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A Simplified Overview of Certain Relations among Infinite Series That Arose in the Context of Fractional Calculus

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An interesting infinite series relation was proven recently by applying a certain known generalization of the Riemann–Liouville and Erdélyi–Kober operators of fractional calculus. In our attempt to give a much simpler proof of this result, without using the generalized fractional calculus operator, we are led here to another class of infinite series relations. Some relevant connections with a number of known results are also indicated. © 1991 Academic Press, Inc.

1. INTRODUCTION

In the theory of the psi (or digamma) function

$$\psi(z)=\frac{d}{dz}\{\log \Gamma(z)\}=\frac{\Gamma'(z)}{\Gamma(z)}, \tag{1}$$

it is a well-known (rather classical) result that

$$\sum_{n=1}^{\infty}\frac{(\nu)_n}{n(\lambda)_n}=\psi(\lambda)-\psi(\lambda-\nu) \qquad (\mathrm{Re}(\lambda-\nu)>0;\ \lambda\neq 0,\ -1,\ -2,\ \cdots), \tag{2}$$

where $(\lambda)_n$ is the familiar Pochhammer symbol defined by

$$(\lambda)_n=\frac{\Gamma(\lambda+n)}{\Gamma(\lambda)}=\begin{cases}1, & \text{if } \ n=0,\\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } \ n\in\mathbb{N}=\{1,2,3,...\}.\end{cases} \tag{3}$$

The summation formula (2) and its various special cases were revived recently as illustrations emphasizing the usefulness of the Riemann–Liouville fractional integral operator $D_z^{-\nu}$ defined by

$$D_z^{-\nu}\{f(z)\}=\frac{1}{\Gamma(\nu)}\int_0^z (z-\zeta)^{\nu-1}f(\zeta)\,d\zeta \qquad (\mathrm{Re}(\nu)>0). \tag{4}$$

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See, for a detailed historical account of (2), and of its numerous consequences and generalizations, one of the latest works on the subject by Nishimoto and Srivastava [1]. From the work of Nishimoto and Srivastava [1], we choose to recall here two interesting consequences of the summation formula (2), which are contained in

THEOREM 1 (Nishimoto and Srivastava [1, p. 104]). *Let* $\{\Omega_n\}_{n=0}^{\infty}$ *be a suitably bounded sequence of complex numbers. Also let the parameters* λ *and* ν *be constrained by*

$$\operatorname{Re}(\lambda-\nu)>0 \qquad (\lambda \neq 0, -1, -2, \ldots). \tag{5}$$

Then

$$\sum_{n=1}^{\infty} \frac{(\nu)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{\Omega_k}{(\lambda+n)_k} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{\Omega_k}{(\lambda)_k} [\psi(\lambda+k)-\psi(\lambda-\nu+k)] \frac{z^k}{k!} \tag{6}$$

and

$$\sum_{n=1}^{\infty} \frac{(\nu)_n}{n(\lambda)_n} \sum_{k=0}^{\infty} \frac{(\nu+n)_k \, \Omega_k}{(\lambda+n)_k} \frac{z^k}{k!} = \sum_{k=0}^{\infty} \frac{(\nu)_k \, \Omega_k}{(\lambda)_k} [\psi(\lambda+k)-\psi(\lambda-\nu)] \frac{z^k}{k!}, \tag{7}$$

provided that the series involved converge absolutely.

As a matter of fact, Nishimoto and Srivastava [1, p. 105] also gave hypergeometric forms of their assertions (6) and (7). Recently, in a sequel to the work of Nishimoto and Srivastava [1], Raina [3] proved an infinite series relation which generalizes the assertion (6) of Theorem 1. With a view to recalling Raina's result, we begin by introducing a generalized hypergeometric ${}_rF_s$ series defined by (cf., e.g., Slater [5, Chap. 2])

$${}_rF_s\left[\begin{matrix}\alpha_1, \ldots, \alpha_r; \\ \beta_1, \ldots, \beta_s;\end{matrix}\; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_r)_n}{(\beta_1)_n \cdots (\beta_s)_n} \frac{z^n}{n!} \tag{8}$$

$$\left(r \leqslant s,\ |z|<\infty;\ r=s+1,\ |z|<1;\ r=s+1,\ |z|=1, \quad \text{and} \right.$$

$$\left. \operatorname{Re}\left(\sum_{j=1}^{s} \beta_j - \sum_{j=1}^{r} \alpha_j\right) > 0\right),$$

provided that no zeros appear in the denominator. A slightly modified form of the main result in Raina's paper [3], involving a special case of (8) when $r=3$, $s=2$, and $z=1$, is given by

THEOREM 2 (cf. Raina [3, p. 173]). *In terms of a suitably bounded sequence* $\{\Lambda_n\}_{n=0}^{\infty}$ *of complex numbers, let*

$$T_n(z)=\sum_{k=0}^{\infty}\Lambda_k\frac{(\mu-\beta)_k\,(\mu-\gamma)_k}{(\mu-\beta-\gamma)_k\,(\mu+n)_k}\frac{z^k}{k!}\,{}_3F_2\left[\begin{matrix}\alpha+n,\beta,\gamma;\\ \alpha,\mu+n+k;\end{matrix}\ 1\right]. \tag{9}$$

Suppose also that

$$\mathrm{Re}(\mu)>\max\{\mathrm{Re}(\alpha),\mathrm{Re}(\beta+\gamma)\}\qquad(\mu\neq 0,-1,-2,\ldots). \tag{10}$$

Then

$$\sum_{n=1}^{\infty}\frac{(\alpha)_n}{n(\mu)_n}T_n(z)=\frac{\Gamma(\mu)\,\Gamma(\mu-\beta-\gamma)}{\Gamma(\mu-\beta)\,\Gamma(\mu-\gamma)}\sum_{k=0}^{\infty}\Lambda_k$$
$$\cdot[\psi(\mu-\beta+k)+\psi(\mu-\gamma+k)-\psi(\mu-\alpha+k)$$
$$-\psi(\mu-\beta-\gamma+k)]\frac{z^k}{k!}, \tag{11}$$

provided that the series involved converge absolutely.

The proof of Theorem 2, detailed by Raina [3], depends rather heavily upon the following generalization of the Riemann–Liouville fractional integral operator (4), as also of the Erdélyi–Kober fractional integral operator,

$$I_{0,z}^{\alpha,\beta,\eta}\{f(z)\}=\frac{z^{-\alpha-\beta}}{\Gamma(\alpha)}\int_0^z(z-\zeta)^{\alpha-1}\,{}_2F_1\left[\begin{matrix}\alpha+\beta,-\eta;\\ \alpha;\end{matrix}\ 1-\frac{\zeta}{z}\right]f(\zeta)\,d\zeta, \tag{12}$$

which has indeed been studied in a number of works by (for example) Saigo [4], Srivastava and Saigo [6], and Owa *et al.* [2]. The present paper stems from our attempt to give a much simpler direct proof of Theorem 2 *without* using the fractional integral operators $D_z^{-\nu}$ and $I_{0,z}^{\alpha,\beta,\eta}$ defined by (4) and (12), respectively. Our direct proof of Theorem 2 is shown here to apply *mutatis mutandis* to derive an analogous result (contained in Theorem 3 below) which generalizes the assertion (7) of Theorem 1.

2. Derivations Without Using Fractional Calculus

There are a number of ways in which Theorem 2 would apply to yield the assertion (6) of Theorem 1. Only one such way was considered by Raina [3, p. 174] who set

$$\text{(i)}\quad \beta=\alpha, \qquad \gamma=0, \qquad \text{and} \qquad \Lambda_n=\frac{\Omega_n}{(\mu)_n} \qquad (n\in\mathbb{N}_0=\mathbb{N}\cup\{0\}).$$

As a matter of fact, the assertion (6) of Theorem 1 would follow from Theorem 2 rather simply when

$$\text{(ii)}\quad \beta=0 \qquad \text{and} \qquad \Lambda_n=\frac{\Omega_n}{(\mu)_n} \qquad (n\in\mathbb{N}_0)$$

or

$$\text{(iii)}\quad \gamma=0 \qquad \text{and} \qquad \Lambda_n=\frac{\Omega_n}{(\mu)_n} \qquad (n\in\mathbb{N}_0).$$

Additionally, in view of the Gauss summation theorem [5, p. 28]

$${}_2F_1\left[\begin{matrix}a, b; \\ c;\end{matrix}\ 1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)} \qquad (\operatorname{Re}(c-a-b)>0), \tag{13}$$

it is not difficult to recover the assertion (6) of Theorem 1 from Theorem 2 when either

$$\text{(iv)}\quad \beta=\alpha, \qquad \Lambda_n=\frac{\Omega_n}{(\mu-\gamma)_n} \qquad (n\in\mathbb{N}_0), \qquad \text{and} \qquad \mu\to\mu+\gamma$$

or

$$\text{(v)}\quad \gamma=\alpha, \qquad \Lambda_n=\frac{\Omega_n}{(\mu-\beta)_n} \qquad (n\in\mathbb{N}_0), \qquad \text{and} \qquad \mu\to\mu+\beta.$$

In each of the five instances described above, Theorem 2 yields the assertion (6) of Theorem 1. A similar generalization of the assertion (7) of Theorem 1 is provided by Theorem 3 below, which we prove here with a view to illustrating our derivations (without using fractional calculus) of all such generalizations of Theorem 1.

Theorem 3. *In terms of a suitably bounded sequence* $\{\Lambda_n\}_{n=0}^{\infty}$ *of complex numbers, let*

$$S_n(z)=\sum_{k=0}^{\infty}\Lambda_k\frac{(\mu-\beta)_k\,(\alpha+n)_k}{(\mu+n)_k}\frac{z^k}{k!}\,{}_3F_2\left[\begin{matrix}\alpha+n+k, \beta, \gamma; \\ \alpha+k, \mu+n+k;\end{matrix}\ 1\right]. \tag{14}$$

Suppose also that the parameters α, β, γ, *and* μ *are constrained by* (10). *Then*

$$\sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\mu)_n} S_n(z) = \frac{\Gamma(\mu)\,\Gamma(\mu-\beta-\gamma)}{\Gamma(\mu-\beta)\,\Gamma(\mu-\gamma)} \sum_{k=0}^{\infty} \Lambda_k \frac{(\mu-\beta-\gamma)_k\,(\alpha)_k}{(\mu-\gamma)_k} \cdot [\psi(\mu-\beta+k)+\psi(\mu-\gamma+k)-\psi(\mu-\alpha) - \psi(\mu-\beta-\gamma+k)] \frac{z^k}{k!}, \tag{15}$$

provided that the series involved converge absolutely.

Proof. For a simple and direct proof of Theorem 3, without using the fractional integral operators defined by (4) and (12), let $\omega(z)$ denote the left-hand side of the assertion (15). Then, substituting for $S_n(z)$ from (14) and applying the definitions (3) and (8), we have

$$\begin{aligned}\omega(z) &= \sum_{n=1}^{\infty} \frac{(\alpha)_n}{n(\mu)_n} \sum_{k,r=0}^{\infty} \Lambda_k \frac{(\mu-\beta)_k\,(\alpha+n)_{k+r}\,(\beta)_r\,(\gamma)_r}{r!\,(\alpha+k)_r\,(\mu+n)_{k+r}} \frac{z^k}{k!} \\ &= \sum_{k,r=0}^{\infty} \Lambda_k \frac{(\mu-\beta)_k\,(\alpha)_k\,(\beta)_r\,(\gamma)_r}{r!\,(\mu)_{k+r}} \frac{z^k}{k!} \sum_{n=1}^{\infty} \frac{(\alpha+k+r)_n}{n(\mu+k+r)_n},\end{aligned} \tag{16}$$

where the inversion of the order of summation can be justified by the absolute convergence of the series involved.

The innermost series in (16) is summable by means of the well-known result (2) with

$$v=\alpha+k+r \qquad \text{and} \qquad \lambda=\mu+k+r \qquad (k, r \in \mathbb{N}_0),$$

and we thus obtain

$$\omega(z) = \sum_{k=0}^{\infty} \Lambda_k \frac{(\mu-\beta)_k\,(\alpha)_k}{(\mu)_k} \frac{z^k}{k!} \sum_{r=0}^{\infty} \frac{(\beta)_r\,(\gamma)_r}{r!\,(\mu+k)_r} [\psi(\mu+k+r)-\psi(\mu-\alpha)], \tag{17}$$

provided that [cf. Eq. (5)]

$$\operatorname{Re}(\mu-\alpha)>0 \qquad (\mu \neq 0, -1, -2, \ldots). \tag{18}$$

Now it is not difficult to see from the definitions (1) and (8) that

$$\frac{\partial}{\partial c}\left\{ {}_2F_1\left[\begin{matrix} a, b; \\ c; \end{matrix}\; z\right]\right\} = -\sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} [\psi(c+n)-\psi(c)] \frac{z^n}{n!}, \tag{19}$$

which, in view of the Gauss summation theorem (13), yields

$$\sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{n! (c)_n} [\psi(c+n) - \psi(c)]$$
$$= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} [\psi(c-a) + \psi(c-b) - \psi(c) - \psi(c-a-b)], \tag{20}$$

provided that the parameters a, b, and c are restricted as in (13).

Applying (20) with

$$a = \beta, \qquad b = \gamma, \qquad \text{and} \qquad c = \mu + k \qquad (k \in \mathbb{N}_0),$$

we find from (17) that

$$\omega(z) = \frac{\Gamma(\mu)\Gamma(\mu-\beta-\gamma)}{\Gamma(\mu-\beta)\Gamma(\mu-\gamma)} \sum_{k=0}^{\infty} \Lambda_k \frac{(\mu-\beta-\gamma)_k (\alpha)_k}{(\mu-\gamma)_k}$$
$$\cdot [\psi(\mu-\beta+k) + \psi(\mu-\gamma+k) - \psi(\mu-\alpha) - \psi(\mu-\beta-\gamma+k)] \frac{z^k}{k!}, \tag{21}$$

provided, in addition to the conditions listed in (18), that

$$\mathrm{Re}(\mu-\beta-\gamma) > 0. \tag{22}$$

This evidently completes our proof of Theorem 3 under the constraints (18) and (22), which are already stated in (10).

3. Remarks and Applications

First of all, we remark that our derivation of Theorem 3 (detailed in the preceding section) applies *mutatis mutandis* to prove Theorem 2 which was obtained by Raina [3] by means of the fractional integral operator $I_{0,z}^{\alpha,\beta,\eta}$ defined by (12). Our direct proof of Theorem 3 (and hence also of Theorem 2) would make use of the well-known results (2) and (13).

Although there are a total of five instances in which Theorem 2 was shown above to reduce to the assertion (6) of Theorem 1, Theorem 3 would yield the assertion (7) of Theorem 1 in its special case when either

$$\beta = 0 \qquad \text{and} \qquad \Lambda_n = \frac{\Omega_n}{(\mu)_n} \qquad (n \in \mathbb{N}_0) \tag{23}$$

or

$$\gamma = 0 \qquad \text{and} \qquad \Lambda_n = \frac{\Omega_n}{(\mu - \beta)_n} \qquad (n \in \mathbb{N}_0). \tag{24}$$

We conclude by remarking that Theorems 2 and 3, *together*, provide an *exhaustive* generalization of Theorem 1.

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