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Circulant tournaments of prime order are tight

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Dedicated to the memory of Víctor Neumann-Lara

Abstract

We say that a tournament is tight if for every proper 3-coloring of its vertex set there is a directed cyclic triangle whose vertices have different colors. In this paper, we prove that all circulant tournaments with a prime number $p \geq 3$ of vertices are tight using results relating to the acyclic disconnection of a digraph and theorems of additive number theory.

Keywords: Circulant tournament; Acyclic disconnection; Tight tournament; 3-coloring; Sumsets; Addition theorems

1. Introduction

The notion of tightness in tournaments was developed from the study of the *acyclic disconnection* $\overrightarrow{\omega}(D)$ (resp., the $\overrightarrow{C}_3$*-free disconnection* $\overrightarrow{\omega}_3(D)$) of a digraph D, defined to be the maximum possible number of connected components of a digraph obtained from D by deleting an acyclic set of arcs (resp., a $\overrightarrow{C}_3$-free set of arcs, where $\overrightarrow{C}_3$ denotes a directed cyclic triangle). These numerical invariants as measures of the cyclic structure of digraphs were introduced in [8], where a variety of results were given, particularly those concerning the determination of the acyclic (resp., $\overrightarrow{C}_3$-free) disconnection of some infinite families of circulant tournaments. It was established in Proposition 2.4 of [8] that $\overrightarrow{\omega}_3^+(D) := \overrightarrow{\omega}_3(D) + 1$ is the minimum number r such that every proper r-coloring (that is, a coloring using all the colors) of the vertices of D leaves at least one directed cyclic triangle with its three vertices of different colors, i.e. one *heterochromatic* $\overrightarrow{C}_3$.

Given a uniform 3-hypergraph H (or briefly, a 3-*graph* H), the *heterochromatic number* $hc(H)$ is the minimum number of colors r such that every proper r-coloring of its vertex set leaves at least one heterochromatic 3-edge. H is called *tight* if and only if $hc(H) = 3$ (see [2,3] for details and properties). For any tournament T, $\overrightarrow{\omega}_3^+(T)$ is the heterochromatic number $hc(H_T)$ of the 3-graph $H_T = (V, E)$ obtained from T by defining $V(H_T) = V(T)$ and the

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set of 3-edges $E(H_T)$ as the set of subsets $S \subseteq V(T)$ such that the induced subdigraph $T[S]$ is isomorphic to $\overrightarrow{C}_3$. We say that T is *tight* if and only if H_T is tight, i.e. $\overrightarrow{\varpi}_3^+(T) = hc(H_T) = 3$.

Some infinite families of tight circulant tournaments are known (see the next section for a catalogue of these families). In this paper, we prove that all circulant tournaments with a prime number $p \geq 3$ of vertices are tight using results relating to the acyclic disconnection of a digraph and theorems of additive number theory.

2. Preliminaries

In this section, we give some other definitions and results not stated in the introduction that will be useful throughout the paper. For the general terminology of graph theory used in what follows, see [4].

If D is a digraph, $V(D)$ and $A(D)$ denote the set of vertices and arcs of D respectively and $|V(D)|$ is the *order* of D. Let D_1 and D_2 be digraphs. We define $D_1 \cup D_2$ as the digraph with $V(D_1 \cup D_2) = V(D_1) \cup V(D_2)$ and $A(D_1 \cup D_2) = A(D_1) \cup A(D_2)$. A digraph D is *acyclic* if it contains no directed cycle. A set of arcs of D not containing a $\overrightarrow{C}_3$ is said to be $\overrightarrow{C}_3$*-free*. Let $\pi = V_1 \mid V_2 \mid \cdots \mid V_k$ $(k \in \mathbb{N})$ be a non-degenerate partition of the set $V(D)$. An equivalence class V_j $(j = 1, 2, \ldots k)$ of π is *singular* if V_j is a singleton. An arc $(u, v) \in A(D)$ is said to be *external* if u and v belong to different equivalence classes of π. We say that π is an *externally* $\overrightarrow{C}_3$*-free partition* if the set of external arcs of D is $\overrightarrow{C}_3$-free. A digraph D is said to be $\overrightarrow{\varpi}_3$*-keen* if there exists an optimal externally $\overrightarrow{C}_3$-free partition π of $V(D)$ having exactly one singular equivalence class and no externally $\overrightarrow{C}_3$-free partition leaves more than one such a class.

We denote by T a tournament. For every tournament T, we have that $\overrightarrow{\varpi}_3(T) \geq 2$ ([8], Remark 4.1.(i)).

Proposition 1 *([8, Corollary 2.6]). If T is a regular tournament, then every externally $\overrightarrow{C}_3$-free partition of T has at most one singular equivalence class.*

Remark 1. (i) As a consequence of this proposition, if a regular tournament T has an optimal externally $\overrightarrow{C}_3$-free partition with a unique singular equivalence class, then T is $\overrightarrow{\varpi}_3$-keen.

(ii) In particular, if $\overrightarrow{\varpi}_3^+(T) = 3$ (that is, T is tight), where T is regular, then T is $\overrightarrow{\varpi}_3$-keen.

We recall that given nonempty subsets $A, B \subseteq \mathbb{Z}_n$, the group of residues modulo a positive integer n, then

$$A + B = \{a + b : a \in A, b \in B\},$$
$$cA = \{ca : a \in A, c \in \mathbb{Z}_n\},$$
$$-A = \{-a : a \in A\} \quad \text{and}$$
$$A - B = A + (-B).$$

If A and B are sets, we denote the set difference by $A \setminus B$.

Let $\mathbb{Z}_p$ be set of residues modulo a prime p $(p \geq 3)$. The following properties are easy consequences of the above definitions (they can be found in [9]).

Proposition 2. *Let A, B, C and D be nonempty subsets of $\mathbb{Z}_p$. Then:*

(i) *If $A - B = C - D$, then $A \cap B = \varnothing$ if and only if $C \cap D = \varnothing$.*
(ii) *$(A + B) \cap C = \varnothing$ if and only if $A \cap (B - C) = \varnothing$.*
(iii) *$(A - B) \cap (C + D) = \varnothing$ if and only if $(A - C) \cap (B + D) = \varnothing$.*

We will use the famous Cauchy–Davenport theorem concerning sumsets in cyclic groups of prime order:

Theorem 1 *([7], Cauchy–Davenport). Let A and B be nonempty subsets of $\mathbb{Z}_p$. Then*

$$|A + B| \geq \min\{|A| + |B| - 1, p\}.$$

In [9], A. G. Vosper classified those sets $A, B \subseteq \mathbb{Z}_p$ for which equality holds in the Cauchy–Davenport theorem. For our purposes, we only need a very special case of this characterization.

Proposition 3 *([9], Corollary to Vosper's Theorem). Let A be a nonempty subset of $\mathbb{Z}_p$ such that $A + A \neq \mathbb{Z}_p$. Then $|A + A| = 2|A| - 1$ if and only if A is an arithmetic progression, i.e.*

$$A = \{a + id : i = 0, 1, \ldots, |A| - 1\},$$

where $a, d \in \mathbb{Z}_p$ and $d \neq 0$.

Now consider the set $\mathbb{Z}_{2n+1}$ of integers modulo $2n + 1$ ($n \in \mathbb{N}$) and a nonempty subset $J \subseteq \mathbb{Z}_{2n+1} \setminus \{0\}$ such that for every $j \in \mathbb{Z}_{2n+1} \setminus \{0\}$, we have that $|\{j, -j\} \cap J| = 1$. Observe that

$$|\{j, -j\} \cap J| = 1 \Leftrightarrow \overline{J} = -J \cup \{0\} \Leftrightarrow 0 \notin J + J \tag{1}$$

and this implies that $|J| = |-J| = n$. In addition, if $2n + 1 = p$ is a prime number, then

$$p - 2 = 2|J| - 1 \leq |J + J| \leq 2|J| = p - 1,$$

where the first part of the inequality is an application of the Cauchy–Davenport theorem and the second one is a consequence of the above chain of equivalences. Observe that $|J + J| = p - 2$ if and only if J is an arithmetic progression (Proposition 3).

The *circulant tournament* $T = \overrightarrow{C}_{2n+1}(J)$ is defined by

$$V(\overrightarrow{C}_{2n+1}(J)) = \mathbb{Z}_{2n+1} \quad \text{and}$$
$$A(\overrightarrow{C}_{2n+1}(J)) = \{(i, j) : i, j \in \mathbb{Z}_{2n+1} \text{ and } j - i \in J\}.$$

Depending on the context, an arc (x, y) of $\overrightarrow{C}_{2n+1}(J)$ is represented by $x \longrightarrow y$ or more specifically by $x \overset{(j)}{\longrightarrow} y$, where $j \in J$ such that $y - x = j$. Note that $\overrightarrow{C}_{2n+1}(J)$ is a circulant tournament if and only if (1) holds. It is well known that a circulant tournament is regular and its automorphism group is vertex-transitive.

Let us denote by I_n the set $\{1, 2, \ldots, n\} \subseteq \mathbb{Z}_{2n+1}$ and $S \subseteq I_n$. The tournament $\overrightarrow{C}_{2n+1}\langle S\rangle$ is defined to be the circulant tournament $\overrightarrow{C}_{2n+1}(J)$, where $J = (I_n \cup (-S)) \setminus S$. In particular, $\overrightarrow{C}_{2n+1}(I_n)$ is called the *cyclic circulant tournament*. It is easy to check that there is only one cyclic circulant tournament up to isomorphism.

Remark 2. In particular, $\overrightarrow{C}_p(J) \cong \overrightarrow{C}_p(I_{\frac{p-1}{2}})$ if and only if $J = cI_{\frac{p-1}{2}}$, where $c \in \mathbb{Z}_p$ and $c \neq 0$. Observe that in this case, J is an arithmetic progression.

The following result due to Alspach [1] will be used in the proof of the main theorem:

Proposition 4. *If T is a regular tournament, then every arc of T is contained in a directed cyclic triangle.*

Finally, we list an update of the infinite families of circulant tournaments proved to be tight:

(i) $\overrightarrow{C}_{2n+1}(I_n)$ ([8], Proposition 4.4.(i)).
(ii) $\overrightarrow{C}_{2n+1}\langle s\rangle$, where $(2n + 1, s) \neq (9, 2)$ ([8], Theorem 4.11).
(iii) $\overrightarrow{C}_{2n+1}\langle s_1, s_2\rangle$, where $1 \leq s_1 < s_2 \leq n$ and

$$\overrightarrow{C}_{2n+1}\langle s_1, s_2\rangle \notin \left\{\overrightarrow{C}_9\langle 1, 4\rangle, \overrightarrow{C}_9\langle 2, 3\rangle, \overrightarrow{C}_{15}\langle 2, 5\rangle, \overrightarrow{C}_{15}\langle 3, 4\rangle\right\}$$

([5], Theorem 5).
(iv) $\overrightarrow{C}_p(J)$, where $p \geq 3$ is a prime number (Section 4, Theorem 4).

3. Circulant tournaments of prime order are $\overrightarrow{\varpi}_3$-keen

We consider the circulant tournament $\overrightarrow{C}_p(J)$, where $p \geq 3$ is a prime number.

Remark 3. $\overrightarrow{C}_3$ is trivially tight. The only circulant tournament of order 5 up to isomorphism, $\overrightarrow{C}_5(I_2)$, is tight by Proposition 4.4.(i) of [8]. Using this result, $\overrightarrow{C}_p(I_{\frac{p-1}{2}})$ is tight for every prime number $p \geq 3$. Up to isomorphism, there exist two circulant tournaments of order 7, $\overrightarrow{C}_7(I_3)$ and $\overrightarrow{C}_7\langle 3\rangle$ which are tight (the last one by virtue of Theorem 4.11 of [8]).

We need some preliminary results in order to prove the main theorem.

An n-partite tournament ($n \geq 2$) is obtained by directing the edges of a complete n-partite graph. The following theorem will be used in the proof of Lemma 1.

Theorem 2 (*[6, Theorem 1]*). *Let T be an n-partite tournament, $n \geq 3$. Then T contains a $\overrightarrow{C}_3$ if and only if there exists a cycle in T that contains vertices from at least three partite sets.*

Lemma 1. *Let $\pi = A \mid B \mid C$ be an externally $\overrightarrow{C}_3$-free partition of $\overrightarrow{C}_p(J)$, where p is a prime number. Then*

$$(((((A + J) \cap B) + J) \cap C) + J) \cap A = \varnothing.$$

Proof. Without loss of generality, we assume that $1 \leq |A| \leq |B| \leq |C|$ (this means that $1 \leq |A| \leq \left\lfloor \frac{p}{3} \right\rfloor$, $1 \leq |B| \leq \left\lfloor \frac{p}{3} \right\rfloor$ if $p \equiv 1 (\mathrm{mod}\ 3)$ or $1 \leq |B| \leq \left\lceil \frac{p}{3} \right\rceil$ if $p \equiv 2 (\mathrm{mod}\ 3)$, and $\left\lceil \frac{p}{3} \right\rceil \leq |C| \leq p - 2$).

1. By contradiction, suppose that

$$x \in (((((A + J) \cap B) + J) \cap C) + J) \cap A.$$

Then $x \in ((((A + J) \cap B) + J) \cap C) + J$ and $x \in A$. There exist

$$z \in (((A + J) \cap B) + J) \cap C$$

and $j \in J$ such that $x = z + j \Leftrightarrow x - z \in J$. Observe that $z \in ((A + J) \cap B) + J$ and $z \in C$. Therefore, there exist $y \in (A + J) \cap B$ and $j' \in J$ such that $z = y + j' \Leftrightarrow z - y \in J$. Observe again that $y \in A + J$ and $y \in B$. So, there exist $x' \in A$ and $j'' \in J$ such that

$$y = x' + j'' \Leftrightarrow y - x' \in J.$$

We have constructed a directed path

$$x' \xrightarrow{(j'')} y \xrightarrow{(j')} z \xrightarrow{(j)} x \quad \text{(P1)}$$

in $\overrightarrow{C}_p(J)$ such that $x, x' \in A$, $y \in B$ and $z \in C$. If $x = x'$, then the vertices x, y and z induce a $\overrightarrow{C}_3$ in $\overrightarrow{C}_p(J)$, a contradiction to the condition of the lemma. Then we can suppose that $x \neq x'$ and consequently $|A| \geq 2$.

2. Consider the 3-partite subtournament M in $\overrightarrow{C}_p(J)$ induced by classes A, B and C. We will prove that there always exists a cycle in M that contains vertices from the three partite sets, and by Theorem 2, there exists a $\overrightarrow{C}_3$ in M contradicting that π is an externally $\overrightarrow{C}_3$-free partition.

3. Since $\overrightarrow{C}_p(J)$ is regular and $|A| \leq \left\lfloor \frac{p}{3} \right\rfloor$, there exists $y' \in B$ or $z' \in C$ such that $x \to y'$ or $x \to z'$ and we can extend the directed path (P1) to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \quad \text{or} \quad \text{(P2)}$$

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z'. \quad \text{(P3)}$$

4. For directed path (P2), if there are no $x'' \in A$ and $z'' \in C$ such that $y' \longrightarrow x''$ and $y' \longrightarrow z''$, then $N^+(y) \subseteq B$ and $C \Longrightarrow y'$ (for every vertex $z \in C$ there exists $z \longrightarrow x$), respectively. We have that $|B| \geq \frac{p-1}{2}$ and $|C| \leq \frac{p-1}{2} - 1$ (the missing arc is $x \longrightarrow y'$), respectively. This yields a contradiction, $|C| \leq |B|$. We conclude that there exists $y' \longrightarrow x''$ or $y' \longrightarrow z''$ and the directed path (P2) can be extended to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow x'' \quad \text{or} \quad \text{(P4)}$$

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow z''. \quad \text{(P5)}$$

Observe that $z \neq z''$, since $x \longrightarrow y' \longrightarrow z \longrightarrow x$ is a directed cyclic triangle and π is an externally $\overrightarrow{C}_3$-free partition. If $x'' = x'$, then

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow x'$$

is a cycle containing vertices from A, B and C, and there exists a $\overrightarrow{C}_3$ in M (see 2).

5. For directed path (P3), if there are no $x'' \in A$ and $y'' \in B$ such that $z' \longrightarrow x''$ and $z' \longrightarrow y''$, then $A \Longrightarrow z'$ and $B \Longrightarrow z'$, respectively. We have two cases:

(a) If $N^+(x) \subseteq A \cup C$, then $A \Longrightarrow N^+(x)$ and $B \Longrightarrow N^+(x)$ (there are no arcs from z' to any vertex of A and B). Notice that every $z' \in N^+(x)$ must belong to C, that is, $N^+(x) + J \subseteq C$. By the Cauchy–Davenport theorem,
$$\left|N^+(x) + J\right| \geq \left|N^+(x)\right| + |J| - 1 = p - 2$$
and this implies that $|C| \geq p - 2$. Since $1 \leq |A| \leq |B| \leq |C|$ and $|A| + |B| + |C| = p$, then $|A| = |B| = 1$ and $|C| = p - 2$, a contradiction to the supposition that $|A| \geq 2$.

(b) If $N^+(x) \not\subseteq A \cup C$, then there exists $y' \in B$ such that $x \longrightarrow y'$ and we have a directed path of type (P2).

So, if the cases (a) and (b) do not occur, directed path (P3) can be extended to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow x'' \quad \text{or} \tag{P6}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow y''. \tag{P7}$$

Once again, if $x'' = x'$ or $y'' = y$, then

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow x' \quad \text{or}$$
$$y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow y$$

is a cycle in M, a contradiction.

6. Using the arguments of 3., directed paths (P4) and (P6) can be extended to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow x'' \longrightarrow y''' \quad \text{or}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow x'' \longrightarrow z''', \quad \text{and}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow x'' \longrightarrow y''' \quad \text{or}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow x'' \longrightarrow z''',$$

respectively. In the case that $y''' \in B$ and $z''' \in C$ coincide with one vertex of its classes used before, then there is a cycle in M, a contradiction.

7. For directed path (P5), if there are no $x''' \in A$ and $y''' \in B$ such that $z'' \longrightarrow x'''$ and $z'' \longrightarrow y'''$, then $A \Longrightarrow z''$ and $B \Longrightarrow z''$, respectively. We have two cases:

(a) If $N^+(y') \subseteq B \cup C$, then $A \Longrightarrow N^+(x)$ and $B \Longrightarrow N^+(x)$. Notice that every $z'' \in N^+(y')$ must belong to C, that is, $N^+(x) + J \subseteq C$. Using the argument of case (a) of 5, $\left|N^+(x) + J\right| \geq p - 2$ and then $|A| = 1$, a contradiction.

(b) If $N^+(x) \not\subseteq B \cup C$, then there exists $x'' \in A$ such that $y' \longrightarrow x''$ and we have a directed path of type (P4).

So, if the cases (a) and (b) do not occur, directed path (P5) can be extended to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow z'' \longrightarrow x''' \quad \text{or}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow y' \longrightarrow z'' \longrightarrow y'''.$$

If $x''' = x$, $x''' = x'$ or $y''' = y$, we have the desired cycle in M, a contradiction.

8. For directed path (P7), if there are no $x''' \in A$ and $z''' \in C$ such that $y'' \longrightarrow x'''$ and $y'' \longrightarrow z'''$, then $N^+(y'') \subseteq B$ and $C \Longrightarrow y''$, respectively. We have that $|B| \geq \frac{p-1}{2}$ and $|C| \leq \frac{p-1}{2}$, respectively. Since $1 \leq |A| \leq |B| \leq |C|$, the only possibility is that $|B| = |C| = \frac{p-1}{2}$ and therefore $|A| = 1$, a contradiction, we supposed that $|A| \geq 2$ (see 1). We conclude that there exists $y'' \longrightarrow x'''$ or $y'' \longrightarrow z'''$ and the directed path (P7) can be extended to

$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow y'' \longrightarrow x''' \quad \text{or}$$
$$x' \longrightarrow y \longrightarrow z \longrightarrow x \longrightarrow z' \longrightarrow y'' \longrightarrow z'''.$$

If $x''' = x$, $x''' = x'$ or $z''' = z$, a cycle appears in M, a contradiction.

Observe that we classified every possible directed path according to the last two vertices (types AB, AC, BA, BC, CA and CB). In all the cases, we obtain a contradiction or the directed path is extended. Since the number of vertices is finite, this procedure continues until a vertex from one class is repeated. We obtain a cycle in M that contains vertices from the three classes, a contradiction. ■

Remark 4. Under the same conditions of Lemma 1, one can similarly prove that

$$(((((A + J) \cap C) + J) \cap B) + J) \cap A = \varnothing.$$

Moreover, the lemma is valid for any circulant tournament $\overrightarrow{C}_{2n+1}(J)$.

Lemma 2. *Let A, C and J be nonempty subsets of $\mathbb{Z}_p$ such that $A \cap C = \varnothing$ and $\overline{J} = -J \cup \{0\}$. Then*

$$(C \cap (A - J)) - J = (C - J) \cap (A - J - J).$$

Proof. First, suppose that $x \in (C \cap (A - J)) - J$. Then there exist $y \in C \cap (A - J)$ and $j \in J$ such that $x = y - j$. But $y \in C$ and $y \in A - J$ and these relations imply that there exist $a \in A$ and $j' \in J$ such that $y = a - j'$. Therefore $x = a - j' - j \in A - J - J$. Since $a - j' \in C$, then $x \in C - J$ and we conclude that $x \in (C - J) \cap (A - J - J)$.

On the other hand, if $x \in (C - J) \cap (A - J - J)$, then $x \in C - J$ and $x \in A - J - J$. There exist $c \in C$, $a \in A$ and $j, j', j'' \in J$ such that

$$x = c - j = a - j' - j''. \tag{2}$$

Since $a \neq c$ (A and C are disjoint sets), there exists $l \in \mathbb{Z}_p \setminus \{0\}$ such that $a = c + l \Leftrightarrow c = a - l$. So in (2), we have that $-j = l - j' - j''$ which is equivalent to $l + j = j' + j''$. Therefore $l + j \in J + J$ and $l + J \subseteq J + J$. Consequently, $l \in J$ (observe that if $l \notin J$, then $l \in -J$ and there exists $k \in J$ such that $l + k = 0 \notin J + J$). This implies that $c = a - l \in A - J$ and $x \in (C \cap (A - J)) - J$. ■

Note that if A, B and C are nonempty subsets of $\mathbb{Z}_p$, then

$$(A \cap B) + C \subseteq (A + C) \cap (B + C)$$

and the equality proved in Lemma 2 is not valid in general. For example, let A, B and C be subsets of $\mathbb{Z}_{17}$ defined by $A = \{1, 2, 3, 4, 5, 6\}$, $B = \{4, 6, 7, 8, 9, 10\}$ and $C = \{0, 1, 4\}$. Then

$$(A \cap B) + C = \{4, 5, 6, 7, 8, 10\} \quad \text{and}$$
$$(A + C) \cap (B + C) = \{4, 5, 6, 7, 8, 9, 10\}.$$

Proposition 5. *Let $\overrightarrow{C}_p(J)$ be a circulant tournament, where $p \geq 11$ and J is not an arithmetic progression. Then $\overrightarrow{C}_p(J)$ is $\overrightarrow{\varpi}_3$-keen.*

Proof. Let us suppose that $\overrightarrow{\varpi}_3(\overrightarrow{C}_p(J)) = k \geq 2$. If $k = 2$, then by Remark 1(ii), $\overrightarrow{C}_p(J)$ is $\overrightarrow{\varpi}_3$-keen. Therefore, we consider the case when $k \geq 3$. By Proposition 1, the tournament $\overrightarrow{C}_p(J)$ has at most one singular equivalence class in every externally $\overrightarrow{C}_3$-free partition. Suppose that there exists an optimal externally $\overrightarrow{C}_3$-free partition of $V(\overrightarrow{C}_p(J))$ without singular equivalence classes. Let $\pi = V_1 \mid V_2 \mid \cdots \mid V_k$ ($k \geq 3$) be this optimal externally $\overrightarrow{C}_3$-free partition of $\mathbb{Z}_p$ such that

$$2 \leq |V_1| \leq |V_2| \leq \cdots \leq |V_k|.$$

and define the sets

$$A = V_1, \quad B = \bigcup_{i=1}^{m} V_{j_i} \quad \text{and} \quad C = \bigcup_{i=m}^{k} V_{j_i},$$

where $1 \leq m \leq k - 1$ and the set family $\{V_{j_i} : i = 2, 3, \ldots k\}$ is a permutation of $\{V_j : j = 2, 3, \ldots k\}$. We can assume that

$$2 \leq |A| \leq |B|, |A| \leq |C| \quad \text{and} \quad |A| + |B| + |C| = p. \tag{3}$$

Since π is an externally $\overrightarrow{C}_3$-free partition, so is $\pi' = A \mid B \mid C$. Then by Lemma 1,

$$(((((A + J) \cap B) + J) \cap C) + J) \cap A = \varnothing$$
$$\Leftrightarrow ((((A + J) \cap B) + J) \cap C) \cap (A - J) = \varnothing$$
$$\Leftrightarrow (((A + J) \cap B) + J) \cap (C \cap (A - J)) = \varnothing$$
$$\Leftrightarrow ((A + J) \cap B) \cap ((C \cap (A - J)) - J) = \varnothing$$

using Proposition 2(iii). By Lemma 2, the last relation is equivalent to

$$((A + J) \cap B) \cap ((C - J) \cap (A - J - J)) = \varnothing$$

$$\Leftrightarrow (A+J)\cap B\cap (C-J)\cap (A-J-J)=\varnothing$$
$$\Leftrightarrow (A+J)\cap B\cap (C-J)\subseteq \overline{A-J-J}, \tag{4}$$

where the complement is relative to $\mathbb{Z}_p$. Applying the Cauchy–Davenport theorem to $A-J-J=A-(J+J)$, we have that

$$|A-J-J|\geq |A|+|-(J+J)|-1$$

and $|A-J-J|\leq p$. These inequalities imply that $|A|\leq 2$ and using (3), we conclude that $|A|=2$. Recall that since J is not an arithmetic progression, $|-(J+J)|=|J+J|=p-1$ by Proposition 3. From this,

$$|A-J-J|=p\Leftrightarrow A-J-J=\mathbb{Z}_p.$$

Therefore (4) is equivalent to

$$(A+J)\cap B\cap (C-J)=\varnothing$$
$$\Leftrightarrow A\cap (B-J)\cap (C-J-J)=\varnothing$$
$$\Leftrightarrow A\cap (B-J)\subseteq \overline{C-J-J} \tag{5}$$

if we again apply Proposition 2(ii) and Lemma 2. Repeating the above reasoning, we obtain that $|C|=2$ and $C-J-J=\mathbb{Z}_p$. So (5) is equivalent to

$$A\cap (B-J)=\varnothing \Leftrightarrow A\subseteq \overline{B-J}.$$

From this relation, we have that

$$2=|A|\leq \left|\overline{B-J}\right| \Leftrightarrow |B-J|\leq p-2.$$

By the Cauchy–Davenport theorem,

$$|B-J|\geq |B|+|J|-1=|B|+\frac{p-1}{2}-1.$$

Combining the last two inequalities, $|B|\leq \frac{p-1}{2}$. Then

$$|A|+|B|+|C|\leq 2+\frac{p-1}{2}+2=\frac{p+7}{2}.$$

By (3), $p\leq \frac{p+7}{2}\Leftrightarrow p\leq 7$, a contradiction to the original supposition that $p\geq 11$. ■

Theorem 3. *For every $p\geq 3$, $\overrightarrow{C}_p(J)$ is $\overrightarrow{\omega}_3$-keen.*

Proof. It is a consequence of Proposition 5, Remark 1(ii) and Remark 3. ■

Using the last theorem and Remark 1(i) and Remark 3, we have the following

Corollary 1. *Every externally $\overrightarrow{C}_3$-free partition of $\overrightarrow{C}_p(J)(p\geq 3)$ has exactly one singular equivalence class.*

4. Main result

Theorem 4. *For every $p\geq 3$, $\overrightarrow{C}_p(J)$ is tight.*

Proof. First, we can suppose that $\overrightarrow{C}_p(J)$ is not isomorphic to the cyclic circulant tournament $\overrightarrow{C}_p(I_{\frac{p-1}{2}})$ which is tight. So, J is not an arithmetic progression. By contradiction, let us suppose that $\overrightarrow{C}_p(J)$ is not tight. Then there exists an externally $\overrightarrow{C}_3$-free partition of $V(\overrightarrow{C}_p(J))$ with 3 classes. By Corollary 1, one such class is singular and since the automorphism group of $\overrightarrow{C}_p(J)$ is vertex-transitive, we can assume without loss of generality that the externally $\overrightarrow{C}_3$-free partition of $V(\overrightarrow{C}_p(J))$ is $\pi=\{0\}\mid B\mid C$, where $|B|+|C|=p-1$. Taking $A=\{0\}$ in (4) of Proposition 5, we have that

$$J_B\cap (C-J)\subseteq \overline{-J-J}=\overline{-(J+J)},$$

where $J_B = J \cap B$ (similarly, $-J_B = (-J) \cap B$, $J_c = J \cap C$ and $-J_C = (-J) \cap C$). Since $0 \notin J + J$ and J is not an arithmetic progression, using Proposition 3 (and subsequent remarks),

$$|J + J| = 2\,|J| = p - 1.$$

Therefore, $0 \notin -(J + J)$ and

$$|-(J + J)| = 2\,|-J| = 2\,|J| = p - 1.$$

From this, $\overline{-(J + J)} = \{0\}$ and consequently $J_B \cap (C - J) \subseteq \{0\}$. We have two cases:

Case 1. $J_B \cap (C - J) = \{0\}$. But then $0 \in J$ and $0 \in B$, which is a contradiction.

Case 2. $J_B \cap (C - J) = \varnothing$. By Proposition 2 (ii), this equality is equivalent to

$$(J_B + J) \cap C = \varnothing$$
$$\Leftrightarrow J_B + J \subseteq \overline{C} = B \cup \{0\} \subset B = J_B \cup (-J_B)$$

(note that $0 \notin J_B + J \subseteq J + J$). Then using the Cauchy–Davenport theorem,

$$|J_B| + |J| - 1 \leq |J_B + J| < |J_B| + |-J_B|$$

and from this inequality, we obtain that

$$|-J_B| > |J| - 1 \Leftrightarrow |-J_B| \geq |J| = \frac{p-1}{2}.$$

Since $-J_B \subseteq -J$, then $|-J_B| \leq |-J| = \frac{p-1}{2}$ and so,

$$|-J_B| = \frac{p-1}{2} \Leftrightarrow -J_C = \varnothing.$$

We conclude that $-J \subseteq B$ and $C \subseteq J$. Then, the tournament $\overrightarrow{C}_p(J)$ contains the arcs $0 \longrightarrow j$ for all $j \in J_C = C$. By Proposition 4, there exists a directed cyclic triangle

$$\begin{array}{ccccc} & \swarrow & 0 & \nwarrow & \\ J \supseteq C \ni j & & \longrightarrow & & j' \in -J \subseteq B \end{array}.$$

We have obtained a contradiction; $\pi = \{0\} \mid B \mid C$ is an externally $\overrightarrow{C}_3$-free partition of $V(\overrightarrow{C}_p(J))$. ■

Since for every tournament T, $2 \leq \overrightarrow{\varpi}(T) \leq \overrightarrow{\varpi}_3(T)$ (see Remark 2.1(iv) and 4.1(i) of [8]), we have the following consequence of Theorem 4:

Corollary 2. *For every* $p \geq 3$, $\overrightarrow{\varpi}(\overrightarrow{C}_p(J)) = 2$.

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