DISCRETE MATHEMATICS

Doubly alternating Baxter permutations are Catalan

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Abstract

The Baxter permutations who are alternating and whose inverse is also alternating are shown to be enumerated by the Catalan numbers. A bijection to complete binary trees is also given. © 2000 Elsevier Science B.V. All rights reserved.

Résumé

Nous montrons que les permutations de Baxter alternantes dont l'inverse est également une permutation de Baxter alternante sont énumérées par les nombres de Catalan. De plus, nous donnons une bijection avec les arbres binaires complets. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Baxter [1] studied a particular class of permutations by studying fixed points of the composite of commuting functions. These permutations can be defined by forbidden subsequences [10–12,14,16]. In fact, the *Baxter permutations* are exactly the permutations π of length n which verify the two following conditions:
for all $1 \leqslant i < j < k < l \leqslant n$,

$$\text{if } \pi_i + 1 = \pi_l \quad \text{and} \quad \pi_j > \pi_l \text{ then } \pi_k > \pi_l,$$

$$\text{if } \pi_l + 1 = \pi_i \quad \text{and} \quad \pi_k > \pi_i \text{ then } \pi_j > \pi_i.$$

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For example, 2413 and 3142 are the only permutations on four elements which are not Baxter permutations.

Chung et al. [2] enumerated analytically the Baxter permutations of length n by the formula $\sum_{r=0}^{n-1} [\binom{n+1}{r} \cdot \binom{n+1}{r+1} \cdot \binom{n+1}{r+2}]/[\binom{n+1}{1} \cdot \binom{n+1}{2}]$. Later, Mallows [13] found a more precise formula whereas Viennot [15] (see also [4]) gave a combinatorial proof of the formula obtained by Chung et al.

A permutation is said to be *alternating* if the permutation starts with a rise and then descents and rises come in turn. More precisely, an alternating permutation of length n is such that $\pi(2i-1) < \pi(2i) > \pi(2i+1)$ for $1 \leqslant i \leqslant \lfloor n/2 \rfloor$ that is to say its rise (respectively, descent) set denoted $\mathrm{Rise}(\pi)$ (respectively $\mathrm{Des}(\pi)$) is equal to odd (respectively, even) index.

Cori et al. [3] established a one-to-one correspondence which proved that alternating Baxter permutations of length $2n$ and $2n+1$ are enumerated by $c_n c_n$ and $c_n c_{n+1}$ where $c_n = [1/(n+1)]\binom{2n}{n}$ is nth Catalan number. More recently, Dulucq and Guibert [5] found a new bijection which unifies [15,3] and gives a combinatorial interpretation of Mallows formula [13].

Considering the definition, it is clear that if π is a Baxter permutation of length n then

1. its *mirror* π^* defined by $\pi^*(i) = \pi(n+1-i)$ for $1 \leqslant i \leqslant n$,
2. its *complement* π^c defined by $\pi^c(i) = n+1-\pi(i)$ for $1 \leqslant i \leqslant n$,
3. its *inverse* π^{-1} defined by $\pi^{-1}(i) = j \Leftrightarrow \pi(j) = i$ for $1 \leqslant i \leqslant n$

are also Baxter permutations.

However, one can find an alternating Baxter permutation whose inverse is not alternating. A permutation is said to be *doubly alternating* if it is alternating and its inverse is alternating. We will use

Remark 1. For enumeration of doubly alternating Baxter permutations it does not make any difference if we define alternating to start with a rise or with a descent.

Proof. Follows easily from 1–3 above. □

Example 2. The permutation 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4 is a doubly alternating Baxter permutation of length 18, see Fig. 1.

The purpose of this note is to establish the following.

Theorem 3 (Main Theorem). *The number of doubly alternating Baxter permutations of length* $2n+\varepsilon$ *where* $\varepsilon = 0$ *or* 1 *denoted* $d_{2n+\varepsilon}$ *is the Catalan number*

$$c_n = \frac{1}{n+1}\binom{2n}{n}.$$

Note that we have $d_0 = 1$ by considering the empty permutation.

Now we consider another way to regard permutations: the *essential set* of a permutation with its *rank function*, introduced by Fulton [9] and further studied by Eriksson and Linusson [6,7]. First, let every permutation π of length n be represented by its dotted permutation matrix, regarded as an $n \times n$-collection of squares in the plane, where square $(i, \pi(i))$ in row i and column $\pi(i)$ has a dot for all $1 \leqslant i \leqslant n$.

We then get the Rote diagram of the permutation by shading the square containing the dot, the squares in that row to the right of the dot, and the squares in that column below the dot.

We call a white (nonshaded) square a *white corner* if the squares directly to the right and below of it are shaded. The essential set of a permutation π is defined to be the set of white corners of the diagram of π. For every square, its rank is defined by the number of dots northwest of the square. We will not need the rank in this article.

Eriksson and Linusson have characterized the essential sets of Baxter permutations.

Proposition 4 (Eriksson and Linusson [7]). *A permutation is Baxter if and only if its essential set has at most one white corner in each column and row.*

Clearly, the essential set of a permutation has a white corner in row i if and only if the permutation has a descent in position i and the essential set has a white corner in column i if and only if the inverse permutation has a descent in position i. Hence we get the following corollary which is the key to the proof of Theorem 3.

Corollary 5. *A permutation is a doubly alternating Baxter permutation if and only if its essential set has exactly one square in each even row that is not the last row and in each even column that is not the last column.*

Example 6. Fig. 1 shows the essential set of the doubly alternating Baxter permutation 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4.

2. Proof

First a technical lemma about a rectangular part R of the matrix of a doubly alternating Baxter permutation. The reader might want to read the proof below of Theorem 3 first. Note that the construction of the Rote diagram and of the essential set works as well for a dot configuration that is not of full rank.

Lemma 7. *Let R be a rectangle with $2n+1$ rows and $s \geqslant 2n+1$ columns. Assume that R has one dot in each row j on square $(j, \pi(j))$ such that π is alternating starting with a descent. Even stronger we demand that the essential set of the dot configuration in R has exactly one white corner in the rows $\{1, 3, 5, \ldots, 2n-1\}$ and none in the other rows. Assume also that if R has a dot in an odd column (different*

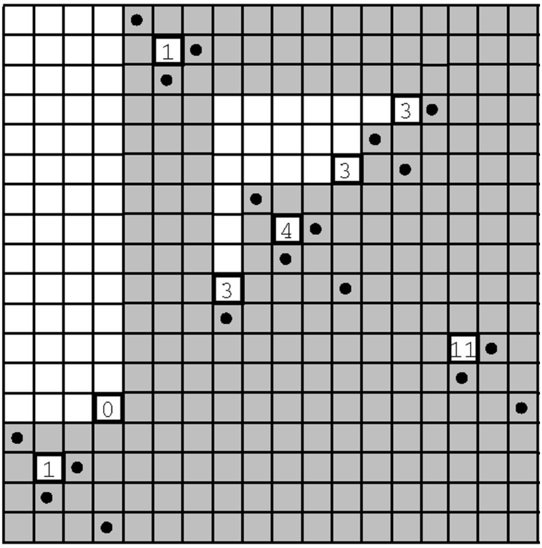

Fig. 1. The essential set (and the ranks of the white corners) of the doubly alternating Baxter permutation 5 7 6 15 13 14 9 11 10 12 8 17 16 18 1 3 2 4.

from $2n+1$) *then there is also a white corner in that column, and that there are no white corners in the even columns.*

Then $\pi(j)\leqslant 2n$ *for each* $1\leqslant j\leqslant 2n$ *and* $\pi(2n+1)=2n+1$.

Proof. The proof will be done by induction on n. It is trivial for $n=0$. For the reasoning below it will help to look at Fig. 2. Let $k:=\pi(2n+1)$. Since we do not have a white corner on row $2n+1$, we know that R has dots in all the columns $1,\dots,k-1$, therefore, it suffices to show $k=2n+1$.

Assume that $k<2n+1$. If k was even then there could not be a white corner in column $k-1$ so k is odd and we have a white corner (i,k) with i odd (i odd implies directly that $k>1$). We have $j:=\pi(i+1)<k$, because (i,k) is a white corner. There can be no dots in the rectangle $[i+1,2n+1]\times[k+1,s]$, since a dot there would imply a white square and hence a white corner in column k below row $i+1$ contradicting Baxter. There can also be no dots in $[1,i]\times[j+1,k]$, since otherwise there would be a second white corner on row i, a contradiction. The latter implies that j has to be even.

Now, let $(i',j-1)$ be the white corner in row $j-1$, where i' is odd, $2n+1>i'>i$. There are no dots in $[i+2,i']\times[1,j-1]$, since such would cause a second white corner in row $j-1$. Hence the rectangle $R':=[i+2,i']\times[j+1,k-1]$, striped in Fig. 2, contains a dot in every row. It fulfills also all the other conditions in the lemma. By induction we get that $\pi(i')=j+i'-i-1\leqslant k-1$ is the last of the dots in R'. Since $j+i'-i-1$ is odd, we should have a white corner in that column, which is impossible, a contradiction. □

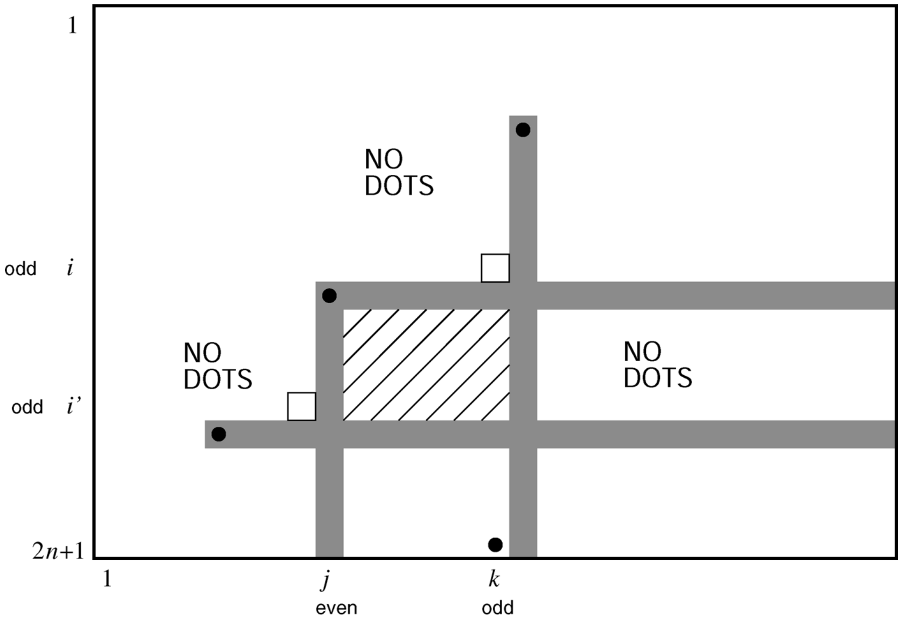

Fig. 2. The striped rectangle has the same properties as the entire rectangle in Lemma 7.

Corollary 8. *If* π *is a Baxter permutation of length* $2n+1$ *with* $\mathrm{Des}(\pi)=\mathrm{Des}(\pi^{-1})=\{1,3,5,\dots,2n-1\}$, *then* $\pi(2n+1)=2n+1$. *In particular, we have* $d_{2n}=d_{2n+1}$.

Proof. Follows directly from Lemma 7 and Remark 1. $\square$

Lemma 9. *If* π *is a Baxter permutation of length* $2n$ *with* $\mathrm{Des}(\pi)=\mathrm{Des}(\pi^{-1})=\{2,4,6,\dots,2n-2\}$, *then*

$$\pi(1)=2k+1 \quad\Rightarrow\quad \pi(2n-2k)=2n \text{ and no dots in } [1,2n-2k]\times[1,2k].$$

Proof. Let $(i,2k)$ be the white corner in column $2k$, i even. The Baxter condition gives that there are no dots in $[1,i]\times[1,2k]$. So the rectangle $[2,i]\times[2k+2,2n]$ fulfills the conditions of Lemma 7 which means that $\pi(i)=2k+1+i-1$, an even number. There are no white squares above the dot in row i and hence it must be in the last column. This means $2k+i=2n$ and the lemma is proved. $\square$

Proof of Theorem 3 (Main Theorem). We will count the number of Baxter permutations of length $2n$ with $\mathrm{Des}(\pi)=\mathrm{Des}(\pi^{-1})=\{2,4,6,\dots,2n-2\}$. From the conditions on π^{-1} it follows that $\pi(1)$ is odd, say $2k+1$. From Lemma 9 we get $\pi(2n-2k)=2n$ which implies that there are dots neither in $[1,2n-2k]\times[1,2k]$ nor in $[2n-2k+1,2n]\times[2k+1,2n]$. We have that $[2,2n-2k-1]\times[2k+2,2n-1]$ is isomorphic to any matrix of the mirrored complement of a doubly alternating Baxter permutation of length $2n-2k-2$, i.e. with descents in odd rows and columns; and we

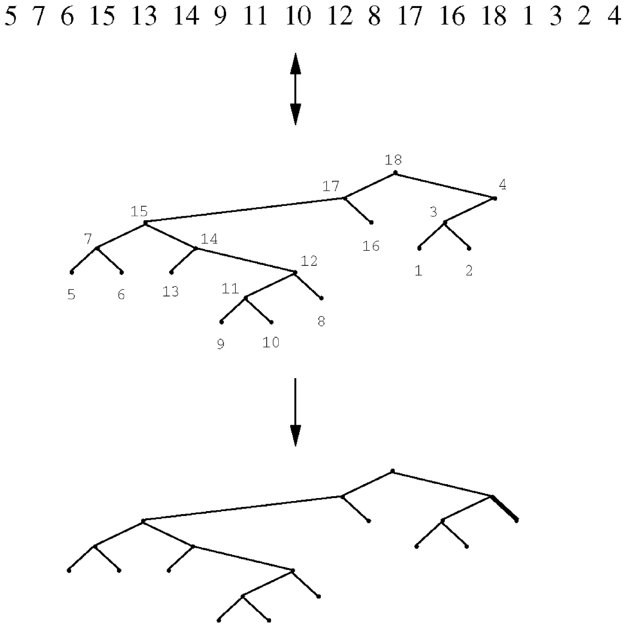

Fig. 3. The mapping of the bijection: from a doubly alternating Baxter permutation to a complete binary tree.

have that $[2n-2k+1,2n]\times[1,2k]$ is isomorphic to any matrix of a doubly alternating Baxter permutation of length $2k$, i.e. with descents in even rows and columns. Using Remark 1, we get the recursion

$$d_{2n}=\sum_{k=0}^{n-1} d_{2n-2k-2}\cdot d_{2k}.$$

This is the well-known recursion for the Catalan numbers $c_n=\sum_{k=0}^{n-1} c_{n-k-1}\cdot c_k$. Using Corollary 8 the theorem is proved. □

3. Example of bijection

Now we exhibit a bijection for doubly alternating Baxter permutations which can be regarded as a specialization of the correspondence of Dulucq and Guibert [5]. The fact that it is a bijection follows from the structural decomposition in Section 2.

Lemma 10. *There is a bijection between doubly alternating Baxter permutations of length $2n$ and complete binary trees with $2n+1$ nodes.*

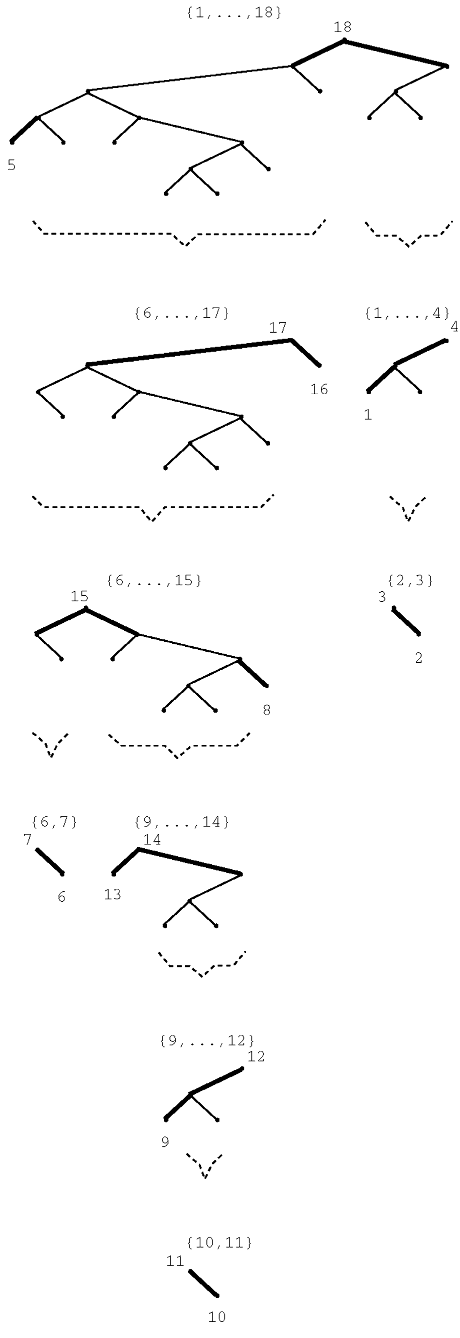

Fig. 4. The inverse mapping of the bijection: labeling a complete binary tree in decreasing order.

Algorithm for a right-truncated tree with the set of labels $\{2i+1,\dots,2j\}$.

- Label $2j$ the root.
- Let $2k-1$ be the number of nodes of the left subtree.
- Label $2(j-k)+1$ the leftmost leaf of the tree.
- Call the algorithm for the left-truncated tree (given in Fig. 6) with the left subtree without its leftmost leaf, and the labels $\{2(j-k)+2,\dots,2j-1\}$ as input.
- Call the algorithm for the right-truncated tree with the right subtree and labels $\{2i+1,\dots,2(j-k)\}$ as input.

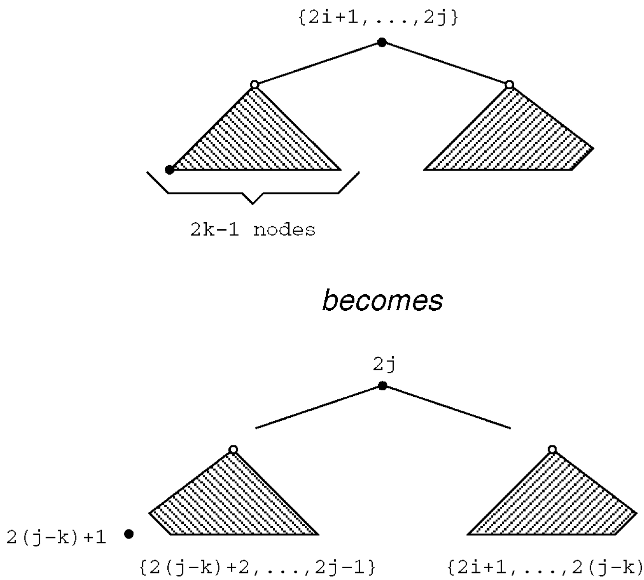

Fig. 5. Algorithm for a right-truncated (complete binary) tree with the set of consecutive labels $\{2i+1,\dots,2j\}$.

Algorithm for a left-truncated tree with the set of labels $\{2i+2,\dots,2j+2\}$.

- Label $2j+1$ the root.
- Let $2k-1$ be the number of nodes of the right subtree.
- Label $2(j-k+1)$ the rightmost leaf of the tree.
- Call the algorithm for the left-truncated tree with the left subtree and labels $\{2i+2,\dots,2(j-k)+1\}$ as input.
- Call the algorithm for the right-truncated tree with the right subtree without its rightmost leaf, and the labels $\{2(j-k+1)+1,\dots,2j\}$ as input.

Fig. 6. Algorithm for a left-truncated (complete binary) tree with the set of consecutive labels $\{2i+2,\dots,2j+1\}$.

The mapping and its inverse can be defined in the following way:

- The mapping (an example is given by Fig. 3) consists in first building the decreasing complete binary tree of the doubly alternating Baxter permutation and then forgetting its labels.

  We briefly recall the correspondence between alternating permutations and decreasing complete binary trees:

  $$\mathrm{decr}(w) = \triangleleft\ \mathrm{decr}(u), x, \mathrm{decr}(v)\ \triangleright$$

  where $w = uxv$ and $x = \max\{w_i\colon w = w_1 w_2 \ldots w_p\}$.
- The inverse mapping (an example is given by Fig. 4) consists in labeling a complete binary tree with $2n+1$ nodes, in decreasing order. The algorithms given in Fig. 5 is called with the complete binary tree where its rightmost leaf is omitted (the tree is said to be right-truncated) and with the set of consecutive labels $\{1,2,\ldots,2n\}$. Thus, we recursively obtain a decreasing labeling of the initial complete binary tree. The doubly alternating Baxter permutation is obtained by projecting in infix order this labeling (Fig. 6).

That the mapping and its inverse are well defined follows from Section 2. Thus, we obtain the following results.

Corollary 11. *The number of doubly alternating Baxter permutations of length* $2n+\varepsilon$ *where* $\varepsilon = 0$ *or* 1 *such that* $\pi(2k) = 2n+\varepsilon$ *where* $1 \leqslant k \leqslant n$ *is*

$$c_{k-1} \cdot c_{n-k} \quad \text{where } c_i = \frac{1}{i+1}\binom{2i}{i}.$$

Proof. We immediatly deduce this result from the recursive decomposition (see Lemma 9). □

Corollary 12. *The number of doubly alternating Baxter permutations of length* $2n+\varepsilon$ *where* $\varepsilon = 0$ *or* 1 *having* k *left-to-right maxima is the ballot number*

$$\binom{2n-k}{n-1} - \binom{2n-k}{n}.$$

Proof. We deduce this result from the enumeration of the complete binary tree according to the number of nodes and the length of the left main branch, called ballot numbers or Delannoy numbers [8]. □

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