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Moment Problems with Functions in Some Köthe Spaces

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1. INTRODUCTION

Let C be a family of nonnegative integrable functions in $[0, 1]$ such that C has the following properties.

(a) $1 \in C$;

(b) if $c(x) \in C$ and for a measurable function $c_1(x)$, $0 \leqslant c_1(x) \leqslant c(x)$ a.e. in $[0, 1]$, then $c_1(x) \in C$;

(c) the integrals $\int_0^1 c(x)\,dx$, $c(x) \in C$ are uniformly bounded.

Define the Köthe space $X(C)$ as the space of all functions $f(x)$ integrable in $[0, 1]$ and such that

$$\|f\| = \sup_{c \in C} \int_0^1 c(x)\,|f(x)|\,dx < \infty.$$

With $\|f\|$ as the norm of f, it was proved by Lorentz ([7], p. 67), who had introduced the space, that $X(C)$ is a complete normed space. For C the family of nonnegative integrable functions $g(x)$ in $[0, 1]$ satisfying $\int_0^1 [g(x)]^q\,dx \leqslant 1$, where $1 \leqslant q < \infty$, the space $X(C)$ is $L^p[0, 1]$ where $1/p + 1/q = 1$. For C the family of nonnegative measurable functions $g(x)$ which are essentially bounded in $[0, 1]$, the space $X(C)$ is $L^1[0, 1]$. For additional examples see Lorentz ([7], Section 3.5).

Let the sequence $\{\lambda_i\}$ $(i \geqslant 0)$ satisfy the following

$$0 \leqslant \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots \uparrow \infty, \qquad \sum_{i=1}^{\infty} \frac{1}{\lambda_i} = \infty. \tag{1.1}$$

It is our purpose in this article to obtain necessary and sufficient conditions on a sequence $\{\mu_n\}$ $(n \geqslant 0)$ so that it should possess the representation

$$\mu_n = \int_0^1 t^{\lambda_n} f(t)\,dt \qquad n = 0, 1, 2, \ldots, \tag{1.2}$$

where $f \in X(C)$.

For the sequence $\lambda_n = n$, $n = 0, 1, 2, \ldots$, necessary and sufficient conditions on $\{\mu_n\}$ $(n \geqslant 0)$ were given by Lorentz ([7], Section 3.8). The problem was discussed recently by Ramanujan [8] for the sequence $\lambda_n = n+1$, $n = 0, 1, 2, \ldots$ and by Jakimovski and Ramanujan [3] for the sequence $\lambda_n = n + \alpha$, $n = 0, 1, 2, \ldots$, where $\alpha \geqslant 0$.

2. Preliminaries and Auxiliary Lemmas

Denote

$$\begin{aligned} \lambda_{nm}(t) &= (-1)^{n-m} \lambda_{m+1} \times \cdots \times \lambda_n \sum_{i=m}^{n} \frac{t^{\lambda_i}}{\omega'_{nm}(\lambda_i)}, \qquad 0 \leqslant m < n = 1, 2, \ldots, \\ \lambda_{nn}(t) &= t^{\lambda_n} \qquad n = 0, 1, 2, \ldots, \end{aligned} \tag{2.1}$$

where

$$\omega_{nm}(x) = (x - \lambda_m) \times \cdots \times (x - \lambda_n), \qquad 0 \leqslant m \leqslant n = 0, 1, 2, \ldots, \tag{2.2}$$

and denote

$$\begin{aligned} \lambda^*_{nm}(t) &= \frac{\lambda_m}{\lambda_n} \lambda_{nm}(t), \qquad 0 \leqslant m < n = 1, 2, \ldots \\ \lambda^*_{nn}(t) &= t^{\lambda_n} \qquad n = 0, 1, 2, \ldots . \end{aligned} \tag{2.3}$$

By [7], p. 46 (10),

$$\lambda_{nm}(t) \geqslant 0 \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad 0 \leqslant m \leqslant n = 0, 1, 2, \ldots . \tag{2.4}$$

If $\lambda_0 = 0$, it follows by [7], p. 46 (11) that

$$\sum_{m=0}^{n} \lambda_{nm}(t) = 1 \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad n = 0, 1, 2, \ldots \tag{2.5}$$

and so by (2.4) and (2.5) it follows that for $\lambda_0 > 0$,

$$\sum_{m=0}^{n} \lambda_{nm}(t) \leqslant 1 \quad \text{for} \quad 0 \leqslant t \leqslant 1 \quad \text{and} \quad n = 0, 1, 2, \ldots . \tag{2.6}$$

Also by [2], Theorem 2.3,

$$\begin{aligned} \sum_{n=m}^{\infty} \lambda^*_{nm}(t) &= 1 \qquad \text{for} \qquad 0 < t \leqslant 1, \\ \sum_{n=m}^{\infty} \lambda^*_{n,m}(0) &= 0 \qquad \text{for} \qquad m \geqslant 1. \end{aligned} \tag{2.7}$$

Let $\alpha_{nm} = [(1 - \lambda_1/\lambda_{m+1}) \cdot \cdots \cdot (1 - \lambda_1/\lambda_n)]^{1/\lambda_1}$, $0 \leqslant m < n = 1, 2, \ldots$, $\alpha_{nn} = 1$ and let $\beta_{nm} = \alpha_{n-1,m-1}$ $1 \leqslant m \leqslant n = 1, 2, \ldots$. Then if $\lambda_0 = 0$, it follows by [7], Theorem 2.2.2 that for every continuous function $f(x)$ in [0, 1]

$$\lim_{n\to\infty} \sum_{m=0}^{n} \lambda_{nm}(x) f(\alpha_{nm}) = f(x) \qquad \text{uniformly in} \qquad 0 \leqslant x \leqslant 1. \tag{2.8}$$

Only a slight modification is needed to show that if $\lambda_0 > 0$ then for every continuous function $f(x)$ in [0, 1]

$$\lim_{n\to\infty} \sum_{m=0}^{n} \lambda_{nm}(x) f(\alpha_{nm}) = f(x) \qquad \text{for} \qquad 0 < x \leqslant 1. \tag{2.9}$$

Again by [2], Theorem 2.3, it follows that for every continuous function $f(x)$ in [0, 1],

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} \lambda_{nm}^{*}(x) f(\beta_{nm}) = f(x) \qquad \text{for} \qquad 0 < x \leqslant 1. \tag{2.10}$$

We make use of the following lemmas.

Lemma 1. *For every* n, m, $0 \leqslant m < n = 1, 2, \ldots$,

$$\int_0^1 \lambda_{nm}(t)\, dt = \frac{\lambda_{m+1} \times \cdots \times \lambda_n}{(\lambda_m + 1) \times \cdots \times (\lambda_n + 1)}$$

and

$$\int_0^1 \lambda_{nn}(t)\, dt = \frac{1}{\lambda_n + 1}, \qquad n = 0, 1, 2, \ldots .$$

Proof. The proof follows readily by induction on $n - m$ using the equality

$$\lambda_{nm}(t) = \frac{\lambda_n \lambda_{n-1,m}(t) - \lambda_{m+1}\lambda_{n,m+1}(t)}{\lambda_n - \lambda_m}, \qquad 0 \leqslant t \leqslant 1, \quad n > m.$$

(See [7], Section 2.7 (1), (8).) This completes our proof.

Lemma 2. *Let*

$$\chi_{nm} = \frac{\lambda_{m+1} \times \cdots \times \lambda_n}{(\lambda_{m+1} + 1) \times \cdots \times (\lambda_n + 1)}$$

and let $n(p)$, $m(p)$ *be functions assuming integral values only. Then* $n(p) \to \infty$ *and* $\chi_{n(p),m(p)} \to x > 0$ *as* $p \to \infty$ *imply* $\alpha_{n(p),m(p)} \to x$.

Proof. For $-1 < x < 1$, $\log 1 - x = -x - (x^2/2)(1 - \theta x)^{-2}$ where $\theta = \theta(x)$, $0 < \theta < 1$. Hence,

$$\chi_{nm}^{-1} = \left(1 + \frac{1}{\lambda_{m+1}}\right) \times \cdots \times \left(1 + \frac{1}{\lambda_n}\right)$$

$$= \exp\left[\sum_{k=m+1}^{n} \lambda_k^{-1}\right] \exp\left[-\tfrac{1}{2} \sum_{k=m+1}^{n} \lambda_k^{-2}(1 + \theta_k \lambda_k^{-1})^{-2}\right]$$

and

$$\alpha_{nm} = \left[\left(1 - \frac{\lambda_1}{\lambda_{m+1}}\right) \times \cdots \times \left(1 - \frac{\lambda_1}{\lambda_n}\right)\right]^{1/\lambda_1}$$

$$= \exp\left[-\sum_{k=m+1}^{n} \lambda_k^{-1}\right] \exp\left[-\tfrac{1}{2}\lambda_1 \sum_{k=m+1}^{n} \lambda_k^{-2}(1 - \theta_k{}^*\lambda_1\lambda_k^{-1})^{-2}\right],$$

where $0 < \theta_k, \theta_k{}^* < 1$. Thus,

$$\alpha_{nm}\chi_{nm}^{-1} = \exp\left[-\tfrac{1}{2}\lambda_1 \sum_{k=m+1}^{n} \lambda_k^{-2}(1 - \theta_k{}^*\lambda_1\lambda_k^{-1})^{-2} - \tfrac{1}{2} \sum_{k=m+1}^{n} \lambda_k^{-2}(1 + \theta_k\lambda_k^{-1})^{-2}\right]. \tag{2.11}$$

Now,

$$0 < \chi_{nm} \leqslant \exp\left[-\sum_{k=m+1}^{n} (\lambda_k + 1)^{-1}\right] \quad \text{and} \quad \chi_{nm} \to x > 0,$$

consequently $\sum_{k=m+1}^{n} (\lambda_k + 1)^{-1}$ remains bounded as $p \to \infty$, whence $\sum_{k=m+1}^{n} \lambda_k^{-1}$ remains bounded as $p \to \infty$. Since $\sum_{k=1}^{\infty} \lambda_k^{-1} = \infty$, it follows that $m(p) \to \infty$ and so $\lambda_{m(p)+1} \to \infty$. Therefore,

$$0 \leqslant \sum_{k=m+1}^{n} \lambda_k^{-2}(1 + \theta_k\lambda_k^{-1})^{-2} \leqslant \lambda_{m+1}^{-1} \sum_{k=m+1}^{n} \lambda_k^{-1} \to 0, \quad \text{as} \quad p \to \infty,$$

and

$$0 \leqslant \sum_{k=m+1}^{n} \lambda_k^{-2}(1 - \theta_k{}^*\lambda_1\lambda_k^{-1})^{-2} \leqslant (1 - \lambda_1\lambda_2^{-1})^{-2}\lambda_{m+1}^{-1} \sum_{k=m+1}^{n} \lambda_k^{-1} \to 0, \quad \text{as} \quad p \to \infty.$$

By (2.11) it follows now that

$$\lim \alpha_{n(p),m(p)}\chi_{n(p),m(p)}^{-1} = 1,$$

and so our proof is complete.

The following lemma, the proof of which is left for the reader, is an extension of [7], p. 79, Theorem 3.8.2, and Remark 1.

LEMMA 3. *Let $f_n(x)$ be integrable functions in* $[0, 1]$ *and suppose that the set functions* $F_n(e) = \int_e f_n(x)\,dx$, $e \subseteq [0, 1]$, *e is measurable are uniformly absolutely continuous. Then there is a subsequence $f_{n_k}(x)$ and an integrable function $f(x)$ such that for every uniformly bounded sequence of integrable functions $g_k(x)$ tending almost everywhere to $g(x)$,*

$$\int_0^1 f_{n_k}(x)\, g_k(x)\, dx \to \int_0^1 f(x)\, g(x)\, dx, \qquad \text{as} \qquad k \to \infty.$$

3. The Moment Problem in $X(C)$

For a sequence $\{\mu_n\}$ $(n \geqslant 0)$, let

$$\lambda_{nm} = (-1)^{n-m} \lambda_{m+1} \times \cdots \times \lambda_n \sum_{i=m}^{n} \frac{\mu_i}{\omega'_{nm}(\lambda_i)}, \qquad 0 \leqslant m < n = 1, 2, \ldots$$

$$\lambda_{nn} = \mu_n, \qquad n = 0, 1, 2, \ldots, \tag{3.1}$$

where $\omega_{nm}(x)$ is defined by (2.2); and let

$$\lambda^*_{nm} = \frac{\lambda_m}{\lambda_n} \lambda_{nm}, \qquad 0 \leqslant m < n = 1, 2, \ldots,$$

$$\lambda^*_{nn} = \mu_n, \qquad n = 0, 1, 2, \ldots . \tag{3.2}$$

Denote

$$\Lambda_{nm} = \sum_{k=0}^{m} \int_0^1 \lambda_{nk}(t)\, dt, \qquad 0 \leqslant m \leqslant n = 0, 1, 2, \ldots,$$

and

$$\Lambda^*_{nm} = \sum_{k=n}^{\infty} \int_0^1 \lambda^*_{km}(t)\, dt, \qquad n \geqslant m = 0, 1, 2, \ldots .$$

Define the functions $f_n(x)$ by

$$f_n(x) = \frac{\lambda_{nm}}{\int_0^1 \lambda_{nm}(t)\, dt} \quad \text{if} \quad \Lambda_{n,m-1} \leqslant x < \Lambda_{nm}, \quad (\Lambda_{n,-1} = 0),$$

$$f_n(x) = 0 \qquad \text{elsewhere}; \tag{3.3}$$

and the functions $f_m{}^*(x)$ by

$$f_m{}^*(x) = \frac{\lambda^*_{nm}}{\int_0^1 \lambda^*_{nm}(t)\, dt} \quad \text{if} \quad \Lambda^*_{n+1,m} < x \leqslant \Lambda^*_{n,m}, \quad f_m{}^*(0) = 0. \tag{3.4}$$

THEOREM 1. *Let $X(C)$ have a rearrangement-invariant norm (see* [7], Section 3.4) *and let the set functions $F(e) = \int_e f(x)\,dx$ $f \in X(C)$, $\|f\| \leqslant 1$ be uniformly absolutely continuous. Then the sequence $\{\mu_n\}$ $(n \geqslant 0)$ possesses the representation* (1.2) *where $f \in X(C)$, $\|f\| \leqslant M$, if, and only if, $\|f_n\| \leqslant M$ for all $n \geqslant 0$.*

For $\lambda_n = n$ $n \geqslant 0$, Theorem 1 reduces to [7], Theorem 3.8.4, and for $\lambda_n = n + \alpha$, $n \geqslant 0$ it reduces to [3], Theorem 4.

Proof. It follows by (1.2), (2.1), and (3.1) that

$$\lambda_{nm} = \int_0^1 \lambda_{nm}(t) f(t)\,dt \qquad 0 \leqslant m \leqslant n = 0, 1, 2,\ldots .$$

Thus, if the kernels $K_n(x, t)$ are defined by

$$K_n(x, t) = \frac{\lambda_{nm}(t)}{\int_0^1 \lambda_{nm}(t)\,dt} \qquad \text{if} \qquad \Lambda_{n,m-1} \leqslant x < \Lambda_{nm}\,, \quad 0 \leqslant t \leqslant 1$$

and

$$K_n(x, t) = 0 \qquad \text{elsewhere,}$$

then by (3.3)

$$f_n(x) = \int_0^1 K_n(x, t) f(t)\,dt, \qquad 0 \leqslant x \leqslant 1, \quad n = 0, 1, 2,\ldots . \tag{3.5}$$

Now,

$$\int_0^1 |K_n(x, t)|\,dt = \frac{\int_0^1 \lambda_{nm}(t)\,dt}{\int_0^1 \lambda_{nm}(t)\,dt} = 1 \qquad \text{for} \qquad \Lambda_{n,m-1} \leqslant x < \Lambda_{nm}$$

and

$$\int_0^1 |K_n(x, t)|\,dt = 0 \qquad \text{elsewhere,}$$

hence,

$$\int_0^1 |K_n(x, t)|\,dt \leqslant 1 \qquad \text{for} \qquad 0 \leqslant x \leqslant 1. \tag{3.6}$$

By (2.5) and (2.6)

$$\int_0^1 |K_n(x, t)|\,dx = \sum_{m=0}^{n} \lambda_{nm}(t) \leqslant 1 \qquad \text{for} \qquad 0 \leqslant t \leqslant 1. \tag{3.7}$$

Since $X(C)$ has a rearrangement-invariant norm, it follows by (1.2) where $f \in X(C), \|f\| \leqslant M$, (3.5), (3.6), (3.7), and [7], Theorem 3.8.3, that $\|f_n\| \leqslant M$ for all $n \geqslant 0$. This completes the proof of the necessary part of the theorem.

Conversely, if $\|f_n\| \leqslant M$ for all $n \geqslant 0$, then since $1 \in C$ we have

$$\sum_{m=0}^{n} |\lambda_{nm}| = \int_0^1 |f_n(x)|\, dx \leqslant \|f_n\| \leqslant M, \qquad \text{for all} \qquad n \geqslant 0.$$

Therefore, by [4], Theorem 2.1, $\{\mu_n\}$ $(n \geqslant 0)$ possesses the representation

$$\mu_n = \int_0^1 t^{\lambda_n}\, d\alpha(t) \qquad n = 0, 1, 2,..., \tag{3.8}$$

where $\alpha(t)$ is of bounded variation in [0, 1].

By (2.5), (2.6), (2.8), and (2.9) it follows that for any $k \geqslant 0$

$$\begin{aligned} \lim_{n\to\infty} \sum_{m=0}^{n} \lambda_{nm} \alpha_{nm}^{\lambda_k} &= \lim_{n\to\infty} \int_0^1 \sum_{m=0}^{n} \lambda_{nm}(t)\, \alpha_{nm}^{\lambda_k}\, d\alpha(t) \\ &= \int_0^1 t^{\lambda_k}\, d\alpha(t) = \mu_k . \end{aligned} \tag{3.9}$$

Let the function $g_n^{(k)}(x)$ be defined by

$$g_n^{(k)}(x) = \alpha_{nm}^{\lambda_k} \qquad \text{for} \qquad \Lambda_{n,m-1} \leqslant x < \Lambda_{nm},$$

$$g_n^{(k)}(x) = 0 \qquad \text{elsewhere.}$$

We will prove that

$$g_n^{(k)}(x) \to x^{\lambda_k} \qquad \text{as} \qquad n \to \infty \quad \text{for} \quad 0 \leqslant x < 1. \tag{3.10}$$

Now, for $x = 0$, (3.10) follows immediately since $\alpha_{n1} \to 0$ as $n \to \infty$. Assume that $0 < x < 1$ and let $m \equiv m(n)$, such that $\Lambda_{n,m-1} \leqslant x < \Lambda_{nm}$. For every fixed $k \geqslant 0$, it follows by Hausdorff ([1], Section 2 and (25)) that $\lim_{n\to\infty} \int_0^1 \lambda_{nk}(t)\, dt = 0$, whence $m(n) \to \infty$, as $n \to \infty$. Consequently, by Lemma 1,

$$\begin{aligned} 0 \leqslant \Lambda_{nm} - \Lambda_{n,m-1} &= \int_0^1 \lambda_{nm}(t)\, dt \\ &= \frac{\lambda_{m+1} \times \cdots \times \lambda_n}{(\lambda_m + 1) \times \cdots \times (\lambda_n + 1)} \\ &\leqslant \frac{1}{\lambda_m} \to 0, \qquad \text{as} \qquad n \to \infty. \end{aligned}$$

Thus, $\Lambda_{n,m(n)} \to x$ as $n \to \infty$.

Again by Lemma 1,

$$\Lambda_{nm} = \sum_{k=0}^{m} \int_0^1 \lambda_{nk}(t)\, dt$$

$$= \sum_{k=0}^{m} \frac{\lambda_{k+1} \times \cdots \times \lambda_n}{(\lambda_k + 1) \times \cdots \times (\lambda_n + 1)}$$

$$= \frac{\lambda_{m+1} \times \cdots \times \lambda_n}{(\lambda_{m+1} + 1) \times \cdots \times (\lambda_n + 1)} \sum_{k=0}^{m} \frac{\lambda_{k+1} \times \cdots \times \lambda_m}{(\lambda_k + 1) \times \cdots \times (\lambda_m + 1)}$$

$$= \chi_{nm} \sum_{k=0}^{m} \int_0^1 \lambda_{mk}(t)\, dt.$$

(χ_{nm} is defined as in Lemma 2.)

By (2.5) or (2.6) and (2.9), for $f(t) = 1$, we have

$$\lim_{m\to\infty} \sum_{k=0}^{m} \int_0^1 \lambda_{mk}(t)\, dt = 1.$$

Hence, $\chi_{n,m(n)} \to x$ as $n \to \infty$. So by Lemma 2, $\alpha_{n,m(n)} \to x$ and, thus, $\alpha_{n,m(n)}^{\lambda_k} \to x^{\lambda_k}$ as $n \to \infty$. This concludes the proof of (3.10).

Rewriting (3.9) we obtain

$$\mu_k = \lim_{n\to\infty} \int_0^1 f_n(x)\, g_n^{(k)}(x)\, dx, \qquad k = 0, 1, 2,\dots . \tag{3.11}$$

By (3.10) and Lemma 3, there is a subsequence $f_{n_i}(x)$ and a function $f(x)$ such that

$$\int_0^1 f_{n_i}(x)\, g_{n_i}^{(k)}(x)\, dx \to \int_0^1 f(x)\, x^{\lambda_k}\, dx, \qquad \text{as} \qquad i \to \infty,$$

whence by (3.11)

$$\mu_k = \int_0^1 x^{\lambda_k} f(x)\, dx, \qquad k = 0, 1, 2,\dots .$$

That $f \in X(C)$ and $\| f \| \leqslant M$ follows by [7], p. 80, Remark 2. This completes our proof.

COROLLARY 1. *The space* $L^p[0, 1]$, $1 < p \leqslant \infty$, *satisfies the assumptions of Theorem* 1, *hence the sequence* $\{\mu_n\}$ $(n \geqslant 0)$ *possesses the representation* (1.2) *where* $f \in L^p[0, 1]$, $1 < p \leqslant \infty$, $\| f \|_p \leqslant M$, *if, and only if,*

$$\sum_{m=0}^{n} \frac{|\Lambda_{nm}|^p}{\left(\int_0^1 \lambda_{nm}(t)\, dt\right)^{p-1}} \leqslant M, \qquad \textit{for all} \qquad n \geqslant 0, \qquad \textit{if} \qquad 1 < p < \infty$$

and

$$\frac{|\lambda_{nm}|}{\int_0^1 \lambda_{nm}(t)\,dt} \leqslant M, \qquad \textit{for all} \qquad n, m, 0 \leqslant m \leqslant n, \qquad \textit{if} \qquad p = \infty.$$

Corollary 1 can also be obtained by results in [4, 5].

Interpreting Theorem 1 for Lorentz's spaces $\Lambda(\phi, p)$ and $M(\phi, p)$, $1 < p < \infty$, is left for the reader. (For $\lambda_n = n$, $n \geqslant 0$, see such interpretation in [7], p. 81.

THEOREM 2. *Let $X(C)$ satisfy the assumptions of Theorem 1. Then the sequence $\{\mu_n\}$ $(n \geqslant 0)$ possesses the representation (1.2) for $n \geqslant 1$, where $f \in X(C)$, $\|f\| \leqslant M$, if, and only if, $\|f_m{}^*\| \leqslant M$ for all $m \geqslant 1$.*

Remark. If $\lambda_0 > 0$, then Theorem 2 can be stated with $n \geqslant 0$, $m \geqslant 0$. This is not the case, however, if $\lambda_0 = 0$, in which case μ_0 may or may not have the same representation even if $\|f_m{}^*\| \leqslant M$ for all $m \geqslant 0$.

Theorem 2 for the sequence $\lambda_n = n + 1$, $n \geqslant 0$, is due to Ramanujan [8]; it is due to Jakimovski and Ramanujan [3] for $\lambda_n = n + \alpha + 1$, $n \geqslant 0$.

Proof. The proof of the necessity part is similar to the proof of that part in Theorem 1, using (2.3), (3.2), and (2.7) instead of (2.1), (3.1), and (2.5) and (2.6), respectively. The kernels that are used in this proof are the following:

$$K_m{}^*(x, t) = \frac{\lambda_{nm}^*(t)}{\int_0^1 \lambda_{nm}^*(t)\,dt} \qquad \text{if} \qquad \Lambda_{n+1,m}^* < x \leqslant \Lambda_{nm}^*, \quad 0 \leqslant t \leqslant 1$$

and

$$K_m{}^*(0, t) = 0, \qquad 0 \leqslant t \leqslant 1.$$

In order to prove the sufficiency part, we see that since $1 \in C$ it follows that

$$\sum_{n=m}^{\infty} |\lambda_{nm}^*| = \int_0^1 |f_m{}^*(x)|\,dx \leqslant \|f_m{}^*\| \leqslant M, \qquad \text{for all} \qquad m \geqslant 1.$$

Therefore, by [6], Theorem 3.1, $\{\mu_n\}$ $(n \geqslant 1)$ has the representation (3.8).

By (2.7) and (2.9) it follows that for every $k \geqslant 1$,

$$\begin{aligned}\lim_{m\to\infty} \sum_{n=m}^{\infty} \lambda_{nm}^* \beta_{nm}^{\lambda_k} &= \lim_{m\to\infty} \int_0^1 \sum_{n=m}^{\infty} \lambda_{nm}^*(t)\, \beta_{nm}^{\lambda_k}\, d\alpha(t) \\ &= \int_0^1 t^{\lambda_k}\, d\alpha(t) = \mu_k .\end{aligned} \tag{3.12}$$

Define the functions $h_m^{(k)}(x)$ by

$$h_m^{(k)}(x) = \beta_{nm}^{\lambda_k} \qquad \text{for} \qquad \Lambda_{n+1,m}^* < x \leqslant \Lambda_{nm}^*, \quad h_m^*(0) = 0.$$

We will prove that for $k \geqslant 1$,

$$h_m^{(k)}(x) \to x^{\lambda_k}, \qquad \text{as} \qquad m \to \infty \qquad \text{for} \qquad 0 \leqslant x \leqslant 1. \tag{3.13}$$

Then the rest of the proof will follow as in the proof of Theorem 1. For $x = 0$, $h_m^{(k)}(0) = 0 = 0^{\lambda_k}$ for all $k \geqslant 1$. If $0 < x \leqslant 1$, let $n \equiv n(m)$ such that $\Lambda_{n+1,m}^* < x \leqslant \Lambda_{nm}^*$. By Lemma 1

$$\begin{aligned} 0 \leqslant \Lambda_{nm}^* - \Lambda_{n+1,m}^* &= \int_0^1 \lambda_{nm}^*(t)\,dt \\ &= \frac{\lambda_m \times \cdots \times \lambda_{n-1}}{(\lambda_m + 1) \times \cdots \times (\lambda_n + 1)} \\ &\leqslant \frac{1}{\lambda_n} \to 0 \qquad \text{as} \qquad m \to \infty. \end{aligned}$$

Thus,

$$\Lambda_{n(m),m}^* \to x \qquad \text{as} \qquad m \to \infty.$$

By (2.7) and Lemma 1,

$$\begin{aligned} \Lambda_{nm}^* &= \sum_{k=n}^{\infty} \int_0^1 \lambda_{km}^*(t)\,dt \\ &= \sum_{k=n}^{\infty} \frac{\lambda_m \times \cdots \times \lambda_{k-1}}{(\lambda_m + 1) \times \cdots \times (\lambda_k + 1)} \\ &= \chi_{n-1,m-1} \sum_{k=n}^{\infty} \int_0^1 \lambda_{kn}^*(t)\,dt \\ &= \chi_{n-1,m-1}. \end{aligned}$$

Hence, $\chi_{n(m)-1,m-1} \to x$ as $m \to \infty$ and by Lemma 2, $\beta_{n(m),n} \to x$ as $m \to \infty$. This completes the proof.

COROLLARY 2. *The sequence* $\{\mu_n\}$ $(n \geqslant 1)$ *possesses the representation* (1.2) *for* $n \geqslant 1$, *where* $f \in L^p[0, 1]$, $1 < p < \infty$, $\|f\|_p \leqslant M$, *if, and only if,*

$$\sum_{n=m}^{\infty} \frac{|\lambda_{nm}^*|^p}{\left(\int_0^1 \lambda_{nm}^*(t)\,dt\right)^{p-1}} \leqslant M \qquad \textit{for all} \qquad m \geqslant 1.$$

4. The Moment Problem in $L^1[0, 1]$

As we noted in the introduction, $L^1[0, 1]$ is an $X(C)$ space and it is readily seen to have a rearrangement-invariant norm. However, the set functions $F(e) = \int_e f(x)\,dx, f \in L^1[0, 1], \|f\| \leqslant 1$, are not uniformly absolutely continuous, hence, Theorems 1, 2 do not apply here. Moreover, the requirement of uniformly absolute continuity is quite a strong one. Our results involve a weaker condition.

Definition. The functions $\alpha_n(x)$ defined in [0, 1] are said to be eventually absolutely continuous if for every $\epsilon > 0$ there is a $\delta(\epsilon) > 0$ such that if π is a finite union of disjoint intervals in [0, 1], $\pi = \bigcup_{i=1}^{k} (x_i, y_i)$, say, with $m(\pi) < \delta$, then there exists $N(\epsilon, \pi)$ such that

$$\sum_{i=1}^{k} | \alpha_n(x_i) - \alpha_n(y_i)| < \epsilon \qquad \text{for all} \qquad n > N.$$

For a sequence $\{\mu_n\}$ $(n \geqslant 0)$ let

$$\alpha_n(0) = 0, \qquad \alpha_n(x) = \sum_{\alpha_{nm} \leqslant x} \lambda_{nm}, \qquad 0 < x \leqslant 1,$$

and

$$\alpha_m{}^*(0) = 0, \qquad \alpha_m{}^*(x) = \sum_{\beta_{nm} \leqslant x} \lambda_{nm}^*, \qquad 0 < x \leqslant 1.$$

Theorem 3. *The sequence $\{\mu_n\}$ $(n \geqslant 0)$ possesses the representation* (1.2), *where $f \in L^1[0, 1]$, if, and only if, the function $\alpha_n(x)$ are of uniformly bounded variations and are eventually absolutely continuous.*

Proof. Assume first that $\{\mu_n\}$ $(n \geqslant 0)$ has the representation (1.2) where $f \in L^1[0, 1]$. Then by (2.4), (2.5), and (2.6),

$$\begin{aligned}\bigvee_0^1 [\alpha_n] = \sum_{m=0}^{n} | \lambda_{nm} | &\leqslant \sum_{m=0}^{n} \int_0^1 \lambda_{nm}(t)\, |f(t)|\, dt \\ &= \int_0^1 \sum_{m=0}^{n} \lambda_{nm}(t)\, |f(t)|\, dt \\ &\leqslant \int_0^1 |f(t)|\, dt < \infty, \qquad \text{for all} \qquad n \geqslant 0.\end{aligned}$$

By (2.8) and (2.9) it is easily seen that for any $0 < x \leqslant 1$,

$$\sum_{\alpha_{nm} \leqslant x} \lambda_{nm}(t) \to \begin{cases} 1 & \text{if} \quad 0 \leqslant t < x \\ 0 & \text{if} \quad x < t \leqslant 1 \end{cases} \qquad \text{as} \qquad n \to \infty. \tag{4.1}$$

Thus, for $0 < x \leqslant 1$,

$$\alpha_n(x) = \sum_{\alpha_{nm} \leqslant x} \lambda_{nm} = \int_0^1 \sum_{\alpha_{nm} \leqslant x} \lambda_{nm}(t) f(t)\, dt \to \int_0^x f(t)\, dt \qquad \text{as} \qquad n \to \infty; \tag{4.2}$$

and for $x = 0$,

$$\alpha_n(0) = 0 = \int_0^0 f(t)\, dt.$$

Now, $F(x) = \int_0^x f(t)\, dt$ is an absolutely continuous function and $\alpha_n(x) \to F(x)$, $0 \leqslant x \leqslant 1$, as $n \to \infty$, whence $\alpha_n(x)$ are eventually absolutely continuous. Conversely, since the functions $\alpha_n(x)$ are of uniformly bounded variations, it follows by [4], Theorem 2.1, that $\{\mu_n\}$ $(n \geqslant 0)$ has the representation (3.8). Also, similar to (3.9), we obtain for each $k \geqslant 0$,

$$\lim_{n\to\infty} \int_0^1 t^{\lambda_k}\, d\alpha_n(t) = \lim_{n\to\infty} \sum_{m=0}^{n} \lambda_{nm} \alpha_{nm}^{\lambda_k} = \mu_k . \tag{4.3}$$

Once again, since $\alpha_n(x)$ are of uniformly bounded variations, it follows by Helly's theorem that there is a subsequence $\{\alpha_{n_i}(x)\}$ and a function of bounded variation $\beta(x)$ such that $\alpha_{n_i}(x) \to \beta(x)$, $0 \leqslant x \leqslant 1$, as $i \to \infty$. Hence, by (4.3) and Helly-Bray's theorem,

$$\mu_k = \int_0^1 t^{\lambda_k}\, d\beta(t) \qquad k = 0, 1, 2, \ldots . \tag{4.4}$$

Now $\alpha_{n_i}(x)$ are eventually absolutely continuous and $\alpha_{n_i}(x) \to \beta(x)$, $0 \leqslant x \leqslant 1$, whence $\beta(x)$ is absolutely continuous; thus,

$$\beta(x) = \int_0^x f(t)\, dt, \qquad 0 \leqslant x \leqslant 1, \qquad f \in L^1[0, 1]. \tag{4.5}$$

Our result follows now by (4.4) and (4.5). This completes our proof.

The following result is proved similarly.

THEOREM 4. *The sequence* $\{\mu_n\}$ $(n \geqslant 1)$ *possesses the representation* (1.2), *for* $n \geqslant 1$, *where* $f \in L^1[0, 1]$, *if, and only if, the function* $\alpha_m^*(x)$ $(m \geqslant 1)$ *are of uniformly bounded variations and are eventually absolutely continuous.*

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