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Stability of Cauchy–Jensen type functional equation in generalized fuzzy normed spaces

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ABSTRACT

In this paper, we establish some stability results concerning the Cauchy–Jensen functional equation in generalized fuzzy normed spaces. The results of the present paper improve and extend some recent results.

1. Introduction

The theory of fuzzy sets was introduced by Zadeh in 1965 [1]. Fuzzy set theory is a powerful hand set for modeling uncertainty and vagueness in various problems arising in the field of science and engineering. It has also very useful applications in various fields, e.g., population dynamics, chaos control, computer programming, nonlinear dynamical systems, fuzzy physics, fuzzy topology, nonlinear operators, statistical convergence, etc. (see [2–17]).

On the other hand, the study of stability problems for functional equations is related to a question of Ulam [18] concerning the stability of group homomorphisms, which was affirmatively answered for Banach spaces by Hyers [19]. Subsequently, the result of Hyers was generalized by Aoki [20] for additive mappings and by Rassias [21] for linear mappings by considering an unbounded Cauchy difference. The paper by Rassias has provided a lot of influence in the development of what we now call the generalized Hyers–Ulam stability or Hyers–Ulam–Rassias stability of functional equations. Rassias [22] considered the Cauchy difference controlled by a product of different powers of norm. The above results have been generalized by Forti [23] and Găvruta [24] who permitted the Cauchy difference to become arbitrary unbounded. For more details about the results concerning such problems, the reader is referred to [25–39]. Also, the stability of some functional equations in the framework of fuzzy and random normed spaces has been established (see e.g., [40–53]).

Recently, interesting results concerning Cauchy–Jensen functional equation

$$f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)=f(x) \tag{1.1}$$

have been obtained in [54–58]. The main purpose of this paper is to establish the stability result concerning the Cauchy–Jensen functional equation in the setting of generalized fuzzy normed spaces and apply obtained results to study the stability of the Cauchy and Jensen functional equations. The achieved results via this paper improve and extend some recent well-known pertinent results.

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2. Preliminaries

In 1899, Hensel [59] discovered the p-adic numbers as a number theoretical analogue of power series in complex analysis. Fix a prime number p. For any nonzero rational number x, there exists a unique integer n_x such that $x = \frac{a}{b}p^{n_x}$, where a and b are integers not divisible by p. Then p-adic absolute value $|x|_p := p^{-n_x}$ defines a non-Archimedean norm on $\mathbb{Q}$. The completion of $\mathbb{Q}$ with respect to the metric $d(x, y) = |x-y|_p$ is denoted by $\mathbb{Q}_p$, and it is called the p-adic number field. In fact, $\mathbb{Q}_p$ is the set of all formal series $x = \sum_{k\geq n_x}^{\infty} a_k p^k$, where $|a_k| \leq p-1$ are integers (see for instance [60,61]). Note that if $p > 2$, then $|2^n|_p = 1$ in for each integer n but $|2|_2 < 1$. During the last three decades p-adic numbers have gained the interest of physicists for their research, in particular in problems coming from quantum physics, p-adic strings and superstrings [62,63]. A key property of p-adic numbers is that they do not satisfy the Archimedean axiom.

Let $\mathbb{K}$ denote a field and function (valuation absolute) $|\cdot|$ from $\mathbb{K}$ into $[0, \infty)$. A non-Archimedean valuation is a function $|\cdot|$ that satisfies the strong triangle inequality; namely, $|x + y| \leq \max\{|x|, |y|\} \leq |x| + |y|$ for all $x, y \in \mathbb{K}$. The associated field $\mathbb{K}$ is referred to as a non-Archimedean field. Clearly, $|1| = |-1| = 1$ and $|n| \leq 1$ for all $n \geq 1$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except 0 into 1 and $|0| = 0$. We always assume in addition that $|\cdot|$ is non trivial, i.e., there is an $z \in \mathbb{K}$ such that $|z| \neq 0, 1$.

Let X be a linear space over a field $\mathbb{K}$ with a non-Archimedean non-trivial valuation $|\cdot|$. A function $\|\cdot\| : X \to [0, \infty)$ is said to be a non-Archimedean norm if it is a norm over $\mathbb{K}$ with the strong triangle inequality (ultrametric); namely, $\|x + y\| \leq \max\{\|x\|, \|y\|\}$ for all $x, y \in X$. Then $(X, \|\cdot\|)$ is called a non-Archimedean space.

Recently, Goleţ in [64] introduced a generalization of the concept of fuzzy normed spaces which includes some earlier defined fuzzy normed spaces as special cases. We briefly recall some definitions and results used later on in the paper. For more details the reader is referred to [64–71].

A triangular norm (shorter t-norm, [72]) is a binary operation $\star : [0, 1] \times [0, 1] \longrightarrow [0, 1]$ which is commutative, associative, non-decreasing in each variable and has 1 as the unit element. Basic examples are the Łukasiewicz t-norm $\star_L$, $\star_L(a, b) = \max\{a + b - 1, 0\}$, the product t-norm $\star_P$, $\star_P(a, b) = a \cdot b$ and the strongest triangular norm $\star_M$, $\star_M(a, b) = \min\{a, b\}$. By an operation $\circ$ on $\mathbb{R}_+ = \{x \in \mathbb{R};\ x \geq 0\}$ we mean a two place function $\circ : [0, \infty) \times [0, \infty) \longrightarrow [0, \infty)$ which is associative, commutative, non-decreasing in each place and such that $a \circ 0 = a$, for all $a \in [0, \infty)$. The most used operations on $\mathbb{R}_+$ are $\circ_a(a, b) = a + b$, $\circ_m(a, b) = max\{a, b\}$ and $\circ_q(a, b) = (a^q + b^q)^{\frac{1}{q}}$ [64,66].

Definition 2.1 (*Goleţ [67]*)**.** Let X be a linear space over the field of real numbers $\mathbb{R}$, and let N be a function defined on $X \times [0, \infty)$ with values into $[0, 1]$ satisfying the following conditions: for all $x, y \in X$ and $s, t \in [0, \infty)$,

(N_1) $N(x, 0) = 0$.
(N_2) $N(x, t) = 1$, for all $t > 0$, if and only if $x = \theta$.
(N_3) $N(x + y, s + t) \geq N(x, s) \star N(y, t)$.
(N_4) $N(x, .) : [0, \infty) \to [0, 1]$ is left continuous.
(N_5) $N(\alpha x, t) = N(x, \frac{t}{|\alpha|})$, for all $x \in X$ and $\alpha \in \mathbb{R} - \{0\}$;

then the triple $(X, N, \star)$ is called a fuzzy normed space. If we define on X the mapping $x \to F_x$, $F_x(t) = N(x, t)$, then the triple $(X, F, \star)$ becomes a generalized probabilistic normed space [72] (the random variable associated to the distribution function F_x can take the value ∞ with a probability greater than 0). If the following condition is satisfied:

(N_6) $\lim_{t\to\infty} N(x, t) = 1$, for all $x \in X$,

then the triple $(X, F, \star)$ is a probabilistic normed space. Conversely, if $(X, F, \star)$ is a probabilistic normed space, and we define $N(x, t) = F_x(t)$ then $(X, N, \star)$ becomes a fuzzy normed space. One can see a generality of fuzzy normed space. The specific for the both fuzzy and probabilistic norms are the distinct areas of applicability and the different interpretation ways.

Let ϕ be a function defined on the real field $\mathbb{R}$ into itself with the following properties:

(a_1) $\phi(-t) = \phi(t)$ for all $t \in \mathbb{R}$;
(a_2) $\phi(1) = 1$;
(a_3) ϕ is strict increasing and continuous on $(0, \infty)$;
(a_4) $\lim_{\alpha\to 0} \phi(\alpha) = 0$ and $\lim_{\alpha\to\infty} \phi(\alpha) = \infty$.

As examples of such functions are: $\phi(\alpha) = |\alpha|$; $\phi(\alpha) = |\alpha|^{\beta}$, $\beta \in \mathbb{R}_+$; $\phi(\alpha) = \frac{2\alpha^n}{|\alpha|+1}$, $n \in \mathbb{N}^+$.

Definition 2.2 (*Goleţ [64]*)**.** Let X be a linear space over the field of real numbers $\mathbb{R}$, and let N be a function defined on $X \times [0, \infty)$ with values into $[0, 1]$ satisfying the following conditions: for all $x, y \in X$ and $s, t \in [0, \infty)$,

(G_{f_1}) $N(x, 0) = 0$.
(G_{f_2}) $N(x, t) = 1$ for all $t > 0$ if and only if, $x = \theta$.
(G_{f_3}) $N(x + y, s \circ t) \geq N(x, s) \star N(y, t)$.
(G_{f_4}) $N(x, .) : [0, \infty) \to [0, 1]$ is left continuous.
(G_{f_5}) $N(\alpha x, t) = N(x, \frac{t}{\phi(\alpha)})$, for all $x \in X$ and $\alpha \in \mathbb{R}$;

then the triple $(X, N, \star)$ is called fuzzy ϕ-normed space under operation '$\circ'$. The axiom (G_{f_5}) gives the connection between the fuzzy norm of the product of a vector x by a real number α and the fuzzy norm of the vector x. This connection is given by the contraction or dilation function ϕ. This axiom is natural with a better fuzzy character and gives a consistent generalization see Example 2.10 of [64].

If $\circ = \circ_a$ then $(X, N, \star)$ is called a fuzzy ϕ-normed space. If $\circ = \circ_m$ then $(X, N, \star)$ is called a non-Archimedean fuzzy ϕ-normed space. If $\circ = \circ_a$ and $\phi(\alpha) = |\alpha|^\beta$, where β is a real number such that $0 < \beta \leq 1$, then $(X, N, \star)$ is called a fuzzy β–normed space. When $\beta = 1$ we say that $(X, N, \star)$ is a fuzzy normed space.

In the following, if $\circ = \circ_a$ or $\circ = \circ_m$ then $(X, N, \star)$ (fuzzy ϕ-normed space under operation ´$\circ$´) is called a generalized fuzzy normed space.

Recall that if $(X, N, \star)$ is a generalized fuzzy normed space, then a sequence $\{x_n\}$ in X is convergent if, for some $x \in X$, $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and we write $N - \lim_{n\to\infty} x_n = x$. The sequence $\{x_n\}$ in X is a Cauchy sequence if for each $t > 0$ and $k > 0$, we have $\lim_{n\to\infty} N(x_{n+k} - x_n, t) = 1$. If every Cauchy sequence in X is convergent, then the space is called a complete generalized fuzzy normed space.

We now study some properties of fuzzy ϕ-normed spaces. For the non-Archimedean case the same results can be similarly obtained.

Definition 2.3. By a ϕ-normed space we mean a pair $(X, \|\cdot\|)$, where X is a linear space, $\|\cdot\|$ is a real valued mapping defined on X such that the following conditions are satisfied:

(n_1) $\|x\| \geq 0$ for all $x \in X$ and $\|x\| = 0$ if and only if $x = \theta$;
(n_2) $\|\alpha . x\| = \phi(\alpha)\|x\|$ for all $\alpha \in \mathbb{R}$ and $x \in X$;
(n_3) $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$.

For $\phi(\alpha) = |\alpha|^\beta$ $(0 < \beta \leq 1)$ one obtains a β-normed space and for $\phi(\alpha) = |\alpha|$ one obtains an ordinary normed space. Each ϕ-norm space can be made a fuzzy ϕ-norm space.

The continuity of the t-norm $\star$ and the inequality (G_{f_3}) imply that the limit of a sequence in a ϕ-normed space is uniquely determined. In what follows we suppose that a fuzzy ϕ-norm N satisfies the following condition:

(G_{f_6}) $\lim_{t\to\infty} N(x, t) = 1$, for all $x \in X$.

Example 2.4. Let $(X, \|\cdot\|)$ be a ϕ-normed space over the field $\mathbb{R}$. Consider the t-norms $\star = \star_M$ and $\star = \star_p$. For all $x \in X$ and $t > 0$ let

$$N(x, t) = \frac{t}{t + \|x\|}.$$

Then the triple $(X, N, \star)$ is a fuzzy ϕ-normed space. It is called a standard fuzzy ϕ-normed space.

Recall that if $X \neq \{0\}$ and $(X, N, \star)$ is a non-Archimedean fuzzy normed space with $\star$ of Hadžić-type [65], then the valuation of $\mathbb{K}$ is non-Archimedean; cf. [42]. Thus, if X is a real linear space and $(X, N, \star)$ is a non-Archimedean fuzzy normed space with $\star$ of Hadžić-type, then $X = \{0\}$; cf. [14].

Example 2.5. Let $(X, \|\cdot\|)$ be a (non-Archimedean) normed space. For all $x \in X$, consider

$$N(x, t) = \begin{cases} \dfrac{t}{t + \|x\|}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Then the triple $(X, N, \star_M)$ (under operation $\circ_m$) under operation $\circ_a$ is a (non-Archimedean) fuzzy normed space.

Note that a non-Archimedean fuzzy normed space $(X, N, \star_M)$ over fields $\mathbb{R}$ or $\mathbb{C}$, is trivial.

Example 2.6. Let $(X, \|\cdot\|)$ be a real normed space. For all $x \in X$, consider

$$N(x, t) = \begin{cases} e^{-\frac{\|x\|}{t}}, & t > 0, \\ 0, & t \leq 0. \end{cases}$$

Then the triple $(X, N, \star_p)$ is a non-Archimedean fuzzy normed space. Moreover, if $(X, \|\cdot\|)$ is complete, then $(X, N, \star_p)$ is complete.

3. Generalized fuzzy approximately Cauchy–Jensen mappings

In this section, unless otherwise explicitly stated, we will assume that $\mathbb{K}$ is a non-Archimedean field, X is real linear space, $(Y, N, \star)$ is a complete generalized fuzzy normed space and $(Z, N', \star)$ is a fuzzy ϕ-normed space.

Definition 3.1. Let φ be a function from $X \times X$ to Z:

(a) a function $f : X \to Y$ is called a φ-approximately Cauchy function, if

$$N(f(x+y) - f(x) - f(y), t) \geq N'(\varphi(x, y), t), \tag{3.1}$$

(b) a function $f : X \to Y$ is called a φ-approximately Jensen function, if

$$N\left(2f\left(\frac{x+y}{2}\right) - f(x) - f(y), t\right) \geq N'(\varphi(x, y), t), \tag{3.2}$$

(c) a function $f : X \to Y$ is called a φ-approximately Cauchy–Jensen function, if

$$N\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), t\right) \geq N'(\varphi(x, y), t) \tag{3.3}$$

for all $x, y \in X$ and $t > 0$.

Theorem 3.2. *Let $\ell \in \{-1, 1\}$ be fixed and let $f : X \to Y$ be a φ-approximately Cauchy–Jensen function. If for some positive real number α and some positive integer k with $\phi(k)\ell < \phi(\alpha)\ell$,*

$$\varphi(k^{-1}x, k^{-1}y) = \alpha^{-1}\varphi(x, y) \tag{3.4}$$

for all $x, y \in X$. Then the limit

$$T(x) = N - \lim_{r\to\infty} k^{\ell r} f(k^{-\ell r}x) \tag{3.5}$$

exists for all $x \in X$ and $T : X \to Y$ is a unique additive function satisfying

$$N\left(f(x) - T(x), \underbrace{t \circ t \circ \cdots \circ t}_{k}\right) \geq M\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \tag{3.6}$$

for all $x \in X$ and $t > 0$, where $\bigodot_{\imath=1}^{j} a_\imath = a_1 \circ a_2 \circ \cdots \circ a_j$ and

$$M(x, t) = N'(\varphi(2x, 0), t) \star N'(\varphi(3x, -x), t) \star \cdots \star N'(\varphi(kx, (2-k)x), t).$$

Proof. By induction on j, we shall show that for each $j \geq 2$,

$$N\left(f(jx) - jf(x), \underbrace{t \circ t \circ \cdots \circ t}_{j-1}\right) \geq M_j(x, t) = N'(\varphi(2x, 0), t) \star N'(\varphi(3x, -x), t) \star \cdots \star N'(\varphi(jx, (2-j)x), t) \tag{3.7}$$

for all $x \in X$ and $t > 0$. Replacing x and y by $2x$ and 0 in (3.3), respectively, we obtain

$$N(f(2x) - 2f(x), t) \geq N'(\varphi(2x, 0), t)$$

for all $x \in X$ and $t > 0$. This proves (3.7) for $j = 2$. Let (3.7) hold for some $j > 2$. Replacing x and y by $(1+j)x$ and $(1-j)x$ in (3.3), respectively, we get

$$N(f(x) + f(jx) - f((1+j)x), t) \geq N'(\varphi((1+j)x, (1-j)x), t)$$

for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N\left(f((j+1)x) - (j+1)f(x), \underbrace{t \circ t \circ \cdots \circ t}_{j}\right) &\geq N(f(x) + f(jx) - f((1+j)x), t) \star N\left(f(jx) - jf(x), \underbrace{t \circ t \circ \cdots \circ t}_{j-1}\right) \\ &\geq N'(\varphi((1+j)x, (1-j)x), t) \star M_j(x, t) = M_{j+1}(x, t) \end{aligned}$$

for all $x \in X$ and $t > 0$. Thus (3.7) holds for each $j \geq 2$. In particular

$$N\left(f(kx) - kf(x), \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M(x, t) \tag{3.8}$$

for all $x \in X$ and $t > 0$. By our assumption

$$M(k^{-1}x, t) = M(x, \phi(\alpha)t) \tag{3.9}$$

for all $x \in X$ and all $t > 0$. It follows from (3.8) and (3.9) that

$$N\left(f(x) - kf\left(\frac{x}{k}\right), \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M(x, \phi(\alpha)t) \tag{3.10}$$

for all $x \in X$ and $t > 0$. Replacing x by $k^{-r}x$ in (3.10), we obtain

$$N\left(k^r f(k^{-r}x) - k^{r+1} f(k^{-(r+1)}x), (\phi(k))^r \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M(k^{-r}x, \phi(\alpha)t) = M(x, (\phi(\alpha))^{r+1}t)$$

for all $x \in X$, $t > 0$ and each integers $r \geq 0$. So

$$N\left(k^r f(k^{-r}x) - k^{r+1} f(k^{-(r+1)}x), \left(\frac{\phi(k)}{\phi(\alpha)}\right)^r \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M(x, \phi(\alpha)t) \tag{3.11}$$

for all $x \in X$, $t > 0$ and each integers $r \geq 0$. Hence

$$N\left(k^{-r} f(k^r x) - k^{-(r+1)} f(k^{(r+1)}x), (\phi(k))^{-(2r+1)} \left(\frac{\phi(k)}{\phi(\alpha)}\right)^r \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right)$$
$$\geq M(k^{2r+1}x, \phi(\alpha)t) = M(x, \phi(\alpha)(\phi(\alpha))^{-(2r+1)}t)$$

for all $x \in X$, $t > 0$ and each integers $r \geq 0$. Therefore

$$N\left(k^{-r} f(k^r x) - k^{-(r+1)} f(k^{(r+1)}x), \left(\frac{\phi(\alpha)}{\phi(k)}\right)^r \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M(x, \phi(k)t) \tag{3.12}$$

for all $x \in X$, $t > 0$ and each integers $r \geq 0$. It follows from (3.11) and (3.12) that

$$\begin{aligned} &N\left(f(x) - k^{\ell r} f(k^{-\ell r}x), \bigodot_{j=\frac{1-\ell}{2}}^{r-\frac{1+\ell}{2}} \left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j} \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \\ &\geq \coprod_{j=\frac{1-\ell}{2}}^{r-\frac{1+\ell}{2}} N\left(k^{\ell j} f(k^{-\ell j}x) - k^{\ell(j+1)} f(k^{-\ell(j+1)}x), \left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j} \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \\ &\geq M\left(x, \left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t\right) \end{aligned} \tag{3.13}$$

for all $x \in X$, $t > 0$ and each integers $r > 0$, where $\coprod_{\iota=1}^{j} a_\iota = a_1 \star a_2 \star \cdots \star a_j$. Replacing x by $k^{-\ell m}x$ in (3.13), we obtain

$$\begin{aligned} &N\left(k^{\ell m} f(k^{-\ell m}x) - k^{\ell(r+m)} f(k^{-\ell(r+m)}x), \bigodot_{j=\frac{1-\ell}{2}}^{r-\frac{1+\ell}{2}} \frac{(\phi(k))^{\ell(j+m)}}{(\phi(\alpha))^{\ell j}} \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \\ &\geq M\left(k^{-\ell m}x, \left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t\right) = M\left(x, \left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)(\phi(\alpha))^{\ell m}t\right) \end{aligned} \tag{3.14}$$

for all $x \in X$, $t > 0$ and each integers $m \geq 0$, $r > 0$. Hence

$$\begin{aligned} &N\left(k^{\ell m} f(k^{-\ell m}x) - k^{\ell(r+m)} f(k^{-\ell(r+m)}x), \bigodot_{j=\ell m+\frac{1-\ell}{2}}^{r+\ell m-\frac{1+\ell}{2}} \left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j} \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \\ &\geq M\left(x, \left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t\right) \end{aligned}$$

for all $x \in X$, $t > 0$ and each integers $m \geq 0$, $r > 0$. By last inequality, we obtain

$$N\left(k^{\ell m}f(k^{-\ell m}x) - k^{\ell(r+m)}f(k^{-\ell(r+m)}x), \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\ell m+\frac{1-\ell}{2}}^{r+\ell m-\frac{1+\ell}{2}}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \tag{3.15}$$

for all $x \in X$, $t > 0$ and each integers $m \geq 0$, $r > 0$. Since $\phi(k)\ell < \phi(\alpha)\ell$, so $\{k^{\ell r}f(k^{-\ell r}x)\}_{r\in\mathbb{N}}$ is a Cauchy sequence in a complete generalized fuzzy normed space. Hence, we can define a mapping $T : X \to Y$ such that

$$\lim_{r\to\infty} N(k^{\ell r}f(k^{-\ell r}x) - T(x), t) = 1$$

for all $x \in X$ and $t > 0$. By (3.15) to obtain

$$N\left(f(x) - k^{\ell r}f(k^{-\ell r}x), \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \geq M\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{r-\frac{1+\ell}{2}}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \tag{3.16}$$

for all $x \in X$, $t > 0$ and all $r > 0$. From which we obtain

$$\begin{aligned} N\left(T(x) - f(x), \underbrace{t \circ t \circ \cdots \circ t}_{k}\right) &\geq N(T(x) - k^{\ell r}f(k^{-\ell r}x), t) \star N\left(k^{r\ell}f(k^{-\ell r}x) - f(x), \underbrace{t \circ t \circ \cdots \circ t}_{k-1}\right) \\ &\geq M\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{r-\frac{1+\ell}{2}}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \end{aligned} \tag{3.17}$$

for all $x \in X$, $t > 0$ and all $r > 0$. Taking the limit as $r \to \infty$ in (3.17), we obtain (3.6).

It follows from (3.3)–(3.5) that

$$\begin{aligned} &N\left(T\left(\frac{x+y}{2}\right) + T\left(\frac{x-y}{2}\right) - T(x), t \circ t \circ t \circ t\right) \\ &\geq N\left(T\left(\frac{x+y}{2}\right) - k^{\ell r}f\left(k^{-\ell r}\left(\frac{x+y}{2}\right)\right), t\right) \star N\left(T\left(\frac{x-y}{2}\right) - k^{\ell r}f\left(k^{-\ell r}\left(\frac{x-y}{2}\right)\right), t\right) \\ &\quad \star N(T(x) - k^{\ell r}f(k^{-\ell r}x), t) \star N\left(k^{\ell r}f\left(k^{-\ell r}\left(\frac{x+y}{2}\right)\right) + k^{\ell r}f\left(k^{-\ell r}\left(\frac{x-y}{2}\right)\right) - k^{\ell r}f\left(k^{-\ell r}x\right), t\right) \\ &\geq N'\left(\varphi(k^{-\ell r}x, k^{-\ell r}y), (\phi(k))^{-\ell r}t\right) = N'\left(\varphi(x,y), \left(\frac{\phi(\alpha)}{\phi(k)}\right)^{\ell r}t\right) \end{aligned}$$

for all $x, y \in X$ and $t > 0$. Letting $r \to \infty$ in the previous inequality, we obtain from $\phi(k)\ell < \phi(\alpha)\ell$ that

$$N\left(T\left(\frac{x+y}{2}\right) + T\left(\frac{x-y}{2}\right) - T(x), t\right) = 1$$

for all $x, y \in X$ and $t > 0$. This means that T satisfies (1.1). Putting $x = x + y$ and $y = x - y$ in (1.1), we get

$$T(x+y) = T(x) + T(y)$$

for all $x, y \in X$. So T is additive. Now, to prove the uniqueness property of T, let $T' : X \to Y$ be another additive function satisfying (3.6). It follows from (3.6) and (3.9) that

$$\begin{aligned} N\left(T(x) - T'(x), \underbrace{t \circ t \circ \cdots \circ t}_{2k}\right) &= N\left(k^{\ell r}T(k^{-\ell r}x) - k^{\ell r}T'(k^{-\ell r}x), \underbrace{t \circ t \circ \cdots \circ t}_{2k}\right) \\ &\geq N\left(k^{\ell r}T(k^{-\ell r}x) - k^{\ell r}f(k^{-\ell r}x), \underbrace{t \circ t \circ \cdots \circ t}_{k}\right) \\ &\quad \star N\left(k^{\ell r}f(k^{-\ell r}x) - k^{\ell r}T'(k^{-\ell r}x), \underbrace{t \circ t \circ \cdots \circ t}_{k}\right) \end{aligned}$$

$$\geq M\left(k^{-\ell r}x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha)+\frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\frac{(\phi(k))^{\ell r}(\phi(k))^{\ell j}}{(\phi(\alpha))^{\ell j}}}\right)$$

$$= M\left(x, \frac{(\phi(\alpha))^{\ell r}\left(\frac{1+\ell}{2}\phi(\alpha)+\frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\frac{(\phi(k))^{\ell r}(\phi(k))^{\ell j}}{(\phi(\alpha))^{\ell j}}}\right)$$

for all $x \in X$ and $t > 0$. Since $\phi(k)\ell < \phi(\alpha)\ell$, we obtain $\lim_{r\to\infty} M\left(x, \frac{(\phi(\alpha))^{\ell r}(\frac{1+\ell}{2}\phi(\alpha)+\frac{1-\ell}{2}\phi(k))t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\frac{(\phi(k))^{\ell r}(\phi(k))^{\ell j}}{(\phi(\alpha))^{\ell j}}}\right) = 1$, completing the proof of uniqueness. □

Corollary 3.3. *Assume that $\circ = \circ_a$ and apply Theorem 3.2, we find that the function $T : X \to Y$ is a unique additive function satisfying the following inequality near the φ-approximate Cauchy–Jensen function $f : X \to Y$,*

$$N(f(x) - T(x), t) \geq M\left(x, \frac{\frac{1+\ell}{2}\phi(\alpha)+\frac{1-\ell}{2}\frac{\phi(k)^2}{\phi(\alpha)}-\phi(k)}{k}t\right) \tag{3.18}$$

for all $x \in X$ and $t > 0$, where $M(x, t)$ is defined as in Theorem 3.2. In this case, Y is a complete fuzzy ϕ-normed space.

In particular, letting $\phi(\alpha) = |\alpha|^\beta$ and $\ell k < \ell\alpha$, we get to a result in fuzzy β-normed space as follows:

$$N(f(x) - T(x), t) \geq M\left(x, \frac{(|\alpha|^\beta - |k|^\beta)\left(\frac{1+\ell}{2}|\alpha|^\beta + \frac{\ell-1}{2}|k|^\beta\right)}{k|\alpha|^\beta}t\right)$$

for all $x \in X$ and $t > 0$. When $\beta = 1$ and $\star = \star_M$ we get to a result in fuzzy normed space which is defined in [2].

Remark 3.4. Let $\phi(k) < \phi(\alpha)$. Suppose that the function $t \longmapsto N(f(x) - T(x), .)$ from $[0, \infty)$ into $[0, 1]$ is right continuous. Then we obtain a better fuzzy (3.18) as follows,

$$N\left(T(x) - f(x), s + \underbrace{t + t + \cdots + t}_{k-1}\right) \geq N(T(x) - k^r f(k^{-r}x), s) \star N\left(k^r f(k^{-r}x) - f(x), \underbrace{t + t + \cdots + t}_{k-1}\right)$$

$$\geq M\left(x, \frac{\phi(\alpha)t}{\sum_{j=0}^{r-1}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^j}\right)$$

tending s to zero we infer that

$$N(f(x) - T(x), t) \geq M\left(x, \frac{\phi(\alpha) - \phi(k)}{k-1}t\right)$$

for all $x \in X$ and $t > 0$.

Corollary 3.5. *Assume that $\circ = \circ_m$ and apply Theorem 3.2, we find that the function $T : X \to Y$ is a unique additive function satisfying the following inequality near the φ-approximate Cauchy–Jensen function $f : X \to Y$,*

$$N(f(x) - T(x), t) \geq M\left(x, \left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t\right)$$

for all $x \in X$ and $t > 0$, where $M(x, t)$ is defined as in Theorem 3.2. In this case, Y is a complete non-Archimedean fuzzy ϕ-normed space.

In the next result, we investigate the generalized fuzzy stability of Cauchy equation.

Proposition 3.6. *Let $\ell \in \{-1, 1\}$ be fixed and let $f : X \to Y$ be a φ-approximately Cauchy function. If for some positive real number α and some positive integer k with $\phi(k)\ell < \phi(\alpha)\ell$ satisfying (3.4). Then there is a unique additive function $T : X \to Y$ such that*

$$N\left(f(x) - T(x), \underbrace{t \circ t \circ \cdots \circ t}_{k}\right) \geq G\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \tag{3.19}$$

for all $x \in X$ *and* $t > 0$, *where*

$$G(x,t) = N'(\varphi(x,x),t) \star N'(\varphi(x,2x),t) \star \cdots \star N'(\varphi(x,(k-1)x),t).$$

Proof. Replacing x and y by $\frac{x+y}{2}$ and $\frac{x-y}{2}$ in (3.1), respectively, we obtain

$$N\left(f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-f(x),t\right) \geq N'\left(\varphi\left(\frac{x+y}{2},\frac{x-y}{2}\right),t\right) \tag{3.20}$$

for all $x, y \in X$ and $t > 0$. By replacing $N'(\varphi(x,y),t)$ in Theorem 3.2 by the right-hand side of (3.20), we find a unique additive function $T : X \to Y$ such that (3.19) holds. $\square$

Corollary 3.7 *([46], Theorem 3.1). Assume that* $\ell = 1$, $\phi(\alpha) = |\alpha|$, $\circ = \circ_a$, $\star = \star_M$ *and* $k = 2$ *and apply Proposition 3.6, we find that the function* $T : X \to Y$ *is a unique additive function satisfying the following inequality near the* φ*-approximate Cauchy function* $f : X \to Y$,

$$N(f(x) - T(x), t) \geq N'\left(\varphi(x,x), \frac{2-\alpha}{2}t\right)$$

for all $x \in X$ *and* $t > 0$.

Corollary 3.8 *([47], Theorem 3.1). In Proposition 3.6, let* X *be a vector space over* $\mathbb{K}$ *and* Y *be a complete non-Archimedean fuzzy* ϕ*-normed space over* $\mathbb{K}$*, then assume that* $\ell = 1$, $\phi(\alpha) = |\alpha|$, $\circ = \circ_m$ *and* $\star = \star_M$ *and apply Proposition 3.6, we find that the function* $T : X \to Y$ *is a unique additive function satisfying the following inequality near the* φ*-approximate Cauchy function* $f : X \to Y$,

$$N(f(x) - T(x), t) \geq G_m(x, \alpha t)$$

for all $x \in X$ *and* $t > 0$, *where*

$$G_m(x,t) = \min\{N'(\varphi(x,x),t), N'(\varphi(x,2x),t), \ldots, N'(\varphi(x,(k-1)x),t)\}.$$

In the next result, we investigate the generalized fuzzy stability of Jensen equation.

Proposition 3.9. *Let* $\ell \in \{-1, 1\}$ *be fixed and let* $f : X \to Y$ *be a* φ*-approximately Jensen function. If for some positive real number* α *and some positive integer* k *with* $\phi(k)\ell < \phi(\alpha)\ell$ *satisfying (3.4). Then there is a unique additive function* $T : X \to Y$ *such that*

$$N\left(f(x) - f(0) - T(x), \underbrace{t \circ t \circ \cdots \circ t}_{2k-1}\right) \geq G_J\left(x, \frac{\left(\frac{1+\ell}{2}\phi(\alpha) + \frac{1-\ell}{2}\phi(k)\right)t}{\bigodot_{j=\frac{1-\ell}{2}}^{\infty}\left(\frac{\phi(k)}{\phi(\alpha)}\right)^{\ell j}}\right) \tag{3.21}$$

for all $x \in X$ *and* $t > 0$, *where*

$$\begin{aligned} G_J(x,t) = {} & N'(\varphi(x,x),t) \star N'(\varphi(x,2x),t) \star \cdots \star N'(\varphi(x,(k-1)x),t) \\ & \star N'(\varphi(2x,0),t) \star N'(\varphi(3x,0),t) \star \cdots \star N'(\varphi(kx,0),t). \end{aligned}$$

Proof. Let $g(x) = f(x) - f(0)$ for all $x \in X$, and then using (3.2), we get

$$N\left(2g\left(\frac{x+y}{2}\right) - g(x) - g(y), t\right) \geq N'(\varphi(x,y),t) \tag{3.22}$$

for all $x, y \in X$ and $t > 0$. Replacing x by $x + y$ and y by 0 in (3.22), respectively, we get

$$N\left(2g\left(\frac{x+y}{2}\right) - g(x+y), t\right) \geq N'(\varphi(x+y,0),t) \tag{3.23}$$

for all $x, y \in X$ and $t > 0$. It follows from (3.22) and (3.23) that

$$N(g(x+y) - g(x) - g(y), t \circ t) \geq N'(\varphi(x,y),t) \star N'(\varphi(x+y,0),t)$$

for all $x, y \in X$ and $t > 0$. Using the same argument as in the proof of Proposition 3.6, we can show that there exists a unique additive function $T : X \to Y$, which satisfies inequality (3.21). $\square$

Corollary 3.10 (*[47], Theorem 4.1*). *In Proposition 3.9, let X be a vector space over $\mathbb{K}$ and Y be a complete non-Archimedean fuzzy ϕ-normed space over $\mathbb{K}$, then assume that $\ell = 1$, $\phi(\alpha) = |\alpha|$, $\circ = \circ_m$ and $\star = \star_M$ and apply Proposition 3.9, we find that the function $T : X \to Y$ is a unique additive function satisfying the following inequality near the φ-approximate Jensen function $f : X \to Y$,*

$$N(f(x) - T(x), t) \geq G_I(x, \alpha t)$$

for all $x \in X$ and $t > 0$, where

$$G_I(x, t) = \min\{N'(\varphi(x, x), t), N'(\varphi(x, 2x), t), \dots, N'(\varphi(x, (k-1)x), t), N'(\varphi(2x, 0), t), N'(\varphi(3x, 0), t), \dots, N'(\varphi(kx, 0), t)\}.$$

4. Applications of generalized fuzzy stability

In this section, we investigate applications of generalized fuzzy stability to the stability of Cauchy–Jensen functional equation in β-normed spaces and non-Archimedean normed spaces.

Corollary 4.1. *Let X be a β-normed space and Y be a complete fuzzy β-normed space. If there exist real numbers $\varepsilon > 0$ and $\lambda \neq \frac{1}{\beta}$ such that a mapping $f : X \to Y$ satisfies the inequality*

$$\left\| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right\| \leq \varepsilon(\|x\|^{\lambda} + \|y\|^{\lambda}) \tag{4.1}$$

for all $x, y \in X$. Then there exists a unique additive mapping $T : X \to Y$ satisfies (1.1) and the inequality

$$\|f(x) - T(x)\| \leq \frac{2^{\beta\lambda+1}}{2^{\beta^2\lambda} - 2^{\beta}} \varepsilon \|x\|^{\lambda} \tag{4.2}$$

for all $x \in X$, where $\beta\lambda > 1$.

Proof. For all $x \in X$ and $t > 0$, define the function N by

$$N(x, t) = \frac{t}{t + \|x\|}.$$

It is easy to see that $(X, N, \star_M)$ is a fuzzy β-normed space and $(Y, N, \star_M)$ is a complete fuzzy β-normed space. Denote $\varphi : X \times X \longrightarrow \mathbb{R}_+$, the function sending each (x, y) to $\varepsilon\ (\|x\|^{\lambda} + \|y\|^{\lambda})$. By assumption

$$N\left(f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x), t\right) \geq N'(\varphi(x, y), t)$$

we note that if $N' : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow [0, 1]$ given by

$$N'(x, t) = \frac{t}{t + |x|},$$

then $(\mathbb{R}_+, N', \star_M)$ is a fuzzy β-normed space. By Corollary 3.3 (for $k = 2$ and $\ell = 1$), there exists a unique additive function $T : X \longrightarrow Y$ satisfies Eq. (1.1) and

$$\begin{aligned} \frac{t}{t + \|f(x) - T(x)\|} &= N(f(x) - T(x), t) \\ &\geq N'\left(2^{\beta\lambda}\varepsilon\|x\|^{\lambda}, \frac{2^{\lambda\beta^2} - 2^{\beta}}{2} t\right) = \frac{2^{\lambda\beta^2} - 2^{\beta} t}{2^{\lambda\beta^2} - 2^{\beta} t + 2^{\beta\lambda+1}\varepsilon\|x\|^{\lambda}}. \end{aligned}$$

Thus, we can find a additive function $T : X \longrightarrow Y$ such that (4.2) holds. □

The following example shows that the above result is not valid in non-Archimedean normed spaces.

Example 4.2. Let $p > 2$ be a prime number and define $f : \mathbb{Q}_p \to \mathbb{Q}_p$ by $f(x) = x + 1$. Since $|2^n|_p = 1$,

$$\left| f\left(\frac{x+y}{2}\right) + f\left(\frac{x-y}{2}\right) - f(x) \right|_p = |1|_p = 1$$

for all $x, y \in X$. Hence the conditions of Corollary 4.1 for $\varepsilon = 1$ and $\lambda = 0$ hold. However for each $r \in \mathbb{N}$, we have

$$|2^r f(2^{-r}x) - 2^{r+1} f(2^{-(r+1)}x)|_p = |2^r|_p |1 - 2|_p = 1$$

for all $x \in \mathbb{Q}_p$. Hence $\{2^r f(2^{-r}x)\}$ is not a Cauchy sequence.

However, we have the following result.

Corollary 4.3. *Let X be a non-Archimedean normed space over $\mathbb{Q}_p$ and Y be a complete non-Archimedean normed space over $\mathbb{Q}_p$, where $p > 2$ is a prime number. Suppose that a Cauchy–Jensen function $f : X \to Y$ satisfies the inequality*

$$\left\|f\left(\frac{x+y}{2}\right)+f\left(\frac{x-y}{2}\right)-f(x)\right\| \leq \begin{cases}\varepsilon(\|x\|^{\lambda}+\|y\|^{\lambda}), & \lambda < 1;\\ \varepsilon \max\{\|x\|^{\lambda}, \|y\|^{\lambda}\}, & \lambda > 1;\end{cases}$$

for all $x, y \in X$, where ε and λ are non-negative real numbers. Then there exists a unique additive function $T : X \to Y$ satisfying

$$\|f(x)-T(x)\| \leq \begin{cases}(1+p^{\lambda})\varepsilon\|x\|^{\lambda}, & \lambda < 1;\\ p^{\lambda}\varepsilon\|x\|^{\lambda}, & \lambda > 1;\end{cases}$$

for all $x \in X$.

Proof. The proof is similar to the proof of Corollary 4.1 and the result follows from Corollary 3.5. □

Corollary 4.4. *Let $(Y, \|\cdot\|)$ be a real complete normed space, $(Y, N, \star_p)$ be a complete non-Archimedean normed space in Example 2.6, $(Z, N', \star_p)$ be the fuzzy normed space and $f : X \to Y$ be a φ-approximately Cauchy–Jensen function. If for some positive real number α and some positive integer k with $k < \alpha$, inequality (3.4) holds, then there is a unique additive function $T : X \to Y$ such that*

$$e^{-\|f(x)-T(x)\|/t} \geq \prod_{\iota=2}^{k} N'(\varphi(\iota x, (2-\iota)x), \alpha t) \tag{4.3}$$

for all $x \in X$ and $t > 0$, where $\prod_{\iota=1}^{j} a_\iota = a_1 \cdot a_2 \ldots a_j$.

Proof. The conclusion follows from Corollary 3.5. □

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