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On a generalization of uniformly convex and related functions

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ABSTRACT

In this paper, we define and study some subclasses of analytic functions by using the concept of k-uniformly convexity. Several interesting properties, coefficients and radius problems are investigated. The behaviour of these classes under a certain integral operator is also studied. We indicate the relevant connections of our results with various known ones.

1. Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disc $E = \{z : |z| < 1\}$. Let S denote the class of all functions in $\mathcal{A}$ which are univalent in E. Also, let S^*_γ, C_γ be the subclasses of S which consist of starlike and convex functions of order $\gamma (0 \le \gamma < 1)$ respectively. For details, see [1]. Kanas and Wisniowska [2,3] studied the classes of k-uniformly convex functions, denoted by $k - UCV$, and the corresponding class of $k - ST$ related by the Alexander type relation.

For $0 \le k < \infty$, define the domain Ω_k as follows: see [4],

$$\Omega_k = \{u + \mathrm{i}v : u > k\sqrt{(u-1)^2 + v^2}\}. \tag{1.2}$$

For fixed k, Ω_k represents the conic region bounded, successively, by the imaginary axis ($k = 0$), the right branch of hyperbola ($0 < k < 1$), a parabola ($k = 1$) and an ellipse ($k > 1$).

Also, we note that, for no choices of k ($k > 1$), Ω_k reduces to a disc. We define the domain $\Omega_{k,\gamma}$, see [5], as

$$\Omega_{k,\gamma} = (1-\gamma)\Omega_k + \gamma, \quad (0 \le \gamma < 1).$$

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The following functions, denoted by $p_{k,\gamma}(z)$, are univalent in E and map E onto $\Omega_{k,\gamma}$ such that $p_{k,\gamma}(0)=1$ and $p'_{k,\gamma}(0)>0$:

$$p_{k,\gamma}(z)=\begin{cases}\dfrac{1+(1-2\gamma)z}{1-z} & (k=0),\\ 1+\dfrac{2(1-\gamma)}{\pi^2}\left(\log\dfrac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2, & (k=1)\\ 1+\dfrac{2(1-\gamma)}{1-k^2}\sin h^2\left[\left(\dfrac{2}{\pi}\arccos k\right)\arctan h\sqrt{z}\right], & (0<k<1)\\ 1+\dfrac{(1-\gamma)}{k^2-1}\sin\left(\dfrac{\pi}{2R(t)}\displaystyle\int_0^{\frac{u(z)}{\sqrt{t}}}\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(tx)^2}}\mathrm{d}x\right)+\dfrac{1-\gamma}{k^2-1}, & (k>1),\end{cases} \tag{1.3}$$

where $u(z)=\frac{z-\sqrt{t}}{1-\sqrt{tz}}$, $t\in(0,1), z\in E$ and z is chosen such that $k=\cosh\left(\frac{\pi R'(t)}{4R(t)}\right)$, $R(t)$ is the Legendre's complete elliptic integral of the first kind and $R'(t)$ is the complementarity integral of $R(t)$, see [6,2,5].

We note that the function $p_{k,\gamma}(z)$ is continuous as regards to k, $k\in[0,\infty)$ and has real coefficients for $k\in[0,\infty)$, see [7,8].

We define a subclass of Caratheodory class P as follows.

Definition 1.1. Let $k-P(\gamma)\subset P$ be the class consisting of functions $p(z)$ which are analytic in E with $p(0)=1$, and which are subordinate to $p_{k,\gamma}(z)$ in E. We write $p\in k-P(\gamma)$ implies $p\prec p_{k,\gamma}$ where $p_{k,\gamma}$ is given by (1.3) That is $p(E)\subset p_{k,\gamma}(E)$. We note that $0-P(0)=P$ and $p\in 0-P(\gamma)=P(\gamma)$ implies that $\mathrm{Re}p(z)>\gamma,\ z\in E$. It is easy to note that the class $k-P(\gamma)$ is a convex set.

We extend the class $k-P(\gamma)$ as follows.

Definition 1.2. Let $p(z)$ be analytic in E with $p(0)=1$. Then $p\in k-P_m(\gamma)$ if and only if, for $m\geq 2, 0\leq\gamma<1,\ k\in[0,\infty), z\in E$,

$$p(z)=\left(\frac{m}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{m}{4}-\frac{1}{2}\right)p_2(z),\quad p_1,p_2\in k-P(\gamma). \tag{1.4}$$

We note that

$$k-P_2(\gamma)=k-P(\gamma)$$

and

$$0-P(0)=P_m,$$

is the class introduced and studied by Pinchuk [9].

We now define the following.

Definition 1.3. Let $f\in\mathcal{A}$. Then $f\in k-UV_m(\gamma), 0\leq\gamma<1,\ k\in[0,\infty)$ and $m\geq 2$, if and only if

$$\left\{1+\frac{zf''(z)}{f'(z)}\right\}\in k-P_m(\gamma),\quad z\in E.$$

We call $k-UV_m(\gamma)$ the class of functions of k-uniform bounded boundary rotation m with order γ. It can easily be seen that $0-UV_m(0)=V_m$ coincides with the class of functions of bounded boundary rotation, see [1,10–12].

The corresponding class $k-UR_m(\gamma)$ is defined as

$$k-UR_m(\gamma)=\{g\in\mathcal{A}: g=zf',\ f\in k-UV_m(\gamma)\}.$$

We have the following special cases.

(i) If $u+\mathrm{i}v=\left(1+\frac{zf''(z)}{f'(z)}\right),\ z\in E$, then $f\in 1-UV_m(0)$ means that the range of the expression $\left(1+\frac{zf''(z)}{f'(z)}\right)$ is the region bounded by a parabola $u=\frac{v^2}{2}+\frac{1}{2}$ and its boundary rotation is bounded by $m\pi$.

(ii) For $k=1, m=2$ and $\gamma=0$, we obtain the class $1-UR_2(0)=1-ST$ and if $f\in 1-ST$, then

$$\mathrm{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>\frac{1}{2}\quad\text{and}\quad\left|\arg\frac{zf'(z)}{f(z)}\right|<\frac{\pi}{4},\quad\text{see [13]}.$$

(iii) $0-UR_m(0)=R_m$ is the class of analytic functions with bounded radius rotation, see [14,2]. Also we denote $k-UR_2(\gamma)$ as $k-ST(\gamma)$.

(iv) For $m=2, \gamma=0$, we have $k-UV_2(0)=k-UCV$, the class of uniformly convex functions.

Remark 1.1. It is known [3] that $k-UCV\subset C\left(\frac{k}{k+1}\right)$ for $k\in[0,\infty)$.

Throughout this paper, we assume that $k\geq 0, 0\leq\gamma<1$ and $m\geq 2$ unless otherwise specified.

2. Preliminary results

Let $V_m(\rho),\ m \geq 2, 0 \leq \rho < 1$ be the class of functions $f(z)$, analytic and locally univalent in E with $f(0) = 0,\ f'(0) = 1$ and satisfying the condition

$$\int_0^{2\pi} \left| \frac{\left(\mathrm{Re}\,\frac{(zf'(z))'}{f'(z)} - \rho\right)}{(1-\rho)} \right| \mathrm{d}\theta \leq m\pi. \tag{2.1}$$

When $\rho = 0$, the class $V_m(0) = V_m$ coincides with the class of functions of bounded boundary rotation.

We shall need the following known results.

Lemma 2.1 ([15]). *An analytic function $f \in V_m(\rho)$ if and only if, there exists $f_1 \in V_m$ such that*

$$f'(z) = (f_1'(z))^{1-\rho}, \quad \text{see [16]}. \tag{2.2}$$

We give an easy extension of a result proved in [4] *as follows.*

Lemma 2.2. *Let $0 \leq k < \infty$ and let β, δ be any complex numbers with $\beta \neq 0$ and $\mathrm{Re}\left(\frac{\beta k}{k+1} + \delta\right) > \gamma$. If $h(z)$ is analytic in E, $h(0) = 1$, and satisfies*

$$\left(h(z) + \frac{zh'(z)}{\beta h(z) + \delta}\right) \prec p_{k,\gamma}(z), \tag{2.3}$$

and $q_{k,\gamma}(z)$ is an analytic solution of

$$q_{k,\gamma} + \frac{zq'_{k,\gamma}(z)}{\beta q_{k,\gamma}(z) + \delta} = p_{k,\gamma}(z), \tag{2.4}$$

then $q_{k,\gamma}$ is univalent,

$$h(z) \prec q_{k,\gamma}(z) \prec p_{k,\gamma}(z),$$

and $q_{k,\gamma}(z)$ is the best dominant of (2.3).

Lemma 2.3 ([15]). *Let $f \in V_m(\rho)$ and let $F_1(z) = \frac{(zf'(z))'}{f'(z)}$ with*

$$F_1(z) = 1 + \sum_{n=1}^{\infty} c_n z^n.$$

Then, with $z = re^{i\theta},\ z \in E$,

(i)

$$\frac{1}{2\pi}\int_0^{2\pi} |F_1(re^{i\theta})|^2 \mathrm{d}\theta \leq \frac{1 - \{m^2(1-\rho^2) - 1\}r^2}{1-r^2}.$$

(ii)

$$\frac{1}{2\pi}\int_0^{2\pi} |F_1'(re^{i\theta})| \mathrm{d}\theta \leq \frac{m(1-\rho)}{1-r^2}.$$

3. Main results

Theorem 3.1. *Let $f \in k - UR_m(\gamma)$. Then there exists $s_1, s_2 \in k - ST(\gamma)$ such that*

$$f(z) = \frac{(s_1(z))^{\frac{m+2}{4}}}{(s_2(z))^{\frac{m-2}{4}}}, \quad m \geq 2,\ k \in [0, \infty).$$

Proof. For $s \in k - ST(\gamma)$, we have

$$\frac{zs'(z)}{s(z)} \prec p_{k,\gamma}(z)$$

and therefore

$$s(z) \prec z \exp \int_0^z \frac{p_{k,\gamma}(t) - 1}{t} \mathrm{d}t.$$

Let μ_m be the class of real-valued functions $\mu(t)$ of bounded variation on $[-\pi, \pi]$ satisfying the conditions

$$\int_{-\pi}^{\pi} \mathrm{d}\mu(t) = 2, \qquad \int_{-\pi}^{\pi} |\mathrm{d}\mu(t)| \leq m.$$

Since $f \in k-UR_m(\gamma)$, $\frac{zf'}{f} \in k-P_m(\gamma)$ and from this, we can easily deduce a representation formula for the class $k-UR_m(\gamma)$ as follows.

$$f(z) = z \exp \int_0^z \frac{p_{k,\gamma}(t) - 1}{t} \mathrm{d}\mu(t), \quad \mu \in \mu_m.$$

We can write the real-valued function of bounded variation as

$$\mu(t) = \mu_1(t) - \mu_2(t),$$

where μ_1 and μ_2 are nonnegative increasing functions. Thus

$$\frac{f(z)}{z} = \frac{\exp \int_0^z \frac{p_{k,\gamma}(t)-1}{t} \mathrm{d}\mu_1(t)}{\exp \int_0^z \frac{p_{k,\gamma}(t)-t}{t} \mathrm{d}t\mu_2(t)} = \frac{N(z)}{D(z)}, \tag{3.1}$$

where

$$\int_{-\pi}^{\pi} \mathrm{d}\mu_1(t) - \mathrm{d}\mu_2(t) = 2, \quad \text{and} \quad \int_{-\pi}^{\pi} \mathrm{d}\mu_1(t) + \mathrm{d}\mu_2(t) \leq m$$

since $\mu \in \mu_m$.

These, in turn, imply that

$$\int_{-\pi}^{\pi} \mathrm{d}\mu_1(t) \leq \frac{m+2}{2}, \qquad \int_{-\pi}^{\pi} \mathrm{d}\mu_2(t) \leq \frac{m-2}{2}. \tag{3.2}$$

From (3.1), we note that $\int_{-\pi}^{\pi} \mathrm{d}\mu_1(t)$ and $\int_{-\pi}^{\pi} \mathrm{d}\mu_2(t)$ are the boundary rotation of the image of E under the mappings

$$w_1(z) = \int_0^z N(\xi)\mathrm{d}\xi \quad \text{and} \quad w_2(z) = \int_0^z D(\xi)\mathrm{d}\xi$$

respectively.

From (3.2), the functions

$$w_1(z) = (N(z))^{\frac{4}{m+2}}$$

and

$$w_2(z) = (D(z))^{\frac{4}{m-2}}$$

are the derivatives of functions whose boundary rotations are 2. In other words, these are the derivatives of functions belonging to $k - UCV(\gamma)$.

Let

$$s_1(z) = z(N(z))^{\frac{4}{m+2}}$$
$$s_2(z) = z(D(z))^{\frac{4}{m-2}}.$$

This means $s_1, s_2 \in k - ST(\gamma)$. Hence

$$f(z) = \frac{(s_1(z))^{\frac{m+2}{4}}}{(s_2(z))^{\frac{m-2}{4}}}, \quad m \geq 2, k \in [0, \infty).$$

The proof is complete. □

Theorem 3.2. *Let $f \in k - UV_m(\gamma)$. Then $f \in k - UR_m(\gamma)$ for $z \in E$.*

Proof. Let $f \in k - UV_m(\gamma)$ and let

$$\frac{zf'(z)}{f(z)} = p(z) = \left(\frac{m+2}{4}\right) p_1(z) - \left(\frac{m-2}{4}\right) p_2(z). \tag{3.3}$$

Then

$$\left\{1+\frac{zf''(z)}{f'(z)}\right\} = \left\{p(z)+\frac{zp'(z)}{p(z)}\right\} \in k-P_m(\gamma). \tag{3.4}$$

Define

$$\phi(z) = \frac{1}{2}\left[\frac{z}{1-z}+\frac{z}{(1-z)^2}\right] = \frac{z\left(1-\frac{z}{2}\right)}{(1-z)^2}.$$

Let $\star$ denote convolution (Hadamard product). Then, using the convolution technique, we note that

$$\left(p(z)\star\frac{\phi(z)}{z}\right) = p(z)+\frac{zp'(z)}{p(z)},$$

and from (3.3) and (3.4), it follows that

$$\left\{p_i(z)+\frac{zp_i'(z)}{p_i(z)}\right\} \in k-P(\gamma), \quad i=1,2.$$

This implies that, for $i = 1, 2$

$$\left(p_i+\frac{zp_i'}{p_i}\right) \prec p_{k,\gamma}.$$

Applying Lemma 2.2 with $\beta = 1$, $\delta = 0$, we have

$$p_i \prec q_{k,\gamma} = \left[\int_0^1\left(\exp\int_t^{tz}\frac{p_{k,\gamma}(u)-1}{u}\mathrm{d}u\right)\mathrm{d}t\right]^{-1} \prec p_{k,\gamma}.$$

This proves that $f \in k-UR_m(\gamma)$. $\square$

We have the following special cases of Theorem 3.2.

Corollary 3.1. *For $k = 0$ and $\gamma = 0$, $f \in V_m$. Then it follows that $f \in R_m\left(\frac{1}{2}\right)$, see [11]. Since $\left(p_i+\frac{zp_i'}{p_i}\right) \prec \frac{1+z}{1-z}$ in E, it implies that*

$$p_i \prec q(z) = \frac{z}{1-z},$$

and $q(-1) = \frac{1}{2}$. This means that $p \in P_m\left(\frac{1}{2}\right)$ and the result follows.

For $m = 2$, we obtain a well-known result that every convex function is starlike of order $\frac{1}{2}$.

Corollary 3.2. *Let $\gamma = 0$, $k \in (1,\infty)$ and $f \in k-UV_m(0)$. Then, from Theorem 3.2 , $f \in k-UR_m(\gamma_1)$, where*

$$\gamma_1 = \frac{1}{(k+1)\log\left(1+\frac{1}{k}\right)},$$

since in this case, for $i = 1, 2$

$$\left(p_i+\frac{zp_i'}{p_i}\right) \prec \frac{k}{k-z}, \quad \text{in } E.$$

This implies

$$p_i(z) \prec q_{k,0}(z) = \frac{z}{(z-k)\log\left(1-\frac{z}{k}\right)}, \quad i=1,2$$

and

$$q_{k,0}(-1) = \frac{1}{(k+1)\log\left(1+\frac{1}{k}\right)},$$

the assertion follows.

For the case $k = 2$, we note that $f \in 2-UV_m(0)$ which implies $f \in 2-UR_m\left(\frac{4}{5}\right)$. In fact, in this case,

$$\left(p_i+\frac{zp_i'}{p_i}\right) \prec q_{2,0}(z) = \frac{1}{1-\frac{z}{2}}, \quad i=1,2,$$

which implies

$$\mathrm{Re}\left[p_i(z)+\frac{zp_i'(z)}{p_i(z)}\right]>\frac{2}{3},\quad i=1,2.$$

This gives us

$$\mathrm{Re}\{p_i(z)\}>\frac{1}{3\log\frac{3}{2}}\approx 0.813,\quad i=1,2.$$

Corollary 3.3. *For $k=1,\gamma=0$, let $f\in 1-UV_m(0)$. Then it follows directly from Theorem 3.2 that $f\in 1-UR_m\left(\frac{1}{2}\right)$. In this case $p_i(z)\prec q_{1,0}(z)$ with*

$$q_{1,0}(z)=1+\frac{2}{\pi^2}\left(\log\frac{1+\sqrt{z}}{1-\sqrt{z}}\right)^2,$$

the branch of $\sqrt{z}$ is chosen such that $\mathrm{Im}\sqrt{z}\geq 0$ and $q_{1,0}(-1)=\frac{1}{2}$.

We now deal with a partial converse case of Theorem 3.2 as follows.

Theorem 3.3. *Let $f\in 0-UR_m(0)\equiv R_m$ for $z\in E$. Then $f\in k-UV_m(0)$ for $|z|<r_k$, where*

$$r_k=\frac{1}{2(k+1)+\sqrt{4k^2+6k+3}}. \tag{3.5}$$

Proof. Let

$$\frac{zf'(z)}{f(z)}=p(z)=\left(\frac{k}{4}+\frac{1}{2}\right)p_1(z)-\left(\frac{k}{4}-\frac{1}{2}\right)p_2(z). \tag{3.6}$$

Since $f\in R_m$, so $p\in P_m$ and $p_i\in P,\ i=1,2$ for $z\in E$. From Theorem 3.1, we can write

$$\frac{zf'(z)}{f(z)}=\left(\frac{m}{4}+\frac{1}{2}\right)\frac{zs_1'(z)}{s_1(z)}-\left(\frac{m}{4}-\frac{1}{2}\right)\frac{zs_2'(z)}{s_2(z)},$$

with

$$p_i(z)=\frac{zs_i'(z)}{s_i(z)},\quad s_i\in S^\star\subset C,\ i=1,2$$

and it is known [3] that $s_i\in k-UCV$ for $|z|<r_k$. This implies that

$$\left(p_i(z)+\frac{zp_i'(z)}{p_i(z)}\right)\in k-P(0)=k-P,\quad i=1,2,\ \text{for } |z|<r_k.$$

Therefore, from (3.6), it follows that

$$\frac{(zf'(z))'}{f'(z)}=\left(p(z)+\frac{zp'(z)}{p(z)}\right)\in k-P_m(0),\quad \text{for } |z|<r_k,$$

which implies that $f\in k-UV_m(0)$ for $|z|<r_k$ where r_k is given by (3.5). This completes the proof. □

Corollary 3.4. *With $k=0$, it follows that $f\in R_m$ is a function of bounded boundary rotation for $|z|<r_0=\frac{1}{2+\sqrt{3}}$, see [11,12].*

When $k=1,\ r_1=\frac{4-\sqrt{13}}{3}$ and this result coincides with that for $m=2$ proved in [17].

Theorem 3.4. *Let $f\in k-UV_m(\gamma)$ and be given by (1.1). Then*

$$a_n=O(1).n^{\beta_1-2},\quad (n\longrightarrow\infty),$$

where

$$\beta_1=\left\{\left(\frac{1-\gamma}{1+k}\right)\left(\frac{m}{2}+1\right)\right\} \tag{3.7}$$

and $O(1)$ is a constant depending only on k, m and γ. The exponent (β_1-2) is best possible when $k=0=\gamma$, see [15].

Proof. Since $k-UCV \subset C\left(\frac{k}{1+k}\right)$ for $k \in [0,\infty)$, it easily follows from Theorem 3.1 that $f \in k-UV_m(0)$ implies that $f \in V_m(\rho),\ \rho=\frac{k}{1+k}$. Also, from Lemma 2.1, we can write

$$f'(z)=(f_1'(z))^{1-\rho},\quad f_1 \in V_m.$$

Now, from Theorem 3.1,

$$\begin{aligned} f'(z) &= \frac{(s_1'(z))^{\frac{m}{4}+\frac{1}{2}}}{(s_2'(z))^{\frac{m}{4}-\frac{1}{2}}},\quad s_1,s_2 \in k-UV_2(\gamma)\\ &= \frac{(g_1'(z))^{\left(1-\frac{k}{k+1}\right)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(g_2'(z))^{\left(1-\frac{k}{k+1}\right)\left(\frac{m}{4}-\frac{1}{2}\right)}},\quad g_1,g_2 \in V_2(\gamma)\\ &= \frac{(\phi_1'(z))^{\left(\frac{1}{k+1}\right)(1-\gamma)\left(\frac{m}{4}+\frac{1}{2}\right)}}{(\phi_2'(z))^{\left(\frac{1}{k+1}\right)(1-\gamma)\left(\frac{m}{4}-\frac{1}{2}\right)}},\quad \phi_1,\phi_2 \in C. \end{aligned} \tag{3.8}$$

Hence, using a result due to Brannan [14], it follows that, for $f \in k-UV_m(\gamma)$, we can write

$$f'(z)=(F_1'(z))^{\frac{1-\gamma}{1+k}},\quad F_1 \in V_m. \tag{3.9}$$

Set

$$\begin{aligned} F(z) &= (z(zf'(z))')'\\ &= (zf'(z)h'(z))',\qquad h(z)=\frac{(zf'(z))'}{f'(z)} \in P_m\left(\frac{\gamma+k}{1+k}\right)\\ &= f'(z)[h^2(z)+zh'(z)]. \end{aligned} \tag{3.10}$$

Now, for $z=re^{i\theta}$,

$$n^3|a_n| = \frac{1}{2\pi r^n}\left|\int_0^{2\pi} F(z)e^{-in\theta}d\theta\right|,$$

and using (3.8)–(3.10), we have

$$n^3|a_n| \le \frac{1}{2\pi r^n}\int_0^{2\pi} \frac{|\phi_1(z)|^{\left(\frac{1-\gamma}{1+k}\right)\left(\frac{m}{4}+\frac{1}{2}\right)}}{|\phi_2(z)|^{\left(\frac{1-\gamma}{1+k}\right)\left(\frac{m}{4}-\frac{1}{2}\right)}}|h^2(z)+zh'(z)|d\theta,$$

where $\phi_1,\phi_2 \in C$ and $h \in P_m\left(\frac{\gamma+k}{1+k}\right)$ for $z \in E$. Thus, on using well-known distortion results for convex functions, Lemma 2.3 with

$$\rho=\frac{\gamma+k}{1+k},\qquad r=\left(1-\frac{1}{n}\right),$$

we have

$$a_n = O(1)n^{\beta_1-2},\quad (n \longrightarrow \infty),$$

where β_1 is given by (3.7), and $O(1)$ depends only on γ, k and m. □

For $\gamma=0=k$ and $m=2$, we note that $f(z)$ is convex in E and $a_n=O(1)$, which is a well known result.
For the class $0-UV_m(\gamma)$, we have

$$a_n = O(1)n^{(1-\gamma)\left(\frac{m}{2}+1\right)-2}.$$

For different choices of γ, k and m, we obtain several new and known results as special cases of Theorem 3.4.

It is known [18] that, for a z_1 with $|z_1|=r$ such that for any univalent function,

$$\max_{|z|=r}|(z-z_1)s(z)| \le \frac{2r^2}{1-r^2}. \tag{3.11}$$

Using (3.11) and a similar technique of Theorem 3.4, we can easily prove the following.

Theorem 3.5. *Let $f \in k - UV_m(\gamma)$ and be given by (1.1). Then, for $m \geq 2\left(\frac{\gamma+k}{1-\gamma}\right)$,*

$$||a_n| - |a_{n+1}|| \leq A(k, m, \gamma) n^{\beta_1 - 3}, \quad (n \longrightarrow \infty),$$

where β_1 is given by (3.7) and $A(k, m, \gamma)$ is a constant depending upon k, m and γ only.

For $\gamma = 0 = k$, the exponent $\left(\frac{m}{2} - 2\right)$ is the best possible, see [19].

We shall now study the behaviour of the class $k - UR_m(\gamma)$ under an integral operator as follows.

Theorem 3.6. *Let $f, g \in k - UR_m(\gamma)$ and let α, c, δ and ν be positively real with $(\nu + \delta) = \alpha$. Then, the function F, defined by*

$$[F(z)]^\alpha = cz^{\alpha - c} \int_0^z t^{(c-\delta-\nu)-1} (f(t))^\delta (g(t))^\nu \mathrm{d}t \tag{3.12}$$

belongs to $k - UR_m(\gamma)$ for $z \in E$.

Proof. First we show that the function F, defined by (3.12), is well-defined. Let

$$G(z) = z^{-(\nu+\delta)} (f(z))^\delta (g(z))^\nu = 1 + d_1 z + d_2 z^2 + \cdots,$$

and choose the branches which equal 1 when $z = 0$. Now, for

$$K(z) = c^{(c-\nu-\delta)-1} (f(z))^\delta (g(z))^\nu = z^{c-1} G(z),$$

we have

$$L(z) = \frac{c}{z^c} \int_0^z K(t) \mathrm{d}t = 1 + \frac{c}{c+1} d_1 z + \frac{c}{c+2} d_2 z^2 + \cdots.$$

This shows that $L(z)$ is well-defined and analytic in E.

We now let

$$F(z) = [z^\alpha L(z)]^{\frac{1}{\alpha}} = z[L(z)]^{\frac{1}{\alpha}}$$

and here we choose the branch of $[L(z)]^{\frac{1}{\alpha}}$ which equals 1 when $z = 0$. Thus F is analytic and satisfies (3.12). Now, from (3.12), we have

$$z^{(c-\alpha)} [F(z)]^\alpha \left[(c - \alpha) + \frac{\alpha z F'(z)}{F(z)} \right] = cz^{(c-\delta-\nu)-1} (f(z))^\delta (g(z))^\nu. \tag{3.13}$$

We write

$$\frac{zF'(z)}{F(z)} = H(z) \tag{3.14}$$

and note that H is analytic in E with $H(0) = 1$. Also, since $f, g \in k - UR_m(\gamma)$, we have

$$\frac{zf'(z)}{f(z)} = H_1(z), \qquad \frac{zg'(z)}{g(z)} = H_2(z), \quad H_1, H_2 \in k - P_m(\gamma). \tag{3.15}$$

Differentiating (3.13) logarithmically and using (3.14) and (3.15), we have

$$\alpha \left[H(z) + \frac{zH'(z)}{(c - \alpha) + \alpha H(z)} \right] = \delta H_1(z) + \nu H_2(z).$$

That is

$$\left[H(z) + \frac{\frac{1}{\alpha} zH'(z)}{H(z) + \frac{c-\alpha}{\alpha}} \right] = \frac{\delta}{\alpha} H_1(z) + \frac{\nu}{\alpha} H_2(z).$$

Since $\delta + \nu = \alpha$ and $k - P_m(\gamma)$ is a convex set, we have

$$\left[H(z) + \frac{\frac{1}{\alpha} zH'(z)}{H(z) + \frac{c-\alpha}{\alpha}} \right] \in k - P_m(\gamma). \tag{3.16}$$

Define

$$\phi_{a,b}(z) = \frac{1}{a+b} \frac{z}{(1-z)^a} + \frac{b}{1+b} \frac{z}{(1-z)^{a+1}},$$

with $a = \frac{1}{\alpha}$, $b = \frac{c-\alpha}{\alpha}$, and let

$$H(z) = \left(\frac{m}{4} + \frac{1}{2}\right) h_1(z) - \left(\frac{m}{4} - \frac{1}{2}\right) h_2(z).$$

Then

$$\left(H(z) \star \frac{\phi_{a,b}(z)}{z}\right) = \left[H(z) + \frac{\frac{1}{\alpha} zH'(z)}{H(z) + \frac{c-\alpha}{\alpha}}\right]$$
$$= \left(\frac{m}{4} + \frac{1}{2}\right)\left[h_1(z) + \frac{\frac{1}{\alpha} zh_1'(z)}{h_1(z) + \frac{c-\alpha}{\alpha}}\right] - \left(\frac{m}{4} - \frac{1}{2}\right)\left[h_2(z) + \frac{\frac{1}{\alpha} zh_2'(z)}{h_2(z) + \frac{c-\alpha}{\alpha}}\right].$$

Thus it follows, from (3.16) that, for $i = 1, 2$

$$\left[h_i(z) + \frac{zh_i'(z)}{\alpha h_i(z) + (c-\alpha)}\right] \prec p_{k,\gamma}(z).$$

Using Lemma 2.2, we have

$$h_i \prec p_{k,\gamma}, \quad i = 1, 2.$$

Consequently, $H \in k - P_m(\gamma)$ and hence $F \in k - UR_m(\gamma)$. This complete the proof. □

We have several interesting special cases by appropriate choosing various permitted values of parameters k, m and γ.

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