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The Reduced Birman–Wenzl Algebra of Coxeter Type B

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We introduce a reduced form of a Birman–Murakami–Wenzl algebra associated to the braid group of Coxeter type B and investigate its semisimplicity, Bratteli diagram, and Markov trace. Applications in knot theory and physics are outlined.

1. INTRODUCTION

To every Coxeter diagram a braid group is associated that has the same presentation as the Coxeter group but without the degree 2 relations for the generators. The braid group ZB_n of Coxeter type B has generators τ_i, $i = 0, 1, \ldots, n-1$. Generators τ_i, $i \geq 1$, satisfy the relations of Artin's braid group (which is the braid group of Coxeter type A):

$$\tau_i \tau_j = \tau_j \tau_i \qquad \text{if } |i-j| > 1 \tag{1}$$

$$\tau_i \tau_j \tau_i = \tau_j \tau_i \tau_j \qquad \text{if } |i-j| = 1. \tag{2}$$

The generator τ_0 has relations

$$\tau_0 \tau_1 \tau_0 \tau_1 = \tau_1 \tau_0 \tau_1 \tau_0 \tag{3}$$

$$\tau_0 \tau_i = \tau_i \tau_0 \qquad \text{if } i \geq 2. \tag{4}$$

This braid group may be interpreted as the group of symmetric braids or cylinder braids (see the graphical interpretation in Section 6).

The group algebras of these braid groups typically have numerous finite dimensional quotients. The most important ones for Coxeter type A are Temperley–Lieb, Hecke, and Birman–Murakami–Wenzl algebras. Hecke algebras of arbitrary Coxeter type are already classics in this field. Temper-

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ley−Lieb algebras of Coxeter type B have been introduced by tom Dieck in [4] as algebras of symmetric tangles without crossings.

The standard Birman−Murakami−Wenzl algebra of type A imposes cubic relations on its generators in a way that enables its interpretation as an algebra of tangles with a skein relation that comes from the Kauffman polynomial.

In full analogy a BMW algebra of Coxeter type B should be an extension by an additional generator Y related to τ_0 which should satisfy a cubic relation as well. It turns out, however, that such an algebra is rather intricate and deserves further study (see [7]).

In this paper we define a reduced BMW algebra of type B where the additional generator Y satisfies a quadratic (Hecke type) relation. This may seem strange at first but from the view of knot theory of B-type it is quite natural. Generalizations of this algebra where Y may obey any polynomial relation are considered in [8].

We now outline the structure of the paper and point out the main results. After a short review of the Birman−Wenzl algebra of A-type in Section 2 we go on to define the reduced BMW algebra of B-type BB_n in Section 3 where a number of fundamental relations are established. They are used extensively in Section 4 to determine normal forms for words in BB_n. An upper bound for the dimension is derived. Section 5 shows how to obtain the B-type Hecke algebra as a quotient of BB_n.

Section 6 introduces the graphical interpretation of our algebra and studies its classical limit. This will also give insight into the relations chosen in the definition of BB_n. The construction of a Markov trace fills Section 7.

The main theorem of this paper is contained in Section 8. We prove that BB_n is semisimple in the generic case and show how its simple components can be enumerated in terms of Young diagrams. The Bratelli diagram is given and we show that the Markov trace is faithful.

T. tom Dieck has found a representation of BB_n on tensor product spaces. In Section 9 we review his representation and show that it allows to calculate the Markov trace as a matrix trace.

The algebra BB_n has interesting applications both in physics and in knot theory. They are outlined at the end of Section 9 and in Section 10. The physical interest comes from the fact that the additional generator Y may be interpreted as describing a boundary reflection in a twodimensional quantum system. The Markov trace allows to define an extension of the Kaufman polynomial to links in the solid torus.

A next goal would be to construct a tensor category [10] where BB_n is the endomorphism set of a n-fold tensor product of a generating simple element.

2. PRELIMINARIES: THE A-TYPE BMW ALGEBRA

We review the definition of the Birman–Murakami–Wenzl algebra [14] in our notation and collect a stock of relations that will be needed later on.

DEFINITION 1. Let R denote an integral domain. Assume that q, λ, and x are units in R and define $\delta := q - q^{-1}$. Assume that the relation

$$x\delta = \delta - \lambda + \lambda^{-1} \tag{5}$$

holds. The Birman—Wenzyl algebra of type A with n strands $\mathrm{BA}_n(R)$ is defined as the algebra generated by invertible $X_1, \ldots, X_{n-1}$. The relations read:

$$X_i X_j = X_j X_i \qquad |i-j| > 1 \tag{6}$$

$$X_i X_j X_i = X_j X_i X_j \qquad |i-j| = 1 \tag{7}$$

$$X_i e_i = e_i X_i = \lambda e_i \tag{8}$$

$$e_i X_{i-1}^{\pm 1} e_i = \lambda^{\mp 1} e_i \tag{9}$$

$$e_i^2 = x e_i \tag{10}$$

$$X_i^{-1} = X_i - \delta + \delta e_i \tag{11}$$

$$X_i^2 = 1 + \delta X_i - \delta\lambda e_i \tag{12}$$

$$X_i^3 = X_i^2(\lambda + \delta) + X_i(1 - \lambda\delta) - \lambda \tag{13}$$

$$X_i^{-2} = 1 + \delta^2 - \delta X_i + \delta(\lambda^{-1} - \delta) e_i = 1 - \delta X_i^{-1} + \delta\lambda^{-1} e_i \tag{14}$$

$$0 = (X_i - \lambda)\left(X_i + q^{-1}\right)(X_i - q) \tag{15}$$

$$e_i e_j = e_j e_i \qquad |i-j| > 1 \tag{16}$$

$$X_i^{-1} X_j^{\pm 1} X_i = X_j X_i^{\pm 1} X_j^{-1} \qquad |i-j| = 1 \tag{17}$$

$$e_i X_j X_i = X_j^{\pm} X_i^{\pm} e_j \qquad |i-j| = 1 \tag{18}$$

$$e_i X_j^{\pm 1} e_i = \lambda^{\mp 1} e_i \qquad |i-j| = 1 \tag{19}$$

$$e_i e_j e_i = e_i \qquad |i-j| = 1 \tag{20}$$

$$X_i^{\pm 1} e_j e_i = X_j^{\mp 1} e_i \qquad |i-j| = 1 \tag{21}$$

$$e_i e_j X_i^{\pm 1} = e_i X_j^{\mp 1} \qquad |i-j| = 1 \tag{22}$$

$$e_i X_j^{\pm} X_i^{\pm} = e_i e_j \qquad |i-j| = 1 \tag{23}$$

$$X_i^{\pm} X_j^{\pm} e_i = e_j e_i \qquad |i-j| = 1 \tag{24}$$

$$X_i e_j X_i^{-1} = X_j^{-1} e_i X_j \qquad |i-j| = 1 \tag{25}$$

$$X_i e_j X_i = X_j^{-1} e_i X_j^{-1} \qquad |i-j| = 1. \tag{26}$$

LEMMA 1. *If δ is invertible one may define*

$$e_i := 1 - \frac{X_i - X_i^{-1}}{\delta} \tag{27}$$

and restrict the relations to (6)–(9).

Proof. We have to show that the remaining relations are implied by this smaller set. The proofs are mostly easy. We only comment on some of them. To show (10) one replaces one of the e_i on the left hand side by its definition (27) and applies (8). Relations (11)–(15) are successive rewritings of (27).

(17) $X_i X_j X_i = X_j X_i X_j \Rightarrow X_j X_i X_j^{-1} = X_i^{-1} X_j X_i \Rightarrow X_i X_j^{-1} X_i^{-1} = X_j^{-1} X_i^{-1} X_j$.

(18) follows from (27) and (17). To show (20) one replaces the e_j in the middle by its definition.

(21) $X_i^{\pm} e_j e_i = X_j^{\mp} X_j^{\pm} X_i^{\pm} e_j e_i \overset{(18)}{=} X_j^{\mp} e_i X_j^{\pm} X_i^{\pm} e_i = \lambda^{\pm} X_j^{\mp} e_i X_j^{\pm} e_i \overset{(19)}{=} X_j^{\mp} e_i$.

(23) Using (20), (21), and (18) we calculate

$$\begin{aligned} e_i X_j^{\pm} X_i^{\pm} &= e_i e_j e_i X_j^{\pm} X_i^{\pm} = e_i X_i^{\mp} X_i^{\pm} e_j e_i X_j^{\pm} X_i^{\pm} \\ &= e_i X_i^{\mp} X_j^{\mp} e_i X_j^{\pm} X_i^{\pm} = e_i X_i^{\mp} X_j^{\mp} X_j^{\pm} e_j = e_i e_j. \end{aligned}$$

∎

3. THE DEFINITION OF THE REDUCED B-TYPE BMW ALGEBRA B

In this section we define the reduced Birman−Murakami−Wenzl algebra of Coxeter type B. The choice of the base ring needs special attention to avoid the algebra from being smaller than expected.

DEFINITION 2. Let R be an integral domain of the kind described in definition 1 with an additional unit $q_0 \in R$ and further elements $A, q_1 \in R$. The reduced Birman−Wenzl algebra of Coxeter B type with n strands

$\mathrm{BB}_n(R)$ is generated by invertible $Y, X_1, \dots, X_{n-1}$. Using the notation from definition 1 the relations are (6) to (9) and in addition:

$$X_1YX_1Y = YX_1YX_1 \tag{28}$$

$$Y^2 = q_1Y + q_0 \tag{29}$$

$$YX_1Ye_1 = e_1 \tag{30}$$

$$YX_i = X_iY \qquad i > 1 \tag{31}$$

$$e_1Ye_1 = Ae_1. \tag{32}$$

In the further development we assume that the algebra is nondegenerate in the sense that e_1 is nonzero and has a vanishing annulator ideal in R and that e_1 and Ye_1 are linearly independent. Otherwise the algebra may not be semisimple.

We study now relations involving Y. The following shortcuts will be useful:

$$Y_i' := X_{i-1}X_{i-2} \cdots X_1YX_1 \cdots X_{i-2}X_{i-1} \tag{33}$$

$$Y_i := X_{i-1}X_{i-2} \cdots X_1YX_1^{-1} \cdots X_{i-2}^{-1}X_{i-1}^{-1}. \tag{34}$$

LEMMA 2.

$$Y^{-1} = q_0^{-1}Y - q_1q_0^{-1} \tag{35}$$

$$Y_i^2 = q_1Y_i + q_0 \tag{36}$$

$$Y_i^{-1} = q_0^{-1}Y_i - q_1q_0^{-1} \tag{37}$$

$$0 = \left[X_1YX_1Y, \{Y, e_1, X_1\}\right] \tag{38}$$

$$Y_i'Y_j' = Y_j'Y_i' \tag{39}$$

$$Y_{i+1}'X_i^{-1} = X_iY_i' \qquad Y_{i+1}X_i = X_iY_i \tag{40}$$

$$0 = \left[Y_i, X_j\right] = \left[Y_i, e_j\right] \qquad j \neq i, i-1 \tag{41}$$

$$0 = \left[Y_i', X_j\right] = \left[Y_i', e_j\right] \qquad j \neq i, i-1 \tag{42}$$

$$e_i = e_iY_iX_iY_i = Y_iX_iY_ie_i \tag{43}$$

$$e_i = e_iY_i'X_iY_i' = Y_i'X_iY_i'e_i \tag{44}$$

$$e_iY_ie_i = Ae_i \tag{45}$$

$$X_iY_iX_iY_i = Y_iX_iY_iX_i \tag{46}$$

$$Y_ie_{i-1} = \lambda^{-1}q_0^{-1}Y_{i-1}e_{i-1} - q_1q_0^{-1}\lambda^{-1}e_{i-1} \tag{47}$$

$$e_{i-1}Y_i = \lambda\left(q_0^{-1} - \delta\right)e_{i-1}Y_{i-1} + \lambda\left(\delta A - q_1 q_0^{-1}\right)e_{i-1} \quad (48)$$

$$X_i Y_{i+1} = Y_i X_i - \delta Y_i + \delta Y_i e_i + \delta Y_{i+1} \quad (49)$$

$$+\left(\delta^2\lambda - \delta\lambda q_0^{-1}\right)e_i Y_i + \left(\delta\lambda q_1 q_0^{-1} - \delta^2\lambda A\right)e_i \quad (50)$$

$$(1 - q_0\delta)X_i Y_i e_i = e_i(q_1\lambda - q_0\delta\lambda A) + q_0 Y_i e_i \quad (51)$$

$$e_{i-1}Y_i' = \lambda e_{i-1}Y_{i-1}'^{-1} \quad (52)$$

$$Y_i' e_{i-1} = \lambda Y_{i-1}'^{-1} e_{i-1} \quad (53)$$

$$e_1 Y_1 X_2 e_1 = q_0 e_1 Y_1 e_2 e_1 + q_1\lambda^{-1}e_1 \quad (54)$$

$$Y_{i+1}Y_i = X_i Y_i X_i Y_i - \delta q_1 X_i Y_i - \delta q_0 X_i + \delta q_0^{-1} Y_i e_i Y_i - \delta q_1 q_0^{-1} e_i Y_i. \quad (55)$$

Proof. (35), (36), and (37) are verified easily.

(38) Using (28) we have $X_1X_1YX_1Y = X_1YX_1YX_1$. Hence X_1YX_1Y commutes with X_1, and also with X_1^{-1}. But then, using (27), we see that it also commutes with e_1.

(39) $[Y, Y_1'] = [Y, Y_2'] = 0$ is trivial. For $i > 1$ the claim follows by induction: $[Y, Y_i'] = 0 \Rightarrow [Y, Y_{i+1}'] = [Y, X_i Y_i' X_i] = 0$. In the general case $[Y_j', Y_i']$ we may assume $j < i$. Then the induction step is shown using (41): $[Y_j', Y_i'] = [X_{j-1}Y_{j-1}'X_{j-1}, Y_i'] = 0$.

(40) trivial

(41, 42) For $j \geq i+1$ follows commutativity from (6, 31) and for $j \leq i-1$ it is an application of equation (7). Commutativity with e_j follows from that with X_j.

(43, 44) The proofs are by induction starting from (30) and its mirror version $e_1 = e_1YX_1Y$, which may be proven easily:

$$\lambda e_1 YX_1Y = e_1X_1YX_1Y \overset{(38)}{=} X_1YX_1Ye_1 = \lambda YX_1Ye_1 = \lambda e_1.$$

The induction step for (43) uses (18) to express e_{i+1} in terms of e_i:

$$\begin{aligned} Y_{i+1}'X_{i+1}Y_{i+1}'e_{i+1} &= X_iY_i'X_iX_{i+1}X_iY_i'X_iX_i^{-1}X_{i+1}^{-1}e_iX_{i+1}X_i \\ &= X_iY_i'X_{i+1}X_iX_{i+1}Y_i'X_{i+1}^{-1}e_iX_{i+1}X_i \\ &= X_iX_{i+1}Y_i'X_iX_{i+1}X_{i+1}^{-1}Y_i'e_iX_{i+1}X_i \\ &= X_iX_{i+1}Y_i'X_iY_i'e_iX_{i+1}X_i = X_iX_{i+1}e_iX_{i+1}X_i = e_{i+1}. \end{aligned}$$

Induction step for (44):

$$\begin{aligned} e_{i+1}Y_{i+1}X_{i+1}Y_{i+1} &= e_{i+1}X_iY_iX_i^{-1}X_{i+1}X_iY_iX_i^{-1} \\ &= e_{i+1}X_iY_iX_{i+1}X_iX_{i+1}^{-1}Y_iX_i^{-1} \\ &= e_{i+1}X_iX_{i+1}Y_iX_iY_iX_{i+1}^{-1}X_i^{-1} \\ &= X_iX_{i+1}e_iY_iX_iY_iX_{i+1}^{-1}X_i^{-1} \\ &= X_iX_{i+1}e_iX_{i+1}^{-1}X_i^{-1} = e_{i+1}. \end{aligned}$$

(45) Induction step:

$$\begin{aligned} e_iY_ie_i &= e_iX_{i-1}Y_{i-1}X_{i-1}^{-1}e_i = e_ie_{i-1}X_i^{-1}Y_{i-1}X_{i-1}^{-1}e_i \\ &= e_ie_{i-1}Y_{i-1}X_i^{-1}X_{i-1}^{-1}e_i = e_ie_{i-1}Y_{i-1}e_{i-1}X_i^{-1}X_{i-1}^{-1} \\ &= Ae_ie_{i-1}X_i^{-1}X_{i-1}^{-1} \overset{(22)}{=} Ae_i. \end{aligned}$$

(46) Again, the proof is by induction. The step is:

$$\begin{aligned} Y_iX_iY_iX_i &= X_{i-1}Y_{i-1}X_{i-1}^{-1}X_iX_{i-1}Y_{i-1}X_{i-1}^{-1}X_i \\ &= X_{i-1}Y_{i-1}X_iX_{i-1}X_i^{-1}Y_{i-1}X_{i-1}^{-1}X_i \\ &= X_{i-1}X_iY_{i-1}X_{i-1}Y_{i-1}X_i^{-1}X_{i-1}^{-1}X_i \\ &= X_{i-1}X_iY_{i-1}X_{i-1}Y_{i-1}X_{i-1}X_i^{-1}X_{i-1}^{-1} \\ &= X_{i-1}X_iX_{i-1}Y_{i-1}X_{i-1}Y_{i-1}X_i^{-1}X_{i-1}^{-1} \\ &= X_iX_{i-1}X_iY_{i-1}X_{i-1}X_i^{-1}Y_{i-1}X_{i-1}^{-1} \\ &= X_iX_{i-1}Y_{i-1}X_iX_{i-1}X_i^{-1}Y_{i-1}X_{i-1}^{-1} \\ &= X_iX_{i-1}Y_{i-1}X_{i-1}^{-1}X_iX_{i-1}Y_{i-1}X_{i-1}^{-1} \\ &= X_iY_iX_iY_i. \end{aligned}$$

(47), (48), (50)

$$\begin{aligned} Y_ie_{i-1} &= X_{i-1}Y_{i-1}X_{i-1}^{-1}e_{i-1} = \lambda^{-1}X_{i-1}Y_{i-1}e_{i-1} \\ &\overset{(44)}{=} \lambda^{-1}Y_{i-1}^{-1}e_{i-1} = \lambda^{-1}q_0^{-1}Y_{i-1}e_{i-1} - q_1q_0^{-1}\lambda^{-1}e_{i-1} \end{aligned}$$

$$\begin{aligned} e_{i-1}Y_i &= e_{i-1}X_{i-1}Y_{i-1}X_{i-1}^{-1} = \lambda e_{i-1}Y_{i-1}X_{i-1}^{-1} \\ &= \lambda e_{i-1}Y_{i-1}X_{i-1} - \delta\lambda e_{i-1}Y_{i-1} + \delta\lambda e_{i-1}Y_{i-1}e_{i-1} \\ &= \lambda e_{i-1}Y_{i-1}^{-1} - \delta\lambda e_{i-1}Y_{i-1} + \delta\lambda Ae_{i-1} \end{aligned}$$

$$= q_0^{-1}\lambda e_{i-1}Y_{i-1}q_1q_0^{-1}\lambda e_{i-1} - \delta\lambda e_{i-1}Y_{i-1} + \delta\lambda A e_{i-1}$$
$$= \lambda\big(q_0^{-1} - \delta\big)e_{i-1}Y_{i-1} + \lambda\big(\delta A - q_1q_0^{-1}\big)e_{i-1}$$

$$X_iY_{i+1} = X_i^2Y_iX_i^{-1}$$
$$= Y_iX_i^{-1} + \delta Y_{i+1} - \delta\lambda e_iY_iX_i^{-1}$$
$$= Y_iX_i - \delta Y_i + \delta Y_ie_i + \delta Y_{i+1} - \delta\lambda e_iY_iX_i + \delta^2\lambda e_iY_i - \delta^2\lambda e_iY_ie_i$$
$$= Y_iX_i - \delta Y_i + \delta Y_ie_i + \delta Y_{i+1} - \delta\lambda q_0^{-1}e_iY_i$$
$$+ \delta\lambda q_1q_0^{-1}e_i + \delta^2\lambda e_iY_i - \delta^2\lambda A e_i$$
$$= Y_iX_i - \delta Y_i + \delta Y_ie_i + \delta Y_{i+1} + \big(\delta^2\lambda - \delta\lambda q_0^{-1}\big)e_iY_i$$
$$+\big(\delta\lambda q_1q_0^{-1} - \delta^2\lambda A\big)e_i.$$

(51)

$$X_iY_ie_i = X_iY_iY_iX_iY_ie_i = q_1X_iY_iX_iY_ie_i + q_0X_1^2Y_ie_i$$
$$= q_1X_ie_i + q_0(1 + \delta X_i - \delta\lambda e_i)Y_ie_i$$
$$= q_1\lambda e_i + q_0Y_ie_i + q_0\delta X_iY_ie_i - q_0\delta\lambda A e_i$$
$$\Rightarrow (1 - q_0\delta)X_iY_ie_i = e_i(q_1\lambda - q_0\delta\lambda A) + q_0Y_ie_i.$$

(54) We prove the following equivalent relation:

$$e_1Y_1e_2e_1 = e_1Y_1X_1X_2e_1 = e_1Y_1X_1Y_1Y_1^{-1}X_2e_1 = e_1Y_1^{-1}X_2e_1$$
$$= q_0^{-1}e_1Y_1X_2e_1 - q_1q_0^{-1}e_1X_2e_1 = q_0^{-1}e_1Y_1X_2e_1 - q_1q_0^{-1}\lambda^{-1}e_1.$$

(52), (53) are proven according to the scheme

$$e_{i-1}Y_i' = e_{i-1}X_{i-1}Y_{i-1}'X_{i-1} = \lambda e_{i-1}Y_{i-1}'X_{i-1}Y_{i-1}'Y_{i-1}'^{-1} = \lambda e_{i-1}Y_{i-1}'^{-1}.$$

(55)

$$Y_{i+1}Y_i = X_iY_iX_i^{-1}Y_i$$
$$= X_iY_iX_iY_i - \delta X_iY_i^2 + \delta X_iY_ie_iY_i$$
$$= X_iY_iX_iY_i - \delta q_1X_iY_i - \delta q_0X_i + \delta Y_i^{-1}e_iY_i$$
$$= X_iY_iX_iY_i - \delta q_1X_iY_i - \delta q_0X_i + \delta q_0^{-1}Y_ie_iY_i - \delta q_1q_0^{-1}e_iY_i.$$

∎

Our nondegeneracy assumptions introduce relations among the parameters.

LEMMA 3. *The assumption that e_1 has nonvanishing annulator ideal leads to the requirement*

$$A(1 - q_0\lambda) = q_1x. \tag{56}$$

The additional assumption that Ye_1 and e_1 are linearly independent leads to the equation

$$q_0 - q_0^{-1} = -\delta. \tag{57}$$

Proof.

$$e_1Ye_1 = e_1YYX_1Ye_1 = q_1e_1YX_1Ye_1 + q_0e_1X_1Ye_1 = q_1xe_1 + q_0\lambda e_1Ye_1$$
$$\Rightarrow (1 - q_0\lambda)e_1Ye_1 = q_1xe_1 \Rightarrow A(1 - q_0\lambda) = q_1x.$$

To obtain the second relation we observe that (30) implies $Ye_1 = X_1^{-1}Y^{-1}e_1$. We multiply by q_0 and calculate

$$\begin{aligned} q_0Ye_1 &= q_0X_1^{-1}Y^{-1}e_1 = (X_1 - \delta + \delta e_1)(Y - q_1)e_1 \\ &= X_1Ye_1 - q_1X_1e_1 - \delta Ye_1 + \delta q_1e_1 + \delta e_1Ye_1 - \delta q_1e_1^2 \\ &= \left(q_0^{-1}Y - q_0^{-1}q_1\right)e_1 - q_1\lambda e_1 - \delta Ye_1 + \delta q_1e_1 + \delta Ae_1 - \delta q_1xe_1 \\ &= e_1\left(-q_1q_0^{-1} - q_1\lambda + \delta q_1 + \delta A - \delta q_1x\right) + Ye_1\left(q_0^{-1} - \delta\right). \end{aligned}$$

The coefficient of Ye_1 is (57). The coefficient of e_1 vanishes when (56) and (57) hold. ∎

From now on we will always assume that these relations hold in the ground ring.

Using the relations of Lemma 2 one sees that the ideal generated by e_1 in BB_2 is spanned by e_1, Ye_1, e_1Y, Ye_1Y. Using the relations of the above lemma one may (by construction of a twodimensional irreducible representation) show that the ideal is indeed four dimensional and hence that the nondegeneracy assumptions imply no further relations among the parameters. We do not go into details of this but see [8] for a detail exposition of such arguments in a more complicated case.

At this stage of the development it is useful to look ahead to the classical limit of the algebra we shall discuss later on. Such a limit should have $X_1 = X_1^{-1}$ which is implied by $q \to 1$. Furthermore, one would expect that Y as well should obey a Coxeter relation $Y^2 = 1$ in the limit. It is therefore reasonable to choose

$$q_0 = q^{-1} \tag{58}$$

among the solutions of (57), as we will do from now on.

The generic ground ring that we will use is:

DEFINITION 3. The ring R_0 is defined to be the quotient of the polynomial ring $\mathbb{C}[q, q^{-1}, q_0, q_0^{-1}, \delta, \delta^{-1}, \lambda, \lambda^{-1}, q_1, A]$ quotiented by the relations (56), $\delta = q - q^{-1}$ and the Laurent style relations $qq^{-1} = 1$ and so on. Its quotient field is denoted by K_0.

Here we have already eliminated q_0. In the quotient ring of R_0 we can solve the equations defining the ideal uniquely. Hence this ideal is primary and therefore R_0 is an integral domain. Therefore R_0 is embedded in K_0.

Remark 1. The algebra BB_n has an involution given by

$$X_i^* := X_i^{-1}, \quad Y^* := Y^{-1}, \quad q^* := q^{-1}, \quad \lambda^* := \lambda^{-1},$$
$$q_0^* := q_0^{-1}, \quad q_1^* := -q_1 q_0^{-1} \tag{59}$$

This implies $\delta^* = -\delta$, $e_i^* = e_i$, $A^* := (A - q_1 x)/q_0$.

A second involution $a \mapsto \bar{a}$ exists that fixes all parameters and generators.

4. THE WORD PROBLEM IN BB_n

In this section we single out a set of words in standard form that linearly generate BB_n. Although this does not lead to a linear basis of BB_n, it allows a tight upper bound for the dimension to be determined.

PROPOSITION 4. *Every element in* BB_n *is a linear combination of words of the form* $w_1 \gamma w_2$, *where* $w_i \in \mathrm{BB}_{n-1}$ *and* $\gamma \in \Gamma_n := \{1, e_{n-1}, X_{n-1}, Y_n\}$.

Proof. We prove the proposition by induction. The case $n = 1$ is trivial and $n = 2$ can also be verified easily.

Let $w_0 \gamma_0 w_1 \gamma_1 \cdots w_k \gamma_k w_{k+1} \in \mathrm{BB}_n$ be an arbitrary word. It suffices to show that any two neighboring γ_i can be combined together. Hence the situation we have to investigate is $w = \gamma_1 w_1 \gamma_2, w_1 \in \mathrm{BB}_{n-1}, \gamma_1, \gamma_2 \in \Gamma_n$. By the induction hypothesis we have $w_1 = u_1 \alpha u_2, u_i \in \mathrm{BB}_{n-2}, \alpha \in \Gamma_{n-1}$, and hence $w = \gamma_1 u_1 \alpha u_2 \gamma_2 = u_1 \gamma_1 \alpha \gamma_2 u_2$. Thus it suffices to investigate $w' = \gamma_1 \alpha \gamma_2$. The cases $\gamma_1 = 1$ or $\gamma_2 = 1$ are trivial. We now investigate in turn the four possible values of α.

1. Case $\alpha = 1$: The following table gives the relation that allows the product $\gamma_1 \gamma_2$ to be reduced to the standard form of the proposition.

$\gamma_1 \setminus \gamma_2$	Y_n	e_{n-1}	X_{n-1}
Y_n	(36)	(47)	(40)
e_{n-1}	(48)	(10)	(8)
X_{n-1}	(50)	(8)	(12)

2. Case $\alpha = X_{n-2}$:

$\gamma_1 \setminus \gamma_2$	Y_n	e_{n-1}	X_{n-1}
Y_n	$= X_{n-2}Y_n^2$ (36)	$= X_{n-2}Y_n e_{n-1}$ (47)	$= X_{n-2}Y_n X_{n-1}$ (40)
e_{n-1}	$= e_{n-1}Y_n X_{n-2}$ (48)	(9)	(23)
X_{n-1}	(50)	(24)	(7)

3. Case $\alpha = e_{n-2}$:

$\gamma_1 \setminus \gamma_2$	Y_n	e_{n-1}	X_{n-1}
Y_n	$= e_{n-2}Y_n^2$ (36)	(47)	(40)
e_{n-1}	(48)	(20)	(22)
X_{n-1}	(48)	(21)	(26)

4. Case $\alpha = Y_{n-1}$. This case requires more complex calculations which are given below:

$\gamma_1 \setminus \gamma_2$	Y_n	e_{n-1}	X_{n-1}
Y_n	(60)	(61)	(62)
e_{n-1}	(63)	(45)	(64)
X_{n-1}	(65)	analog. (64)	(66)

$$
\begin{aligned}
Y_n Y_{n-1} Y_n &= X_{n-1}Y_{n-1}X_{n-1}^{-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} \\
&\overset{(46)}{=} X_{n-1}Y_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1}X_{n-1}^{-1} \\
&= q_1 X_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-2} + q_0 X_{n-1}^2 Y_{n-1}X_{n-1}^{-2} \\
&= q_1 Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} + q_0 X_{n-1}^2 Y_{n+1}X_{n-1}^{-2} \\
&= q_1 Y_{n-1}Y_n \\
&\quad + q_0(1 + \delta X_{n-1} - \delta\lambda e_{n-1}) \\
&\qquad \times Y_{n-1}\big(1 + \delta^2 - \delta X_{n-1} + \delta(\lambda^{-1} - \delta)e_{n-1}\big). \quad (60)
\end{aligned}
$$

This reduces the problem to the other cases.

$$\begin{aligned}
&Y_nY_{n-1}e_{n-1}\\
&\quad\overset{(55)}{=} X_{n-1}Y_{n-1}X_{n-1}Y_{n-1}e_{n-1} - \delta q_1 X_{n-1}Y_{n-1}e_{n-1} - \delta q_0 X_{n-1}e_{n-1}\\
&\qquad + \delta q_0^{-1}Y_{n-1}e_{n-1}Y_{n-1}e_{n-1} - \delta q_1 q_0^{-1}e_{n-1}Y_{n-1}e_{n-1}\\
&\quad = X_{n-1}e_{n-1} - \delta q_1 Y_{n-1}^{-1}e_{n-1} - \delta q_0\lambda e_{n-1}\\
&\qquad + \delta q_0^{-1}AY_{n-1}e_{n-1} - \delta q_1 q_0^{-1}Ae_{n-1}\\
&\quad = \lambda e_{n-1} - \delta q_1 Y_{n-1}^{-1}e_{n-1} - \delta q_0\lambda e_{n-1}\\
&\qquad + \delta q_0^{-1}AY_{n-1}e_{n-1} - \delta q_1 q_0^{-1}Ae_{n-1}. \qquad (61)
\end{aligned}$$

$$\begin{aligned}
Y_nY_{n-1}X_{n-1} &\overset{(55)}{=} X_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1} - \delta q_1 X_{n-1}Y_{n-1}X_{n-1}\\
&\quad - \delta q_0 X_{n-1}X_{n-1} + \delta q_0^{-1}Y_{n-1}e_{n-1}Y_{n-1}X_{n-1}\\
&\quad - \delta q_1 q_0^{-1}e_{n-1}Y_{n-1}X_{n-1}\\
&= Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^2 - \delta q_1 Y_n X_{n-1}^2 - \delta q_0 X_{n-1}^2\\
&\quad + \delta q_0^{-1}Y_{n-1}e_{n-1}Y_{n-1}^{-1} - \delta q_1 q_0^{-1}e_{n-1}Y_{n-1}^{-1}. \qquad (62)
\end{aligned}$$

Only the first and second term are not yet reduced.

$$Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^2 = Y_{n-1}Y_nX_{n-1}^3.$$

This is reduced using (13, 47).

$$\begin{aligned}
Y_nX_{n-1}^{-2} &= Y_n(1+\delta^2) - \delta Y_nX_{n-1} + \delta(\lambda^{-1}-\delta)Y_ne_{n-1}\\
&= Y_n(1+\delta^2) - \delta X_{n-1}Y_{n-1} + \delta(\lambda^{-1}-\delta)Y_ne_{n-1}.
\end{aligned}$$

This can be reduced using (47).

$$e_{n-1}Y_{n-1}Y_n = e_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} = \lambda^{-1}e_{n-1} \qquad (63)$$

$$e_{n-1}Y_{n-1}X_{n-1} = e_{n-1}Y_{n-1}X_{n-1}Y_{n-1}Y_{n-1}^{-1} = e_{n-1}Y_{n-1}^{-1} \qquad (64)$$

$$X_{n-1}Y_{n-1}Y_n = X_{n-1}Y_{n-1}X_{n-1}Y_{n-1}X_{n-1}^{-1} = Y_{n-1}X_{n-1}Y_{n-1} \qquad (65)$$

$$\begin{aligned}
X_{n-1}Y_{n-1}X_{n-1} &= Y_nX_{n-1}^2 = Y_n + \delta Y_nX_{n-1} - \delta\lambda Y_ne_{n-1} \qquad (66)\\
&= Y_n + \delta X_{n-1}Y_{n-1} - \delta\lambda Y_ne_{n-1}.
\end{aligned}$$

The last term can be reduced using (47). ∎

This shows that BB_n is finite dimensional.

Remark 2. It is obvious that similar propositions hold if Y_n or X_{n-1} or both in Γ_n are replaced by their inverses.

PROPOSITION 5. *In Proposition* 4 *one may replace* Γ_n *by* $\Gamma_n' := \{1, e_{n-1}, X_{n-1}, Y_n'\}$.

Proof. It suffices to show that Y_n can be expressed using words in normal form with Y_n'. For $n = 1$ this is trivial. Induction step: Express Y_n in $Y_{n+1} = X_n Y_n X_n^{-1}$ in terms of normal form words. If they are build with 1, X_{n-1}, or e_{n-1} as γ there is nothing to show. The only remaining case is:

$$\begin{aligned} X_n Y_n' X^{-n} &= X_n Y_n' X_n X_n^{-2} = Y_{n+1}'\left(1 - \delta X_n^{-1} + \delta \lambda^{-1} e_n\right) \\ &= Y_{n+1}' - \delta Y_{n+1}' X_n^{-1} + \delta \lambda^{-1} Y_{n+1}' e_n \\ &\overset{(53)}{=} Y_{n+1}' - \delta X_n Y_n' + \delta Y_n'^{-1} e_n . \end{aligned}$$

This shows that terms of this kind can be brought to the normal form as well. ∎

The aim of the rest of this section is to determine an upper bound for the dimension of BB_n.

LEMMA 6. BB_n *is spanned linearly by the set* S_n *defined recursively by*:

$$\begin{aligned} S_1 &:= \{1, Y\} \\ S_n &:= \Gamma_1' \cdots \Gamma_n' S_{n-1}. \end{aligned}$$

More strongly, *of the elements of* $\Gamma_1' \cdots \Gamma_n'$ *only those of the following form are needed*:

$$Y_i' X_i \cdots X_j e_{j+1} \cdots e_n, \quad X_i \cdots e_{j+1} \cdots e_n .$$

Here $1 \le i \le n$ *and* $i - 1 \le j \le n$. *Thus the strings of* X *and* e *may be empty*.

Proof. Proposition 5 yields the following decomposition of BB_n which implies the claim:

$$\begin{aligned} \mathrm{BB}_n &= \mathrm{BB}_{n-1} \Gamma_n' \mathrm{BB}_{n-1} \\ &= \mathrm{BB}_{n-2} \Gamma_{n-1}' \mathrm{BB}_{n-2} \Gamma_n' \mathrm{BB}_{n-1} = \mathrm{BB}_{n-2} \Gamma_{n-1}' \Gamma_n' \mathrm{BB}_{n-1} \\ &= \Gamma_1' \cdots \Gamma_n' \mathrm{BB}_{n-1}. \end{aligned}$$

To show the second statement assume that Y'_j appears in the middle of a chain $Z_i \cdots Z_{j-1} Y'_j Z_{j+1} \cdots Z_n$ where $Z_s \in \Gamma'_s$. Then $Z_i \cdots Z_{j-1}$ commutes with the rest of the chain and thus can be absorbed in the right BB_{n-1}. Similarly, assume that there appears a $e_i X_{i+1}$ in such a chain. Then one can rewrite this as $e_i X_{i+1} = e_i e_{i+1} X_i^{-1}$ and now the X_i^{-1} can be absorbed in the right BB_{n-1}. Thus all X must appear to the left of all e in the chain. This completes the proof of the given form. ∎

PROPOSITION 7. *There is a basis of* BB_n *consisting of elements of the form* $\alpha\beta\gamma$ *where* α *is a product of* Y', γ *is a product of* Y'^{-1} *and* β *is an element of a basis of the A-type algebra* BA_n. *Together* α *and* γ *contain at most n factors* Y', Y'^{-1}.

The dimension of BB_n *is* $\leq 2^n(2n-1)!!$.

Proof. The proof is by induction on n. For $n = 1$ it is trivial. Now, assume the claim is already shown for $n-1$. To show the first statement it suffices to show that we can move all Y'_i that appear on the left hand side of our basis of BB_{n-1} through the outer Γ' chain to the left or, alternatively, even to the right of BB_{n-1}. We investigate the various arising cases. First assume that we have $e_{n-1} Y'_{n-1}$. Then we rewrite this as

$$e_{n-1} Y'_{n-1} = e_{n-1} Y'_{n-1} X_{n-1} Y'_{n-1} Y'^{-1}_{n-1} X^{-1}_{n-1} = e_{n-1} Y'^{-1}_{n-1} X^{-1}_{n-1}$$
$$= \lambda e_{n-1} X^{-1}_{n-1} Y'^{-1}_{n-1} X^{-1}_{n-1} = \lambda e_{n-1} Y'^{-1}_n .$$

If we have $e_i e_{i+1} Y'_i = e_i Y'_i e_{i+1}$ we may apply the same reasoning twice to obtain $Y'_{i+2} e_i e_{i+1}$. The remaining cases are such that we have $X_i Y'_i = Y'_{i+1} X_i^{-1} = Y'_{i+1} X_i - \delta Y'_{i+1} + \delta Y'_{i+1} e_i$. The first summand is of the desired form. In the second there may be a chain of X left to the Y'_{i+1} which may be commuted to the right and absorbed in the BB_{n-1}. The third summand is either of the desired form, or it may violate the rule that no e_i should appear in a chain on the left of a X_i. But if this rule is violated, it may be restored by the same argument as in the proof of the previous lemma.

None of our rewritings changed the number of Y' and so we can't have more than n of them, at most one coming from each recursion in the construction of S_n. By induction assumption the dimension of BB_{n-1} is less than $2^{n-1}(2n-3)!!$ and we have brought the Y' safely outside the region of BA_n elements. From the theory of BA_n it follows that $2n-1$ different chains $Z_i \cdots Z_n, Z_j \in \{e_{i-1}, X_{i-1}\}$ are needed. Each of these chains may have a Y'_i at its front. Hence we conclude that the dimension increases at most by a factor $2(2n-1)$. Thus the claim follows. ∎

5. RELATION TO THE B TYPE HECKE ALGEBRAS

DEFINITION 4. Let HB_n denote the Hecke algebra of Coxeter type B with generators $X_0, X_1, \ldots, X_{n-1}$ and parameters Q, Q_0 and relations:

$$X_0 X_1 X_0 X_1 = X_1 X_0 X_1 X_0 \tag{67}$$

$$X_i X_j = X_j X_i \qquad |i-j| > 1 \tag{68}$$

$$X_i X_j X_i = X_j X_i X_j \qquad |i-j| = 1 \tag{69}$$

$$X_i^2 = (Q-1) X_i + q \qquad i \geq 0 \tag{70}$$

$$X_0^2 = (Q_0 - 1) X_0 + Q_0. \tag{71}$$

LEMMA 8. *Let I_n be the ideal generated by e_{n-1} in* BB_n. *Every other e_i generates the same ideal and the quotient algebra is isomorphic to HB_n.*

Proof. The first relation follows from (25) which allows to express any e_i in terms of any other e_j. The isomorphism $BB_n / I_n \to HB_n$ is given by $X_i \mapsto q^{-1} X_i$, $Q = q^2$, $Y \mapsto -X_0 q^{-1}(q q_1 + \sqrt{4q + q^2 q_1^2})/2$, $2Q_0 = 2 + q q_1^2 - q_1 \sqrt{4q + q^2 q_1^2}$. ∎

Of course one can avoid square roots by using a different normalization of the generators.

LEMMA 9. $I_n = BB_{n-1} e_{n-1} BB_{n-1}$.

Proof. The ideal is defined to be $I_n = BB_n e_{n-1} BB_n$. If we apply proposition 4 we obtain

$$\begin{aligned} I_n &= BB_{n-1} \Gamma_n' BB_{n-1} e_{n-1} BB_{n-1} \Gamma_n' BB_{n-1} \\ &= BB_{n-1} \Gamma_n' BB_{n-2} \Gamma_{n-1}' BB_{n-2} e_{n-1} BB_{n-2} \Gamma_{n-1}' BB_{n-2} \Gamma_n' BB_{n-1} \\ &= BB_{n-1} \Gamma_n' \Gamma_{n-1}' e_{n-1} BB_{n-2} \Gamma_{n-1}' \Gamma_n' BB_{n-1}. \end{aligned}$$

Hence it suffices to establish that $\Gamma_n' \Gamma_{n-1}' e_{n-1} \subset BB_{n-1} e_{n-1}$. This is done easily using the relations from Lemmas 1 and 2. ∎

6. GRAPHICAL INTERPRETATION AND THE CLASSICAL LIMIT

The definition of BB_n is inspired by B type knot theory. This section supplies the precise definition of the graphical version of the algebra.

Let R be an integral domain. Consider the free R algebra generated by isotopy classes of ribbons in $(\mathbb{R}^2 - \{0\}) \times [0,1]$ between n upper and n

lower intervals imbedded on the line $\mathbb{R}^+ \times 0 \times 1$ resp. $\mathbb{R}^+ \times 0 \times 1$. There may be ribbon components that are not connected to these endpoints. Multiplication is given by putting the graphs on top of each other. Next, restrict the attention to the subalgebra that consists of those isotopy classes that have a representation as a product of the generators $X_i^{(G)}, e_i^{(G)}, Y^{(G)}$, $1 \le i \le n-1$ from Fig. 1. We define $G\mathrm{BB}_n(R)$ (where R is as in the definition of BB_n with (for the moment) δ invertible) to be the quotient of this algebra by the relations (8), (9), (29), (32). The remaining relations in the definition of BB_n have obvious graphical interpretations. Hence, we have a surjective morphism $\Psi_n : \mathrm{BB}_n(R) \to G\mathrm{BB}_n(R)$. It is important to note that $G\mathrm{BB}_n$ is, in contrast to, say, the Temperley–Lieb algebra, not defined by giving a linear basis. It is, rather, an algebra defined by generators and relations where not all relations are stated explicitly. The existence of Ψ_n tells us that $2^n(2n-1)!!$ is an upper bound for the dimension of $G\mathrm{BB}_n$ as well. Furthermore, versions of Propositions 4 and 5 hold as well for this algebra.

The classical limit of a tangle algebra is defined by forgetting over and under crossings. In our situation this should only be applied to the crossings $X_i^{(G)}$. Then, one has $X_i^{(G)} = X_i^{(G)-1}$ and we demand that we have $Y^{(G)^2} = 1$ in the limit as well. Thus $\Psi_n(Y_i') = \Psi_n(Y_i)$ in the limit. This shows that in the limit $Y^{(G)}$ behaves natural with respect to crossings and may therefore be represented by a dot on the arc. Relation (43) together with $Y_i^{(G)} = Y_i^{(G)'} = Y_i^{(G)-1}$ shows that in the classical limit one has $\Psi_n(e_i Y_i) = \Psi_n(e_i Y_{i+1})$.

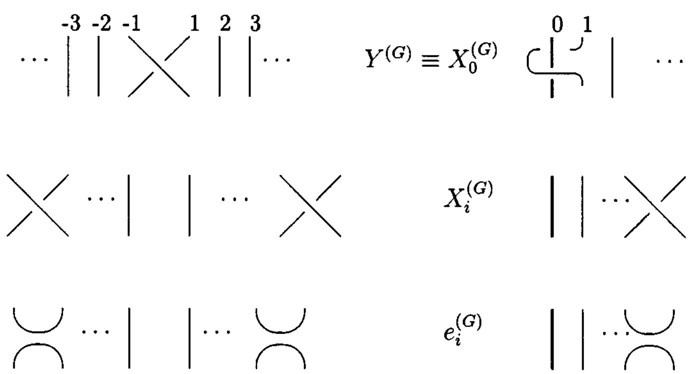

FIG. 1. The graphical interpretation of the generators as symmetric tangles (on the left) and as cylinder tangles (on the right).

The classical limit may be obtained by specializing the parameters of the algebra. It is given by

$$\mathrm{BB}_n^c := \mathrm{BB}_n(R_0) \otimes_{R_0} R_c \tag{72}$$

$$R_c := R_0/(\lambda - 1, q - 1, q_1). \tag{73}$$

It is obvious that $\Psi_n(\mathrm{BB}_n^c)$ is an algebra of dotted Brauer graphs. Each arc may have none or one dot on it. Upon multiplication the number of dots is reduced modulo 2 and a dotted cycle is eliminated at the expense of a factor A. At the moment, however, we do not know if one obtains the full $2^n(2n-1)!!$ dimensional dotted Brauer algebra since it may be that BB_n is too small.

7. CONDITIONAL EXPECTATION AND TRACE ON BB_n

The graphical interpretations suggest that a Markov trace should exist on BB_n. It will be defined as iteration of the conditional expectation which, graphically speaking, closes the last strand

We will need the following assumption:

Hypothesis 5. The inclusion $i: \mathrm{BB}_n \to \mathrm{BB}_{n+2}, a \mapsto x^{-1}ae_{n+1}$ is injective.

LEMMA 10. *This hypothesis is valid for* $G\mathrm{BB}_n(R)$, *that is the morphism* $i^{(G)}: \mathrm{BB}_n^{(G)} \to \mathrm{BB}_{n+2}^{(G)}, a \mapsto x^{-1}ae_{n+1}^{(G)}$ *is injective.*

Proof. Assume that a lies in the kernel of $i^{(G)}$. Now, we deform the nth strand of a above and below of a in the way indicated in Fig. 2. Thus we have an isotopy to a graph that looks locally like ae_{n+1}. So $ae_{n+1} = 0$ implies $a = 0$. ∎

Consider $w = w_1 \gamma w_2 \in \mathrm{BB}_{n+1}$ with $w_i \in \mathrm{BB}_n, \gamma \in \Gamma_{n+1}$. Then we have $e_{n+1}we_{n+1} = w_1 e_{n+1}\gamma e_{n+1} w_2 = s w_1 w_2 e_{n+1}$, with a factor s which assumes

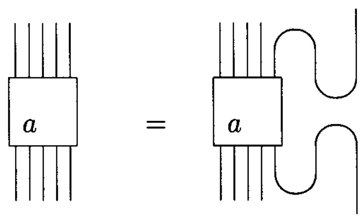

FIGURE 2

the value $s = x, \lambda^{-1}, A$ if $\gamma = 1, e_n, X_n, Y_{n+1}$. Thanks to hypothesis 5 we can give the following definition of the conditional expectation.

DEFINITION 6. $\epsilon_n : \mathrm{BB}_{n+1} \to \mathrm{BB}_n$ is defined by $e_{n+1} a e_{n+1} =: x\epsilon_n(a) e_{n+1}$.

Obviously, $\epsilon_n(w_1 a w_2) = w_1 \epsilon_n(a) w_2$ if $w_i \in \mathrm{BB}_n$. Furthermore, it follows from (20) that $e_{n+1} = e_{n+1} e_n e_{n+1} = x\epsilon_n(e_n) e_{n+1}$ thus $\epsilon_n(e_n) = x^{-1}$. Similarly one derives from (19) the relation $e_{n+1} = \lambda^{\pm} e_{n+1} X_n^{\pm} e_{n+1} = \lambda^{\pm} x \epsilon_n(X_n^{\pm}) e_{n+1}$ thus $\epsilon_n(X_n^{\pm}) = x^{-1}\lambda^{\mp}$ and from (45) it follows that $\epsilon_{n+1} = A^{-1} e_{n+1} Y_{n+1} e_{n+1} = A^{-1} x \epsilon_n(Y_{n+1}) e_{n+1}$ thus $\epsilon_n(Y_{n+1}) = Ax^{-1}$.

The itarated application of the conditional expectation yields a map to the ground ring that will turn out to be a trace.

DEFINITION 7. $\mathrm{tr}(a) := \mathrm{tr}(\epsilon_{n-1}(a)), \mathrm{tr}(1) := 1$.

LEMMA 11. $\mathrm{tr}(e_n) = \epsilon_n(e_n) = x^{-1}$, $\mathrm{tr}(X_n^{\pm}) = \epsilon_n(X_n^{\pm}) = x^{-1}\lambda^{\mp}$, $\mathrm{tr}(Y_{n+1}) = \epsilon_n(Y_{n+1}) = Ax^{-1}$.

LEMMA 12. $\forall w_1, w_2 \in \mathrm{BB}_n, \gamma \in \Gamma_{n+1}$ *we have* $\mathrm{tr}(w_1 \gamma w_2) = \mathrm{tr}(\gamma)\mathrm{tr}(w_1 w_2)$ *and* $\epsilon_n(w_1 \gamma w_2) = \mathrm{tr}(\gamma) w_1 w_2$.

Proof. The first statement is a consequence of the second, which is established in the following calculation:

$$\begin{aligned} x\epsilon_n(w_1 \gamma w_2) e_{n+1} &= e_{n+1} w_1 \gamma w_2 e_{n+1} = w_1 e_{n+1} \gamma e_{n+1} w_2 \\ &= w_1 x \epsilon_n(\gamma) e_{n+1} w_2 = w_1 w_2 x \epsilon_n(\gamma) e_{n+1}. \end{aligned}$$

∎

LEMMA 13. *For all* $a \in \mathrm{BB}_n$ *the following equations hold*:

$$\epsilon_n\left(X_n^{-1} a Y'_{n+1}\right) = \epsilon_n\left(X_n^{-1} Y'_{n+1}\right) a = x^{-1}\lambda^{-1} Y'_n a \tag{74}$$

$$\epsilon_n(X_n Y'_{n+1}) = x^{-1}\lambda^{-1} Y'_n + \delta A x^{-1} - \delta x^{-1} Y_n'^{-1}. \tag{75}$$

Proof.

$$\begin{aligned} \epsilon_n\left(X_n^{-1} a Y'_{n+1}\right) &= \epsilon_n\left(X_n^{-1} Y'_{n+1} a\right) \\ &= \epsilon_n\left(X_n^{-1} Y'_{n+1}\right) a = \epsilon_n(Y'_n X_n) a = Y'_n \epsilon(X_n) a = x^{-1}\lambda^{-1} Y'_n a \\ \epsilon_n(X_n Y'_{n+1}) &= g e_n\left(X_n^2 Y'_n X_n\right) \\ &= \epsilon_n(Y'_n X_n) + \delta\epsilon_n(X_n Y'_n X_n) - \delta\lambda\epsilon_n(e_n Y'_n X_n) \\ &= Y'_n \epsilon_n(X_n) + \delta\epsilon_n(Y'_{n+1}) - \delta\lambda\epsilon_n\left(e_n Y_n'^{-1}\right) \\ &= x^{-1}\lambda^{-1} Y'_n + \delta A x^{-1} - \delta\lambda x^{-1} Y_n'^{-1}. \end{aligned}$$

∎

LEMMA 14. $\forall a \in \mathrm{BB}_n \quad \epsilon_n(X_n^{-1} a X_n) = \epsilon_n(X_n a X_n^{-1}) = \epsilon_n(e_n a e_n) = \epsilon_{n-1}(a)$.

Proof. By linearity and Proposition 4 it is enough to show:

$$\begin{aligned} e_{n+1}\left(X_n^{-1}\gamma X_n\right)e_{n+1} &= e_{n+1}\left(X_n \gamma X_n^{-1}\right)e_{n+1} \\ &= e_{n+1}(e_n \gamma e_n)e_{n+1} = x \operatorname{tr}(\gamma)e_{n+1}. \end{aligned}$$

This is obviously true for $\gamma = 1$. For $\gamma = e_{n-1}$ one obtains

$$\begin{aligned} e_{n+1}\left(X_n^{-1}e_{n-1}X_n\right)e_{n+1} &= e_{n+1}\left(X_n e_{n-1} X_n^{-1}\right)e_{n+1} \\ &= e_{n+1}(e_n e_{n-1} e_n)e_{n+1} = xx^{-1}e_{n+1} \\ \Leftrightarrow e_{n+1}\left(X_{n-1}e_n X_{n-1}^{-1}\right)e_{n+1} &= e_{n+1}\left(X_{n-1}^{-1}e_n X_{n-1}\right)e_{n+1} \\ &= e_{n+1}e_n e_{n+1} = e_{n+1}. \end{aligned}$$

This is true by (25).

If $\gamma = Y_n$ one has

$$\begin{aligned} e_{n+1}\left(X_n^{-1}Y_n X_n\right)e_{n+1} &= e_{n+1}\left(X_n Y_n X_n^{-1}\right)e_{n+1} \\ &= e_{n+1}(e_n Y_n e_n)e_{n+1} = x \operatorname{tr}(Y_n)e_{n+1} \\ \Leftrightarrow e_{n+1}\left(X_n^{-1}Y_n X_n\right)e_{n+1} &= e_{n+1}Y_{n+1}e_{n+1} \\ &= e_{n+1}(e_n Y_n e_n)e_{n+1} = Ae_{n+1}. \end{aligned}$$

That this is true may be seen by transforming the first expression

$$\begin{aligned} e_{n+1}X_n^{-1}Y_n X_n e_{n+1} &= e_{n+1}e_n X_{n+1}Y_n X_n e_{n+1} = e_{n+1}e_n Y_n X_{n+1} X_n e_{n+1} \\ &= e_{n+1}e_n Y_n e_n X_{n+1}X_n = Ae_{n+1}e_n X_{n+1}X_n = Ae_{n+1}. \end{aligned}$$

The last case is $\gamma = X_{n-1}$:

$$\begin{aligned} e_{n+1}\left(X_n^{-1}X_{n-1}X_n\right)e_{n+1} &= e_{n+1}\left(X_n X_{n-1} X_n^{-1}\right)e_{n+1} \\ &= e_{n+1}(e_n X_{n-1} e_n)e_{n+1} = x \operatorname{tr}(X_{n-1})e_{n+1} \\ \Leftrightarrow e_{n+1}\left(X_{n-1}X_n X_{n-1}^{-1}\right)e_{n+1} &= e_{n+1}\left(X_{n-1}^{-1}X_n X_{n-1}\right)e_{n+1} \\ &= e_{n+1}\left(\lambda^{-1}e_n\right)e_{n+1} = \lambda^{-1}e_{n+1} \\ \Leftrightarrow X_{n-1}e_{n+1}X_n e_{n+1}X_{n-1}^{-1} &= X_{n-1}^{-1}e_{n+1}X_n e_{n+1}X_{n-1} \\ &= \lambda^{-1}e_{n+1} = \lambda^{-1}e_{n+1} \\ \Leftrightarrow X_{n-1}\lambda^{-1}e_{n+1}X_{n-1}^{-1} &= X_{n-1}^{-1}\lambda^{-1}e_{n+1}X_{n-1} = \lambda^{-1}e_{n+1}. \end{aligned}$$

∎

Now we show that tr is really a trace, i.e., $\operatorname{tr}(ab) = \operatorname{tr}(ba)$.

LEMMA 15. *Assume* I_{n+1} *to be semisimple and* tr *to be a trace on* BB_n *Then* tr *is a trace on* BB_{n+1}.

Proof. It suffices to show that $\mathrm{tr}(uv) = \mathrm{tr}(vu) \forall u, v \in \mathrm{BB}_{n+1}$. If one of the factors, u say, is actually in BB_n this follows from a simple calculation: $\mathrm{tr}(uv) = \mathrm{tr}(\epsilon_n(uv)) = \mathrm{tr}(u\epsilon_n(v)) = \mathrm{tr}(\epsilon_n(v)u) = \mathrm{tr}(\epsilon_n(vu)) = \mathrm{tr}(vu)$.

Using Proposition 4 one can write arbitrary elements $u, v \in \mathrm{BB}_{n+1}$ in the form

$$u = u_1 + u_2 Y'_{n+1} + u_3 e_n u_4 + u_5 X_n u_6 \tag{76}$$

$$v = v_1 + v_2 Y'_{n+1} + v_3 e_n v_4 + v_5 X_n^{-1} v_6. \tag{77}$$

Since tr is linear it suffices to prove the proposition for all combinations. We have already dealt with the cases $u \in \mathrm{BB}_n$ or $v \in \mathrm{BB}_n$ so only nine cases remain. We investigate symmetric combinations first and write a (resp. b) for one of the summands of u (resp. v) and rename the u_i, v_i in a handy way.

First Case. $a = a_1 e_n a_2$, $b = b_1 e_n b_2$, $a_i, b_i \in \mathrm{BB}_n$.

$$\begin{aligned}
\mathrm{tr}(ab) &= \mathrm{tr}\big(\epsilon_n(a_1 e_n a_2 b_1 e_n b_2)\big) = \mathrm{tr}\big(a_1 \epsilon_n(e_n a_2 b_1 e_n) b_2\big) \\
&= \mathrm{tr}\big(a_1 \epsilon_{n-1}(a_2 b_1) b_2\big) = \mathrm{tr}\big(b_2 a_1 \epsilon_{n-1}(a_2 b_1)\big) \\
&= \mathrm{tr}\big(\epsilon_{n-1}(b_2 a_1) \epsilon_{n-1}(a_2 b_1)\big) = \mathrm{tr}\big(\epsilon_{n-1}(a_2 b_1) \epsilon_{n-1}(b_2 a_1)\big) \\
&= \mathrm{tr}\big(a_2 b_1 \epsilon_{n-1}(b_2 a_1)\big) = \mathrm{tr}\big(b_1 \epsilon_{n-1}(b_2 a_1) a_2\big) \\
&= \mathrm{tr}\big(b_1 \epsilon_n(e_n b_2 a_1 e_n) a_2\big) = \mathrm{tr}\big(\epsilon_n(b_1 e_n b_2 a_1 e_n a_2)\big) = \mathrm{tr}(ba)
\end{aligned}$$

Second Case. $a = a_1 X_n a_2$, $b = B_1 X_n^{-1} b_2$

$$\begin{aligned}
\mathrm{tr}(ab) &= \mathrm{tr}\big(a_1 X_n a_2 b_1 X_n^{-1} b_2\big) = \mathrm{tr}\big(a_1 \epsilon_n\big(X_2 a_2 b_1 X_n^{-1}\big) b_2\big) \\
&= \mathrm{tr}\big(a_1 \epsilon_{n-1}(a_2 b_1) b_2\big) = \mathrm{tr}(a_1 a_2 b_1 b_2) \\
&= \mathrm{tr}(b_1 b_2 a_1 a_2) = \mathrm{tr}(ba)
\end{aligned}$$

Third Case. $a = a_1 Y'_{n+1}$, $b = b_1 Y'_{n+1}$.

$$\mathrm{tr}(ab) = \mathrm{tr}(a_1 Y'_{n+1} b_1 Y'_{n+1}) = \mathrm{tr}\big(a_1 \epsilon_n(Y'_{n+1} b_1 Y'_{n+1})\big)$$

$$\mathrm{tr}\big(a_1 \epsilon_n(Y'_{n+1} Y'_{n+1} b_1)\big) = \mathrm{tr}\Big(a_1 \epsilon_n\Big(Y'^2_{n+1}\Big) b_1\Big)$$

$$\mathrm{tr}\Big(b_1 a_1 \epsilon_n\Big(Y'^2_{n+1}\Big)\Big) = \mathrm{tr}\Big(b_1 \epsilon_n\Big(Y'^2_{n+1}\Big) a_1\Big)$$

$$\mathrm{tr}\Big(a_1 b_1 \epsilon_n\Big(Y'^2_{n+1}\Big)\Big) = \mathrm{tr}(ba).$$

Here we used the fact that $\epsilon_n(Y_{n+1}'^2)$ commutes with a_2 since for all $c \in \mathrm{BB}_n$ one has

$$c\epsilon_n\left(Y_{n+1}'^2\right)e_{n+1} = cx^{-1}e_{n+1}Y_{n+1}'^2 e_{n+1} = x^{-1}e_{n+1}Y_{n+1}'^2 e_{n+1}c$$
$$= \epsilon_n\left(Y_{n+1}'^2\right)e_{n+1}c.$$

Fourth Case. $a = a_1 Y'_{n+1}$, $b = a_3 X_n^{-1} a_4$

$$\begin{aligned}\operatorname{tr}(ab) &= \operatorname{tr}\left(a_1\epsilon_n\left(Y'_{n+1}a_3X_n^{-1}\right)a_4\right) = \operatorname{tr}\left(a_1a_3\epsilon_n\left(Y'_{n+1}X_n^{-1}\right)a_4\right)\\ &= x^{-1}\lambda^{-1}\operatorname{tr}(a_1a_3Y'_na_4) = x^{-1}\lambda^{1}\operatorname{tr}(a_3Y'_na_4a_1)\\ &= \operatorname{tr}\left(a_3\epsilon_n\left(X_n^{-1}Y'_{n+1}\right)a_4a_1\right) = \operatorname{tr}\left(\epsilon_n\left(a_3X_n^{-1}a_4a_1Y'_{n+1}\right)\right) = \operatorname{tr}(ba).\end{aligned}$$

Sixth Case. $a = a_1 X_n a_2$, $b = a_3 Y'_{n+1}$.

$$\begin{aligned}\operatorname{tr}(ab) &= \operatorname{tr}\left(a_1\epsilon_n(X_na_2a_3Y'_{n+1})\right) = \operatorname{tr}\left(a_1\epsilon_n(X_nY'_{n+1})a_2a_3\right)\\ &= x^{-1}\lambda^{-1}\operatorname{tr}(a_1Y'_na_2a_3) + \delta Ax^{-1}\operatorname{tr}(a_1a_2a_3) - \delta\lambda x^{-1}\operatorname{tr}\left(a_1Y_n'^{-1}a_2a_3\right)\\ &= x^{-1}\lambda^{-1}\operatorname{tr}(a_3a_1Y'_na_2) + \delta Ax^{-1}\operatorname{tr}(a_3a_1a_2) - \delta\lambda x^{-1}\operatorname{tr}\left(a_3a_1Y_n'^{-1}a_2\right)\\ &= \operatorname{tr}\left(a_3a_1\epsilon_n(Y'_{n+1}X_n)a_2\right) = \operatorname{tr}(ba).\end{aligned}$$

Seventh Case. $a = a_1 e_n a_2$, $b = a_3 Y'_{n+1}$.

$$\begin{aligned}\operatorname{tr}(ab) &= \operatorname{tr}\left(a_1\epsilon_n(e_na_2a_3Y'_{n+1})\right) = \operatorname{tr}\left(a_1\epsilon_n(e_nY'_{n+1})a_2a_3\right)\\ &= \lambda\operatorname{tr}\left(a_1\epsilon_n\left(e_nY_n'^{-1}\right)a_2a_3\right) = \lambda\operatorname{tr}\left(a_1\epsilon_n(e_n)Y_n'^{-1}a_2a_3\right)\\ &= \lambda x^{-1}\operatorname{tr}\left(a_1Y_n'^{-1}a_2a_3\right) = \lambda x^{-1}\operatorname{tr}\left(a_2a_3a_1Y_n'^{-1}\right)\\ &= \lambda\operatorname{tr}\left(a_2a_3a_1\epsilon_n(e_n)Y_n'^{-1}\right) = \operatorname{tr}\left(a_2a_3a_1\epsilon_n(Y'_{n+1}e_n)\right)\\ &= \operatorname{tr}\left(\epsilon_n(a_3Y'_{n+1}a_1e_n)a_2\right) = \operatorname{tr}(ba).\end{aligned}$$

The case $b = a_1 e_n a_2$, $a = a_3 Y'_{n+1}$ is similar. The only remaining cases are nonsymmetric with one occurrence of e_n. Since we assume I_{n+1} to be semisimple there is an idempotent $z \in \mathrm{BB}_{n+1}$ such that $z\mathrm{BB}_{n+1} \cong I_{n+1}$. Now assume that a contains e_n, hence $a \in I_{n+1}$ i.e. $a = az$. Then we have $ab = azb = a(zb)$, which shows that we might as well assume $b \in I_{n+1}$. But $a, b \in I_{n+1}$ implies that a, b are linear combinations of the form $a = \sum_i a_i e_n a'_i$, $b = \sum_i b_i e_n b'_i$ with $a_i, a'_i, b_i, b'_i \in \mathrm{BB}_n$. Thus we are back in a case that was already treated. ∎

8. THE STRUCTURE THEOREM

We only need a few definitions on Young diagrams before we can state the structure theorem for BB_n.

A Young diagram λ of size n is a partition of the natural number n. $\lambda = (\lambda_1, \dots, \lambda_k)$, $\sum_i \lambda_i = n$, $\lambda_i \geq \lambda_{i+1}$. In the following we use ordered pairs of Young diagrams (cf. [1]). The size of a pair of Young diagrams is the sum of sizes of its components. Let $\hat{\Gamma}_n$ be the set of all pairs of Young diagrams of sizes $n, n-2, \dots$.

PROPOSITION 16. *The following statements hold for the algebra* $\mathrm{BB}_n(K_0)$ *over the quotient field* K_0.

1. BB_n *is isomorphic to* $G\mathrm{BB}_n$ *and it is semisimple. The simple components are indexed by* $\hat{\Gamma}_n$.

$$\mathrm{BB}_n = \bigoplus_{(\mu, \lambda) \in \hat{\Gamma}_n} \mathrm{BB}_{n,(\mu,\lambda)}. \tag{78}$$

2. *The Bratelli rule for restrictions of modules: A simple* $\mathrm{BB}_{n,(\nu,\rho)}$ *module* $V_{(\nu,\rho)}$, $(\nu, \rho) \in \hat{\Gamma}_n$ *decomposes into* BB_{n-1} *modules such that the* BB_{n-1} *module* $(\mu, \lambda) \in \hat{\Gamma}_{n-1}$ *occurs iff* (μ, λ) *may be obtained from* (ν, ρ) *by adding or removing a box. See Fig.* 3.

3. tr *is a faithful trace. To every pair of Young diagrams* $(\mu, \lambda) \in \hat{\Gamma}_n$ *there is a minimal idempotent* $p_{(\mu,\lambda)}$ *and a non vanishing, rational function* $Q_{(\mu,\lambda)}$ *which does not depend on n and satisfies* $\mathrm{tr}(p_{(\mu,\lambda)}) = Q_{(\mu,\lambda)}/x^n$.

For the proof of the structure theorem we need some facts from Jones–Wenzl theory of inclusions of finite dimensional semisimple algebras.

Let $A \subset B \subset C$ be a unital imbedding of finite dimensional semisimple algebras and let tr be a trace on A, B that is compatible with the inclusion. The associated conditional expectation is denoted by $\epsilon_A : B \to A, \mathrm{tr}(ab) = \mathrm{tr}(a\epsilon_A(b))$. It is assumed that there is an idempotent $e \in C$ such that $e^2 = e, ebe = e\epsilon_A(b)\, \forall b \in B$ and $\varphi : A \to C, a \mapsto ae$ is injective.

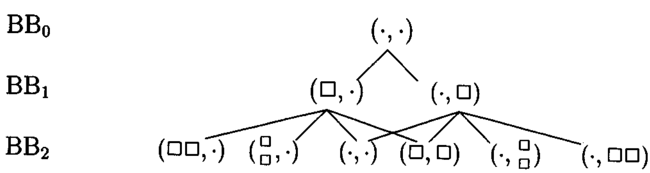

FIG. 3. The Bratteli diagram of BB_n.

Such a situation can be realized starting from an inclusion pair $A \subset B$ with a common faithful trace tr and conditional expectation ϵ_A. We set $\hat{C} := \{\alpha : B \to B | \alpha \text{ linear}, \ \alpha(ba) = \alpha(b)a \ \forall a \in A, b \in B\}$. The inclusion $B \subset \hat{C}$ is given by $b \mapsto \alpha_b, \alpha_b(b_1) := bb_1$. Here e is given by $e_A = \epsilon_a : B \to B$. The subalgebra of $\hat{C}$ generated by B and e_A is denoted by $\langle B, e_A \rangle$. For this setup Wenzyl has obtained the following results [15, Theorem 1.1].

1. $\langle B, e_A \rangle \cong \mathrm{End}_A(B)$.

2. The simple components of A and $\langle B, e_A \rangle$ are in 1-1 correspondence. The inclusion matrices of $A \subset B \subset \langle B, e_A \rangle$ are relatively transposed. If p is a minimal idempotent in A then pe_A is minimal idempotent in $\langle B, e_A \rangle$.

3. $\langle B, e_A \rangle \cong Be_A B$.

4. $\langle B, e \rangle \cong \langle B, e_A \rangle \oplus \tilde{B}$ where $\tilde{B}$ is a subalgebra of B.

5. 4 implies that the ideal generated by e in C is isomorphic to $\langle B, e_A \rangle$.

We now give the proof of the main theorem.

Proof. BB_0 is simply the ground ring. Thus the proposition is true with $\mathrm{tr}(p_{(\cdot,\cdot)}) = \mathrm{tr}(1) = Q_{(\cdot,\cdot)}/x^0, Q_{(\cdot,\cdot)} = 1$. The algebra BB_1 is two-dimensional and has a basis $\{1, Y\}$.

Assume the proposition is shown by induction for BB_n.

By the induction assumptions we have $\mathrm{BB}_n = G\mathrm{BB}_n$. Using this we show that the inclusion $i : \mathrm{BB}_n \to \mathrm{BB}_{n+2}$ of Section 7 is injective. Assume we have $i(a) = 0$, then $0 = \Psi_{n+2}(i(a)) = i^{(G)}(a)$ and the claim follows from injectivity of $i^{(G)}$.

We apply the Jones–Wenzyl theory to the following situation: $A = \mathrm{BB}_{n-1}, B = \mathrm{BB}_n, \mathrm{IC} = \mathrm{BB}_{n+1}, e = x^{-1}e_n, \epsilon_A = \epsilon_{n-1}$. This is possible because A, B are semisimple algebras with a faithful trace by induction assumption. All properties needed for e have already been established. Statement 1 of Jones–Wenzl theory asserts the semisimplicity of $\mathrm{End}_A(B) \cong \langle B, e_A \rangle$ which is by 5 the ideal generated by e. Thus I_{n+1} is semisimple. The quotient algebra by $\mathrm{BB}_{n+1}/I_{n+1}$ is the Hecke algebra HB_{n+1} and is semisimple according to [1]. Now, in general if A is a finite dimensional algebra over some field with a semisimple ideal I such that A/I is semisimple as well then A is semisimple itself: The map $A \to A/I$ maps the radical $\mathrm{Rad}(A)$ into the radical of A/I which is trivial, hence $\mathrm{Rad}(A) \subset I$ and thus $\mathrm{Rad}(A) = I \cap \mathrm{Rad}(A) \subset \mathrm{Rad}(I) = \{0\}$. For finite dimensional algebras over a field vanishing of the radial is equivalent to semisimplicity.

Thus BB_{n+1} is semisimple and is a direct sum $BB_{n+1} = I_{n+1} \oplus BB_{n+1}/I_{n+1}$. Now, the same reasoning can be applied to the algebra GBB_n. In this case the quotient $GBB_{n+}/I^{(G)}_{n+1}$ arises. Imposing the reaction $e^{(G)}_i = 0$ obviously annihilates all tangles that are not ribbon braids of B-type. But then standard knowledge about the graphical interpretation of Hecke algebras shows that $HB_{n+1} = GBB_{n+1}/I^{(G)}_{n+1}$ as well, Jones–Wenzyl theory then implies $GBB_{n+1} = BB_{n+1}$.

Statement 2 asserts that the simple components of I_{n+1} are indexed by $\hat{\Gamma}_{n-1}$. The simple components of HB_{n+1} are indexed by pairs of Young diagrams of size $n + 1$ (see [1]). This completes the proof of point 1 of the theorem.

The inclusion matrix for the part I_{n+1} is the transpose of the inclusion matrix of $BB_{n-1} \subset BB_n$. For the part HB_{n+1} the Bratteli rule follow from [1].

The results proven so far and Lemma 15 imply that tr is a trace. To show its faithfulness one has to show that the Q functions do not vanish. If $p_{(\mu,\lambda)} \in BB_{n-1}$ is a minimal idempotent in $BB_{n-1,(\mu,\lambda)}$ then $x^{-1}p_{(\mu,\lambda)}e_n$ is a minimal idempotent in BB_{n+1}. The trace of this idempotent is $\mathrm{tr}(x^{-1}p_{(\mu,\lambda)}e_n) = x^{-2}\,\mathrm{tr}(p_{(\mu,\lambda)}) = Q_{(\mu,\lambda)}/x^{n-1+2}$. Obviously, this is nonvanishing (using the induction assumption). The idempotents of this kind are those of I_{n+1}. For the other idempotents (which are those of BB_{n+1}/I_{n+1}) the function Q is defined by $\mathrm{tr}(p_{(\mu,\lambda)}) = Q_{(\mu,\lambda)}/x^n$.

To establish faithfulness of the trace we use the classical limit. A minimal idempotent $p_{(\lambda,\mu)}$ of BB_n yields an idempotent in the classical limit. On this algebra the trace in known to be nondegenerate [13]. Thus the function $Q_{(\lambda,\mu)}$ has a nonvanishing limit. ∎

In the rest of this section we sketch a second proof of the semisimplicity of $BB_n(K_0)$. It is based on a different approach to the Markov trace which is based on a different realization of process that may graphically interpreted as closing tangles.

We start with some definitions:

$$X(i,j) := X_i X_{i+1} \cdots X_j \tag{79}$$

$$X^{-1}(i,j) := X_i^{-1} X_{i+1}^{-1} \cdots X_j^{-1} \tag{80}$$

$$E(i,j) := e_i e_{i+2} \cdots e_j \tag{81}$$

$$H_1 := e_1 \tag{82}$$

$$H_{n+1} := e_{n+1} X(n+2, 2n+1) X(n+1, 2n) H_n. \tag{83}$$

The following properties can be shown by straightforward (inductive) calculations.

LEMMA 17.

$$H_n = E(n,n)E(n-1,n+1)\cdots E(1,2n-1) \tag{84}$$

$$H_{n+1} = e_{n+1}X^{-1}(n+2,2n+1)X^{-1}(n+1,2n)H_n \tag{85}$$

$$X_i^{\pm}H_n = X_{2n-i}^{\pm}H_n, \qquad e_iH_n = e_{2n-i}H_n \tag{86}$$

$$e_{2n-1} = X(n,2n-2)^{-1}X(n+1,2n-1)^{-1}e_n \; X(n+1,2n-1)X(n,2n-2) \tag{87}$$

$$e_n = X(n+1,n+k)X(n,n+k-1)e_{n+k} \; X(n,n+k-1)^{-1}X(n+1,n+k)^{-1} \tag{88}$$

$$H_{n+1} = e_{n+1}X(n+2,2n)X(n+1,2n-1)X_{2n+1}^{-1}X_{2n}^{-1}H_n \tag{89}$$

$$Y^{\pm 1}H_n = \lambda^{\pm 1}Y_{2n}'^{\mp 1}H_n \tag{90}$$

$$\overline{H_n}X_i^{\pm} = \overline{H_n}X_{2n-i}^{\pm} \tag{91}$$

$$\overline{H_n}e_i = \overline{H_n}e_{2n-i} \tag{92}$$

$$\overline{H_n}Y^{\pm 1} = \lambda^{\pm 1}\overline{H_n}Y_{2n}'^{\mp 1} \tag{93}$$

$$\overline{H_n}abH_n = \overline{H_n}baH_n, \qquad \forall a,b \in \mathrm{BB}_n \tag{94}$$

$$x^n \operatorname{tr}(a)E(1,2n-1) = \overline{H_n}aH_n, \qquad \forall a \in \mathrm{BB}_n \tag{95}$$

$$0 = x^n(\operatorname{tr}(ab) - \operatorname{tr}(ba))E(1,2n-1). \tag{96}$$

Recall that e_1 does not vanish and has vanishing annulator ideal in $\mathrm{BB}_2(R_0)$. Similarly, the same is true for $e_1^{(G)}$. By induction using Lemma 10 it follows that the same is true for $E(1,2n-1) \in G\mathrm{BB}_{2n}(R_0)$. This shows that tr is a trace on $G\mathrm{BB}_n$.

We now investigate properties of the trace in the classical limit. Let a be a dotted Brauer graph and let $n_i(a)$, $i = 0,1$ be the number of cycles in its closure with i dots on it. The trace of a may easily be seen to be given by

$$\operatorname{tr}(a) = x^{-n}x^{n_0(a)}A^{n_1(a)}. \tag{97}$$

See Fig. 4.

PROPOSITION 18. tr *is nondegenerate and hence* $G\mathrm{BB}_n(K_0)$ *is semisimple. Furthermore,* $G\mathrm{BB}_n(K_0) = \mathrm{BB}_n(K_0)$.

Proof. Let $S_n = \{v_i | 1 \le i \le 2^n(2n-1)!!\}$ be a set of elements that generate $G\mathrm{BB}_n(R_0)$ and yield a basis of dotted Brauer graphs in the classical limit.

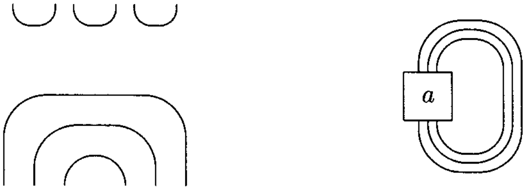

FIG. 4. The graphical interpretations of H_3 (on the left) and of $\mathrm{tr}(a)$ (on the right).

To prove the first statement of the proposition it is enough to show that $0 \neq \det(\mathrm{tr}(v_i v_j^*)_{i,j}) \in R_0$. We tensor this element with R_c to pass to the classical limit. The involution $a \mapsto a^*$ maps graphs to their top-down mirrored image while keeping dots. Due to the reduction of dots modulo 2 there are no dots in the closre of aa^*. Assume a has s upper (and s lower) horizontal arcs. Then aa^* has s cycles. When closing to calculate the trace another s cycles arise from the s lower and s upper horizontal arcs of a and a^*. The vertical arcs of a describe a permutation and a^* contains the inverse permutation. Thus, upon closing, these vertical arcs yield another $n - 2s$ cycles. We conclude that $\mathrm{tr}(aa^*) = 1$. Now, we specify $A = x^{-1}$ by forming a further tensor product. The trace will then be a Laurent polynomial in x. The choice of A lets dots on arcs decrease the degree of the trace polynomial. Now, denote by β an arc in a and let b be another graph which does not contain an arc that is the involutive image of β. Investigating the cases that β is horizontal or vertical one observes that the cycle in $\mathrm{tr}(ab)$ containing β must contain more than two arcs of a and b. The trace of ab thus is of lower degree in x than the trace of aa^*. We conclude that $b = a^*$ is the unique graph of highest degree of x in $\mathrm{tr}(ab)$.

Using this we can establish that

$$\det\bigl(\mathrm{tr}(v_i v_j^*)_{i,j}\bigr) = x^{-nk^n(2n-1)!!}\det\bigl((x^{n_0(v_i v_j^*)}x^{-n_1(v_i v_j^*)})_{i,j}\bigr)$$

does not vanish. The diagonal elements in this matrix are those of highest x-degree in each row. Evaluation of the determinant thus yields only one term with highest x-degree and hence the determinant cannot vanish. But then the original determinant of the trace on $G\mathrm{BB}_n(R_0)$ has to be non zero.

The inclusion image of S_n in $G\mathrm{BB}_n(K_0)$ generates this algebra as a K_0 vector space and the determinant of the trace is the same nonvanishing element of $R_0 \subset K_0$ as before. Existence of a nondegenerate trace on an algebra over a field of characteristic zero implies its semisimplicity.

A further consequence is that the dimension of $G\mathrm{BB}_n(K_0)$ is actually equal to $2^n(2n-1)!!$. The surjection $\Psi_n : G\mathrm{BB}_n(K_0) \to \mathrm{BB}_n(K_0)$ is thus an isomorphism. ∎

9. TENSOR REPRESENTATIONS

Tensor representations of BB_n were found by tom Dieck [6]. We review their definition and show that they can be used to calculate the trace on BB_n as a matrix trace. The ground field K is either the function field $\mathbb{C}(q)$ or $\mathbb{C}$ with an element $q \in \mathbb{C}$. The construction uses the R-matrix of the quantum group $U_q(so_N)$, $N = 2m+1, m \in \mathbb{N}$. The N-dimensional defining representation operates on $V = \{v_i | i \in I\}$. The index set is $I = \{-N+2, -N+4, \ldots, -3, -1, 0, 1, 3, \ldots, N-2\}$. The permuting R-matrix is

$$B = \sum_{i\neq 0} \left(q f_{i,i} \otimes f_{i,i} + q^{-1} f_{i,-i} \otimes f_{-i,i} \right) + f_{0,0} \otimes f_{0,0} + \sum_{i \neq j, -j} f_{i,j} \otimes f_{j,i} + (q - q^{-1}) \left(\sum_{i<j} f_{i,i} \otimes f_{j,j} - \sum_{j<-i} q^{i+j/2} f_{i,j} \otimes f_{-i,-j} \right). \tag{98}$$

Here $f_{i,j}$ is the $N \times N$ matrix with a 1 at position (i,j) and 0 elsewhere.

$E := 1 - (B - B^{-1})/\delta$ is given by

$$E = \sum_{i,j} q^{i+j/2} f_{i,j} \otimes f_{-i,-j}. \tag{99}$$

This implies $E^2 = xE$ with $x = \sum_i q^i$ and hence $\lambda = q^{1-N}$.

Tom Dieck has found the following representation matrix for Y:

$$F = -f_{0,0} + q^{-1/2} \sum_{i \neq 0} f_{-i,i} + \left(q^{-1} - 1\right) \sum_{i>0} f_{i,i}. \tag{100}$$

It satisfies $F^2 = (q^{-1}-1)F + q^{-1}$, $(F \otimes 1)B(F \otimes 1)B = B(F \otimes 1)B(F \otimes 1)$, $E = E(F \otimes 1)B(F \otimes 1)$. Hence $\phi : \mathrm{BB}_n \to \mathrm{End}(V^{\otimes n})$, $Y \mapsto F \otimes 1 \cdots \otimes 1$, $X_i \mapsto 1 \otimes \cdots \otimes 1 \otimes B \otimes 1 \cdots \otimes 1$ defines a representation of BB_n. The parameters are $q_1 = (q^{-1} - 1)$, $\lambda = q^{1-N}$.

Let D be the matrix $D_{i,i} := q^i$ and define $\Psi : \mathrm{End}(V^{\otimes n}) \to K$, $\Psi(a) := \mathrm{Tr}(a(D^{\otimes n}))/\mathrm{Tr}(D^{\otimes n})$. Here Tr is the usual trace of matrices.

LEMMA 19. $\operatorname{tr} = \Psi \circ \phi$.

Proof. Using the parameters of the tensor representation we obtain

$$\operatorname{tr}(Y) = \frac{A}{x} = \frac{q_1}{1 - q_0\lambda} = \frac{q^{-1} - 1}{1 - q^{-1}q^{1-N}} = \frac{q^{-1} - 1}{1 - q^{-N}}.$$

We now calculate $\Psi(Y)$:

$$\operatorname{Tr}(D) = \sum_{I \ni i > 0} q^i + q^0 + \sum_{i \ni i < 0} = 1 + q^{-1} \sum_{i=1}^{m} (q^2)^i + q \sum_{i=1}^{m} (q^{-2})^i$$

$$= 1 + \frac{q - q^N}{1 - q^2} + \frac{q^{-1} - q^{-N}}{1 - q^{-2}}$$

$$\operatorname{Tr}(DF) = -1 + \sum_{I \ni i > 0} (q^{-1} - 1) q^i = (q^{-2} - q^{-1}) \sum_{i=1}^{m} (q^2)^i - 1$$

$$= \frac{q^{-2} - q^{-1}}{1 - q^2}(q^2 - q^{N+1}) - 1$$

$$\Psi(Y) = \operatorname{Tr}(DF)/\operatorname{Tr}(D) = \frac{-q^{N+1} + q^{2N} - q^{2N-1} + q^{N+2}}{q^N - q^{N+2} - q^{2N} + q^2}$$

$$= -\frac{q - 1}{q - q^{-N+1}}.$$

The rest of the proof coincides with the proof of [15, Lemma 5.4]. ∎

A physical application of tensor representations of BB_n has been found in [9]. Two-dimensional integrable systems are described by solutions of the spectral parameter dependent Yang–Baxter equation (YBE) that reads with $R \in \operatorname{End}(V \otimes V)$):

$$R_1(t_1) R_2(t_1 t_2) R_1(t_2) = R_2(t_2) R_1(t_1 t_2) R_2(t_1) \qquad \forall t_1, t_2. \quad (101)$$

If the system is restricted to a half plane an additional matrix $K(t) \in \operatorname{End}(V)$ is needed to describe reflections. It has to fulfill Skylanin's reflection equation [12]:

$$R(t_1/t_2)(K(t_1) \otimes 1) R(t_1 t_2)(K(t_2) \otimes 1)$$
$$= (K(t_2) \otimes 1) R(t_1 t_2)(K(t_1) \otimes 1) R(t_1/t_2). \quad (102)$$

It is possible to obtain solutions of the YBE by Baxterization from the A-type BWM algebra [3]:

$$R_i(t) = -\delta t(t + q\lambda^{-1}) + (t - 1)(t + q\lambda^{-1}) X_i + \delta t(t - 1) e_i. \quad (103)$$

Under the additional generator Y of BB_n one can extend this to obtain solutions of the reflection equation:

PROPOSITION 20. *$K(t) = (t^2 q_1(1 - t^2)^{-1} + Y)f_1(t)$ is for arbitrary f_1 a solution of the reflection equation* (102).

It is a remarkable fact that no similar solution exists for the Hecke algebra HB_n.

10. APPLICATION: INVARIANTS OF LINKS IN A SOLID TORUS

The Markov trace can be used to define a link invariant for links of B-type which are links in a solid torus. There is an analog of Markov's theorem for type B links found by S. Lambrodopoulou in [11]. It takes the same form as the usual Markov theorem, i.e., two B-braids β_1, β_2 have isotopic closures $\hat{\beta}_1, \hat{\beta}_2$ if β_1, β_2 may transformed in one another by a finite sequence of moves of the following two kinds: (I) conjugation $\beta \sim \alpha\beta\alpha^{-1}$ and (II) $\alpha \sim \alpha\tau_n$ for $\alpha \in ZB_n$.

This theorem implies that there exists an extension of the Kauffman polynomial to braids of B-type. Denote by $\pi : ZB_n \to BB_n$ the morphism $\tau_i \mapsto X_i, \tau_0 \mapsto Y$. Then we obtain without any further proof an invariant of the B-type link $\hat{\beta}$ that is the closure of a B-braid $\beta \in ZB_n$ by the following definition:

DEFINITION 8. The B-type Kauffman polynomial of a B-link $\hat{\beta}$ is defined to be

$$L(\hat{\beta}, n) := x^{n-1}\lambda^{e(\beta)}\,\mathrm{tr}(\beta) \qquad \beta \in ZB_n \tag{104}$$

$e : ZB_n \to \mathbb{Z}$ is the exponential sum with $e(X_i) = 1$, $e(Y) = 0$.

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