots + n_k = n$, then A has no mate, and they also raised the question whether there are other doubly stochastic matrices which have no mate. Answering this question, Gibson [2] showed that

$$B = \begin{bmatrix} x & 1-x \\ 1-x & x \end{bmatrix} \oplus 1, \quad 0 \leq x \leq 1,$$

has no mate.

Recently, Hwang [3] investigated some properties of the set of all mates of $A \in \Omega_n$ and proposed the conjecture that, for $n \geq 4$, no matrix in the interior of Ω_n has a mate. We disprove this conjecture for all $n \geq 4$, by providing a class of counter examples. To develop this class, we first, characterize the set of all matrices in Ω_3 and Ω_4 having their transpose as mates and then generalize these results.

In this paper, for brevity, let us use the notation $M(a, b; c, d)$ to denote the 3×3 doubly stochastic matrix

$$\begin{bmatrix} a & b & 1-a-b \\ c & d & 1-c-d \\ 1-a-c & 1-b-d & a+b+c+d-1 \end{bmatrix}.$$

Let A^T denote the transpose of A, and let the trace of A be denoted by $\operatorname{tr} A$. Let $s(A)$ denote the sum of the entries of A. An $n \times n$ doubly stochastic matrix $A = (a_{ij})$ is called a k-matching matrix if there exists precisely one $k\,(1 \leq k \leq n)$ such that $a_{kj} = a_{jk}$ for $j = 1, 2, \ldots, n$. Let $M_{4,k}$ denote the set of all k-matching matrices in Ω_4, and let T denote a real valued function on $M_{4,k}$ defined by

$$T(A) = 4a_{kk}^2 - 2a_{kk} \operatorname{tr} A + 1 - 2\sum_{\substack{i=1 \\ i \neq k}}^{4} a_{ki}^2$$

for $A = (a_{ij}) \in M_{4,k}$. Following essentially the same notation used by Minc [4], for integers $r, n\,(1 \leq r \leq n)$, let $Q_{r,n}$ denote the set of all sequences $(i_1, \ldots, i_r)$ such that $1 \leq i_1 < \cdots < i_r \leq n$. For fixed $\alpha, \beta \in Q_{r,n}$, let $A(\alpha \mid \beta)$ be the submatrix of A obtained by deleting the rows α and columns β of A, and let $A[\alpha \mid \beta]$ denote the submatrix of A formed by the rows α and columns β of A.

In this paper, we frequently use the following result (Theorem 1.4 in Minc [4]).

THEOREM 1.1. *Let $A = (a_{ij})$ and $B = (b_{ij})$ be $n \times n$ matrices. Then*

$$\operatorname{per}(A+B) = \sum_{r=0}^{n} \sum_{\alpha,\beta \in Q_{r,n}} S_r(A,B),$$

where

$$S_r(A,B) = \operatorname{per} A[\alpha \mid \beta] \operatorname{per} B(\alpha \mid \beta)$$

and

$$\operatorname{per} A[\alpha \mid \beta] = 1 \quad \textit{when} \quad r = 0 \textit{ and } \quad \operatorname{per} B(\alpha \mid \beta) = 1 \textit{ when } r = n.$$

2. MATRICES HAVING THE TRANSPOSE AS MATES

In this section, we investigate the possibility for $A \in \Omega_n$ to have A^T as its mate. We are able to complete the analysis for the cases $n = 3$ and 4. First we find a necessary and sufficient condition for A^T to be a mate of $A \in \Omega_3$.

THEOREM 2.1. *Let $A \in \Omega_3$ and let A be asymmetric. Then A^T is a mate of A if and only if $\operatorname{tr} A = \frac{3}{2}$.*

PROOF. Let $A = M(a,b;c,d)$ where $b \neq c$. Let $\operatorname{per} A = f(a,b,c,d)$. Then straightforward calculations show that

$$\begin{aligned} f(a,b,c,d) &= a+b+c+d-2(a+d)(b+c) \\ &\quad + 2ad\left(a+d+2b+2c-\tfrac{5}{2}\right) + 2bc\left(2a+2d+b+c-\tfrac{5}{2}\right). \end{aligned}$$

Let $\phi(t) = \operatorname{per}\left[tA + (1-t)A^T\right]$. Clearly, $\phi(t) = f(a, c+\delta, b-\delta, d)$, where $\delta = t(b-c)$. A^T is a mate of A iff $\phi(t)$ is constant for all $t \in [0,1]$. Hence

$$\phi(t) = \phi(1) = \operatorname{per} A,$$

i.e.,

$$f(a, c+\delta, b-\delta, d) = f(a,b,c,d).$$

On simplification, this condition reduces to

$$[(c+\delta)(b-\delta) - bc](2a+2d+b+c-\tfrac{5}{2}) = 0,$$

i.e.,

$$t(1-t)(b-c)^2(\operatorname{tr} A - \tfrac{3}{2}) = 0.$$

Since $b \neq c$, this is true for all $t \in [0,1]$ iff $\operatorname{tr} A = \frac{3}{2}$. ■

The fact that permanental mates of the form (A, A^T) exist for all $n \geq 4$ is easily seen by taking A as the direct sum of any collection of the following three types of matrices including at least one matrix of type (a):

(a) Asymmetric matrices in Ω_3 with trace $\frac{3}{2}$.
(b) Matrices in Ω_2.
(c) The identity matrix of order 1.

Are these the only matrices in Ω_n $(n \geq 4)$, up to permutations of rows and columns, forming permanental pairs with their transpose? We answer this question negatively by considering the following example:

$$A = \begin{bmatrix} \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} & b & c \\ \frac{1}{4} & c & \frac{1}{2} & b \\ \frac{1}{4} & b & c & \frac{1}{2} \end{bmatrix}, \tag{2.1}$$

where $b + c = \frac{1}{4}$ and $b \neq c$. By straightforward calculations, we find that

$$\operatorname{per}[tA + (1-t)A^T] = \tfrac{15}{128}.$$

Next, we shall derive necessary and sufficient conditions for $A \in \Omega_n$ $(A \neq A^T)$ to have A^T as its mate. For this, we need the following two lemmas.

LEMMA 2.1. *If B is a nonzero skew symmetric matrix of order 4 such that each row has sum 0 and* $\operatorname{per} B = 0$, *then precisely one row (and therefore one corresponding column) of B is zero.*

PROOF. Let

$$B = \begin{bmatrix} 0 & x & y & -(x+y) \\ -x & 0 & z & x-z \\ -y & -z & 0 & y+z \\ x+y & z-x & -(y+z) & 0 \end{bmatrix}.$$

Expanding $\operatorname{per} B$ along the last row, we have

$$\operatorname{per} B = 2\left[(x+y)^2 z^2 + (z-x)^2 y^2 + (y+z)^2 x^2\right].$$

Since $\operatorname{per} B = 0$, it follows that

$$(x+y)z = (z-x)y = (y+z)x = 0. \tag{2.2}$$

Since $B \neq 0$, it is easy to see from (2.2) that precisely one pair among x, y, z is zero or $x + y = z - x = 0$. This is equivalent to saying that precisely one row of B is zero. ∎

LEMMA 2.2. *Let*

$$A = \begin{bmatrix} a & b & c & d \\ b & b_2 & b_3 & b_4 \\ c & c_2 & c_3 & c_4 \\ d & d_2 & d_3 & d_4 \end{bmatrix} \in M_{4,1} \tag{2.3}$$

and

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & -1 & 0 \end{bmatrix}. \tag{2.4}$$

Then we have

$$\operatorname{per} B = 0, \tag{2.5}$$

$$S_1(A^T, B) = 0, \tag{2.6}$$

$$S_2(A^T, B) = T(A). \tag{2.7}$$

PROOF. Obviously, $\operatorname{per} B = 0$ and $S_1(A^T, B) = 0$. By direct computation,

$$\begin{aligned} S_2(A^T, B) = & -(ab_2 + b^2) + (ac_2 + bc) + (ad_2 + bd) \\ & + (ab_3 + bc) - (ac_3 + c^2) + (ad_3 + cd) \\ & + (ab_4 + bd) + (ac_4 + cd) - (ad_4 + d^2). \end{aligned}$$

In view of the fact that each row sum of A is 1, we have

$$\begin{aligned} S_2(A^T, B) &= a(3a + 2 - 2\operatorname{tr} A) + (2bc + 2cd + 2db - b^2 - c^2 - d^2) \\ &= a(3a + 2 - 2\operatorname{tr} A) + (1 - a)^2 - 2(b^2 + c^2 + d^2) \\ &= 4a^2 - 2a\operatorname{tr} A + 1 - 2(b^2 + c^2 + d^2) \\ &= T(A). \end{aligned}$$

∎

Now we state and prove our main result of this section.

THEOREM 2.2. *Let $A = (a_{ij}) \in \Omega_4$ and $A \neq A^T$. Then A^T is a mate of A iff $A \in M_{4,k}$ and $T(A) = 0$.*

PROOF. A necessary condition for A and A^T to be mates is $\operatorname{per}(A - A^T) = 0$ (see Hwang [3]). Since $A - A^T$ is nonzero skew symmetric, it follows from Lemma 2.1 that precisely one row, say the kth row of $A - A^T$, is 0, which is equivalent to saying that $A \in M_{4,k}$.

Without loss of generality, we can assume $k = 1$, since PAP^T and $(\mathrm{PAP}^T)^T$ are mates iff A and A^T are mates for some permutation matrix P. Therefore, we can take A as defined in (2.3). Let

$$f(t) = \operatorname{per}[tA + (1 - t)A^T]. \tag{2.8}$$

Since A is doubly stochastic, $A - A^T$ can be expressed as $(b_3 - c_2)B$, where B is as given in (2.4). By applying Theorem 1.1 and making use of (2.5) and (2.6), we find

$$\begin{aligned} f(t) &= \operatorname{per}[A^T + t(b_3 - c_2)B] \\ &= \operatorname{per} A^T + t(b_3 - c_2)S_3(A^T, B) \\ &\quad + t^2(b_3 - c_2)^2 S_2(A^T, B). \end{aligned} \tag{2.9}$$

Since $b_3 \neq c_2$, a necessary condition for $f(t)$ to be constant is $S_2(A^T, B) = 0$, which implies $T(A) = 0$ in view of (2.7).

We claim that vanishing of $S_2(A^T, B)$ is sufficient for $f(t)$ to be a constant. If we put $S_2(A^T, B) = 0$ in (2.9), we have

$$f(1) = \operatorname{per} A^T + (b_3 - c_2)S_3(A^T, B), \tag{2.10}$$

whereas from (2.8)

$$f(1) = \operatorname{per} A = \operatorname{per} A^T. \tag{2.11}$$

From (2.10) and (2.11), it follows that $S_3(A^T, B) = 0$. Therefore, $f(t)$ is constant. This completes the proof. ■

From the above theorem, we can say that the necessary and sufficient condition for $A \in M_{4,1}$ given by (2.3) to form a permanental pair with A^T is

$$T(A) = 4a^2 - 2a \operatorname{tr} A + 1 - 2(b^2 + c^2 + d^2) = 0. \tag{2.12}$$

Next, we shall use the above condition in order to get another simple necessary condition for $A \in \Omega_4$ to have A^T as its mate.

COROLLARY 2.1. *If $A \in \Omega_4$ forms a permanental pair with A^T, then* $\operatorname{tr} A \leq \frac{5}{2}$.

PROOF. By Theorem 2.2, A should be in $M_{4,k}$. Thus by taking A as in (2.3),

$$\begin{aligned} \operatorname{tr} A &= a + b_2 + c_3 + d_4 \\ &\leq a + (1-b) + (1-c) + (1-d) \\ &= 2 + 2a. \end{aligned}$$

Therefore,

$$a \geq \frac{\operatorname{tr} A}{2} - 1. \tag{2.13}$$

On the contrary, if we assume $\operatorname{tr} A > \frac{5}{2}$, then from (2.13), $a > \frac{1}{4}$. Also, using the condition (2.12),

$$\begin{aligned} 0 \leq b^2 + c^2 + d^2 &= 2a^2 - a \operatorname{tr} A + \tfrac{1}{2} \\ &< 2a^2 - \tfrac{5}{2}a + \tfrac{1}{2} \\ &= 2(a-1)(a - \tfrac{1}{4}), \end{aligned}$$

which is a contradiction, since $\frac{1}{4} < a \leq 1$. The proof is complete. ∎

It may be noted that $\operatorname{tr} A \leq \frac{5}{2}$ is a necessary condition and not sufficient for A, A^T to be mates when $A \in \Omega_4$. For example, consider

$$A = \begin{bmatrix} 0.8 & 0 & 0.2 & 0 \\ 0 & 0.4 & 0 & 0.6 \\ 0.2 & 0.3 & 0.3 & 0.2 \\ 0 & 0.3 & 0.5 & 0.2 \end{bmatrix} \in M_{4,1}.$$

Simple calculations show that $T(A) = 0.96 \neq 0$. Hence, (A, A^T) is not a permanental pair.

Next, we shall deduce Theorem 2.1 from Theorem 2.2.

Let $A = 1 \oplus B$, where $B \in \Omega_3$ and $B \neq B^T$. It may be noted that $A \in M_{4,1}$. If $\operatorname{tr} B = \frac{3}{2}$, then $\operatorname{tr} A = \frac{5}{2}$ and we see that the condition (2.12) is satisfied. Hence A^T is a mate of A which implies that B^T is a mate of B. Conversely, if B^T is a mate of B, then A^T is a mate of A. Hence, applying the condition $T(A) = 0$, we get $\operatorname{tr} A = \frac{5}{2}$, which implies that $\operatorname{tr} B = \frac{3}{2}$. From this argument, Theorem 2.1 follows.

Next, we investigate the existence of $A \in \Omega_4$ forming a permanental pair with its transpose for any given $k \leq \frac{5}{2}$ such that $\operatorname{tr} A = k$. Let us consider three cases.

Case (*i*). Let $0 \leq k \leq 1$. Choose

$$A = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{k}{2} & \frac{5}{6} - \frac{k}{2} & 0 \\ \frac{1}{6} & \frac{1}{2} - \frac{k}{2} & \frac{k}{2} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & 0 & 0 \end{bmatrix}.$$

It is easy to verify that $A \in M_{4,1}$, satisfying $T(A) = 0$. Hence A^T is a mate of A.

Case (*ii*). Let $1 \leq k < 2$. Choose

$$A = \begin{bmatrix} 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ \frac{1}{6} & \frac{k}{3} + \frac{1}{6} & \frac{2}{3} - \frac{k}{3} & 0 \\ \frac{1}{6} & 0 & \frac{k}{3} + \frac{1}{6} & \frac{2}{3} - \frac{k}{3} \\ \frac{2}{3} & \frac{2}{3} - \frac{k}{3} & 0 & \frac{k}{3} - \frac{1}{3} \end{bmatrix}.$$

Since $A \in M_{4,1}$ satisfying $T(A) = 0$, A forms a permanental pair with A^T.

Case (*iii*). Let $2 \leq k \leq \frac{5}{2}$. Let us try to find A of the form

$$A = \begin{bmatrix} a & b & b & b \\ b & \alpha & \beta & 0 \\ b & 0 & \alpha & \beta \\ b & \beta & 0 & \alpha \end{bmatrix}$$

where

$$b = \frac{1-a}{3}, \qquad \alpha = \frac{k-a}{3}, \qquad \beta = \frac{2+2a-k}{3}.$$

Let us choose a such that $T(A) = 0$. Hence

$$4a^2 - 2ak + 1 = 6b^2.$$

Substituting for b in terms of a and solving for a, we get

$$a = \frac{(3k-2) \pm (9k^2 - 12k - 6)^{1/2}}{10}.$$

Choosing $a = [(3k-2) + (9k^2 - 12k - 6)^{1/2}]/10$, it can be verified that $a, b, \alpha \geq 0$ and $\beta > 0$ (if $\beta = 0$, then $A = A^T$).

For $k \in [2, \frac{5}{2}]$, we have $9k^2 - 12k - 6 > 0$ and $3k - 2 > 0$. Therefore, $a \geq 0$. By routine calculations, we have $b \geq 0$ iff $a \leq 1$, i.e., iff $(9k^2 - 12k - 6)^{1/2}$

$\leq 12 - 3k$, i.e., iff $k \leq \frac{5}{2}$, which is true. Similarly, we find that $k \geq a$ and $2 + 2a > k$, and conclude that $\alpha \geq 0$ and $\beta > 0$.

This completes our investigation of the possibility of finding $A \in \Omega_4$ $(A \neq A^T)$ having A^T as its mate for any given $k \leq \frac{5}{2}$ such that $\operatorname{tr} A = k$.

It may be observed that if $A \in M_{4,1}$ is of the form (2.3) with $\operatorname{tr} A = \frac{5}{2}$, then the condition $T(A) = 0$ implies

$$2(b^2 + c^2 + d^2) = (4a - 1)(a - 1), \tag{2.14}$$

and from the condition (2.13), that $a \geq \frac{1}{4}$. If we choose $a \in (\frac{1}{4}, 1)$, the condition (2.14) is not satisfied. Hence, $a = \frac{1}{4}$ or 1. If $a = \frac{1}{4}$ then $b^2 + c^2 + d^2 = 0$, which implies $b = c = d = 0$, a contradiction. Hence $a = 1$. Thus, if $\operatorname{tr} A = \frac{5}{2}$, then A can form a permanental pair with A^T iff $A = 1 \oplus B$, where $B \in \Omega_3, B \neq B^T$ with $\operatorname{tr} B = \frac{3}{2}$, up to permutations of rows and columns.

Thus we conclude that a matrix $A \in \Omega_3$ having A^T as its mate can be found only when $\operatorname{tr} A = \frac{3}{2}$, whereas a matrix $A \in \Omega_4$ having A^T as its mate can be found whenever $0 \leq \operatorname{tr} A \leq \frac{5}{2}$.

From the result we derived in Theorem 2.2, we have the following question. For $A \in \Omega_n, n \geq 5$ $(A \neq A^T)$, is it necessary that $n-3$ rows (or at least one row) of A be identical with the corresponding columns, in order for A^T to be a mate of A? For $n \geq 6$, which is a multiple of 3, it is not necessary, because if A is a matrix which is a direct sum of asymmetric matrices in Ω_3, each with trace $\frac{3}{2}$, then A and A^T are mates without having even a single row of A identical with the corresponding column. If n is not a multiple of 3, it may be necessary that at most two rows of A be identical with the corresponding columns. We do not go into details about this problem.

However, we state and prove a generalized version of Theorem 2.2.

THEOREM 2.3. *Let*

$$A = \begin{bmatrix} X & U \\ U^T & Y \end{bmatrix} \in \Omega_n,$$

where X is symmetric and of order $(n-3) \times (n-3)$, U is $(n-3) \times 3$, and Y is $3 \times 3 (Y \neq Y^T)$. A necessary and sufficient condition for A and A^T to be mates is

$$(\operatorname{per} X)[s(Y) - 2 \operatorname{tr} Y] + \sum_{i,j=1}^{n-3} \langle u_i, u_j \rangle \operatorname{per} X(i \mid j) = 0, \tag{2.15}$$

where $s(Y)$ denotes the sum of all elements in Y, and where

$$\langle u_i, u_j \rangle = u_i(3J_3 - 2I_3)u_j^T$$

for the ith *row u_i of U.*

PROOF. Let $f(t) = \operatorname{per}[tA + (1-t)A^T]$. For the given form of A, row sums and column sums of $Y = (y_{ij})$ are equal. Therefore, row sums and column sums of the skew symmetric matrix $Y - Y^T$ are zero. Hence $Y - Y^T$ can be expressed as δB, where $\delta = (y_{12} - y_{21}) \neq 0$ and

$$B = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Now $f(t) = \operatorname{per}(t\delta D + A^T)$, where D is the direct sum of the $(n-3) \times (n-3)$ null matrix and B. Therefore, by applying Theorem 1.1, we get

$$f(t) = \operatorname{per} A^T + t\delta S_1(D, A^T) + t^2\delta^2 S_2(D, A^T),$$

since $S_3(D, A^T) = 0$ because $\operatorname{per} B = 0$, and $S_k(D, A^T) = 0$ for $k = 4, 5, \dots n$ because D has only three nonzero rows.

A necessary condition for $f(t)$ to be a constant is $S_2(D, A^T) = 0$. Let $U^{(s)}$ denote the sth column of $U = (u_{ij})$. By direct calculations, we have

$$S_2(D, A^T) = \sum_{\substack{r,s=1 \\ r\neq s}}^{3} \operatorname{per}\begin{pmatrix} X & U^{(s)} \\ (U^{(r)})^T & y_{sr} \end{pmatrix} - \sum_{r=1}^{3} \operatorname{per}\begin{pmatrix} X & U^{(r)} \\ (U^{(r)})^T & y_{rr} \end{pmatrix}.$$

Expanding each of the permanents in the summation along the last column and then each subpermanent of order $n-3$, except $\operatorname{per} X$, along the last row,

$$\begin{aligned} S_2(D, A^T) &= (\operatorname{per} X)\{(-y_{11} + y_{21} + y_{31}) + (y_{12} - y_{22} + y_{32}) \\ &\qquad + (y_{13} + y_{23} - y_{33})\} \\ &\quad + \sum_{i,j=1}^{n-3} \{u_{i1}(-u_{j1} + u_{j2} + u_{j3}) + u_{i2}(u_{j1} - u_{j2} + u_{j3}) \\ &\qquad + u_{i3}(u_{j1} + u_{j2} - u_{j3})\} \operatorname{per} X(i \mid j) \\ &= (\operatorname{per} X)\{s(Y) - 2 \operatorname{tr} Y\} \\ &\quad + \sum_{i,j=1}^{n-3} [u_{i1}\, u_{i2}\, u_{i3}] \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_{j1} \\ u_{j2} \\ u_{j3} \end{bmatrix} \operatorname{per} X(i \mid j) \\ &= (\operatorname{per} X)\{s(Y) - 2 \operatorname{tr} Y\} + \sum_{i,j=1}^{n-3} \langle u_i, u_j \rangle \operatorname{per} X(i \mid j). \end{aligned}$$

Therefore, the condition (2.15) follows.

We claim that the vanishing of $S_2(D, A^T)$ is sufficient for $f(t)$ to be a constant, by the same argument adopted in the proof of Theorem 2.2. This completes the proof. ■

We conclude this section by noting that the above generalized result can be used to derive the conditions for $A \in \Omega_n\,(n = 3, 4)$ to form a permanental pair with A^T.

If we take $X = I_{n-3}$ and $U = 0$ in Theorem 2.3, we have $A = I_{n-3} \oplus Y$, and now the condition for A^T to be a mate of A and hence for Y^T to be a mate of Y reduces to $s(Y) - 2\,\mathrm{tr}\,Y = 0$, i.e., $\mathrm{tr}\,Y = \frac{3}{2}$.

For $n = 4$, let us take $X = [a]$, a 1×1 matrix, and $U = [b \quad c \quad d]$. Using the convention that $\mathrm{per}\,X(\alpha \mid \beta) = 1$ when $\alpha, \beta \in Q_{n,n}$, we have $\mathrm{per}\,X(1 \mid 1) = 1$. From (2.15), we find, after simple calculations, that the condition reduces to

$$4a^2 - 2a\,\mathrm{tr}\,A + 1 = 2(b^2 + c^2 + d^2),$$

which is the one we got earlier in (2.12).

3. COUNTEREXAMPLES TO HWANG'S CONJECTURE

In this section we disprove a recent conjecture on permanental mates due to Hwang [3]. The instrument behind the class of counterexamples is our generalized result in the previous section.

Hwang [3] denotes the set of all mates of A for $A \in \Omega_n$ by Mte(A), and the interior of Ω_n by Int Ω_n. In concluding remarks, Hwang [3] asked whether there exists an $A \in \mathrm{Int}\,\Omega_n$ with a nontrivial mate; he answered the question by saying yes for the case $n = 3$ and gave the following example. Let

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \qquad C_t = \frac{1}{4}\begin{bmatrix} 2 & t & 2-t \\ 2-t & 2 & t \\ t & 2-t & 2 \end{bmatrix}, \quad 0 \le t \le 2.$$

Then $B \in \mathrm{Int}\Omega_3$ and $C_t \in \mathrm{Mte}(B)$ for all $t, 0 \le t \le 2$. Of course, for $t = 1, C_t$ is the trivial mate of B.

For large numbers, Hwang thought that an affirmative answer to the question regarding the existence of matrices in Int Ω_n having nontrivial mates was unlikely. Therefore, Hwang proposed the following conjecture: For $n \ge 4$, if $A \in \mathrm{Int}\,\Omega_4$, then Mte($A$) is a trivial set.

It may be noted that $A \in \Omega_4$ given in (2.1) with $b, c > 0$ is itself a class of counterexamples to Hwang's conjecture, since $A \in \mathrm{Int}\,\Omega_4$ and $\mathrm{per}[tA + (1-t)A^T] = \frac{15}{128}$.

Now, we disprove Hwang's conjecture for $n \geq 4$. Let us construct

$$A = (a_{ij}) = \begin{bmatrix} X & U \\ U^T & Y \end{bmatrix} \in \Omega_n$$

as follows. Let $X = (x_{ij})$ be of order $(n-3) \times (n-3)$ with

$$x_{ij} = x = \frac{1}{(1+\varepsilon)(n-3)}, \qquad i,j = 1,\ldots,n-3, \tag{3.1}$$

where $\varepsilon > 0$ and

$$9 - (3n-18)\varepsilon - (n-3)^2\varepsilon^2 > 0. \tag{3.2}$$

We can always choose $\varepsilon > 0$ and arbitrarily small satisfying (3.1), and this condition is crucial for our construction of A. Let $U = (u_{ij})$ be of order $(n-3) \times 3$ with

$$u_{ij} = u = \frac{\varepsilon}{3(1+\varepsilon)} \qquad \text{for } i = 1,\ldots,(n-3), \quad j = 1,2,3.$$

Clearly, $x > 0, u > 0$, and $(n-3)x + 3u = 1$, so that the row sums and the column sums are 1 for the first $n-3$ rows and $n-3$ columns of A.

Let us construct Y as a 3×3 positive asymmetric matrix with

$$\operatorname{tr} Y = \frac{9 - (3n-18)\varepsilon + (n-3)^2\varepsilon^2}{6(1+\varepsilon)}. \tag{3.3}$$

In view of (3.2), $\operatorname{tr} Y > 0$. Also,

$$\begin{aligned} s(Y) - \operatorname{tr} Y &= \left[3 - \frac{(n-3)\varepsilon}{1+\varepsilon}\right] - \frac{9 - (3n-18)\varepsilon + (n-3)^2\varepsilon^2}{6(1+\varepsilon)} \\ &= \frac{9 - (3n-18)\varepsilon - (n-3)^2\varepsilon^2}{6(1+\varepsilon)} > 0. \end{aligned}$$

Since $s(Y) > \operatorname{tr} Y > 0$, it is possible for us to construct Y as a 3×3 positive asymmetric matrix with $\operatorname{tr} Y$ as given in (3.3) so that the matrix A is in the interior of Ω_n. By suitable choice of ε and Y, we can obtain a class of matrices A in the interior of Ω_n satisfying all the conditions of Theorem 2.3.

For such A's, let us verify that the condition (2.15) is satisfied. We give below the details of the calculations:

$$\begin{aligned} \operatorname{per} X &= (n-3)!x^{n-3}, \\ s(Y) - 2\operatorname{tr} Y &= \frac{-(n-3)^2\varepsilon^2}{3(1+\varepsilon)}. \end{aligned}$$

For $i,j = 1, 2, \ldots, n-3$,

$$\langle u_i, u_j \rangle = 3u^2 = \frac{\varepsilon^2}{3(1+\varepsilon)^2}$$

and

$$\operatorname{per} X\langle i \mid j \rangle = x^{n-4}(n-4)!.$$

Substituting these values in the left side of the equation (2.15), we have

$$-x^{n-3}(n-3)!\frac{(n-3)^2\varepsilon^2}{3(1+\varepsilon)} + (n-3)^2\frac{\varepsilon^2}{3(1+\varepsilon)^2}x^{n-4}(n-4)!,$$

which reduces to 0. Hence, we conclude by Theorem 2.3 that A^T is a mate of A. Thus we have shown that Hwang's conjecture is false for any $n \geq 4$, and we arrive at the following theorem.

THEOREM 3.1. *For every $n \geq 3$, there exist infinitely many matrices $A \in$* Int Ω_n *such that* Mte(A) *is nontrivial.*

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