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Uniformities on Fuzzy Topological Spaces

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In this paper we extend the notions of quasi-uniformities on topological spaces to fuzzy topological space. We prove theorems corresponding to many of the usual theorems. In particular we show every fuzzy topological space is quasi-uniformizable and characterize uniformizability in terms of a type of complete regularity. We also construct a natural uniformity on the fuzzy unit interval.

1. INTRODUCTION

In [3] we generalized normality to fuzzy topological spaces as introduced in [1], and characterized it by a sort of Urysohn's lemma. In the process we constructed an interesting fuzzy topological space, the fuzzy unit interval.

In this paper we generalize the notions of quasi-uniformities and uniformities on topological spaces to fuzzy topological spaces. We prove theorems corresponding to many of the usual theorems. In particular we show that every fuzzy topological space is quasi-uniformizable. The fuzzy unit interval plays an essential part in a characterization of uniformizability in terms of a type of complete regularity. To achieve this we construct a natural uniformity on the fuzzy unit interval.

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2. PRELIMINARIES

Throughout this paper $(L, \leqslant, ')$ will be a completely distributive lattice with order reversing involution $'$. An L-fuzzy set on a set X is any map $A: X \to L$. We interpret L as a set of truth values, and $A(x)$ as the degree of membership of x in the fuzzy set A. When L is the lattice $\{0, 1\}$ then the collection of fuzzy sets corresponds to the characteristic functions of ordinary sets.

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We define the union, intersection, and complement of fuzzy sets as follows:

$$\left(\bigcup_i A_i\right)(x) = \bigvee_i A_i(x) \qquad \text{for } x \in X,$$

$$\left(\bigcap_i A_i\right)(x) = \bigwedge_i A_i(x) \qquad \text{for } x \in X,$$

$$A'(x) = A(x)' \qquad \text{for } x \in X.$$

We define a fuzzy topological space as a pair (X, τ), where $\tau \subseteq L^X$ (all maps from X to L) and τ is closed under arbitrary unions and finite intersections. A set is called open if it is in τ, and closed if its complement is in τ. If (X, τ_1) and (Y, τ_2) are fuzzy topological spaces, then a map $f: X \to Y$ is said to be continuous if for every τ_2 open set U, $f^{-1}(U) \in \tau_1$, where $f^{-1}(U)(x) = U(f(x))$ for $x \in X$. The interior and closure of fuzzy sets are defined in the obvious way (see [1]).

3. Quasi-Uniformities

Consider a quasi-uniformity on X in the usual topological sense. An element D is a subset of $X \times X$. We may define $D: 2^X \to 2^X$ by $D(V) = \{y \mid x \in V$ and $(x, y) \in D\}$. It is obvious that $V \subseteq D(V)$ and $D(\bigcup V_\lambda) = \bigcup D(V_\lambda)$ for V and V_λ in 2^X. Conversely, given $D: 2^X \to 2^X$ satisfying $V \subseteq D(V)$ and $D(\bigcup V_\lambda) = \bigcup D(V_\lambda)$ for V and V_λ in 2^X, we may define $D \subseteq X \times X$ such that D contains the diagonal by $D = \{(x, y) \mid y \in D(\{x\})\}$. Thus in defining a quasi-uniformity for a fuzzy topology, we take our basic elements of the quasi-uniformity to be elements of the set $\mathcal{D}$ of maps $D: L^X \to L^X$ which satisfy:

(A1) $V \subseteq D(V)$ for $V \in L^X$.

(A2) $D(\bigcup V_\lambda) = \bigcup D(V_\lambda)$ for $V_\lambda \in L^X$.

Before we define what we mean by a quasi-uniformity we need some preliminary results.

Lemma 1. *Suppose L is a completely distributive lattice and $\alpha \in L$. Then there exists a set $B \subseteq L$ such that* $\sup B = \alpha$ *and if $A \subseteq L$ satisfies* $\sup A = \alpha$ *then for every $\beta \in B$ there exists $\gamma \in A$ such that $\beta \leqslant \gamma$.*

Proof. Consider all possible sets $A \subseteq L$ such that $\sup A = \alpha$. Index these sets $\{A_j \mid j \in J\}$ and index the elements in the sets by $A_j = \{\alpha_{ij} \mid i \in I_j\}$. Consider

$$B = \left\{\bigwedge_{j \in J} \alpha_{i(j)j} \mid i \in \prod I_j\right\}.$$

Then

$$\begin{aligned}\sup B &= \bigvee_{i \in \Pi I_j} \Big(\bigwedge_{j \in J} \alpha_{i(j)j} \Big) \\ &= \bigwedge_{j \in J} \Big(\bigvee_{i(j) \in I_j} \alpha_{i(j)j} \Big) \\ &= \bigwedge_{j \in J} \alpha \\ &= \alpha.\end{aligned}$$

Also $\bigwedge_{j \in J} \alpha_{i(j)j} \leqslant \alpha_{i(k)k}$ for $k \in J$. Thus for every $\beta \in B$ there exists $\gamma \in A_k$ such that $\beta \leqslant \gamma$ (for any $k \in J$).

LEMMA 2. *Suppose L is a completely distributive lattice and $f: L \to L$ satisfies*

(a1) $\alpha \leqslant f(\alpha)$ *for* $\alpha \in L$.
(a2′) $\alpha \leqslant \beta$ *implies* $f(\alpha) \leqslant f(\beta)$ *for* $\alpha, \beta \in L$.

Then $f^*: L \to L$ defined by

$$f^*(\alpha) = \bigwedge_{\sup \Gamma = \alpha} \Big(\bigvee_{\gamma \in \Gamma} f(\gamma) \Big)$$

is the greatest $g: L \to L$ which takes values less than or equal to f and satisfies

(a1) $\alpha \leqslant g(\alpha)$ *for* $\alpha \in L$.
(a2) $g(\bigvee_i \alpha_i) = \bigvee_i g(\alpha_i)$ *for* $\alpha_i \in L$.

Also $f^*(\alpha) = \bigvee_{\beta \in B} f(\beta)$ for B as in Lemma 1.

Proof. Clearly f^* satisfies (a1) and (a2′). Also $f^*(\alpha) \leqslant f(\alpha)$ for $\alpha \in L$.

Choose B as in Lemma 1. If $\sup \Gamma = \alpha$ then for every $\beta \in B$ there exists $\gamma \in \Gamma$ such that $\beta \leqslant B$. Hence

$$\bigvee_{\beta \in B} f(\beta) \leqslant \bigvee_{\gamma \in \Gamma} f(\gamma)$$

which implies

$$f^*(\alpha) = \bigvee_{\beta \in B} f(\beta).$$

Suppose $\bigvee_i \alpha_i = \alpha$. Then we may find B_i such that $\sup B_i = \alpha_i$ and

$$f^*(\alpha_i) = \bigvee_{\beta_i \in B_i} f(\beta_i),$$

so

$$\bigvee_i f^*(\alpha_i) = \bigvee_i \bigvee_{\beta_i \in B_i} f(\beta_i),$$

$$\geqslant f^*(\alpha) \qquad \text{since} \qquad \sup \bigvee_i B_i = \alpha.$$

That f^* is the greatest such g is obvious

DEFINITION 1. Let $f_1 : L \to L$ and $f_2 : L \to L$ satisfy (a1) and (a2) (as in Lemma 2). Let $g: L \to L$ be defined by $g(\alpha) = f_1(\alpha) \wedge f_2(\alpha)$ (so g satisfies (a1) and (a2′)). Then we define $f_1 \wedge f_2 : L \to L$ by $f_1 \wedge f_2 = g^*$ (so $f_1 \wedge f_2$ satisfies (a1) and (a2)).

LEMMA 3. *Suppose $f_1 : L \to L$ and $f_2 : L \to L$ satisfy* (a1) *and* (a2). *Then*

$$(f_1 \wedge f_2)(\alpha) = \bigwedge_{\alpha_1 \vee_2 \alpha = \alpha} \Big(f_1(\alpha_1) \vee f_2(\alpha_2)\Big).$$

Proof.

$$(f_1 \wedge f_2)(\alpha) = \bigwedge_{\sup \Gamma = \alpha} \Big(\bigvee_{\gamma \in \Gamma} (f_1(\gamma) \wedge f_2(\gamma))\Big)$$

$$= \bigwedge_{\sup \Gamma = \alpha} \Big(\bigwedge_{A \subseteq \Gamma} \Big(\Big[\bigvee_{\gamma \in A} f_1(\gamma)\Big] \vee \Big[\bigvee_{\gamma \notin A} f_2(\gamma)\Big]\Big)\Big)$$

$$= \bigwedge_{\sup \Gamma = \alpha} \Big(\bigwedge_{A \subseteq \Gamma} \Big(f_1\Big(\bigvee_{\gamma \in A} \gamma\Big) \vee f_2\Big(\bigvee_{\gamma \notin A} \gamma\Big)\Big)\Big)$$

$$= \bigwedge_{\alpha_1 \vee \alpha_2 = \alpha} (f_1(\alpha_1) \vee f_2(\alpha_2)).$$

Note that L^X is a completely distributive lattice if L is. Hence Lemmas 1, 2, and 3 may be applied with L replaced by L^X. Thus (a1) and (a2) are now conditions (A1) and (A2). For $D: L^X \to L^X$ and $E: L^X \to L^X$ we denote $D \wedge E$ by $D \cap E$. We say $D \subseteq E$ if $D(V) \subseteq E(V)$ for every $V \in L^X$. We define $D \circ E$ by composition of functions. We are now in a position to define a quasi-uniformity.

DEFINITION 2. A (fuzzy) quasi-uniformity on a set X is a subset $\mathscr{D}$ of $\mathscr{Q}$ (the set of all maps satisfying (A1) and (A2)) such that:

(Q1) $\mathscr{D} \neq \varnothing$.

(Q2) $D \in \mathscr{D}$ and $D \subseteq E \in \mathscr{Q}$ implies $E \in \mathscr{D}$.

(Q3) $D \in \mathscr{D}$ and $E \in \mathscr{D}$ implies $D \wedge E \in \mathscr{D}$.

(Q4) $D \in \mathscr{D}$ implies there exists $E \in \mathscr{D}$ such that $E \circ E \subseteq D$.

Note that this definition agrees with the usual definition for $L = \{0, 1\}$. Note that (Q3) may be replaced by

(Q3′) $D_1 \in \mathscr{D}$ and $D_2 \in \mathscr{D}$ imply there exists $D \in \mathscr{D}$ such that $D \subseteq D_1$ and $D \subseteq D_2$.

Also note that any subset $\mathscr{B}$ of $\mathscr{Q}$ which satisfies (Q4) generates a fuzzy quasi-uniformity in the sense that the collection of all $D \in \mathscr{Q}$ which contain a finite intersection of elements of $\mathscr{B}$ is a quasi-uniformity. Such a set $\mathscr{B}$ is called a sub-basis for the quasi-uniformity generated. If $\mathscr{B}$ also satisfies (Q3′) then $\mathscr{B}$ is called a basis.

Before we define the Fuzzy topology generated by a quasi-uniformity, we state the following trivial proposition.

PROPOSITION 4. *Suppose a map* $i\colon L^X \to L^X$ *satisfies the interior axioms*:

(I1) $i(X) = X$.

(I2) $i(V) \subseteq V$ *for* $V \in L^X$.

(I3) $i(i(V)) = i(V)$ *for* $V \in L^X$.

(I4) $i(V \cap W) = i(V) \cap i(W)$ *for* $V, W \in L^X$.

Then $\tau = \{V \in L^X \mid i(V) = V\}$ *is a fuzzy topology and* $i(V) = \mathrm{Int}(V)$.

DEFINITION 3. Let $(X, \mathscr{D})$ be a quasi-uniformity. Define $\mathrm{Int}\colon L^X \to L^X$ by $\mathrm{Int}(V) = \bigcup \{U \in L^X \mid D(U) \subseteq V \text{ for some } D \in \mathscr{D}\}$.

PROPOSITION 5. Int *satisfies the interior axioms.*

Proof. (I1) and (I2) are trivially satisfied.

(I3) is satisfied since:

If U and V are fuzzy sets and $D \in \mathscr{D}$ is such that $D(U) \subseteq V$, then we can find $E \in D$ such that $E \circ E \subseteq D$. So in particular $E(E(U)) \subseteq V$. Thus $E(U) \subseteq \mathrm{Int}(V)$, which implies $U \subseteq \mathrm{Int}(\mathrm{Int}\ V))$. Hence $\mathrm{Int}(V) \subseteq \mathrm{Int}(\mathrm{Int}(V))$, and since the other inclusion follows by (I2) we have $\mathrm{Int}(V) = \mathrm{Int}(\mathrm{Int}(V))$.

(I4) follows by (Q3).

DEFINITION 4. The fuzzy topology generated by D is the fuzzy topology generated by Int.

Hence in particular we note that $D(U)$ is a neighborhood of U in the topology generated by D.

LEMMA 6. *Let* (X, τ) *be a fuzzy topological space. Suppose* $D_i \in \mathscr{Q}$ *and* $D_i(U)$ *is a neighborhood of* U *for any fuzzy set* U $(i = 1, 2)$. *Then* $(D_1 \cap D_2)(U)$ *is a neighborhood of* U *for any fuzzy set* U.

Proof. By Lemmas 1, 2, and 3

$$(D_1 \cap D_2)(U) = \bigcup_j (D_1(U_j) \cap D_2(U_j))$$

for some fuzzy sets U_j whose union is U. But $D_i(U_j)$ is a neighborhood of U_j $(i = 1, 2)$ and hence $D_1(U_j) \cap D_2(U_j)$ is a neighborhood of U_j. So there exist open sets W_j such that $U_j \subseteq W_j \subseteq D_1(U_j) \cap D_2(U_j)$. Hence $W = \bigcup_j W_j$ is open and $U \subseteq W \subseteq \bigcup_j (D_1(U_j) \cap D_2(U_j))$.

THEOREM 7. *Every fuzzy topology is fuzzy quasi-uniformizable.*

Proof. Let (X, τ) be a fuzzy topological space. Let G be any open set in τ. Define

$$\begin{aligned} D_G(V) &= X \quad \text{for} \quad V \not\subseteq G, \\ &= G \quad \text{for} \quad V \subseteq G. \end{aligned}$$

So $D_G \circ D_G = D_G$. Thus by Proposition 6, $\{D_G \mid D \in \tau\}$ forms a sub-base for a quasi-uniformity which generates the topology.

We may define quasi-uniform continuity between quasi-uniform spaces.

DEFINITION 5. Let $(X, \mathscr{D})$ and $(Y, \mathscr{E})$ be quasi-uniform spaces. A map $f: X \to Y$ is said to be quasi-uniformly continuous if for every $E \in \mathscr{E}$, there exists a $D \in \mathscr{D}$ such that $D \subseteq f^{-1}(E)$. That is, for $V \in L^X$, $D(V) \subseteq f^{-1}E(f(V))$.

PROPOSITION 8. *Every quasi-uniformly continuous function is continuous in the induced fuzzy topologies.*

Proof. Let $f: X \to Y$ be quasi-uniformly continuous. Consider an open set V in the fuzzy topology generated by $\mathscr{E}$. So $V = \bigcup \{U \mid E(U) \subseteq V \text{ for some } E \in \mathscr{E}\}$. If $E(U) \subseteq V$ then there exists $D \in \mathscr{D}$ such that

$$Df^{-1}(U) \subseteq f^{-1}(E(ff^{-1}(U)) \subseteq f^{-1}(E(U)) \subseteq f^{-1}(V) \qquad \text{(Ref. [1])}.$$

So $f^{-1}(U) \subseteq \operatorname{Int} f^{-1}(V)$, and hence

$$\bigcup \{f^{-1}(U) \mid E(U) \subseteq V \text{ for some } E \in \mathscr{E}\} \subseteq \operatorname{Int}(f^{-1}(V)).$$

But $f^{-1}(\bigcup U_\lambda) = \bigcup f^{-1}(U_\lambda)$ and hence $f^{-1}(V) \subseteq \operatorname{Int}(f^{-1}(V))$. That is, $f^{-1}(V)$ is open, which is the definition of continuity.

We now prove a theorem corresponding to the characterization of quasi-pseudo metrizability in terms of quasi-uniformities. We effectively define quasi-pseudo metrizability in terms of a special sort of base for a quasi-uniformity. There is a way of converting this quasi-uniformity into a map satisfying the triangle inequality from $X \times X$ to a monoid, but this is no more than a notational change. The description in terms of this special base appears to be more intuitively pleasing at the moment and so we leave it like this.

THEOREM 9 (Quasi-pseudo metrization). *Let $(X, \mathscr{D})$ be a quasi-uniformity. Then $\mathscr{D}$ has a base $\{D_r \mid r \in \mathbb{R}, r > 0\}$ such that $D_r \circ D_s \subseteq D_{r+s}$ for r and s positive reals if and only if $\mathscr{D}$ has a countable base.*

Proof. $(\Rightarrow)$ is trivial.

$(\Leftarrow)$ conversely; suppose $\mathscr{D}$ has a countable base $\{U_n \mid n = 1, 2, 3,...\}$.

We may rechoose $\{U_n\}$ such that $U_n \circ U_n \circ U_n \subseteq U_{n-1}$ (see, for example, [4]).

Define $\phi_\epsilon \in \mathscr{D}$ for $\epsilon > 0$ by $\phi_\epsilon = U_n$ if $2^{-n} \leqslant \epsilon \leqslant 2^{-(n-1)}$ and $\phi_\epsilon(V) = X$ if $1 \leqslant \epsilon$ (so that $\phi_\epsilon \circ \phi_\epsilon \circ \phi_\epsilon \subseteq \phi_{2\epsilon}$).

Define

$$D_\epsilon = \bigcup_{\sum_{i=1}^{k} \epsilon_i = \epsilon} \phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_k}.$$

D_ϵ is obviously in $\mathscr{Q}$ since ϕ_ϵ is. Now

(1) $\phi_\epsilon \subseteq D_\epsilon$ trivially.

(2) $D_\epsilon \subseteq \phi_{2\epsilon}$ since:

If $k = 1$ $\phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_k} \subseteq \phi_{2\epsilon}$ trivially. Assume that $k > 1$ and if $l < k$ and $\epsilon > 0$ then $\phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_l} \subseteq \phi_{2\epsilon}$ (where $\epsilon_1 + \cdots + \epsilon_l = \epsilon$). Consider $\phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_k}$ (where $\epsilon_1 + \cdots + \epsilon_k = \epsilon$). Choose the largest j such that $\epsilon_1 + \cdots + \epsilon_j \leqslant \frac{1}{2}\epsilon$. Thus $\epsilon_{j+2} + \cdots + \epsilon_k \leqslant \frac{1}{2}\epsilon$. By induction

$$\phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_j} \subseteq \phi_\epsilon,$$

$$\phi_{\epsilon_{j+1}} \subseteq \phi_\epsilon,$$

$$\phi_{\epsilon_{j+2}} \circ \cdots \circ \phi_{\epsilon_k} \subseteq \phi_\epsilon.$$

Hence

$$\phi_{\epsilon_1} \circ \cdots \circ \phi_{\epsilon_k} \subseteq \phi_\epsilon \circ \phi_\epsilon \circ \phi_\epsilon, \\ \subseteq \phi_{2\epsilon}.$$

Hence $\{D_s \mid \epsilon > 0\}$ generates the same quasi-uniformity as $\{\phi_\epsilon\}$ and hence as $\{U_n\}$. The family $\{D_s \mid \epsilon > 0\}$ obviously satisfy $D_r \circ D_s \subseteq D_{r+s}$.

4. Uniformities

Consider a uniformity in the usual topological sense. We define D^{-1} by $(x, y) \in D^{-1}$ if and only if $(y, x) \in D$. Equivalent statements are:

$$\begin{array}{lll} y \in D^{-1}(\{x\}) & \text{if and only if} & x \in D)\{y\}), \\ y \notin D^{-1}(\{x\}) & \text{if and only if} & x \notin D(\{y\}), \\ D^{-1}(\{x\}) \subseteq \{y\}' & \text{if and only if} & D(\{y\}) \subseteq \{x\}', \\ D^{-1}(V) \subseteq U' & \text{if and only if} & D(U) \subseteq V'. \end{array}$$

This suggests the following.

Definition 6. Let L be a completely distributive lattice with order reversing involution $'$. Let $f: L \to L$ satisfy (a1) and (a2). Then we define $f^{-1}: L \to L$ by $f^{-1}(\alpha) = \inf\{\beta \mid f(\beta') \leqslant \alpha'\}$.

Proposition 10. (1) $f(\alpha) \leqslant \beta$ *if and only if* $f^{-1}(\beta') \leqslant \alpha'$.

(2) f^{-1} *satisfies* (a1) *and* (a2).

(3) $(f^{-1})^{-1} = f$.

(4) $f \leqslant g$ *if and only if* $f^{-1} \leqslant g^{-1}$.

(5) $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$.

Proof. (1) $(\Rightarrow)$ is trivial.

$(\Leftarrow)$ holds since if $f^{-1}(\beta') \subseteq \alpha'$ then

$$\begin{aligned} f(\alpha) &\leqslant f(f^{-1}(\beta')') \\ &= f\Big(\bigvee\{\gamma \mid f(\gamma) \leqslant \beta\}\Big) \\ &\bigvee\{f(\gamma) \mid f(\gamma) \leqslant \beta\} \\ &\leqslant \beta. \end{aligned}$$

(2) (a1) is trivially satisfied.

(a2) is true since

$$\begin{aligned} \bigvee_i f^{-1}(\alpha_i) \subseteq \beta &\Leftrightarrow f^{-1}(\alpha_i) \subseteq \beta \quad \text{for all } i, \\ &\Leftrightarrow f(\beta') \subseteq \alpha_i' \quad \text{for all } i, \\ &\Leftrightarrow f(\beta') \subseteq \Big(\bigvee \alpha_i\Big) \\ &\Leftrightarrow f^{-1}\Big(\bigvee_i \alpha_i\Big) \subseteq \beta. \end{aligned}$$

Hence $f^{-1}(\bigvee_i \alpha_i) = \bigvee_i f^{-1}(\alpha_i)$.

(3), (4), and (5) follow by a similar argument.

PROPOSITION 11.

$$(f_1 \wedge f_2)^{-1} = f_1^{-1} \wedge f_2^{-1}.$$

Proof. Let $\Delta \subseteq L$ be such that $\sup \Delta = \alpha$ and if $\sup \Gamma = \alpha$ then for every $\delta \in \Delta$ there exists $\gamma \in \Gamma$ such that $\delta \leqslant \gamma$ (as in Lemma 1).

Then

$$\begin{aligned}
&(f_1 \wedge f_2)^{-1}(\alpha) \\
&= \bigwedge \{\beta \mid (f_1 \wedge f_2)(\beta') \leqslant \alpha'\} \\
&= \bigwedge \left\{\beta \,\Big|\, \bigwedge_{\beta_1 \wedge \beta_2 = \beta} (f_1(\beta_1') \vee f_2(\beta_2')) \leqslant \alpha'\right\} \\
&= \bigwedge \{\beta \mid \forall\, \delta \in \Delta,\ \exists \beta_1, \beta_2 \text{ such that } \beta_1 \bigwedge \beta_2 = \beta \text{ and } f_1(\beta_1') \vee f_2(\beta_2') \leqslant \delta'\}, \\
&= \bigvee_{\delta \in \Delta} \left(\bigwedge \{\beta \mid \exists \beta_1, \beta_2 \text{ such that } \beta_1 \wedge \beta_2 = \beta \text{ and } f_1(\beta_1') \leqslant \delta', f_2(\beta_2') \leqslant \delta'\}\right), \\
&= \bigvee_{\delta \in \Delta} \left(\bigwedge \{\beta_1 \wedge \beta_2 \mid f_1(\beta_1') \leqslant \delta' \quad \text{and} \quad f_2(\beta_2) \leqslant \delta'\right) \\
&= \bigvee_{\delta \in \Delta} (f_1^{-1}(\delta) \wedge f_2^{-1}(\delta)) \\
&= (f_1^{-1} \wedge f_2^{-1})(\alpha) \qquad \text{(by Lemma 3).}
\end{aligned}$$

Now if L is a completely distributive lattice with order reversing involution $'$, then so is L^X. Hence for every $D: L^X \to L^X$ in $\mathcal{Q}$ we can define D^{-1} as in Definition 6. Note that if $\mathcal{D}$ is a quasi-uniformity on X then so is $\mathcal{D}^{-1} = \{D^{-1} \mid D \in \mathcal{D}\}$. Also note that if $D \in \mathcal{Q}$ then $(D \cap D^{-1})^{-1} = D \cap D^{-1}$, that is, $D \cap D^{-1}$ is symmetric.

We are now able to define uniform spaces.

DEFINITION 7. A quasi-uniformity $\mathcal{D}$ is a uniformity if it also satisfies

(Q5) $D \in \mathcal{D}$ implies $D^{-1} \in \mathcal{D}$,

or equivalently

(Q5′) $\mathcal{D}$ has a base of symmetric elements.

We introduced the fuzzy unit interval in [3]. It is defined as follows.

DEFINITION 8. The fuzzy unit interval $[0, 1](L)$ is the set of all monotonic decreasing maps $\lambda: \mathbb{R} \to L$ for which

$$\lambda(t) = 1 \quad \text{for} \quad t < 0,$$
$$\lambda(t) = 0 \quad \text{for} \quad t > 0,$$

after the identification of $\lambda: \mathbb{R} \to L$ and $\mu: \mathbb{R} \to L$ if

$$\lambda(t-) = \mu(t-) \quad \text{for} \quad t \in \mathbb{R},$$

and

$$\lambda(t+) = \mu(t+) \quad \text{for} \quad t \in \mathbb{R}$$

where

$$\lambda(t+) = \sup_{s>t} \lambda(s), \text{ etc.}$$

We define a fuzzy topology on $[0, 1](L)$ as the topology generated by the subbase $\{L_t, R_t \mid t \in \mathbb{R}\}$, where

$$L_t : [0, 1](L) \to L$$

and

$$R_t : [0, 1](L) \to L$$

are defined by

$$L_t(\lambda) = \lambda(t-)',$$
$$R_t(\lambda) = \lambda(t+).$$

The fuzzy topology $\{R_t \mid t \in \mathbb{R}\}$ is called the right-hand topology.

We can construct a uniform structure on $[0, 1](L)$ as follows:

DEFINITION 9. We define $B_\epsilon : L^X \to L^X$ by

$$B_\epsilon(U) = R_{t-\epsilon} \quad \text{where } t \text{ is the greatest } s \in \mathbb{R} \text{ such that } U \subseteq L_s',$$
$$= \bigcap \{R_{s-\epsilon} \mid U \subseteq L_s'\}.$$

PROPOSITION 12. (1) B_ϵ *satisfies* (A1) *and* (A2).

(2) $B_\epsilon^{-1} = \bigcap \{L_{s+\epsilon} \mid U \subseteq R_s'\}$.

(3) $B_\epsilon \circ B_\delta \subseteq B_{\epsilon+\delta}$ (*so in particular* $B_\epsilon \circ B_\epsilon \subseteq B_{2\epsilon}$).

Proof. (1) (A1) is true since $L_s' \subseteq R_{s-\epsilon}$.

(A2) is true since

$$\begin{aligned}
U_\lambda \subseteq L'_{s_\lambda} &\Rightarrow \forall \delta > 0 \quad U_\lambda && \subseteq R_{s_\lambda - \delta} \\
&\Rightarrow \forall \delta > 0 \quad \bigcup U_\lambda && \subseteq \bigcup_\lambda R_{s_\lambda - \delta} \\
&\Rightarrow \forall \delta > 0 \quad \bigcup U_\lambda && \subseteq R_{\wedge s_\lambda - \delta} \\
&\Rightarrow \forall \delta > 0 \quad \bigcup U_\lambda && \subseteq R_{\wedge s_\lambda - \delta} \\
&\Rightarrow \quad \bigcup U_\lambda && \subseteq L'_{\wedge s_\lambda} \\
&\Rightarrow \quad B_\epsilon\left(\bigcup U_\lambda\right) && \subseteq R_{\wedge s_\lambda - \epsilon} \\
& && = \bigcup_\lambda B_\epsilon(U_\lambda)
\end{aligned}$$

and the other inclusion is trivial.

(2) $B_\epsilon^{-1}(U) = \bigcap\{V \mid B_\epsilon(V') \subseteq U'\}$
$= \bigcap\{L_t \mid B_\epsilon(L_t') \subseteq U'\}$
$= \bigcap\{L_t \mid R_{t-\epsilon} \subseteq U'\}$
$= \bigcap\{L_{t+\epsilon} \mid U \subseteq R_t'\}$
$= L_{t+\epsilon}$, where t is the smallest s such that $U \subseteq R_s'$.

(3) $B_\epsilon(B_\delta(U)) = B_\epsilon(R_{t-\delta})$, where t is the greatest s such that $U \subseteq L_s'$,
$= R_{t-\delta-\epsilon}$
$= B_{\epsilon+\delta}(U)$.

COROLLARY 13. *The set $\{B_\epsilon \mid \epsilon > 0\}$ is a basis for a quasi-uniformity which generates the right-hand topology.*

Proof. Every open set in the topology generated by $\{B_\epsilon\}$ is open in the usual right-hand topology since it is a union of sets of the form $B_\epsilon(U)$ $(=R_t$ for some $t)$.

Conversely R_t is in the topology generated by $\{B_\epsilon\}$ since $B_\epsilon(L_{t+\epsilon}) = R_t$ and $\bigcup_{\epsilon>0} L_{t+\epsilon} = R_t$.

COROLLARY 14. *The set $\{B_\epsilon, B_\epsilon^{-1} \mid \epsilon > 0\}$ is a sub-basis for a uniformity on* $[0, 1](L)$. *The topology generated by the uniformity is the usual (fuzzy) topology.*

This uniformity is called the usual uniformity for the usual fuzzy topology on $[0, 1](L)$.

We are now in a position to characterize uniformizability.

THEOREM 15. *Let $(X, \mathscr{D})$ be a uniform space and let $D \in \mathscr{D}$. Suppose $D(U) \subseteq V$. Then there exists a uniformly continuous function $f: X \to [0, 1](L)$ such that*

$$U(x) \leqslant f(x)(1-) \leqslant f(x)(0+) \leqslant V(x) \qquad \textit{for} \quad x \in X.$$

Proof. Construct fuzzy sets $\{A_r \mid r \in \mathbb{R}\}$ such that

(1) $A_r = X$ for $r < 0$,
(2) $A_r = \varnothing$ for $r > 1$,
(3) $A_0 = V$,
(4) $A_1 = U$

and symmetric elements $\{D_\epsilon \mid \epsilon > 0\}$ of the uniformity such that

$$D_\epsilon(A_r) \subseteq A_{r-\epsilon} \qquad \text{for} \quad r \in \mathbb{R}.$$

Since D_ϵ is symmetric we have

$$D_\epsilon(A_r') \subseteq A'_{r+\epsilon}.$$

Now define $f: X \to [0, 1](L)$ by $f(x)(r) = A_r(x)$. Clearly f is well defined and satisfies

$$U(x) \leqslant f(x)(1-) \leqslant f(x)(0+) \leqslant V(x) \qquad \text{for} \quad x \in X.$$

Hence we only need to show f is uniformly continuous. Clearly

$$f^{-1}(R_t) = \bigcup_{s>t} A_s \qquad \text{and} \qquad f^{-1}(L_t') = \bigcap_{s<t} A_s .$$

Hence

$$\begin{aligned} D_\epsilon(f^{-1}(L_t')) &\subseteq D_\epsilon(A_{t-\delta}) \qquad \text{for any} \quad \delta > 0, \\ &\subseteq A_{t-\delta-\epsilon} \\ &\subseteq \bigcup_{s>t} A_{s-2\delta-\epsilon} \\ &\subseteq f^{-1}(B_{\epsilon+2\delta}(L_t')). \end{aligned}$$

Letting $\delta = \frac{1}{2}\epsilon$ we have $D_\epsilon \subseteq f^{-1}(B_{2\epsilon})$. Similarly $D_\epsilon \subseteq f^{-1}(B_{2\epsilon}^{-1})$ and so f is uniformly continuous.

COROLLARY 16. *Let $(X, \mathscr{D})$ be a uniform space and U be an open set in the fuzzy topology generated by U. Then there exists a collection $\{W_\lambda\}$ of sets such that $\bigcup W_\lambda = U$ and continuous functions $f_\lambda : X \to [0, 1](L)$ such that*

$$W_\lambda(x) \leqslant f_\lambda(x)(1-) \leqslant f_\lambda(x)(0+) \leqslant U(x) \qquad \textit{for} \quad x \in X.$$

Proof. Since U is open then $U = \bigcup \{W \mid D(W) \subseteq U\}$. Apply the previous theorem.

Note. The condition that for any open set U there exist a collection of sets $\{W_\lambda\}$ and continuous functions as above is equivalent to complete regularity for the topological case (since $\{W_\lambda\}$ might as well be all singletons contained in U).

We use this as our definition of complete regularity for fuzzy topological spaces.

We now show the converse to Corollary 16 is true.

THEOREM 17. *Suppose (X, τ) is a completely regular fuzzy topological space. Then (X, τ) is uniformizable.*

Proof. The set $\{f^{-1}(B_\epsilon), f^{-1}(B_\epsilon^{-1}) \mid f: X \to [0, 1](L) \text{ is continuous and } \epsilon > 0\}$ forms a sub-base for a uniformity $\mathscr{D}$ since

(1) $f^{-1}(B_\epsilon) \circ f^{-1}(B_\delta) \subseteq f^{-1}(B_{\epsilon+\delta})$.

(2) $f^{-1}(B_\epsilon^{-1}) = f^{-1}(B_\epsilon)^{-1}$.

The uniformity induces the topology since

(1) Any continuous function $f: X \to [0, 1](L)$ is uniformly continuous in $(X, \mathscr{D})$ and so is continuous in the topology generated by $\mathscr{D}$ and so the topology induced by $\mathscr{D}$ is coarser than τ.

(2) However, suppose U is open in τ. We may find $\{W_\lambda\}$ such that $\bigcup W_\lambda = U$ and $f_\lambda : X \to [0, 1](L)$ such that

$$W_\lambda(x) \leqslant f_\lambda(x)(1-) \leqslant f_\lambda(x)(0+) \leqslant U(x).$$

Hence in particular $f_\lambda^{-1}(B_{1/2})(W_\lambda) \subseteq U$ and so $W_\lambda \subseteq \text{Int}(U)$ in the topology generated by $\mathscr{D}$. Thus U is open in the topology generated by $\mathscr{D}$.

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