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A note on the Banaś modulus of smoothness in the Bynum space

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ABSTRACT

Recently, the Banaś modulus of smoothness for the Bynum space $b_{2,\infty}$ was obtained by Zuo and Cui (Z. Zuo, Y. Cui, Some modulus and normal structure in Banach space, J. Inequal. Appl. 2009 (2009) 15. doi:10.1155/2009/676373. Article ID 676373). It is however not true in general. In this note, we will present the exact value for this modulus in the $b_{2,\infty}$ space.

1. Introduction

Let X be a Banach space and denote by S_X the unit sphere of X. In connection with the Pythagorean theorem, Gao [1] introduced the following parameter

$$f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : x, y \in S_X\}.$$

This parameter has been considered by several authors. Gao–Saejung [2] and Wang–Yang [3] obtained some sufficient conditions for normal structure by this parameter, which in turn implies the fixed point property of a Banach space. Cui–Wang [4] also considered such a parameter in the Lorentz sequence space. It is known that

- X is uniformly nonsquare if and only if $f(X) > 2$;
- X has normal structure if $f(X) > 4(3-\sqrt{5})$;
- $f(\ell_p) = \min(2^{2/p+1}, 2^{2/q+1})$, where $1/p + 1/q = 1$;
- $f(X) = \inf\{\|x+y\|^2 + \|x-y\|^2 : x, y \in X, \|x\|, \|y\| \geq 1\}$.

Recently, Zuo–Cui [6] stated a formula:

$$f(X) = \inf_{0\leq\tau\leq 2}\left\{\tau^2 + 4(1-\rho_X(\tau))^2\right\}, \tag{1}$$

where the modulus of smoothness [5] $\rho_X(\tau) : [0, 2] \to [0, 1]$ is defined as

$$\rho_X(\tau) = \sup\left\{1 - \frac{\|x+y\|}{2} : x, y \in S_X, \|x-y\| \leq \tau\right\}.$$

From this and the following equality

$$\rho_{b_{2,\infty}}(\tau) = \tau/2\sqrt{2} \quad \text{for } \tau \in [0, 2], \tag{2}$$

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they obtained the value of $f(X)$ for the Bynum space $b_{2,\infty}$ [6, Example 3.4]. However, the equality (2) does not always hold for every $\tau \in [0,2]$. In fact from the definition of $\rho_X(\tau)$ it follows that $\rho_{b_{2,\infty}}(2)=1$, while (2) gives $\rho_{b_{2,\infty}}(2)=1/\sqrt{2}$, a contradiction. In this note, we will show that the exact value for $\rho_{b_{2,\infty}}$ may be the following

$$\rho_X(\tau)=\max\left(\frac{\tau}{2\sqrt{2}},\frac{\tau}{\sqrt{2}}+1-\sqrt{2}\right).$$

2. Main results

Let us begin with some definitions. The James constant is defined as

$$J(X)=\sup\{\min(\|x+y\|,\|x-y\|):x,y\in S_X\},$$

while the Schäffer constant is defined as

$$g(X)=\inf\{\max(\|x+y\|,\|x-y\|):x,y\in S_X\}.$$

It is well known that $J(X)g(X)=2$; see for instance [7]. To simplify the notation, we denote $g:=g(X)$ in the following. Also it is proved that

$$g=2(1-\rho_X(g)), \tag{3}$$

(see for instance [8, Proposition 3]).

Proposition 1. *Let X be the $b_{2,\infty}$ space, i.e., ℓ_2 space with the norm*

$$\|x\|=\max\{\|x^+\|_2,\|x^-\|_2\},$$

where x^+ and x^- are positive and negative parts of $x\in\ell_2$. Then

$$\rho_X(\tau)=\max\left(\frac{\tau}{2\sqrt{2}},\frac{\tau}{\sqrt{2}}+1-\sqrt{2}\right).$$

Proof. In view of [9], we know that

$$g(X)=\frac{2}{J(X)}=2(2-\sqrt{2}).$$

Since $\rho_X(\tau)$ is convex [8], we have from (3) that

$$\rho_X(\tau)\le\frac{\rho_X(g)}{g}\tau=\frac{2-g}{2g}\tau=\frac{\tau}{2\sqrt{2}}$$

for $\tau\in(0,2(2-\sqrt{2})]$, and also that

$$\begin{aligned}\rho_X(\tau)&\le\frac{2-\tau}{2-g}\rho_X(g)+\frac{\tau-g}{2-g}\rho_X(2)\\&=\frac{2-\tau}{2}+\frac{\tau-g}{2-g}\\&=\frac{\tau}{\sqrt{2}}+1-\sqrt{2}\end{aligned}$$

for $\tau\in[2(2-\sqrt{2}),2]$.

On the other hand, if $\tau\in(0,2(2-\sqrt{2})]$, let $\sigma=\sqrt{2}\tau/(2\sqrt{2}-\tau)$ and let

$$x=\left(1-\frac{2\sqrt{2}}{\sqrt{2}+\sigma},1,0,\ldots\right),\qquad y=\left(-1,1-\frac{2\sigma}{\sqrt{2}+\sigma},0,\ldots\right).$$

It is easy to check that $0\le\sigma<\sqrt{2}$, $\|x\|=\|y\|=1$, $\|x-y\|=\tau$ and $\|x+y\|=2-\tau/2\sqrt{2}$, which implies that

$$\rho_X(\tau)\ge\tau/2\sqrt{2}$$

for $\tau\in(0,2(2-\sqrt{2})]$. If $\tau\in[2(2-\sqrt{2}),2]$, let

$$x=(1-\tau,1,0,\ldots),\qquad y=(1,1-\tau,0,\ldots).$$

Then we have $x, y \in S_X$, $\|x-y\| = \tau$, $\|x+y\| = \sqrt{2}(2-\tau)$ and therefore

$$\rho_X(\tau) \geq \frac{\tau}{\sqrt{2}} + 1 - \sqrt{2}$$

for $\tau \in [2(2-\sqrt{2}), 2]$. Combining all the cases, we get the result as desired. □

The above result together with (1) immediately yields $f(X) = 8/3$. It is worthwhile noting that Proposition 1 is also true for the Day–James space $\ell_2 - \ell_\infty$. The parameter $f(X)$ for the Day–James space $\ell_2 - \ell_\infty$ is also considered in [6]. We now consider another Day–James space $\ell_2 - \ell_1$. Our method is due to [10, Example 3].

Proposition 2. *Let X be the $\ell_2 - \ell_1$ space, i.e., $\mathbb{R}^2$ with the norm*

$$\|x\| = \begin{cases} \|x\|_2 & \text{if } x_1x_2 \geq 0, \\ \|x\|_1 & \text{if } x_1x_2 \leq 0. \end{cases}$$

Then we have $f(X) = 3$.

Proof. Let $x = (1/\sqrt{2}, 1/\sqrt{2})$, $y = (-1/2, 1/2)$. It is easy to check that $\|x\| = \|y\| = 1$ and $\|x \pm y\| = \sqrt{3/2}$, which yields $f(X) \geq 3$. To show the converse, let $x = (x_1, x_2), y = (y_1, y_2) \in S_X$. For the symmetry, we only consider three cases for $x, y \in S_X$.

Case 1. $0 \leq x_i \leq 1, 0 \leq y_i \leq 1, i = 1, 2$. Obviously, $\|x\| = \|x\|_2, \|y\| = \|y\|_2, (x_1+y_1)(x_2+y_2) \geq 0$ and $(x_1-y_1)(x_2-y_2) \leq 0$. Thus

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \|x+y\|_2^2 + \|x-y\|_1^2 \\ &= \|x+y\|_2^2 + \|x-y\|_2^2 - 2(x_1-y_1)(x_2-y_2) \\ &\geq 2(\|x\|_2^2 + \|y\|_2^2) = 4. \end{aligned}$$

Case 2. $0 \leq x_2 \leq 1/\sqrt{2} \leq x_1 \leq 1, 0 \leq -y_1 \leq 1/2 \leq y_2 \leq 1$. Obviously, $\|x\| = \|x\|_2, \|y\| = \|y\|_1, (x_1+y_1)(x_2+y_2) \geq 0$ and $x_1 - y_1 \geq 0$. If $x_2 - y_2 \geq 0$, then

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \|x+y\|_2^2 + \|x-y\|_2^2 \\ &= 2(\|x\|_2^2 + \|y\|_2^2) \geq 2\|x\|_2^2 + \|y\|_1^2 = 3. \end{aligned}$$

If not, we have

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \|x+y\|_2^2 + \|x-y\|_1^2 \\ &= \|x+y\|_2^2 + \|x-y\|_2^2 - 2(x_1-y_1)(x_2-y_2) \\ &\geq 2(\|x\|_2^2 + \|y\|_2^2) \geq 2\|x\|_2^2 + \|y\|_1^2 = 3. \end{aligned}$$

Case 3. $-1 \leq x_1 \leq 0 \leq x_2 \leq 1, -1 \leq y_1 \leq 0 \leq y_2 \leq 1$. Obviously, $\|x\| = \|x\|_1, \|y\| = \|y\|_1, (x_1+y_1)(x_2+y_2) \leq 0$ and $(x_1-y_1)(x_2-y_2) = (x_1-y_1)^2 \geq 0$. Thus

$$\begin{aligned} \|x+y\|^2 + \|x-y\|^2 &= \|x+y\|_1^2 + \|x-y\|_2^2 \\ &= 4 + 2(x_1-y_1)^2 \geq 4. \end{aligned}$$

Combining all the cases above, we get $f(X) \geq 3$. □

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