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Extrapolation theory: new results and applications

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Abstract

Extending earlier work by Jawerth and Milman, we develop in detail $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods of extrapolation. As an application we prove general forms of Yano's extrapolation theorem. Applications to logarithmic Sobolev inequalities, integrability of maps of finite distortion and logarithmic Sobolev spaces are given.

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1. Introduction

Many problems in analysis can be formulated as the study of parametrized families of estimates for suitable operators. For each specific family of estimates it is usually of fundamental importance to determine the maximal range of the parameters for which those estimates are valid, and the corresponding analysis usually requires a deep understanding of the problem at hand. Interpolation methods allow us to create parametrized families of

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estimates from a given pair of initial estimates. Conversely, extrapolation methods allow us to extrapolate "end point'' results from a given family of estimates.

Following earlier work by Marcinkiewicz, Titchmarsh, Yano and others (cf. [17] for a historical perspective) a general theory of extrapolation was developed by Jawerth and Milman (cf. [17,21]). In [17] the $\Sigma^{(p)}$ and $\Delta^{(p)}$ extrapolation methods were introduced and applied to construct suitable "end point extrapolation spaces'' as well as to prove new extrapolation estimates. It was also shown, in a very general context, that the usual rearrangement inequalities for the classical operators of analysis are in fact equivalent to families of norm inequalities with a given rate of blow-up.

In [17] only the $\Sigma^{(1)}$ and $\Delta^{(\infty)}$ methods were studied extensively. The purpose of this paper is to provide a more extensive study of the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods of extrapolation for $p > 0$, [2] as well as to give new applications of these methods to classical analysis.

It is of interest to point out that the $\Sigma^{(2)}$ construction was independently considered by Donaldson and Sullivan in [12]. Indeed, in their work [12], Donaldson and Sullivan consider spaces of the form

$$\hat{L}^p(\Omega) = \left\{ f = \sum_{i=1}^{\infty} f_i : \sum_{i=1}^{\infty} \rho^{-i} \, \|f_i\|^2_{L^{p+\varepsilon^i}(\Omega)} < \infty \right\},$$

where Ω is a finite measure space, and ε, ρ are fixed numbers in the interval $(0, 1)$. Equipped with

$$\|f\|_{\hat{L}^p(\Omega)} = \inf \left\{ \left(\sum_{i=1}^{\infty} \rho^{-i} \, \|f_i\|^2_{L^{p+\varepsilon^i}(\Omega)} \right)^{1/2} : \; f = \sum_{i=1}^{\infty} f_i \right\},$$

$\hat{L}^p(\Omega)$ becomes a Banach space. In [12] this construction plays a crucial role: it allows the authors to construct the Sobolev spaces $\hat{L}^4_{1,c}(D)$, based on $\hat{L}^4(D)$, where D is a domain in R^4, with the crucial property

$$\hat{L}^4_{1,c}(D) \subset C^0_c(D). \tag{1}$$

The space "$\hat{L}^4(D)$'' depends on the choices of the parameters: indeed for (1) to hold the correct choice is to select $\rho < \varepsilon^{3/4}$ (cf. [12, Lemma 3.8]). The background of this choice of parameters is indeed an extrapolation result since the selection is achieved by a careful examination of the deterioration of the norm of the embeddings $W^{1,p}_0(D) \subset C^0(D)$ for $p = 4 + \varepsilon^i, i \to \infty$.

In [12] the precise identification of the $\hat{L}^4_{1,c}(D)$ spaces was not important; the authors just needed suitable spaces, where the crucial property (1) was valid, in order to develop their theory. In this paper, on the other hand, a good deal of our effort centers in the explicit computation of extrapolation spaces. In particular, as a consequence of our results, we show that for suitable choices of the parameters, consistent with the validity of (1), we can replace $\hat{L}^4_{1,c}(D)$ with the logarithmic Sobolev space $W^1_0 L^4(\log L)_b$, where $b > \frac{3}{4}$. To have a more explicit family of spaces simplifies some of the analysis in [12]. For example, the elliptic

[2] Some properties of $\Sigma^{(p)}$ method are studied in [27].

theory (cf. [12, Lemma 2.16]) follows from Sneiberg's extrapolation lemma for the real method (cf. [7,30]). [3]

A prototype of the Yano type extrapolation theorems that follow using the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods (cf. Section 5.1) is given by the following

Theorem 1.1. (i) *Let* $0 < s \leqslant 1$. *Let* $\|f\|_{L^{q,s}} = \left\{\frac{s}{q}\int_0^\infty [t^{1/q} f^*(t)]^s dt/t\right\}^{1/s}$, *and* $L^{q,\infty} := (L^s, L^\infty)_{\theta,\infty},\ 1/q = (1-\theta)/s$. *Suppose that* T *is a sublinear operator such that*

$$\|Tf\|_{L^{q,\infty}} \leqslant c(q-s)^{-a}\,\|f\|_{L^{q,s}}\,, \quad 0 < s < q < p,\ \ a > 0.$$

Then

$$T : L^s(\log L)_a + L^{p,s} \to L^s + L^{p,s}.$$

(ii) *Let* $L^{q,\infty} := (L^r, L^\infty)_{\theta,\infty},\ 1/q = (1-\theta)/r,\ r = \min(p,s)$. *Suppose that* T *is an operator such that*

$$\|Tf\|_{L^{q,\infty}} \leqslant cq^a\,\|f\|_{L^{q,s}}\,, \quad 0 < p \leqslant q < \infty,\ \ s > 0,\ a > 0.$$

Then

$$T : L^p \cap L^\infty \to L^r \cap L^\infty(\log L)_{-a}, \quad p \leqslant s$$

and

$$T : L^{p,\infty} \cap L^\infty \to L^{p,\infty} \cap L^\infty(\log L)_{-a}, \quad p > s.$$

In this paper we consider two type of applications. On the one hand we will exhibit spaces of current usage in analysis (e.g. "Lorentz–Zygmund spaces'', "Donaldson–Sullivan'' spaces, "Logarithmic Sobolev spaces'', etc.) as extrapolation spaces for the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods; while on the other we shall consider in detail specific classical operators (e.g. semigroups associated with the theory of logarithmic Sobolev inequalities) to develop concrete applications of extrapolation theory to classical analysis.

In Section 5.3 we apply extrapolation methods to study the modulus of continuity of maps of finite distortion (cf. [16]). Let $S = S(0,R)$ be the ball of radius R centered at the origin in R^n . Let $f = (f_1, f_2, \ldots, f_n) : S \to R^n$ be a map in the Sobolev class $W^{1,1}_{\mathrm{loc}}(S)$, and let $J(x,f) = \det\, Df(x)$ denote the Jacobian of f. We say that the map f is of finite distortion if there exists a real valued measurable function $K(x) \geqslant 1$ such that for all $x \in S$

$$|Df(x)|^n \leqslant K(x)J(x,f).$$

Let $WL^n =$ be the closure of L^n under the norm

$$\|u\|_{WL^n} := \sup_{0<s<1/2} s^{1/n}\,\|u\|_{L^n(\log L)_{-(s+1)/n}}\,.$$

[3] We shall discuss elsewhere the extension of Sneiberg's theorem to the context of extrapolation theory.

For $u \in WL^n$ we also let

$$\varpi_n(u,s) := s^{1/n} \|g\|_{L^n(\log L)_{-(s+1)/n}}.$$

Using extrapolation we give a slight extension of earlier results in [16] and show the following (cf. Theorem 5.8 below)

Theorem 1.2. *Let f be a map of finite distortion such that $|Df(x)| \in WL^n(S)$. Then f is continuous. Moreover, if $|x-y|$ is small and $x, y \in S(0, R/2)$, then there exists a constant $c = c(n,R)$ such that*

$$|f(x) - f(y)| \leqslant c\varpi_n\left(Df(x), \frac{1}{\ln|\ln|x-y||}\right).$$

It is instructive to consider a brief and informal comparison of interpolation and extrapolation methods. One version of the classical Lions–Peetre construction of interpolation spaces (the "J-method") can be described as follows. We are given an initial compatible pair of Banach spaces (X_0, X_1), and we consider those elements in $X_0 + X_1$ that can be represented by integrals (or sums)

$$f = \int_0^\infty u(s)\frac{ds}{s} \quad \text{in } X_0 + X_1,$$

in such a way that

$$\int_0^\infty (s^{-\theta}\|u(s)\|_{X_0 \cap sX_1})^p \frac{ds}{s} < \infty.$$

The norm of the element f in $(X_0, X_1)_{\theta,p;J}$ is given by

$$\|f\|^p_{(X_0,X_1)_{\theta,p;J}} = \inf\left\{\int_0^\infty (s^{-\theta}\|u(s)\|_{X_0\cap sX_1})^p \frac{ds}{s} : f = \int_0^\infty u(s)\frac{ds}{s} \quad \text{in } X_0 + X_1\right\}.$$

Note that we control the norm of f in terms of an average of the norms of the representing functions $u(s)$ in the intersection of the *original pair* (X_0, X_1). On the other hand, given a rate of decay $w(\theta)$ (typically $w(\theta) = \theta^\alpha(1-\theta)^\beta$), the extrapolation spaces $\Sigma^{(p)}w(\theta)X_\theta$, associated with a scale of spaces $\{X_\theta\}$, where $X_\theta \subset X_0 + X_1$ uniformly, consist of elements $f \in X_0 + X_1$ that can be represented by integrals (or sums)

$$f = \int_0^1 u_\theta \, d\theta \tag{2}$$

with

$$\int_0^1 (w(\theta)\|u_\theta\|_{X_\theta})^p \, d\theta < \infty. \tag{3}$$

Likewise the $\Delta^{(p)}$ spaces are a generalization of intersections. For example, suppose that Ω is σ-finite measure space and let $0 < r_0 < q < r \leqslant \infty$, $b > 0$, $0 < p \leqslant \infty$, and let $\Delta^{(p)}$

be the extrapolation space $\Delta^{(p)}((1/q-1/r)^{b-1/p}L^{q,p})$ with quasi-norm

$$\|f\| = \left\{ \int_{r_0}^{r} \left(\frac{1}{q}-\frac{1}{r}\right)^{bp-1} \|f\|_{L^{q,p}}^{p} \frac{dq}{q^2} \right\}^{1/p}.$$

The change of variables $\frac{1}{q}-\frac{1}{r}=\sigma$ and Fubini's theorem give

$$\begin{aligned} \|f\|^p &\approx \int_0^\infty [f^*(t)]^p t^{p/r} \int_0^{\sigma_0} e^{p\sigma \ln t} \sigma^{bp-1}\, d\sigma \\ &\approx \int_0^1 t^{p/r}(1-\ln t)^{-bp}[f^*(t)]^p \frac{dt}{t} + \int_1^\infty t^{p/r_0}(1+\ln t)^{-1}[f^*(t)]^p \frac{dt}{t}. \end{aligned}$$

Hence,

$$\Delta^{(p)}((1/q-1/r)^{b-1/p}L^{q,p}) = L^{r_0,p}(\log L)_{-1/p} \cap L^{r,p}(\log L)_{-b}. \tag{4}$$

This result corresponds to our Theorem 4.7 below for the case $a_0 = 1/p$ (cf. also Theorem 4.5 for more general results). The corresponding computations for the $\Sigma^{(p)}$ method are much more involved since we need to construct suitable decompositions (cf. Theorem 2.1). In [14] formula (4) is proved under the assumptions that Ω is a finite measure space, and $r = p$. The scale $\{L^{q,p}\}_{r_0<q<p}$ can be replaced by $\{L^q\}_{r_0<q<p}$ provided $q \to p$ in a specific way (cf. [14, p. 69], and our Theorems 3.4 and 3.1). The Σ characterization of $L(\log L)_b$ was first given in [17], while the $\Sigma^{(p)}$ characterization of the logarithmic Lorentz spaces $L^p(\log L)_b$, $1<p<\infty$, was given in [14] using (4) and duality arguments.

As illustrated by the Donaldson–Sullivan spaces mentioned above, representations of form (2) occur rather naturally in a number of problems in analysis. For a different perspective on how these representations arise when studying specific operators, let A be a given self-adjoint positive operator in L^2 (provided with Gaussian measure); moreover assume that the semigroup $P_t = e^{-tA}$, $t \geqslant 0$, generated by A, is an hypercontractive semigroup on L^p, $1<p<\infty$. More precisely, this means that for some constant $c>0$ we have

$$P_t : L^p \to L^p \text{ is bounded for all } t \geqslant 0, \quad 1<p<\infty \text{ and } \|P_t\|_p \leqslant ce^{-ct} \tag{5}$$

and

$$P_t : L^p \to L^{q(t)},\ q(t)-1 = e^t(p-1), \text{ is bounded uniformly for all } t \geqslant 0. \tag{6}$$

In the analysis of logarithmic Sobolev inequalities the following (fractional integral) operators

$$Q_z f = \int_0^\varepsilon \theta^{z-1} P_\theta f\, d\theta$$

play a crucial role. Now, from (5) and (6), it is not difficult to see that the study of these operators falls naturally into the scheme of Σ-extrapolation methods (for more details on how to implement this observation see Section 5.2).

Finally let us say a few words about how the calculation of $\Sigma^{(p)}$ and $\Delta^{(p)}$ spaces can be achieved. As we indicated above the computation of $\Delta^{(p)}$ spaces for K-spaces $(X_0, X_1)_{\theta,q}$, $q = p$, can be achieved directly by Fubini. Likewise the computation of $\Sigma^{(p)}$ for J-spaces in the case $q = p$ can also be achieved using a "Fubini type of argument'' in (3). However, the computation of these spaces for $q \neq p$ requires considerable more work. Therefore a great deal of our work in this paper goes into devising effective methods to compute weighted averages of norms.

In comparing our methods with those of [17] we note that in this paper we usually assume that the extrapolations occur "inside'' the interpolation scales. While the theory that we obtain is somewhat less general than the one in [17] [4] this extra assumption is automatically verified for many of the familiar scales of spaces we use in analysis. Thus, for example, while in [17] to extrapolate near L^1, say, we used $\{c(p)L^p\}$ scales for $p > 1$, in this paper we consider L^1 as an interpolation space between L^{p_0} and L^∞, for some $p_0 < 1$, and extrapolate using this information. In this fashion we are able to avoid the failure, at the end points, of the equivalence between the J and K methods of interpolation. In contrast, in [17], which deals with the cases $p = 1$ (resp. $p = \infty$), K-divisibility, or rather its equivalent formulations as strong forms of the fundamental lemma of interpolation theory, [5] is a crucial tool. We hope this simplification will make the reading of the paper easier for those readers who are not familiar with the deeper parts of real interpolation theory.

These choices, and the desire to keep the size of an already long paper under some control, also lead us not to develop, in the context of the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods, the corresponding theory of K/J inequalities (for a treatment in the case $p = 1$, $p = \infty$; cf. [17]). We hope to return to this subject elsewhere.

The paper is organized as follows. In Sections 2 and 3 we develop some basic facts about the $\Sigma^{(p)}$ and $\Delta^{(p)}$ methods of extrapolation focussing on effective tools for their computation. These results are illustrated with the explicit computations of extrapolation spaces in Section 4, where in particular we exhibit the Lorentz–Zygmund spaces as extrapolation spaces. The last part of the paper Section 5 is devoted to applications. In Section 5.1 the computations of Section 4 are used to prove extended forms of "Yano type'' extrapolation theorems. The last three sections are devoted to applications of our methods in other areas of analysis. Using extrapolation methods we prove some general forms of logarithmic Sobolev inequalities (Section 5.2) and improve on certain recent results concerning estimates for the modulus of continuity of maps of finite distortion (Section 5.3). Finally in Section 5.4 we investigate the relation with Donaldson–Sullivan spaces and their theory [12]. A preliminary version of the results was announced in [18].

To conclude this introduction we should mention a number of recent contributions to extrapolation theory, and its applications, that could be of interest to the reader: [8–10,13,15,19,23–25,28].

[4] The extra assumption that the extrapolations occur inside the interpolation scale is equivalent to the assumption that the interpolation scales considered in [17] can be extended.

[5] The original proof of the result can be found in [6]. The formulation given in [11] is particularly useful in extrapolation.

2. $\Sigma^{(p)}$ method of extrapolation

In this section we develop the $\Sigma^{(p)}$ method of extrapolation originally introduced in [17], but studied in detail only in the case $p = 1$.

2.1. Background

Let $\vec{A} = (A_0, A_1)$ be (a compatible) pair of quasi-Banach spaces, i.e. we suppose that A_0 and A_1 are quasi-Banach spaces continuously embedded in some quasi-Banach space Σ_A. For $0 < \theta < 1$, $0 < p \leqslant \infty$, we let $\vec{A}_{\theta,p}$ denote the real interpolation spaces of Lions and Peetre [5,6], provided with the K-method norm,

$$\|f\|_{\vec{A}_{\theta,p}} = \left\{ \int_0^\infty [s^{-\theta} K(s, f; \vec{A})]^p \frac{dt}{t} \right\}^{1/p}.$$

Let $0 \leqslant \theta_0 < \theta_1 \leqslant 1$ be fixed, and let Θ denote the interval (θ_0, θ_1). The K and J methods of interpolation give equivalent quasi-norms on $\vec{A}_{\theta,p}$, $\theta \in \Theta$. Moreover, if $0 < \theta_0 < \theta_1 < 1$, the equivalence of the K and J quasi-norms is uniform (cf. [5]).

Our characterization of extrapolation spaces as interpolation spaces requires spaces that fall outside the classical Lions–Peetre spaces. In particular, our characterization requires the replacement of power weights $t^{-\theta}$ by more general weights w. Note that given a weight w one can define in the familiar way the $\vec{A}_{w,p}$ and $\vec{A}_{w,p,J}$ spaces associated with the K and J methods (for a more systematic study see [3] and the references therein). The corresponding K and J norms are then given (respectively) by

$$\|f\|_{\vec{A}_{w,p}} = \left\{ \int_0^\infty \left[w(t) K(t, f; \vec{A}) \right]^p \frac{dt}{t} \right\}^{1/p}$$

and

$$\|f\|_{\vec{A}_{w,p;J}} = \inf \left\{ \left\{ \sum_{\nu=-\infty}^{\infty} [w(2^\nu) J(2^\nu, u_\nu; \vec{A})]^p \right\}^{1/p} : f = \sum_{\nu=-\infty}^{\infty} u_\nu \right\}.$$

We shall often assume that the weights $w(t)$ satisfy the following condition: There exist positive constants c_1, c_2, such that

$$c_1 w(2^\nu) \leqslant w(t) \leqslant c_2 w(2^\nu) \quad \text{for all} \quad 2^\nu \leqslant t \leqslant 2^{\nu+1}, \nu \in Z. \tag{7}$$

If (7) holds then we can "discretize'' the $\vec{A}_{w,p}$ norm (cf. [5, Lemma 3.1.3]), and obtain

$$\|f\|_{\vec{A}_{w,p}} = \left\{ \sum_{\nu=-\infty}^{\infty} [w(2^\nu) K(2^\nu, f; \vec{A})]^p \right\}^{1/p}.$$

Suppose that A_i satisfies a κ_i-triangle inequality, i.e.,

$$\|f + g\|_{A_i} \leqslant \kappa_i (\|f\|_{A_i} + \|g\|_{A_i}), \quad i = 0, 1.$$

The Aoki–Rolewicz Lemma (cf. [5, Lemma 3.10.1, p. 59]), provides us with equivalent quasi-norms which satisfy the γ-triangle inequality. In fact, let $\kappa = \max\{\kappa_0, \kappa_1\}$, and let γ be defined by

$$(2\kappa)^\gamma = 2, \tag{8}$$

then

$$\|f + g\|^\gamma \leqslant \|f\|^\gamma + \|g\|^\gamma . \tag{9}$$

Spaces satisfying (9) are called γ-Banach spaces. The largest possible γ for the pair $\vec{A} = (A_0, A_1)$ will be denoted by $\gamma_{\vec{A}}$.[6] Given $0 < p \leqslant \infty$ let p^+ be defined by

$$\frac{1}{p^+} = \begin{cases} \frac{1}{\gamma} - \frac{1}{p} & \text{if } p > \gamma, \\ 0 & \text{if } p \leqslant \gamma. \end{cases} \tag{10}$$

We also use the notation p^*, where $1/p^* + 1/p = 1$ if $1 < p \leqslant \infty$, and $p^* = \infty$ if $0 < p \leqslant 1$. We always have $p^+ \leqslant p^*$.

2.2. $\Sigma^{(p)}$ spaces

Let $M(\theta)$ be a positive continuous function on the interval $\Theta = (\theta_0, \theta_1)$, such that $\frac{1}{M(\theta)}$ is bounded. The $\Sigma^{(p)}$ sum of the scale $\{M(\theta)\vec{A}_{\theta,p}\}_{\theta\in\Theta}$ is defined by

$$\Sigma^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \Sigma^{(p)}_{\Theta}(M(\theta)\vec{A}_{\theta,p})$$
$$= \left\{ f \in \Sigma_A : f = \sum_{\theta\in\Theta} g(\theta), g(\theta) \in \vec{A}_{\theta,p} \text{ and } \|f\|_{\Sigma^{(p)}(M(\theta)\vec{A}_{\theta,p})} < \infty \right\},$$

where

$$\|f\|_{\Sigma^{(p)}(M(\theta)\vec{A}_{\theta,p})} = \inf \left\{ \left\{ \sum_{\theta\in\Theta} [M(\theta)\|g(\theta)\|_{\vec{A}_{\theta,p}}]^p \right\}^{1/p} : f = \sum_{\theta\in\Theta} g(\theta) \right\}.$$

Remark 2.1. We are using the *notation* of summation over uncountable sets. In this paper this should be understood as follows. Suppose that $N(\theta)$ is a continuous function on $\Theta = (\theta_0, \theta_1)$ such that $N(\theta) \to 0$ as $\theta \to \theta_0$ and $N(\theta) \to 0$ as $\theta \to \theta_1$. We fix a discretization say $\theta_n = \theta_0 + 2^{-n}$ if $n \geqslant n_1 > 0$, and $\theta_n = \theta_1 - 2^n$ if $n \leqslant n_0 < 0$, where n_0 and n_1 are chosen sufficiently large so that $\theta_1 - \theta_0 > 2^{-n_1} + 2^{n_0}$. Then

$$\sum_{\theta\in\Theta} N(\theta) := \sum_{n\in I} N(\theta_n),$$

where $I = I_+ \cup I_-$, and $I_+ = \{n \in Z : n \geqslant n_1\}$, $I_- = \{n \in Z : n \leqslant n_0\}$.

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[6] Evidently we always have $\gamma_{\vec{A}} \leqslant 1$ and $\gamma_{\vec{A}} = 1$ iff A_i, $i = 0, 1$, are normed spaces.

It is also convenient to use the notation

$$\sum_{\theta\in(\theta_0,\alpha]} N(\theta) := \sum_{n\in I_+} N(\theta_n),\ \alpha := \theta_0 + 2^{-n_1},$$

if $N(\theta) \to 0$ as $\theta \to \theta_0$, and

$$\sum_{\theta\in[\beta,\theta_1)} N(\theta) := \sum_{n\in I_-} N(\theta_n),\ \beta := \theta_1 - 2^{n_0},$$

if $N(\theta) \to 0$ as $\theta \to \theta_1$.

Remark 2.2. In the same fashion the $\Sigma^{(p)}$ construction can be applied to other compatible scales $\{A_\theta\}_{\theta\in\Theta}$ of quasi-Banach spaces, where by "compatible'' we mean scales such that there exists a constant $c > 0$ such that for all $\theta \in \Theta$ we have

$$\|f\|_{\Sigma_A} \leqslant c\,\|f\|_{A_\theta}.$$

Analogously, we can define "one sided'' $\Sigma^{(p)}$ spaces:

$$\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p}) = \Sigma^{(p)-}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p})$$

$$= \left\{ f \in \Sigma_A : f = \sum_{\theta\in(\theta_0,\alpha]} g(\theta),\ g(\theta) \in \vec{A}_{\theta,p} \text{ and } \|f\|_{\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p})} < \infty \right\},$$

where

$$\|f\|_{\Sigma^{(p)-}(M(\theta)\vec{A}_{\theta,p})} = \inf\left\{ \left\{ \sum_{\theta\in(\theta_0,\alpha]} [M(\theta)\|g(\theta)\|_{\vec{A}_{\theta,p}}]^p \right\}^{1/p} : f = \sum_{\theta\in(\theta_0,\alpha]} g(\theta) \right\}.$$

Likewise we let

$$\Sigma^{(p)+}(M(\theta)\vec{A}_{\theta,p}) = \Sigma^{(p)+}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p})$$

$$= \left\{ f \in \Sigma_A : f = \sum_{\theta\in[\beta,\theta_1)} g(\theta),\ g(\theta) \in \vec{A}_{\theta,p} \text{ and } \|f\|_{\Sigma^{(p)+}(M(\theta)\vec{A}_{\theta,p})} < \infty \right\},$$

where

$$\|f\|_{\Sigma^{(p)+}(M(\theta)\vec{A}_{\theta,p})} = \inf\left\{ \left\{ \sum_{\theta\in[\beta,\theta_1)} [M(\theta)\|g(\theta)\|_{\vec{A}_{\theta,p}}]^p \right\}^{1/p} : f = \sum_{\theta\in[\beta,\theta_1)} g(\theta) \right\}.$$

Remark 2.3. For any $\theta_0 < \alpha < \beta < \theta_1$ we have

$$\Sigma^{(p)}_{\theta_0,\theta_1}(M(\theta)\vec{A}_{\theta,p}) = \Sigma^{(p)-}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) + \Sigma^{(p)+}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}).$$

More generally, we have

Remark 2.4. Suppose the scale $\{A_\theta\}$ satisfies

$$A_\theta \subset A_\alpha + A_\beta, \alpha < \theta < \beta.$$

Then

$$\Sigma^{(p)}_{\theta_0,\theta_1}(M(\theta)A_\theta) = \Sigma^{(p)-}_{\theta_0,\alpha}(M(\theta)A_\theta) + \Sigma^{(p)+}_{\beta,\theta_1}(M(\theta)A_\theta).$$

Remark 2.5. When dealing with Banach pairs we can replace sums by integrals in the definition of the $\Sigma^{(p)}$ spaces. This corresponds to the familiar equivalence between the so-called "continuous" and "discrete" definitions of the J and K methods of interpolation. For future reference we discuss in more detail a special case of this equivalence. Suppose that $\vec{A} = (A_0, A_1)$ is a Banach pair and moreover suppose that for some small positive ε we have $\{\int_0^\varepsilon [M(\sigma)]^{-p^*} \frac{d\sigma}{\sigma}\}^{1/p^*} < \infty$, where $1/p^* + 1/p = 1$ if $p > 1$, and $p^* = \infty$ if $0 < p \leqslant 1$. Let us say that $f \in \int_{p,0,\varepsilon}(M(\sigma)\vec{A}_{\sigma+\theta_0,p})$ if and only if there exists a representation

$$f = \int_0^\varepsilon g(\sigma)\frac{d\sigma}{\sigma} \quad \text{with } \; g(\sigma) \in \vec{A}_{\sigma+\theta_0,p},$$

with

$$\int_0^\varepsilon [M(\sigma)\, \|g(\sigma)\|_{\vec{A}_{\sigma+\theta_0,p}}]^p \frac{d\sigma}{\sigma} < \infty.$$

Let

$$\|f\|_{\int_{p,0,\varepsilon}(M(\sigma)\vec{A}_{\sigma+\theta_0,p})} = \inf\left\{\left(\int_0^\varepsilon [M(\sigma)\, \|g(\sigma)\|_{\vec{A}_{\sigma+\theta_0,p}}]^p \frac{d\sigma}{\sigma}\right)^{1/p} : \; f = \int_0^\varepsilon g(\sigma)\frac{d\sigma}{\sigma}\right\}.$$

Suppose that $A_1 \subset A_0$.[7] Suppose in addition that $M(\sigma)$ is a positive, continuous function such that for some $c_1, c_2 > 0$,

$$c_1 M(2^{-n}) \leqslant M(\sigma) \leqslant c_2 M(2^{-n}) \quad \text{for all } \; 2^{-n} \leqslant \sigma \leqslant 2^{-n+1}, n \in I_+.$$

Then

$$\int_{p,0,\varepsilon}(M(\sigma)\vec{A}_{\sigma+\theta_0,p}) = \Sigma^{(p)-}_{(0,\varepsilon)}(M(\sigma)\vec{A}_{\sigma+\theta_0,p}).$$

The proof of this fact is analogous to the usual proof of the discretization of the J-method (cf. [5]).

[7] Such pairs are usually called "ordered" pairs.

2.3. Characterization of $\Sigma^{(p)}$ spaces

2.3.1. Banach case

In this section we show the following characterization of the $\Sigma^{(p)}$ spaces in terms of K and J spaces (cf. [17] for the case $p = 1$).

Theorem 2.1. *Let $\vec{A} = (A_0, A_1)$ be a Banach pair. Suppose that $0 < \theta_0 < \theta_1 < 1$, let Θ be the interval (θ_0, θ_1), and furthermore let $p > 0$. Define the weight w^* by*

$$\frac{1}{w^*(t)} = \left\{ \sum_{\theta \in \Theta} \left[\frac{t^\theta}{M(\theta)} \right]^{p^*} \right\}^{1/p^*}, \quad t > 0. \tag{11}$$

Then

$$\Sigma_\Theta^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{w^*,p} = \vec{A}_{w^*,p;J}, \quad p > 0.$$

Remark 2.6. If $\vec{A}$ is a Banach pair, $0 \leqslant \theta_0 < \theta_1 \leqslant 1$, then (cf. [17] for the case $p = 1$)

$$\Sigma_\Theta^{(p)}(M(\theta)\vec{A}_{\theta,p;J}) = \vec{A}_{w^*,p;J}, \quad p > 0.$$

Analogously we have

Remark 2.7. If $\vec{A}$ is a Banach pair, then

$$\int_{p,0,\varepsilon} (M(\sigma)\vec{A}_{\sigma+\theta_0,p;J}) = \vec{A}_{v,p;J},$$

where

$$\frac{1}{v(t)} = \left\{ \int_0^\varepsilon \left[\frac{t^{\sigma+\theta_0}}{M(\sigma)} \right]^{p^*} \frac{d\sigma}{\sigma} \right\}^{1/p^*}.$$

Theorem 2.1 follows from Theorems 2.2 and 2.3, and Remark 2.8. As it will be useful in what follows, these auxiliary embedding theorems are proved in the more general setting of quasi-Banach spaces.

We also note that Theorem 2.1 also holds, with the same proof, for the "one sided'' extrapolation spaces $\Sigma^{(p)-}$, $\Sigma^{(p)+}$.

Theorem 2.2. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair. Suppose that $0 \leqslant \theta_0 < \theta_1 \leqslant 1$, let Θ be the interval (θ_0, θ_1), let p^+ be defined as in (10), and let w be the weight defined by*

$$\frac{1}{w(t)} = \left\{ \sum_{\theta \in \Theta} \left[\frac{t^\theta}{M(\theta)} \right]^{p^+} \right\}^{1/p^+}, \quad t > 0. \tag{12}$$

Then

$$\Sigma_{\Theta}^{(p)}(M(\theta)A_{\theta,p}) \subset \vec{A}_{w,p}.$$

Analogously,

$$\Sigma_{\Theta}^{(p)}(M(\theta)A_{\theta,p;J}) \subset \vec{A}_{w,p;J}.$$

Proof. Recall the conventions of Remark 2.1: $\theta_n = \theta_0 + 2^{-n}$ if $n \in I_+$, $\theta_n = \theta_1 - 2^n$ if $n \in I_-$. Let $M_n = M(\theta_n)$ and let $f \in \Sigma^{(p)}(M_n \vec{A}_{\theta_n,p})$. Select a decomposition $f = \sum_{n\in I} g_n$, with $g_n \in \vec{A}_{\theta_n,p}$, and such that [8]

$$\|f\|_{\Sigma^{(p)}(M(\theta)\vec{A}_{\theta,p})} \approx \left\{\sum_{n\in I}\left[M_n\,\|g_n\|_{\vec{A}_{\theta_n,p}}\right]^p\right\}^{1/p}.$$

Using the Aoki–Rolewicz Lemma (cf. the discussion of (9) above), we derive

$$K(2^\nu, f; \vec{A})^\gamma \leqslant c\sum_{n\in I}\left[M_n 2^{-\nu\theta_n} K(2^\nu, g_n; \vec{A})\right]^\gamma \left[\frac{2^{\nu\theta_n}}{M_n}\right]^\gamma.$$

Therefore, using Hölder's inequality if $\gamma < p$ or the inclusion $l^p \subset l^\gamma$ if $\gamma \geqslant p$, we obtain

$$K(2^\nu, f; \vec{A}) \leqslant c\left\{\sum_{n\in I}\left[M_n 2^{-\nu\theta_n} K(2^\nu, g_n; \vec{A})\right]^p\right\}^{1/p} \frac{1}{w(2^\nu)}.$$

Thus

$$\begin{aligned}\|f\|_{\vec{A}_{w,p}} &\leqslant c\left\{\sum_{\nu=-\infty}^{\infty}\sum_{n\in I}[2^{-\nu\theta_n} M_n K(2^\nu, g_n; \vec{A})]^p\right\}^{1/p}\\ &\leqslant c\left\{\sum_{n\in I}[M_n\|g_n\|_{\vec{A}_{\theta_n,p}}]^p\right\}^{1/p}\\ &\leqslant c\,\|f\|_{\Sigma^{(p)}(M(\theta)\vec{A}_{\theta,p})}. \qquad \square\end{aligned}$$

Theorem 2.3. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair. Suppose that $0 \leqslant \theta_0 < \theta_1 \leqslant 1$, let Θ be the interval (θ_0, θ_1) and let w^* be the weight defined by (11). Then*

$$\vec{A}_{w^*,p;J} \subset \Sigma_{\theta\in\Theta}^{(p)}(M(\theta)A_{\theta,p;J}).$$

Proof. Let $f \in A_{w^*,p,J}$. Represent $f = \sum_\nu u_\nu$ with

$$\|f\|^p_{\vec{A}_{w^*,p;J}} \leqslant \sum_\nu [w^*(2^\nu) J(2^\nu, u_\nu; \vec{A})]^p.$$

[8] In what follows we use the symbol $A \approx B$ to indicate that A and B are equivalent modulo constants.

Let $\phi_{n,v} = \frac{w^*(2^v)2^{v\theta_n}}{M(\theta_n)}$, then $\sum \phi_{n,v}^{p^*} = 1$ if $p > 1$. If on the other hand $0 < p \leqslant 1$, then $\sup_{n\in I} \phi_{n,v} = 1$, and therefore in this case we can find $m(v)$ such that $\phi_{m(v),v} \approx 1$. To show that $f \in \Sigma_{\Theta}^{(p)}(M(\theta)\vec{A}_{\theta,p;J})$, we now exhibit a suitable representation of f. We define a partition of the unity as follows

$$\psi_{n,v} = \phi_{n,v}^{p^*} \quad \text{if } \ p > 1$$

and

$$\psi_{n,v} = \delta_{n,m(v)} \quad \text{if } \ p \leqslant 1,$$

where $\delta_{n,m}$ stands for the delta Kroenecker index. In either case we have $\Sigma_{n\in I}\psi_{n,v} = 1$. Let $g_n = \sum_{v=-\infty}^{\infty} u_v\psi_{n,v}$, then $f = \Sigma_{n\in I}\, g_n$, and moreover

$$\begin{aligned}\|f\|^p_{\Sigma_{\Theta}^{(p)}(M(\theta)\vec{A}_{\theta,p;J})} &\leqslant c\sum_{n\in I}[M(\theta_n)\|g_n\|_{\vec{A}_{\theta_n,p;J}}]^p \\ &\leqslant c\sum_{n\in I}\sum_{v=-\infty}^{\infty}[2^{-v\theta_n}M(\theta_n)J(2^v,u_v\psi_{n,v};\vec{A})]^p \\ &\leqslant c\sum_{v=-\infty}^{\infty}[J(2^v,u_v;\vec{A})]^p\sum_{n\in I}[M(\theta_n)2^{-v\theta_n}\psi_{n,v}]^p.\end{aligned}$$

Using the definition of $\phi_{n,v}$ we get

$$\|f\|^p_{\Sigma_{\Theta}^{(p)}(M(\theta)\vec{A}_{\theta,p;J})} \leqslant c\sum_{v=-\infty}^{\infty}[w^*(2^v)J(2^v,u_v;\vec{A})]^p\sum_{n\in I}\left[\frac{\psi_{n,v}}{\phi_{n,v}}\right]^p.$$

Observe that for $p > 1$ we have

$$\left\{\sum_{n\in I}\left[\frac{\psi_{n,v}}{\phi_{n,v}}\right]^p\right\}^{1/p} = \left\{\sum_{n\in I}[\phi_{n,v}]^{p^*}\right\}^{1/p} = 1,$$

while if $p \leqslant 1$,

$$\left\{\sum_{n\in I}\left[\frac{\psi_{n,v}}{\phi_{n,v}}\right]^p\right\}^{1/p} \approx 1.$$

Therefore

$$\|f\|^p_{\Sigma_{\Theta}^{(p)}(M(\theta)\vec{A}_{\theta,p;J})} \leqslant c\sum_{v=-\infty}^{\infty}[w^*(2^v)J(2^v,u_v;\vec{A})]^p \leqslant c\|f\|^p_{\vec{A}_{w^*,p;J}},$$

as we wished to show. □

Remark 2.8. Let $0 < \theta_0 < \theta_1 < 1$, $p > 0$, and let w be the weight defined by the formula (12). Then

$$\vec{A}_{w,p;J} = \vec{A}_{w,p}.$$

Proof. We first show the embedding $\vec{A}_{w,p} \subset \vec{A}_{w,p;J}$. Note that w is monotone, and satisfies

$$c_1 t^{-\theta_0} < w(t) < c_2 t^{-\theta_1} \quad \text{if } 0 < t < 1,$$

and

$$c_1 t^{-\theta_1} < w(t) < c_2 t^{-\theta_0} \quad \text{if } t > 1.$$

It follows that if $f \in \vec{A}_{w,p}$ then $K(t, f) \to 0$ as $t \to 0$, and $\frac{K(t,f)}{t} \to 0$ as $t \to \infty$. Therefore we can apply the fundamental lemma of interpolation theory in the usual fashion (cf. [5]) to establish that $f \in \vec{A}_{w,p;J}$ with norm estimates.

We now show that, conversely, $\vec{A}_{w,p;J} \subset \vec{A}_{w,p}$. For Banach pairs $\vec{A}$, $p \geqslant 1$, it is well known (cf. [3,6]) that the embedding $\vec{A}_{w,p;J} \subset \vec{A}_{w,p}$ is equivalent to the boundedness of the Calderón operator

$$Sf(t) = \int_0^\infty \min\left\{1, \frac{t}{s}\right\} f(s)\frac{ds}{s} = \int_0^\infty \min\{1, u\} f\left(\frac{t}{u}\right)\frac{du}{u},$$

on $L^p(w(t)^p \frac{dt}{t})$ [9]. More generally, if $\vec{A}$ is a quasi-Banach pair, and $p > 0$, an analogous characterization holds using the following discrete version of the Calderón operator (with $\gamma \leqslant \min\{p, \gamma_{\vec{A}}\}$):

$$S_\gamma(\{f_\mu\}) = \left\{\sum [\min\{1, 2^{\mu-\nu}\}|f_\nu|]^\gamma\right\}^{1/\gamma}.$$

It then follows readily that $\vec{A}_{w,p;J} \subset \vec{A}_{w,p}$ (cf. [5, Theorem 3.11.3]) iff the Calderón operator S_γ is bounded on the sequence space $l^p([w(2^\mu)]^p)$.

To prove that w is a Calderón weight we write

$$\frac{1}{w(t)} = \left\{\sum_{\theta\in\Theta}\left[\frac{\alpha^{-\theta}\alpha^{\theta}t^{\theta}}{M(\theta)}\right]^{p^+}\right\}^{1/p^+}.$$

Then we see that

$$w(\alpha t) \leqslant \begin{cases} \alpha^{-\theta_0} w(t), & \alpha \geqslant 1, \\ \alpha^{-\theta_1} w(t) & \alpha < 1. \end{cases} \tag{13}$$

Combining (13) with Minkowski's inequality we get

$$\begin{aligned} \left\| S_\gamma(\{f_\mu\})\right\|^\gamma_{l^p([w(2^\mu)]^p)} &= \left\{\sum |S_\gamma(\{f_\mu\}) w(2^\mu)|^p\right\}^{\gamma/p} \\ &\leqslant \sum \min\{1, 2^\nu\}^\gamma \left\{\sum \left[\left|f_\mu\right| w(2^{\mu+\nu})\right]^p\right\}^{\gamma/p} \\ &\leqslant \left(\sum \min\{1, 2^\nu\}^\gamma \max\{2^{-\nu\theta_0}, 2^{-\nu\theta_1}\}^\gamma\right) \left\{\sum \left[|f_\mu| w(2^\mu)\right]^p\right\}^{\gamma/p} \\ &\leqslant c \left\|\{f_\mu\}\right\|^\gamma_{l^p([w(2^\mu)]^p)}, \end{aligned}$$

as desired. □

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[9] Such weights are called Calderón weights in [3].

Theorem 2.1 now follows by combining Theorems 2.2, 2.3 and Remark 2.8.

2.3.2. Quasi-Banach case

We start summarizing the results of the previous section that hold for quasi-Banach spaces.

Theorem 2.4. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair. Suppose that $0 < \theta_0 < \theta_1 < 1$, let Θ be the interval (θ_0, θ_1), and furthermore let $p > 0$. If $0 < p \leqslant \gamma_{\vec{A}}$ then*

$$\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{w^*,p}.$$

If $p > \gamma_{\vec{A}}$, then we have the inclusions

$$\vec{A}_{w^*,p} \subset \Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) \subset \vec{A}_{w,p}.$$

It turns out that if the growth of $M(\theta)$ is tempered in a suitable sense then the weights w and w^* are equivalent. As a consequence we shall be able to show that

$$\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{w,p} = \vec{A}_{w^*,p}. \tag{14}$$

We shall say that a positive continuous function $N(\sigma)$, defined on the interval $(0, 1)$, is "tempered'' in the sense of [17] if there exist $\varepsilon_1 > 0$, $\varepsilon_2 > 0$ such that

$$\begin{aligned} &N(\sigma/2) \approx N(\sigma) \ \text{ for } \ \sigma \in (0, \varepsilon_1) \ \text{ and} \\ &N((1+\sigma)/2) \approx N(\sigma) \ \text{for} \ \sigma \in (1-\varepsilon_2, 1). \end{aligned} \tag{15}$$

Since on any compact subinterval $N(\sigma) \approx 1$ the equivalence in (15) is fulfilled on the whole interval $(0, 1)$. Further, we say that a positive continuous function $N(\theta)$, defined on the interval $(\theta_0, \theta_1) \subset (0, 1)$, is tempered if $\sigma \to N(\sigma + \theta_0)$ and $\sigma \to N(\theta_1 - \sigma)$ are tempered for σ near zero (i.e. $\sigma \sim 0$.) Finally, N is strictly tempered if for some constants $0 < c < d < 1$

$$cN(\sigma) \leqslant N(\sigma/2) \leqslant dN(\sigma), \quad 0 < \sigma < \sigma_0. \tag{16}$$

For example, the function $N(\sigma) := \sigma^a(1 + |\log\sigma|)^b$ is strictly tempered if $a > 0$ and b are arbitrary real numbers. On the other hand, the function $N_0(\sigma) := (\log 1/\sigma)^{-b}$, $b > 1$ is tempered but not strictly tempered.

Suppose that $N(\sigma)$ is strictly tempered. To prove that the weights w^* and w are equivalent it is sufficient to verify that the following two sequences are equivalent for large positive v:

$$f_v := \sum_{n>n_0} 2^{-v\sigma_n} N(\sigma_n), \tag{17}$$

$$g_v := \sup_{n>n_0} 2^{-v\sigma_n} N(\sigma_n), \tag{18}$$

where $\sigma_n = 2^{-n}$, $N(\sigma_n) := \frac{1}{M(\theta_0+\sigma_n)}$. In other words we will show that w^* and w are equivalent to the weight

$$\tilde{w}(t) := \sup_{n\in I} \frac{t^{\theta_n}}{M(\theta_n)}.$$

Theorem 2.5. *If $N(\sigma)$ is strictly tempered, then the sequences f_v and g_v defined by* (17) *and* (18) *are equivalent. In other words, if $N(\sigma)$ is strictly tempered then w^* and w are equivalent.*

Proof. First we shall prove that if $\{g_v\}$ is given by (18) then

$$g_v \approx 2^{-v\sigma_{m(v)}} N(\sigma_{m(v)}), \tag{19}$$

where $m(v) := [\log v]$, $\log = \log_2$. To establish (19) it is enough to prove the upper estimate. Let

$$s_n := [-\log N(\sigma_n)], \quad l_n := v\sigma_n + s_n, \quad k := [\log v].$$

Then

$$g_v \approx 2^{-\inf l_n}.$$

Since $2^k \leqslant v < 2^{k+1}$ we have

$$2^{k+1-n} + s_n > l_n \geqslant 2^{k-n} + s_n;$$

in particular,

$$2 + s_k > l_k \geqslant 1 + s_k.$$

Hence

$$l_{k+j} \geqslant 2^{-j} + s_{k+j}, \quad j > 1,$$

and

$$l_{k-j} \geqslant 2^j + s_{k-j}, \quad 1 \leqslant j < k - n_0.$$

On the other hand,

$$c^j N(\sigma_{n-j}) = c^j N(2^j \sigma_n) \leqslant N(\sigma_n) \leqslant d^j N(\sigma_{n-j}) = d^j N(2^j \sigma_n).$$

Then

$$j \log c + \log N(\sigma_{n-j}) \leqslant \log N(\sigma_n) \leqslant j \log d + \log N(\sigma_{n-j}),$$

and, using $[a] + [b] \leqslant [a+b] < [a] + [b] + 2$, we get

$$s_{n-j} - 2 - j \log d \leqslant s_n \leqslant s_{n-j} + 1 - j \log c.$$

In particular, since $d < 1$,

$$s_{k+j} \geqslant s_k - 2,$$

and

$$s_{k-j} \geqslant s_k - 1 + j \log c > s_k - 2^j - a_1$$

for some positive constant a_1. Then

$$l_{k+j} > l_k - 4, \;\; l_{k-j} > l_k - 2 - a_1.$$

Hence

$$\inf l_n > l_k - a$$

or

$$g_v < c2^{-l_k}.$$

This implies

$$g_v < c2^{-v\sigma_k} N(\sigma_k),$$

proving (19).

Next we notice that if $\varepsilon > \log d$ and $\varepsilon \neq 0$, then the functions

$$\tilde{N}(\sigma) := \sigma^{\varepsilon} N(\sigma)$$

are also strictly tempered, therefore the previous argument applied to the sequence

$$\tilde{g}_v := \sup_{n > n_0} 2^{-v\sigma_n} \tilde{N}(\sigma_n),$$

yields

$$\tilde{g}_v \approx 2^{-v\sigma_{m(v)}} \tilde{N}(\sigma_{m(v)}).$$

Therefore for $0 < \varepsilon < \log 1/d$,

$$\begin{aligned}\sum_{n > m(v)} \sigma_n^{\varepsilon} 2^{-v\sigma_n} N(\sigma_n) \sigma_n^{-\varepsilon} &< c\sigma_{m(v)}^{-\varepsilon} 2^{-v\sigma_{m(v)}} N(\sigma_{m(v)}) \sum_{n > m(v)} \sigma_n^{\varepsilon} \\ &< c\sigma_{m(v)}^{-\varepsilon} 2^{-v\sigma_{m(v)}} N(\sigma_{m(v)}) \sigma_{m(v)}^{\varepsilon} \\ &< cg_v.\end{aligned}$$

Similarly,

$$\begin{aligned}\sum_{n < m(v)} \sigma_n^{-\varepsilon} 2^{-v\sigma_n} N(\sigma_n) \sigma_n^{\varepsilon} &< c\sigma_{m(v)}^{\varepsilon} 2^{-v\sigma_{m(v)}} N(\sigma_{m(v)}) \sum_{n < m(v)} \sigma_n^{-\varepsilon} \\ &< c\sigma_{m(v)}^{\varepsilon} 2^{-v\sigma_{m(v)}} N(\sigma_{m(v)}) \sigma_{m(v)}^{-\varepsilon} \\ &< cg_v.\end{aligned}$$

It follows that

$$f_\nu < c g_\nu,$$

and the desired result is proven. □

Remark 2.9. If N is tempered but not strictly tempered then w and $\tilde{w}$ may not be equivalent. We give a counterexample. Let $N_0(\sigma) := (\log 1/\sigma)^{-b}$, $b > 1$. Then N is tempered, but not strictly tempered. It is easy to see that in this case

$$f_\nu := \sum_{n > n_0} 2^{-\nu\sigma_n} N_0(\sigma_n)$$

satisfies

$$f_\nu \approx (\log \nu)^{1-b}.$$

On the other hand,

$$g_\nu := \sup_{n > n_0} 2^{-\nu\sigma_n} N_0(\sigma_n)$$

satisfies

$$g_\nu \approx (\log \nu)^{-b}.$$

Therefore $\tilde{w}$ and w are not equivalent. Moreover, we also have that w^* and w are not equivalent since (say $\theta_0 = 0$) we have $w^*(2^{-\nu}) \approx (\log \nu)^{b-1/p^*}$, $w(2^{-\nu}) \approx (\log \nu)^{b-1/p^+}$ for $b > 1/p^+$.

For weights of type w^* we have the following result

Theorem 2.6. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair. Suppose that $0 < \theta_0 < \theta_1 < 1$, let Θ be the interval (θ_0, θ_1), $\vec{B} = (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})$ and let $p = \gamma_{\vec{B}}$. Then*

$$\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{w^*,p}.$$

Proof. By Theorem 2.3 it suffices to show that $\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) \subset \vec{A}_{w^*,p}$. By reiteration we see that $\vec{A}_{w^*,p}$ is a p-Banach space. Let $f \in \Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p})$, and select a decomposition $f = \sum f_\theta$ with

$$\sum [M(\theta)\|f_\theta\|_{\vec{A}_{\theta,p}}]^p \approx \|f\|^p_{\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p})}.$$

Since $\vec{A}_{w^*,p}$ is p-Banach and $w^*(t) \leqslant M(\theta)t^{-\theta}$, we get

$$\|f\|^p_{\vec{A}_{w^*,p}} \leqslant \sum \|f_\theta\|^p_{\vec{A}_{w^*,p}} \leqslant \sum [M(\theta)\|f_\theta\|_{\vec{A}_{\theta,p}}]^p \approx \|f\|^p_{\Sigma^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p})}. \qquad \square$$

Remark 2.10. Let $0 < \theta_0 < \theta_1 < 1$, $\vec{B} = (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})$. If either $0 < p \leqslant \gamma_{\vec{A}}$, or $p = \gamma_{\vec{B}}$, then

$$\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,p}) = \vec{A}_{\theta_0,p} + \vec{A}_{\theta_1,p}.$$

Proof. By Theorem 2.4 or Theorem 2.6

$$\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,p}) = \vec{A}_{w^*,p},$$

where

$$w^*(t) = \begin{cases} t^{-\theta_0} & \text{if } t \in (0,1) \\ t^{-\theta_1} & \text{if } t > 1 \end{cases}.$$

Thus

$$\|f\|_{\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,p})} \approx \left\{ \int_0^1 [t^{-\theta_0} K(t,f;\vec{A})]^p \frac{dt}{t} + \int_1^\infty [t^{-\theta_1} K(t,f;\vec{A})]^p \frac{dt}{t} \right\}^{1/p}.$$

Therefore, by Holmstedt's formula,

$$\|f\|_{\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,p})} \approx \|f\|_{\vec{A}_{\theta_0,p}+\vec{A}_{\theta_1,p}}. \qquad \square$$

Remark 2.11. We now give an example showing that for $p > \gamma_{\vec{B}}$ the space $\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,p})$ can be strictly larger than $\vec{A}_{\theta_0,p} + \vec{A}_{\theta_1,p}$. Indeed, let

$$g_n(t) = t^{-1/q_n}(1 - \ln t)^{-\alpha} n^{-\alpha} h_n(t), \quad 0 < t < 1,$$

where h_n is the characteristic function of the interval $(\sigma_n, \sqrt{\sigma_n})$, $\sigma_n = 2^{-n}$, $\frac{1}{p} < \alpha \leqslant \frac{1}{2p} + \frac{1}{2}$, $p > 1$, $\frac{1}{q_n} = \frac{1}{p} - \frac{1}{2^n}$. Then $(L^1, L^\infty)_{2^{-n}+\theta_0,p} = L^{q_n,p}$ (uniformly w.r.t. $n > 2$), and $\|g_n\|_{L^{q_n,p}} \leqslant cn^{-\alpha p}$. Hence

$$f(t) := \Sigma g_n(t) \in \Sigma^{(p)}_{(0,\varepsilon)}(L^1, L^\infty)_{2^{-n}+\theta_0,p}.$$

On the other hand, for $t \sim 0$ we have

$$f(t) \approx t^{-1/p}(1 - \ln t)^{1-2\alpha}.$$

Hence

$$f \notin L^p \text{ and in fact } f \notin L^p(\log L)_{-1/p} \text{ for } \alpha = 1/2,\ p > 2.$$

Remark 2.12. Let $\vec{B} = (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})$. If $0 < p \leqslant \gamma_{\vec{B}} \leqslant q < \infty$ and $0 < \theta_0 < \theta_1 < 1$ then

$$\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,q}) = \vec{A}_{\theta_0,q} + \vec{A}_{\theta_1,q}.$$

Proof. According to (46) below,

$$\vec{A}_{\theta,q} = \langle (B_0, B_1)_{\sigma,q} \rangle \ , \theta = (1-\sigma)\theta_0 + \sigma\theta_1.$$

Therefore

$$\vec{A}_{\theta,q} \subset B_0 + B_1.$$

Since $p \leqslant \gamma_{\vec{B}}$, we readily see that

$$\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,q}) \subset \vec{A}_{\theta_0,q} + \vec{A}_{\theta_1,q}.$$

Conversely, we notice that for all $\eta \in \Theta$,

$$\vec{A}_{\eta,q} \subset \Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,q})$$

with

$$\|f\|_{\Sigma^{(p)}_{\theta\in\Theta}(\vec{A}_{\theta,q})} \leqslant \|f\|_{\vec{A}_{\eta,q}} . \tag{20}$$

Since $q < \infty$ there is no loss of generality if we assume that $f \in A_0 \cap A_1$. Then from $K(t, f; \vec{A}) \leqslant \min\{1, t\} \|f\|_{A_0 \cap A_1}$, we immediately deduce that

$$t^{-\eta} K(t, f; \vec{A}) \leqslant h(t)$$

where $h(t) = t^{1-\theta_1}\chi_{(0,1)}(t) + t^{-\theta_0}\chi_{(1,\infty)}(t)$. Therefore we can take limits in (20) using dominated convergence. □

2.4. *Reiteration and $\Sigma^{(p)}$ spaces*

2.4.1. *Formula for the quasi-norm in the sum $\vec{A}_{w_0,p} + \vec{A}_{w_1,p}$*

As we already know from Theorems 2.1, 2.3, 2.6, the $\Sigma^{(p)}$ space can be identified with the sum

$$\vec{A}_{w_0,p} + \vec{A}_{w_1,p},$$

where

$$\frac{1}{w_0(t)} = \left\{ \sum_{\theta_0 < \theta \leqslant \alpha} \left[\frac{t^\theta}{M(\theta)} \right]^{p^+} \right\}^{1/p^+}$$

and

$$\frac{1}{w_1(t)} = \left\{ \sum_{\beta \leqslant \theta < \theta_1} \left[\frac{t^\theta}{M(\theta)} \right]^{p^+} \right\}^{1/p^+} .$$

The pair of weights $\{w_0, w_1\}$ satisfy the following properties:

$$w_0(t) \leqslant c w_1(t) \ \text{ if } 0 < t < 1; \quad w_1(t) \leqslant c w_0(t) \ \text{ if } t > 1, \tag{21}$$

$$\left\{ \int_0^\infty [\min(1,t) w_j(t)]^p \frac{dt}{t} \right\}^{1/p} < \infty, \quad j = 0, 1. \tag{22}$$

We now show that when computing the quasi-norm of the sum space

$$\vec{A}_{w_0,p} + \vec{A}_{w_1,p},$$

the values of $w_0(t)$ are important only when t is in the range $0 < t < 1$, while $w_1(t)$ is relevant only when $t > 1$. This is easy to see using a variant of Holmstedt's formula for the $\vec{A}_{w_j,p}$ spaces, where w_j are Calderón weights (compare with formula (3.9.8) in [6]).

Theorem 2.7. *Let* $B_j = \vec{A}_{w_j,p}$ *and let* $K(t, f) = K(t, f; \vec{A})$.
(i) *Suppose the weights* $\{w_0, w_1\}$ *satisfy* (21), *then*

$$\left\{ \int_0^1 [w_0(t) K(t,f)]^p \frac{dt}{t} + \int_1^\infty [w_1(t) K(t,f)]^p \frac{dt}{t} \right\}^{1/p} \leqslant c K(1, f; \vec{B}). \tag{23}$$

(ii) *Suppose the weights* $\{w_0, w_1\}$ *satisfy* (22), *then*

$$\left\{ \int_0^1 [w_0(t) K(t,f)]^p \frac{dt}{t} + \int_1^\infty [w_1(t) K(t,f)]^p \frac{dt}{t} \right\}^{1/p} \geqslant c K(1, f; \vec{B}). \tag{24}$$

Proof. Although the proof of (23), (24) is a standard modification of Holmstedt's proof (cf. [5]) we shall give the details for the sake of completeness. Let $f = f_0 + f_1,\ f_j \in B_j,\ j = 0, 1$. Then by (H_1) we get:

$$I^p := \int_0^1 [w_0(t) K(t,f)]^p \frac{dt}{t} \leqslant c[\| f_0 \|_{B_0}^p + \| f_1 \|_{B_1}^p]$$

and

$$J^p := \int_1^\infty [w_1(t) K(t,f)]^p \frac{dt}{t} \leqslant c[\| f_0 \|_{B_0}^p + \| f_1 \|_{B_1}^p],$$

and (23) follows. Suppose that the weights $\{w_0, w_1\}$ satisfy (H_2). Let $f \in A_0 + A_1$ and select a decomposition $f = f_0 + f_1$ such that $K(1, f) \approx \| f_0 \|_{A_0} + \| f_1 \|_{A_1}$. Then

$$K(s, f_0) \leqslant \| f_0 \|_{A_0} \leqslant c K(1, f), \ \ K(s, f_1) \leqslant s \| f_1 \|_{A_1} \leqslant cs K(1, f).$$

By the quasi-triangle inequality,

$$[K(1, f; \vec{B})]^p \leqslant c[\| f_0 \|_{B_0}^p + \| f_1 \|_{B_1}^p]$$

and

$$\begin{aligned} c\| f_0 \|_{B_0}^p \leqslant & \int_0^1 [w_0(s) K(s,f)]^p \frac{ds}{s} + \int_0^1 [w_0(s) K(s, f_1)]^p \frac{ds}{s} \\ & + \int_1^\infty [w_0(s) K(s, f_0)]^p \frac{ds}{s}. \end{aligned}$$

From

$$\int_0^1 [w_0(t)K(t,\, f_1)]^p \frac{dt}{t} \leqslant c \int_0^1 [sw_0(s)]^p \frac{ds}{s} [K(1,\, f)]^p \leqslant c \int_0^1 [w_0(t)K(t,\, f)]^p \frac{dt}{t}$$

and

$$\int_1^\infty [w_0(t)K(t,\, f_0)]^p \frac{dt}{t} \leqslant c \int_1^\infty [w_0(s)]^p \frac{ds}{s} [K(1,\, f)]^p \leqslant c \frac{\int_1^\infty [w_0(t)]^p \frac{dt}{t}}{\int_0^1 [tw_0(t)]^p \frac{dt}{t}} I^p,$$

we obtain

$$c\| f_0\|_{B_0} \leqslant I.$$

Analogously,

$$c\| f_1\|_{B_1}^p \leqslant \int_0^1 [w_1(s)K(s,\, f_1)]^p \frac{ds}{s} + \int_1^\infty [w_1(s)K(s,\, f)]^p \frac{ds}{s} + \int_1^\infty [w_1(s)K(s,\, f_0)]^p \frac{ds}{s}.$$

Since

$$\int_1^\infty [w_1(t)K(t,\, f_0)]^p \frac{dt}{t} \leqslant c \int_1^\infty [w_1(s)]^p \frac{ds}{s} [K(1,\, f)]^p \leqslant c \int_1^\infty [w_1(t)K(t,\, f)]^p \frac{dt}{t}$$

and

$$\int_0^1 [w_1(t)K(t,\, f_1)]^p \frac{dt}{t} \leqslant c \int_0^1 [sw_1(s)]^p \frac{ds}{s} [K(1,\, f)]^p \leqslant c \frac{\int_0^1 [tw_1(t)]^p \frac{dt}{t}}{\int_1^\infty [w_1(t)]^p \frac{dt}{t}} J^p,$$

we obtain

$$c\| f_1\|_{B_1} \leqslant J.$$

The proof is complete. □

In our applications we shall need a variant of Theorem 2.7. Consider the weights $u_j(t) = t^{-\theta_j}(1+|\ln t|)^{c_j},\ 0 < \theta_0 < \theta_1 < 1,\ c_j \in \mathbf{R},\ \ j = 0, 1$. These special weights satisfy the following conditions:

$$\int_0^1 \left[\frac{u_0(t)}{u_1(t)}\right]^{p_0} \frac{dt}{t} < \infty,\ \int_1^\infty \left[\frac{u_1(t)}{u_0(t)}\right]^{p_1} \frac{dt}{t} < \infty,\ 0 < p_0, p_1 < \infty, \tag{25}$$

$$\int_0^\infty [\min(1, t)u_j(t)]^{p_j} \frac{dt}{t} < \infty, \quad j = 0, 1, \tag{26}$$

$$K(t,\, f; \vec{A}) \leqslant c[u_j(t)]^{-1} \| f\|_{B_j}, \quad j = 0, 1, \tag{27}$$

where $B_j = \vec{A}_{u_j, p_j}$.

Theorem 2.8. *Let* $B_j = \vec{A}_{u_j,p_j}$, *let* $K(t, f) = K(t, f; \vec{A})$.

(i) *Suppose that the weights* $\{u_0, u_1\}$ *satisfy* (25), (27), *then*

$$\left\{\int_0^1 [u_0(t)K(t, f)]^{p_0} \frac{dt}{t}\right\}^{1/p_0} + \left\{\int_1^\infty [u_1(t)K(t, f)]^{p_1} \frac{dt}{t}\right\}^{1/p_1} \leqslant cK(1, f; \vec{B}). \tag{28}$$

(ii) *Suppose that the weights* $\{u_0, u_1\}$ *satisfy* (26), *then*

$$\left\{\int_0^1 [u_0(t)K(t, f)]^{p_0} \frac{dt}{t}\right\}^{1/p_0} + \left\{\int_1^\infty [u_1(t)K(t, f)]^{p_1} \frac{dt}{t}\right\}^{1/p_1} \geqslant cK(1, f; \vec{B}). \tag{29}$$

Proof. To prove (28) we argue as above. Let $f = f_0 + f_1$, $f_j \in B_j$, $j = 0, 1$. Then using (25),(27) we find

$$c\int_0^1 [u_0(t)K(t, f)]^{p_0} \frac{dt}{t} \leqslant \|f_0\|_{B_0}^{p_0} + \|f_1\|_{B_1}^{p_0} \int_0^1 \left[\frac{u_0(t)}{u_1(t)}\right]^{p_0} \frac{dt}{t}$$

and

$$c\int_1^\infty [u_1(t)K(t, f)]^{p_1} \frac{dt}{t} \leqslant \|f_1\|_{B_1}^{p_1} + \|f_0\|_{B_0}^{p_0} \int_1^\infty \left[\frac{u_1(t)}{u_0(t)}\right]^{p_1} \frac{dt}{t}.$$

The proof of formula (29) is exactly the same as the proof of formula (24) and we omit the details. The proof is complete. □

2.4.2. Normalization and uniform formulae for the K-functional

In previous sections the uniform equivalence of norms of $\vec{A}_{\theta,p}$ and $\vec{A}_{\theta,p;J}$, $\theta \in \Theta = (\theta_0, \theta_1)$, $0 < \theta_0 < \theta_1 < 1$, plays a fundamental role in the calculations of $\Sigma^{(p)}$. In particular it allows us to use a Fubini type of argument for the computation of $\Sigma_\Theta^{(p)}(M(\theta)\vec{A}_{\theta,p})$.

We can compute $\Sigma_\Theta^{(p)}(M(\theta)\vec{A}_{\theta,q})$ for some $q \neq p$ if the function M is tempered. Using reiteration the problem is reduced to the case $\theta_0 = 0, \theta_1 = 1$. Then it is very useful to normalize the norms of the interpolation spaces as in [17]:

$$\left\langle \vec{A}_{\sigma,p} \right\rangle := c_{\sigma,p}\vec{A}_{\sigma,p}, \quad 0 < \sigma < 1, \quad 0 < p \leqslant \infty,$$

where

$$c_{\sigma,p} = [\sigma(1-\sigma)p]^{1/p}, \quad \|f\|_{\left\langle \vec{A}_{\sigma,p} \right\rangle} = c_{\sigma,p}\|f\|_{\vec{A}_{\sigma,p}},$$

with the convention $\infty^{1/\infty} = 1$.

We have the norm one embeddings (see [5,17, p. 19] [10]):

$$\left\langle \vec{A}_{\sigma,p} \right\rangle \subset \left\langle \vec{A}_{\sigma,r} \right\rangle, \; r \geqslant p. \tag{30}$$

Example 2.1 (*cf. Milman [22]*). Note that with this normalization the L^p spaces can be obtained by the real method

$$\langle (L^1, L^\infty)_{1/p^*,p} \rangle = L^p \tag{31}$$

with norm equivalence independent of p.

The following sharp version of Holmstedt's reiteration formula (cf. [17, formula (3.15), p. 33]) will be useful in what follows. [11]

Lemma 2.1. *Let* $\vec{A} = (A_0, A_1)$ *be a quasi-Banach pair, let* $0 < \theta_0 < \theta_1 < 1$, $0 < q_0 \leqslant q \leqslant \infty$, $B_j = \left\langle \vec{A}_{\theta_j,q} \right\rangle$, $j = 0, 1$, $\alpha = \theta_1 - \theta_0$. *Then, with constants of equivalence independent of* θ_0, θ_1, q,

$$\begin{aligned} K(t, f; \vec{B}) \approx\; & c_{\theta_0,q} \left\{ \int_0^{t^{1/\alpha}} (s^{-\theta_0} K(s, f; \vec{A}))^q \frac{ds}{s} \right\}^{1/q} \\ & + t c_{\theta_1,q} \left\{ \int_{t^{1/\alpha}}^\infty (s^{-\theta_1} K(s, f; \vec{A}))^q \frac{ds}{s} \right\}^{1/q} \\ & + t^{-\theta_0/\alpha} K(t^{1/\alpha}, f; \vec{A}). \end{aligned} \tag{32}$$

The following two variants of Lemma 2.1 are also needed in the sequel.

---

[10] For example, to prove the case $p > 0$, $r = \infty$, which we use in the proof of Theorem 2.9 below, we write $I^p = \int_0^\infty [s^{-\theta} K(s, f)]^p \frac{ds}{s}$, $I_1 = \int_0^t [s^{-\theta} K(s, f)]^p \frac{ds}{s}$, $I_2 = \int_t^\infty [s^{-\theta} K(s, f)]^p \frac{ds}{s}$. Then,

$$[(1-\theta)^{1/p} \theta^{1/p} p^{1/p} I]^p = (1-\theta)\theta p (I_1 + I_2).$$

Moreover since $K(t, f)/t$ decreases,

$$(1-\theta)\theta p I_1 \geqslant \theta K(t, f)^p t^{-\theta p},$$

and since $K(t, f)$ increases,

$$(1-\theta)\theta p I_2 \geqslant (1-\theta) K(t, f)^p t^{-\theta p}.$$

Thus for all $t > 0$ we have

$$(1-\theta)\theta p I_1 + (1-\theta)\theta p I_2 \geqslant K(t, f)^p t^{-\theta p},$$

so that for all $t > 0$,

$$[(1-\theta)^{1/p} \theta^{1/p} p^{1/p} I]^p \geqslant [K(t, f) t^{-\theta}]^p,$$

and the required inequality follows.

[11] The proof follows the original Holmstedt argument.

Lemma 2.2. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair, let $0 < \theta < 1, \alpha = 1 - \theta$, $B_0 = \left\langle \vec{A}_{\theta,q} \right\rangle$, $B_1 = A_1$. Then, with constants of equivalence independent of θ, q,*

$$K(t, f; \vec{B}) \approx c_{\theta,q} \left\{ \int_0^{t^{1/\alpha}} (s^{-\theta} K(s, f; \vec{A}))^q \frac{ds}{s} \right\}^{1/q} + t^{-\theta/\alpha} K(t^{1/\alpha}, f; \vec{A}). \tag{33}$$

Lemma 2.3. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair, let $0 < \alpha < 1$, $B_1 = \left\langle \vec{A}_{\alpha,q} \right\rangle$, $B_0 = A_0$. Then, with constants of equivalence independent of α, q,*

$$K(t, f; \vec{B}) \approx c_{\alpha,q} t \left\{ \int_{t^{1/\alpha}}^{\infty} (s^{-\alpha} K(s, f; \vec{A}))^q \frac{ds}{s} \right\}^{1/q} + K(t^{1/\alpha}, f; \vec{A}). \tag{34}$$

Proofs of Lemmas 2.2 and 2.3. We only give the proof of Lemma 2.3 since the proof of Lemma 2.2 follows the same argument *mutatis mutandis*. Let

$$I := c_{\alpha,q} t \left\{ \int_{t^{1/\alpha}}^{\infty} (s^{-\alpha} K(s, f; \vec{A}))^q \frac{ds}{s} \right\}^{1/q}.$$

We now estimate each of the terms on the right-hand side of (34). Consider an arbitrary decomposition of $f = f_0 + f_1,\ f_j \in A_j,\ j = 0, 1$. Then

$$\begin{aligned} cI &\leqslant c_{\alpha,q} t \left\{ \int_{t^{1/\alpha}}^{\infty} (s^{-\alpha} K(s, f_0; \vec{A}))^q \frac{ds}{s} \right\}^{1/q} \\ &\quad + c_{\alpha,q} t \left\{ \int_{t^{1/\alpha}}^{\infty} (s^{-\alpha} K(s, f_1; \vec{A}))^q \frac{ds}{s} \right\}^{1/q}. \end{aligned}$$

It is plain that $K(s, f_0; \vec{A}) \leqslant \|f_0\|_{A_0}$, whence

$$\begin{aligned} cI &< t\|f_1\|_{\langle \vec{A}_{\alpha,q} \rangle} + c_{\alpha,q} t \left\{ \int_{t^{1/\alpha}}^{\infty} s^{-\alpha q} \frac{ds}{s} \right\}^{1/q} \|f_0\|_{A_0} \\ &\leqslant \|f_0\|_{A_0} + t\|f_1\|_{\langle \vec{A}_{\alpha,q} \rangle}. \end{aligned}$$

Taking infimum over all decompositions we get

$$cI \leqslant K(t, f; \vec{B}). \tag{35}$$

To estimate $K(t^{1/\alpha}, f; \vec{A})$ we use the trivial estimate (cf. [5])

$$K(t, f; \vec{A}) \leqslant t^{\alpha} \|f\|_{\langle \vec{A}_{\alpha,q} \rangle}$$

combined with the triangle inequality. We get

$$\begin{aligned} cK(t^{1/\alpha}, f; \vec{A}) &\leqslant K(t^{1/\alpha}, f_0; \vec{A}) + K(t^{1/\alpha}, f_1; \vec{A}) \\ &\leqslant \|f_0\|_{A_0} + t\|f_1\|_{\langle \vec{A}_{\alpha,q} \rangle}. \end{aligned}$$

Taking infimum yields

$$cK(t^{1/\alpha}, f; \vec{A}) \leqslant K(t, f; \vec{B}). \tag{36}$$

Combining (35) and (36) we obtain the required lower estimate for $K(t, f; \vec{B})$. To prove the upper estimate we use Holmstedt's argument. Select $f_j \in A_j$, $j = 0, 1$, such that $f = f_0 + f_1$ and $K(t, f; \vec{A}) \approx \|f_0\|_{A_0} + t\|f_1\|_{A_1}$. Then

$$K(s, f_0; \vec{A}) \leqslant \|f_0\|_{A_0} \leqslant cK(t^{1/\alpha}, f; \vec{A}), \tag{37}$$
$$K(s, f_1; \vec{A}) \leqslant s\|f_1\|_{A_1} \leqslant cst^{-1/\alpha}K(t^{1/\alpha}, f; \vec{A}).$$

Therefore,

$$\begin{aligned} t\|f_1\|_{\langle \vec{A}_{\alpha,q}\rangle} &= tc_{\alpha,q}\left\{\int_0^\infty s^{-\alpha q}K(s, f_1; \vec{A})^q \frac{ds}{s}\right\}^{1/q} \\ &\leqslant c_{\alpha,q}t\left\{c\int_0^{t^{1/\alpha}} s^{(1-\alpha)q}\frac{ds}{s}\right\}^{1/q} t^{-1/\alpha}K(t^{1/\alpha}, f; \vec{A}) \\ &\quad + c_{\alpha,q}t\left\{c\int_{t^{1/\alpha}}^\infty s^{-\alpha q}[K^q(s, f; \vec{A}) + K^q(s, f_0; \vec{A})]\frac{ds}{s}\right\}^{1/q}. \end{aligned}$$

Combining with (37) we get

$$t\|f_1\|_{\langle \vec{A}_{\alpha,q}\rangle} \leqslant c(I + K(t^{1/\alpha}, f; \vec{A})),$$

concluding the proof. □

The next corollary will be useful in the applications.

Corollary 2.1. *If* $0 < \sigma < \eta < 1$, *then*

$$\left\langle (A_0, A_1)_{\sigma,p}\right\rangle \subset A_0 + \left\langle (A_0, A_1)_{\eta,p}\right\rangle, \tag{38}$$

In particular, if $p \leqslant \gamma_{\vec{A}}$ *then*

$$\Sigma^{(p)-}_{(0,\eta)}\left(\left\langle (A_0, A_1)_{\sigma,p}\right\rangle\right) \subset A_0 + \left\langle (A_0, A_1)_{\eta,p}\right\rangle.$$

Proof. Using (34) we derive

$$\left(A_0, \left\langle (A_0, A_1)_{\eta,p}\right\rangle\right)_{\sigma/\eta,p} = c_{\eta,p}(1 - \sigma/\eta)^{-1/p}(A_0, A_1)_{\sigma,p},$$

whence

$$\left\langle (A_0, A_1)_{\sigma,p}\right\rangle \subset \left\langle \left(A_0, \left\langle (A_0, A_1)_{\eta,p}\right\rangle\right)_{\sigma/\eta,p}\right\rangle \subset A_0 + \left\langle (A_0, A_1)_{\eta,p}\right\rangle.$$

We now show that for the normalized real interpolation scales the second index is not important in the computation of $\Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle \vec{A}_{\sigma,r}\right\rangle\right)$ (cf. [17] for the case $p = 1$).

Theorem 2.9. *Let* $\vec{A} = (A_0, A_1)$ *be a quasi-Banach pair. Suppose that* $M(\sigma)$ *is tempered. Then*

$$\Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle \vec{A}_{\sigma,r}\right\rangle\right) = \Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle \vec{A}_{\sigma,q}\right\rangle\right).$$

Proof. By symmetry it is enough to consider the case $r > q$. Under this assumption the embedding $\Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,q}\right\rangle\right) \subset \Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,r}\right\rangle\right)$ follows directly from (30). It remains to prove

$$\Sigma^{(p)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,r}\right\rangle\right) \subset \Sigma^{(p)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,q}\right\rangle\right). \tag{39}$$

It is enough to consider the extreme case: $r = \infty$ and $q > 0$. We will show below that

$$\left\langle\vec{A}_{\sigma,\infty}\right\rangle \subset \left\langle\vec{A}_{\sigma/2,q}\right\rangle + \left\langle\vec{A}_{2\sigma,q}\right\rangle, \quad \sigma \sim 0, \tag{40}$$

$$\left\langle\vec{A}_{\sigma,\infty}\right\rangle \subset \left\langle\vec{A}_{2\sigma-1,q}\right\rangle + \left\langle\vec{A}_{(1+\sigma)/2,q}\right\rangle, \quad \sigma \sim 1. \tag{41}$$

Since M is tempered, these inclusions combined with Remark 2.3 yield [12]

$$\begin{aligned}\Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,\infty}\right\rangle\right) &\subset \Sigma^{(p)-}_{\sigma\sim 0}\left(M(\sigma)\left\langle\vec{A}_{\sigma/2,q}\right\rangle\right) + \Sigma^{(p)-}_{\sigma\sim 0}\left(M(\sigma)\left\langle\vec{A}_{2\sigma,q}\right\rangle\right)\\ &\quad + \Sigma^{(p)+}_{\sigma\sim 1}\left(M(\sigma)\left\langle\vec{A}_{2\sigma-1,q}\right\rangle\right) + \Sigma^{(p)+}_{\sigma\sim 1}\left(M(\sigma)\left\langle\vec{A}_{(1+\sigma)/2,q}\right\rangle\right)\\ &= \Sigma^{(p)}_{(0,1)}\left(M(\sigma)\left\langle\vec{A}_{\sigma,q}\right\rangle\right).\end{aligned}$$

Therefore the theorem is proved modulo (40) and (41).

To prove (40) suppose that $\sigma \sim 0$, and let $\theta_0 = \sigma/2$, $\theta_1 = 2\sigma$, and $t = 1$ in (32); then

$$\begin{aligned}\|f\|_{\left\langle\vec{A}_{\sigma/2,q}\right\rangle+\left\langle\vec{A}_{2\sigma,q}\right\rangle} &\approx c_{\sigma/2,q}\left\{\int_0^1 [s^{-\sigma/2}K(s,f;\vec{A})]^q \frac{ds}{s}\right\}^{1/q}\\ &\quad + c_{2\sigma,q}\left\{\int_1^\infty [s^{-2\sigma}K(s,f;\vec{A})]^q \frac{ds}{s}\right\}^{1/q} + \|f\|_{A_0+A_1}.\end{aligned} \tag{42}$$

We estimate each of the three terms on the right-hand side of (42). It is clear that

$$\|f\|_{A_0+A_1} \leqslant \|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle}. \tag{43}$$

Moreover,

$$\begin{aligned}c_{\sigma/2,q}\left\{\int_0^1 [s^{-\sigma/2}K(s,f;\vec{A})]^q \frac{ds}{s}\right\}^{1/q} &= c_{\sigma/2,q}\left\{\int_0^1 [s^{-\sigma}K(s,f;\vec{A})]^q s^{\sigma q/2}\frac{ds}{s}\right\}^{1/q}\\ &\leqslant c_{\sigma/2,q}\|f\|_{\vec{A}_{\sigma,\infty}}(q\sigma/2)^{-1/q}\\ &= (1-\sigma/2)^{1/q}\|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle}.\end{aligned}$$

[12] By abuse of notation we use $\Sigma_{\sigma\sim 0}$ to indicate $\Sigma_{0<\sigma<\varepsilon}$.

For the second term we have

$$c_{2\sigma,q}\left\{\int_1^{\infty}[s^{-2\sigma}K(s,\ f;\ \vec{A})]^q\frac{ds}{s}\right\}^{1/q} = c_{2\sigma,q}\left\{\int_1^{\infty}[s^{-\sigma}K(s,\ f;\ \vec{A})]^q s^{-\sigma q}\frac{ds}{s}\right\}^{1/q}$$
$$\leqslant c_{2\sigma,q}\,\|f\|_{\vec{A}_{\sigma,\infty}}(\sigma q)^{-1/q}$$
$$= [2(1-2\sigma)]^{1/q}\,\|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle}.$$

Inserting these estimates in (42) we find

$$\|f\|_{\left\langle\vec{A}_{\sigma/2,q}\right\rangle+\left\langle\vec{A}_{2\sigma,q}\right\rangle}\leqslant c\,\|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle}.$$

To prove (41) we proceed similarly. Let $\sigma\sim 1$, $\theta_0 = 2\sigma-1$, $\theta_1=(1+\sigma)/2$, and $t=1$ in (32). Then,

$$\|f\|_{\left\langle\vec{A}_{2\sigma-1,q}\right\rangle+\left\langle\vec{A}_{(1+\sigma)/2,q}\right\rangle}\approx c_{2\sigma-1,q}\left\{\int_0^1[s^{-2\sigma+1}K(s,\ f;\ \vec{A})]^q\frac{ds}{s}\right\}^{1/q}$$
$$+c_{(1+\sigma)/2,q}\left\{\int_1^{\infty}[s^{-(1+\sigma)/2}K(s,\ f;\ \vec{A})]^q\frac{ds}{s}\right\}^{1/q}$$
$$+\|f\|_{A_0+A_1}.$$

Each of these terms can be estimated as above. For example, the second term yields

$$c_{(1+\sigma)/2,q}\left\{\int_1^{\infty}[s^{-(1+\sigma)/2}K(s,\ f;\ \vec{A})]^q\frac{ds}{s}\right\}^{1/q}$$
$$= c_{(1+\sigma)/2,q})\left\{\int_1^{\infty}[s^{-\sigma}K(s,\ f;\ \vec{A})]^q s^{q(\sigma/2-1/2)}\frac{ds}{s}\right\}^{1/q}$$
$$\leqslant((1+\sigma)/2)^{1/q}\,\|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle}$$

So that all in all we have

$$\|f\|_{\left\langle\vec{A}_{1/2,q}\right\rangle+\left\langle\vec{A}_{(1+\sigma)/2,q}\right\rangle}\leqslant c\,\|f\|_{\left\langle\vec{A}_{\sigma,\infty}\right\rangle},$$

and we have established (41). The theorem follows. □

The same proof gives the following result

Theorem 2.10. *Let* $M(\theta)$ *be tempered on* $(0,1)$. *Then*

$$\Sigma^{(p)-}_{(0,\alpha)}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right)=\Sigma^{(p)-}_{(0,\alpha)}\left(M(\theta)\left\langle\vec{A}_{\theta,q}\right\rangle\right)$$
$$\Sigma^{(p)+}_{(\beta,1)}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right)=\Sigma^{(p)+}_{(\beta,1)}\left(M(\theta)\left\langle\vec{A}_{\theta,q}\right\rangle\right).$$

2.4.3. Characterization of $\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)\vec{A}_{\theta,q})$

To handle the case $0<\theta_0<\theta_1<1$, we use reiteration and pay attention to the dependence on the parameters.

Theorem 2.11 (*Case* $r \geqslant p$). *Let* $0 < \theta_0 < \theta_1 < 1$. *Let* $\sigma \in (0,1)$, $\theta = (1-\sigma)\theta_0 + \sigma\theta_1$ *then, uniformly on* σ, *we have*

$$\vec{A}_{\theta,r} \subset [\sigma(1-\sigma)]^{1/p} (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,r}, \tag{44}$$

and

$$\vec{A}_{\theta,r} \supset [\sigma(1-\sigma)]^{1/r} (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,r}. \tag{45}$$

In particular,

$$\vec{A}_{\theta,p} = \langle (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,p} \rangle. \tag{46}$$

The following analogs will be also useful in the sequel.

Remark 2.13. Let $0 < \theta_0 < \theta_1 < 1$, $\sigma \in (0,1)$, $\theta = (1-\sigma)\theta_0 + \sigma\theta_1$, $0 < p_0, p_1 < \infty$, $1/p = (1-\sigma)/p_0 + \sigma/p_1$. Then, uniformly with respect to σ,

$$\vec{A}_{\theta,p} = \left\langle (\vec{A}_{\theta_0,p_0}, \vec{A}_{\theta_1,p_1})_{\sigma,p} \right\rangle. \tag{47}$$

Remark 2.14. Let $0 < \theta_0 < 1$, $\sigma \in (0,1)$, $\theta = (1-\sigma)\theta_0 + \sigma$. Then, uniformly with respect to σ,

$$(1-\theta)^{1/p} \vec{A}_{\theta,p} = \left\langle (\vec{A}_{\theta_0,p}, A_1)_{\sigma,p} \right\rangle. \tag{48}$$

Example 2.2. (i) Let L^p be the Lebesgue space on arbitrary σ-finite measure space. Then

$$\left\langle (L^{p_0}, L^{p_1})_{\sigma,p} \right\rangle = L^p,\ 1/p = (1-\sigma)/p_0 + \sigma/p_1,\ 0 < p_0, p_1 < \infty. \tag{49}$$

(ii) Let $L^{q,p}$ be the Lorentz space equipped with the quasi-norm

$$\|f\|_{L^{q,p}} = \left\{ \frac{p}{q} \int_0^\infty [t^{1/q} f^*(t)]^p \frac{dt}{t} \right\}^{1/p}.$$

Then

$$\left\langle (L^r, L^\infty)_{\sigma,p} \right\rangle = L^{q,p}, \tag{50}$$

where $1/q = (1-\sigma)/r$, $0 < r < \infty$, $p = p(\sigma)$, $|1/r - 1/p| \leqslant \frac{c}{|\log \sigma(1-\sigma)|}$.

(iii) The same proof also gives

$$\left\langle (L^r, L^\infty)_{\sigma,p} \right\rangle \subset L^{q,p}, \quad 1/q = (1-\sigma)/r, \quad 0 < r \leqslant p. \tag{51}$$

Proof of Theorem 2.11. Let $B_j = \vec{A}_{\theta_j,p}$, $\alpha = \theta_1 - \theta_0$. By Holmstedt's formula (with constants of equivalence depending on θ_j, p):

$$K(t, f; \vec{B}) \approx \left\{ \int_0^{t^{1/\alpha}} (s^{-\theta_0} K(s, f; \vec{A}))^p \frac{ds}{s} \right\}^{1/p} + t \left\{ \int_{t^{1/\alpha}}^\infty (s^{-\theta_1} K(s, f; \vec{A}))^p \frac{ds}{s} \right\}^{1/p}. \tag{52}$$

Then, using Minkowski's inequality for $r > p$ and a change of variables, we get

$$\begin{aligned}
&\int_0^\infty t^{-\sigma r} K^r(t, f; \vec{B})\frac{dt}{t} \\
&\approx \int_0^\infty t^{-\sigma r}\left\{\int_0^{t^{1/\alpha}} (s^{-\theta_0} K(s, f; \vec{A}))^p \frac{ds}{s}\right\}^{r/p} \frac{dt}{t} \\
&\quad + \int_0^\infty t^{(1-\sigma)r}\left\{\int_{t^{1/\alpha}}^\infty (s^{-\theta_1} K(s, f; \vec{A}))^p \frac{ds}{s}\right\}^{r/p} \frac{dt}{t} \\
&= \int_0^\infty t^{-\sigma r}\left\{\int_0^1 (st^{1/\alpha})^{-\theta_0 p} K(st^{1/\alpha}, f; \vec{A})^p \frac{ds}{s}\right\}^{r/p} \frac{dt}{t} \\
&\quad + \int_0^\infty t^{(1-\sigma)r}\left\{\int_1^\infty (st^{1/\alpha})^{-\theta_1 p} K(st^{1/\alpha}, f; \vec{A})^p \frac{ds}{s}\right\}^{r/p} \frac{dt}{t} \\
&\leqslant \left\{\int_0^1 \left(\int_0^\infty t^{-\sigma r}(st^{1/\alpha})^{-\theta_0 r} K(st^{1/\alpha}, f; \vec{A})^r \frac{dt}{t}\right)^{p/r} \frac{ds}{s}\right\}^{r/p} \\
&\quad + \left\{\int_1^\infty \left(\int_0^\infty t^{(1-\sigma)r}(st^{1/\alpha})^{-\theta_1 r} K(st^{1/\alpha}, f; \vec{A})^r \frac{dt}{t}\right)^{p/r} \frac{ds}{s}\right\}^{r/p} \\
&= \alpha\left[\left\{\int_0^1 s^{\sigma\alpha p}\frac{ds}{s}\right\}^{r/p} + \left\{\int_1^\infty s^{-(1-\sigma)\alpha p}\frac{ds}{s}\right\}^{r/p}\right]\int_0^\infty u^{-\theta r} K(u, f; \vec{A})^r \frac{du}{u} \\
&= \alpha\left[\left(\frac{1}{\alpha\sigma p}\right)^{r/p} + \left(\frac{1}{\alpha(1-\sigma)p}\right)^{r/p}\right]\|f\|^r_{\vec{A}_{\theta,r}} \\
&\leqslant c[\sigma(1-\sigma)]^{-r/p}\,\|f\|_{\vec{A}_{\theta,r}}.
\end{aligned}$$

Thus (44) is proven.

To prove the embedding (45), we write

$$\int_0^\infty t^{-\sigma r} K^r(t, f; \vec{B})\frac{dt}{t} \approx I + J,$$

where

$$I = \int_0^\infty t^{-\alpha\sigma r}[g(t)]^{r/p}\frac{dt}{t},\ \ g(t) = \int_0^t (s^{-\theta_0} K(s, f; \vec{A}))^p \frac{ds}{s}$$

and

$$J = \int_0^\infty t^{\alpha(1-\sigma)r}[h(t)]^{r/p}\frac{dt}{t},\ \ h(t) = \int_t^\infty (s^{-\theta_1} K(s, f; \vec{A}))^p \frac{ds}{s}.$$

If $f \in (B_0, B_1)_{\sigma,r}$ we can integrate by parts:

$$I = \frac{1}{\alpha\sigma p}\int_0^\infty [g(t)]^{r/p-1} g'(t) t^{-\alpha\sigma r}\,dt.$$

From

$$g'(t) = t^{-\theta_0 p-1} K^p(t, f; \vec{A})$$

and

$$g(t) \geqslant \frac{1}{(1-\theta_0)p} t^{-\theta_0 p} K^p(t, f; \vec{A}),$$

we get

$$I \geqslant c\sigma^{-1} \|f\|^r_{\vec{A}_{\theta,r}}.$$

Analogously,

$$J \geqslant c(1-\sigma)^{-1} \|f\|^r_{\vec{A}_{\theta,r}}.$$

Thus (45) is proven. □

An analogous result is valid for $r < p$:

Theorem 2.12 (*Case* $r < p$). *Let* $0 < \theta_0 < \theta_1 < 1, \sigma \in (0, 1), \theta = (1-\sigma)\theta_0 + \sigma\theta_1$. *Then, with constants that are bounded w.r. to* σ, *we have the following embeddings*

$$\vec{A}_{\theta,r} \subset [\sigma(1-\sigma)]^{1/r} (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,r},$$

$$\vec{A}_{\theta,r} \supset [\sigma(1-\sigma)]^{1/p} (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,r}.$$

As a corollary of Theorems 2.11 and 2.12 we get

Corollary 2.2. *Let* $0 < \theta_0 < \theta_1 < 1, \sigma \in (0, 1), \theta = (1-\sigma)\theta_0 + \sigma\theta_1$. *Then for* $|1/r(\sigma) - 1/p| \leqslant \frac{c}{|\log \sigma(1-\sigma)|}$ *we have, uniformly with respect to* σ:

$$\vec{A}_{\theta,r(\sigma)} = \left\langle (\vec{A}_{\theta_0,p}, \vec{A}_{\theta_1,p})_{\sigma,r(\sigma)} \right\rangle. \tag{53}$$

Remark 2.15. If $r \neq p$ and r is independent σ, then we do not have equality in (53). To see this let us consider the case $r = \infty$. Let f be such that $K(t, f; \vec{A}) \approx t^\theta$, where $\theta = (1-\sigma)\theta_0 + \sigma\theta_1$. Then $\|f\|_{\vec{A}_{\theta,\infty}} \approx 1$ and we can calculate $K(t, f; \vec{B}) \approx t^\sigma [\sigma(1-\sigma)]^{-1/p}$. It follows that $\|f\|_{\vec{B}_{\sigma,\infty}} \approx [\sigma(1-\sigma)]^{-1/p}$.

Now we are ready to prove an analog of Theorem 2.9 for the case $0 < \theta_0 < \theta_1 < 1$. The following [13] corollary of Theorems 2.9 and 2.11 is only a partial analog of Theorem 2.9.

Corollary 2.3. *Let* $M(\theta)$ *be tempered on the interval* (θ_0, θ_1) $0 < \theta_0 < \theta_1 < 1$, *and let* $r > p$. *Then*

$$\Sigma^{(p)}_{(\theta_0,\theta_1)} (M(\theta)[(\theta - \theta_0)(\theta_1 - \theta)]^{1/r-1/p} \vec{A}_{\theta,r}) \subset \Sigma^{(p)}_{(\theta_0,\theta_1)} (M(\theta) \vec{A}_{\theta,p}).$$

[13] Note that according to Remark 3.5 and Proposition 3.1 below, the embedding can not be reversed.

Theorem 2.13. *Let $\vec{A} = (A_0, A_1)$ be a quasi-Banach pair, let $0 < \theta_0 < \theta_1 < 1$, and let M be tempered on the interval (θ_0, θ_1). Suppose that $|1/r(\theta) - 1/p| \leqslant \frac{c}{|\log(\theta-\theta_0)(\theta_1-\theta)|}$, then*

$$\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)\vec{A}_{\theta,r(\theta)}) = \Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)\vec{A}_{\theta,p}).$$

Analogous results are valid for one-sided extrapolation spaces.

Proof. Let $\theta = (1-\sigma)\theta_0 + \sigma\theta_1$ and $\bar{M}(\sigma) := M(\theta)$, $\bar{r}(\sigma) := r(\theta)$, and $B_j := \vec{A}_{\theta_j,p}$, $j = 0, 1$. Then

$$\begin{aligned}\Sigma^{(p)}_{(\theta_0,\theta_1)}&(M(\theta)\vec{A}_{\theta,r(\theta)}) = \Sigma^{(p)}_{(0,1)}(\bar{M}(\sigma)\vec{A}_{\theta,\bar{r}(\sigma)})\\ &= \Sigma^{(p)}_{(0,1)}\left(\bar{M}(\sigma)\left\langle\vec{B}_{\sigma,p}\right\rangle\right) \text{ (by Corollary 2.2 and Theorem 2.9)}\\ &= \Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)\vec{A}_{\theta,p}) \text{ (by (46)).} \qquad \square\end{aligned}$$

Remark 2.16. In this example we show that in Theorem 2.13 we cannot replace $r(\sigma)$ by a fixed r, $r > p$. In fact, for $0 < p < r \leqslant \infty$, $a > 0$, $\varepsilon > 0$, $\theta_0 > 0$, $\varepsilon + \theta_0 < 1$, $p_0 = (1-\theta_0)p$, $L^p := L^p(0,1)$, we have

$$\Sigma^{(p)-}_{(0,\varepsilon)}(\sigma^{-a}(L^{p_0}, L^\infty)_{\sigma+\theta_0,p}) \neq \Sigma^{(p)-}_{(0,\varepsilon)}(\sigma^{-a}(L^{p_0}, L^\infty)_{\sigma+\theta_0,r}).$$

Proof. Let $n \geqslant n_1$, $0 < \sigma < \varepsilon = 2^{-n_1}$, $1/q_n = 1/p - 2^{-n}/p_0$, , $\beta > a$, $1/r < \alpha < 1/p + a - \beta$, $\alpha < 1/p - \varepsilon/p_0$. Define $g(2^{-n}, t) = t^{-1/q_n}(1-\ln t)^{-\alpha}2^{-n\beta}$, $0 < t < 1$. We have $\|g(2^{-n}, .)\|_{L^{q,r}} = 2^{-n\beta}(\alpha r - 1)^{-1/r}$ if $\alpha > 1/r$. Recall that $(L^{p_0}, L^\infty)_{2^{-n}+\theta_0,r} = L^{q,r}$ (uniformly w.r.t. $n \geqslant n_1$), therefore if we let

$$f(t) := \sum g(2^{-n}, t) \approx t^{-1/p}(1-\ln t)^{-\beta-\alpha},$$

we see that

$$f \in \Sigma^{(p)-}_{(0,\varepsilon)}(2^{na}(L^{p_0}, L^\infty)_{2^{-n}+\theta_0,r}).$$

On the other hand, according to Theorem 4.4 below

$$\Sigma^{(p)-}_{0,\varepsilon}(2^{na}(L^{p_0}, L^\infty)_{2^{-n}+\theta_0,p}) = L^p(\log L)_a.$$

Moreover, since $f(t)$ is equivalent to a decreasing function on $0 < t < t_0 \leqslant 1$, and, since $\alpha < 1/p + a - \beta$, we see that

$$\|f\|^p_{L^p(\log L)_a} \geqslant c\int_0^1 (1-\ln t)^{(a-\beta-\alpha)p}\frac{dt}{t} = \infty.$$

Our claim follows. $\square$

2.5. Extrapolation theorems for $\Sigma^{(p)}$ method

The spaces $\Sigma^{(p)}(M(\theta)A_\theta)$ are extrapolation spaces in the following sense.

Theorem 2.14 (Σ *extrapolation theorem*). *Suppose that* $\{A_\theta\}_{\theta\in\Theta}$ *and* $\{B_\theta\}_{\theta\in\Theta}$ *are compatible scales of quasi-Banach spaces and let* $T(\Sigma_A) \subset \Sigma_B$ *be a continuous linear operator such that its restriction* $T : A_\theta \to B_\theta$ *is bounded with* $\|T\|_{A_\theta\to B_\theta} \leqslant 1$ *for all* $\theta \in \Theta$. *Then* T *defines a bounded linear operator*

$$T : \Sigma^{(p)}(M(\theta)A_\theta) \to \Sigma^{(p)}(M(\theta)B_\theta),$$

with $\|T\|_{\Sigma^{(p)}(M(\theta)A_\theta)\to\Sigma^{(p)}(M(\theta)B_\theta)} \leqslant 1$.

Theorem 2.14 requires that the operator T be ab initio defined on some larger space than the extrapolation space $\Sigma^{(p)}$. However if this assumption does not hold we can, under suitable conditions, extend the operator. This is the content of the next result

Theorem 2.15. *Let* T *be a bounded linear operator* $T : \vec{A}_{\theta,p} \to B_\theta$, *with* $\|T\|_{\vec{A}_{\theta,p}\to B_\theta} \leqslant 1$, $0 < \theta_0 < \theta < \theta_1 < 1$, $0 < p < \infty$. *Then* T *can be extended as a bounded linear operator*

$$T : \vec{A}_{w^*,p} \to \Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)B_\theta).$$

Remark 2.17. In addition, if the space $\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)B_\theta)$ has the lattice property:

$$|f| \leqslant |g| \Rightarrow \|f\| \leqslant \|g\|,$$

then the previous extrapolation theorem holds for sublinear operators, i.e. we only need to assume that the operator T satisfies $|T(f)| \leqslant \sum |T(f_n)|$, whenever $f = \sum f_n$ with $f_n \in A_0 \cap A_1$.

Proof of Theorem 2.15. We remind the reader that w^* is a Calderón weight. Therefore for $p < \infty$, $A_0 \cap A_1$ is dense in $\vec{A}_{w^*,p}$. We may thus assume without loss of generality that $f \in A_0 \cap A_1$. Then we can find a decomposition $f = \sum_{v=-\infty}^{\infty} f_v$, where $f_v \in A_0 \cap A_1$, is such that $J(2^v, f_v; \vec{A}) \leqslant cK(2^v, f; \vec{A})$, $v \in \mathbf{Z}$. Consider the sequence $f^N = \sum_{|v|<N} f_v$, and the partition of the unity $\{\psi_{n,v}\}$ we have used in the proof of Theorem 2.3. In particular, we have $\sum_{n\in I} \psi_{n,v} = 1$. Then,

$$T(f^N) = \sum_{|v|<N} T(f_v) = \sum_{n\in I} T\left(\sum_{|v|<N} \psi_{n,v} f_v\right).$$

Therefore,

$$\begin{aligned}\|T(f^N)\|^p_{\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)B_\theta)} &\leqslant c \sum_{n\in I}\left[M_n \Big\|\sum_{|v|<N} \psi_{n,v} f_v\Big\|_{\vec{A}_{\theta_n,p;J}}\right]^p \\ &\leqslant c \sum_{n\in I}\sum_{|v|<N} [2^{-v\theta_n} M_n J(2^v, f_v\psi_{n,v}; \vec{A})]^p\end{aligned}$$

$$\leqslant c \sum_{|\nu|<N} [K(2^\nu, f; \vec{A})]^p \sum_{n\in I} [M_n 2^{-\nu\theta_n} \psi_{n,\nu}]^p$$
$$\leqslant c \sum_{|\nu|<N} [w^*(2^\nu) K(2^\nu, f; \vec{A})]^p \sum_{n\in I} [\psi_{n,\nu}/\phi_{n,\nu}]^p.$$

As in the proof of Theorem 2.3, the sum over the set of indices I is bounded by some constant, therefore

$$\|T(f^N)\|^p_{\Sigma^{(p)}_{\theta_0,\theta_1}(M(\theta)B_\theta)} \leqslant c \sum_{|\nu|<N} [w^*(2^\nu) K(2^\nu, f; \vec{A})]^p.$$

A similar estimate also holds for the difference $T(f^{N_1}) - T(f^{N_2})$. Consequently, $T(f^N) \to g$ in the $\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)B_\theta)$ quasi-norm. On the other hand, $f^N \to f$ in $\vec{A}_{\theta,p}$ for some fixed θ, whence $T(f^N) \to T(f)$ in B_θ, and therefore $T(f^N) \to T(f)$ also in $\Sigma^{(p)}_{(\theta_0,\theta_1)}(M(\theta)B_\theta)$. Thus $g = T(f)$ and the theorem is proved. □

The following corollary will be useful in applications.

Corollary 2.4. *Let $0 < \theta_0 < 1, 0 < \sigma < \alpha < 1, 0 < p \leqslant \gamma_{\vec{B}}$. Let $T : \vec{A}_{\theta_0+\sigma,p} \to \vec{B}_{\sigma,\infty}$, be a bounded linear operator with $\|T\|_{\vec{A}_{\theta_0+\sigma,p}\to\vec{B}_{\sigma,\infty}} \leqslant M(\sigma)$ for all σ. Let $w^*_\eta, 0 < \sigma < \eta < \alpha$, be the corresponding weight function w^*, i.e.*

$$\frac{1}{w^*_\eta(t)} = \left\{ \sum_{0<\sigma\leqslant\eta} \left[\frac{t^{\theta_0+\sigma}}{M(\sigma)} \right]^{p^*} \right\}^{1/p^*}, t > 0. \tag{54}$$

Then T can be extended as a bounded linear operator

$$T : \vec{A}_{w^*_\eta,p} \to B_0 + \left\langle (B_0, B_1)_{\eta,p} \right\rangle. \tag{55}$$

Proof. Direct consequence of Theorem 2.15 and Corollary 2.1. □

3. $\Delta^{(p)}$ methods of extrapolation

In this section we turn to the construction of the $\Delta^{(p)}$ methods of extrapolation. As in the case of the $\Sigma^{(p)}$ methods, we shall consider continuous and discrete definitions.

Let $0 < p \leqslant \infty$, $0 \leqslant \theta_0 < \theta_1 \leqslant 1$, $\Theta = (\theta_0, \theta_1)$, and suppose that $\{\int_\Theta [M(\theta)]^p d\theta\}^{1/p} < \infty$, where $M(\theta)$ is positive and continuous on the interval Θ. Then we let

$$\Delta^{(p)}_{\theta\in\Theta}(M(\theta)\vec{A}_{\theta,p}) = \Delta^{(p)}(M(\theta)\vec{A}_{\theta,p})$$
$$= \left\{ f \in \bigcap_{\theta\in\Theta} \vec{A}_{\theta,p} : \|f\|_{\Delta^{(p)}(M(\theta)\vec{A}_{\theta,p})} < \infty \right\},$$

where

$$\|f\|_{\Delta^{(p)}(M(\theta)\vec{A}_{\theta,p})} := \left\{\int_{\Theta}[M(\theta)\|f\|_{\vec{A}_{\theta,p}}]^p d\theta\right\}^{1/p}.$$

It follows that for $0 < \theta_0 < \theta_1 < 1$ (in general the space $\Delta^{(p)}$ could be trivial),

$$A_0 \cap A_1 \subset \Delta^{(p)}(M(\theta)\vec{A}_{\theta,p}).$$

Replacing integrals by series we can give a "discrete'' definition of $\Delta^{(p)}$ methods. Let $\alpha = \theta_0 + 2^{-n_1}$, $\beta = \theta_1 - 2^{n_0}$ be sufficiently close to θ_0 and θ_1 respectively. Suppose that $\left\{\sum_{\theta\in\Theta}[M(\theta)]^p\right\}^{1/p} < \infty$. Let

$$\delta_{\Theta}^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \left\{f \in \bigcap_{n\in I}\vec{A}_{\theta_n,p} : \|f\|_{\delta^{(p)}(M(\theta)\vec{A}_{\theta,p})} < \infty\right\},$$

where

$$\|f\|_{\delta^{(p)}(M(\theta)\vec{A}_{\theta,p})} := \left\{\sum_{\theta\in\Theta}[M(\theta)\|f\|_{\vec{A}_{\theta,p}}]^p\right\}^{1/p}.$$

In a similar manner we define one-sided spaces. Indeed suppose that $\left\{\sum_{\theta\in(\theta_0,\alpha]}[M(\theta)]^p\right\}^{1/p} < \infty$. Then we let

$$\delta_{\theta_0,\alpha}^{(p)-}(M(\theta)\vec{A}_{\theta,p}) = \left\{f \in \bigcap_{n\geqslant n_1}\vec{A}_{\theta_n,p} : \|f\|_{\delta^{(p)-}(M(\theta)\vec{A}_{\theta,p})} < \infty\right\},$$

where

$$\|f\|_{\delta^{(p)-}(M(\theta)\vec{A}_{\theta,p})} := \left\{\sum_{\theta\in(\theta_0,\alpha]}[M(\theta)\|f\|_{\vec{A}_{\theta,p}}]^p\right\}^{1/p}.$$

Analogously, suppose that $\left\{\sum_{\theta\in[\beta,\theta_1)}[M(\theta)]^p\right\}^{1/p} < \infty$. Then we let

$$\delta_{\beta,\theta_1}^{(p)+}(M(\theta)\vec{A}_{\theta,p}) = \left\{f \in \bigcap_{n\leqslant n_0}\vec{A}_{\theta_n,p} : \|f\|_{\delta^{(p)+}(M(\theta)\vec{A}_{\theta,p})} < \infty\right\},$$

where

$$\|f\|_{\delta^{(p)+}(M(\theta)\vec{A}_{\theta,p})} := \left\{\sum_{\theta\in[\beta,\theta_1)}[M(\theta)\|f\|_{\vec{A}_{\theta,p}}]^p\right\}^{1/p}.$$

As was the case for the $\Sigma^{(p)}$ method, we now show that only the behavior of $M(\theta)$ at the end points is important.

Remark 3.1. For any $\theta_0 < \alpha < \beta < \theta_1$ we have

$$\varDelta^{(p)}_{\theta_0,\theta_1}(M(\theta)\vec{A}_{\theta,p}) = \varDelta^{(p)}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) \cap \varDelta^{(p)}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}). \tag{56}$$

Proof. It is plain that

$$\varDelta^{(p)}_{\theta_0,\theta_1}(M(\theta)\vec{A}_{\theta,p}) = \varDelta^{(p)}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) \cap \varDelta^{(p)}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}) \cap \varDelta^{(p)}_{\alpha,\beta}(M(\theta)\vec{A}_{\theta,p}).$$

Hence it is sufficient to prove that

$$\varDelta^{(p)}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) \cap \varDelta^{(p)}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}) \subset \varDelta^{(p)}_{\alpha,\beta}(M(\theta)\vec{A}_{\theta,p}). \tag{57}$$

Since $M(\theta)$ is continuous we see that

$$\varDelta^{(p)}_{\alpha,\beta}(M(\theta)\vec{A}_{\theta,p}) = \varDelta^{(p)}_{\alpha,\beta}(\vec{A}_{\theta,p})$$

and the space on the right-hand side can be computed explicitly by Remark 3.6 below:

$$\varDelta^{(p)}_{\alpha,\beta}(\vec{A}_{\theta,p}) = \vec{A}_{W_\alpha,p} \cap \vec{A}_{W_\beta,p}, \;\; W_\alpha(t) = t^{-\alpha}(1+|\ln t|)^{-1/p}. \tag{58}$$

Therefore,

$$\varDelta^{(p)}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) \subset \varDelta^{(p)}_{\alpha-\varepsilon,\alpha}(\vec{A}_{\theta,p}) = \vec{A}_{W_{\alpha-\varepsilon},p} \cap \vec{A}_{W_\alpha,p}$$

and

$$\varDelta^{(p)}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}) \subset \varDelta^{(p)}_{\beta,\beta+\varepsilon}(\vec{A}_{\theta,p}) = \vec{A}_{W_{\beta+\varepsilon},p} \cap \vec{A}_{W_\beta,p}.$$

Hence

$$\varDelta^{(p)}_{\theta_0,\alpha}(M(\theta)\vec{A}_{\theta,p}) \cap \varDelta^{(p)}_{\beta,\theta_1}(M(\theta)\vec{A}_{\theta,p}) \subset \vec{A}_{W_\alpha,p} \cap \vec{A}_{W_\beta,p}.$$

Thus (57) and hence (56) are proven. □

Analogous result is valid for the discrete method.

Remark 3.2.

$$\delta^{(p)}_{\Theta}(M(\theta)\vec{A}_{\theta,p}) = \delta^{(p)-}_{(\theta_0,\alpha)}(M(\theta)\vec{A}_{\theta,p}) \cap \delta^{(p)+}_{(\beta,\theta_1)}(M(\theta)\vec{A}_{\theta,p}).$$

Proof. We proceed as in the proof of Remark 3.1, but now instead of (58) we use

$$\delta^{(p)}_{\alpha,\beta}(\vec{A}_{\theta,p}) = \vec{A}_{\alpha,p} \cap \vec{A}_{\beta,p}. \quad \square$$

Remark 3.3. We can apply the $\varDelta^{(p)}$ construction to any scale $\{A_\theta\}_{\theta\in\Theta}$ of compatible quasi-Banach spaces, i.e., such that there exist quasi-Banach spaces $\varDelta_A$ and Σ_A such that $\varDelta_A \subset A_\theta \subset \Sigma_A$, and the quasi-norms of the embeddings are uniformly bounded with respect to $\theta \in \Theta$. In this fashion it follows that $\varDelta^{(p)}(M(\theta)A_\theta) \supset \varDelta_A$.

Remark 3.4. Let $\{A_\theta\}$ be a scale of quasi-Banach spaces satisfying

$$A_\theta \supset A_\alpha \cap A_\beta, \quad \alpha < \theta < \beta.$$

Then

$$\delta_{\Theta}^{(p)}(M(\theta)A_\theta) = \delta_{(\theta_0,\alpha)}^{(p)-}(M(\theta)A_\theta) \cap \delta_{(\beta,\theta_1)}^{(p)+}(M(\theta)A_\theta).$$

Analogously to Remark 2.5 we have the following result about equivalency of continuous and discrete definitions. Namely, if the pair (A_0, A_1) is ordered and the weight $M(\sigma)$ satisfies the same property as in Remark 2.5, then

$$\Delta_{0,\alpha}^{(p)}(M(\sigma)\sigma^{-1/p}\vec{A}_{\sigma,r}) = \delta_{0,\alpha}^{(p)-}(M(\sigma)\vec{A}_{\sigma,r}).$$

3.1. Characterization of $\Delta^{(p)}$ spaces

Using Fubini and the definition of the K-method of interpolation, it is readily seen that

$$\Delta^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{W,p}, \tag{59}$$

where the weight function W is defined by the formula

$$W(t) = \left\{\int_\Theta [t^{-\theta}M(\theta)]^p d\theta\right\}^{1/p}, \quad p < \infty, \tag{60}$$

$$W(t) = \sup_\theta t^{-\theta} M(\theta), \quad p = \infty.$$

Note also that for a constant scale we have

$$\Delta^{(p)}(M(\theta)A) = A.$$

Analogously for the discrete constructions we have

$$\delta^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{V,p}, \tag{61}$$

where the weight function V is defined by the formula

$$V(t) = \left\{\sum_\Theta [t^{-\theta}M(\theta)]^p\right\}^{1/p}. \tag{62}$$

Of course, if $p \neq q$, Fubini is not available for the computation of $\Delta^{(p)}(M(\theta)\vec{A}_{\theta,q})$, but we can get around this obstacle if the weights $M(\theta)$ are tempered and the scales under consideration are normalized. Our next result extends a result in [17] for the case $p = \infty$.

Theorem 3.1. *Let $M(\theta)$ be tempered on the interval $(0, 1)$. Then*

$$\Delta_{0,1}^{(p)}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right) = \Delta_{0,1}^{(p)}\left(M(\theta)\left\langle\vec{A}_{\theta,q}\right\rangle\right).$$

Proof. Without loss of generality assume that $q > r$. In view of (30) it is sufficient to prove the embedding

$$\Delta_{0,1}^{(p)}\left(M(\theta)\left\langle \vec{A}_{\theta,\infty}\right\rangle\right) \subset \Delta_{0,1}^{(p)}\left(M(\theta)\left\langle \vec{A}_{\theta,r}\right\rangle\right).$$

We have

$$\begin{aligned}\|f\|^p_{\Delta^{(p)}_{\theta\in\Theta}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right)} &\approx \int_0^1 \left[M(\theta)\left(\int_1^\infty t^{-\theta r}K^r(t,f)dt/t\right)^{1/r}[\theta(1-\theta)r]^{1/r}\right]^p d\theta \\ &+ \int_0^1 \left[M(\theta)\left(\int_0^1 t^{-\theta r}K^r(t,f)dt/t\right)^{1/r}[\theta(1-\theta)r]^{1/r}\right]^p d\theta \\ &= I + II.\end{aligned}$$

To estimate I we make the change of variables $\theta = 2\xi$

$$\begin{aligned} I &= 2\int_0^{1/2}\left[M(2\xi)[(1-2\xi)2\xi r]^{1/r}\left(\int_1^\infty t^{-2\xi r}K(t,f;\vec{A})^r\frac{dt}{t}\right)^{1/r}\right]^p d\xi \\ &\leqslant 2\int_0^{1/2}\left[M(2\xi)[(1-2\xi)2\xi r]^{1/r}\left[\sup_t t^{-\xi}K(t,f;\vec{A})\right]\left(\int_1^\infty t^{-\xi r}\frac{dt}{t}\right)^{1/r}\right]^p d\xi \\ &= 2\int_0^{1/2}\left[M(2\xi)[(1-2\xi)2]^{1/r}\|f\|_{\vec{A}_{\xi,\infty}}\right]^p d\xi. \end{aligned} \tag{63}$$

Since M is tempered and continuous we can replace $M(2\xi)$ by $M(\xi)$ in (63). In this fashion we see that

$$I \leqslant c\int_0^1 [M(\theta)\|f\|_{\vec{A}_{\theta,\infty}}]^p d\theta.$$

The term II can be estimated using a similar analysis. First we split $II = L_1 + L_2$, where

$$L_1 = \int_0^{1/2}\left[M(\theta)\left(\int_0^1 t^{-\theta r}K^r(t,f)dt/t\right)^{1/r}[\theta(1-\theta)r]^{1/r}\right]^p d\theta,$$

$$L_2 = \int_{1/2}^{1}\left[M(\theta)\left(\int_0^1 t^{-\theta r}K^r(t,f)dt/t\right)^{1/r}[\theta(1-\theta)r]^{1/r}\right]^p d\theta.$$

The estimate of L_1 is the same as for I. To estimate L_2 we use the change of variables $\theta = 2\xi - 1$. Thus we obtain

$$II \leqslant c\int_0^1 [M(\theta)\|f\|_{\vec{A}_{\theta,\infty}}]^p d\theta.$$

Combining these estimates we find

$$\|f\|_{\Delta^{(p)}_{\theta\in\Theta}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right)} \leqslant c\,\|f\|_{\Delta^{(p)}_{\theta\in\Theta}\left(M(\theta)\left\langle\vec{A}_{\theta,\infty}\right\rangle\right)}$$

as we wished to show. □

For future applications we now state and prove a discrete version of Theorem 3.1 (cf. Theorem 2.9 above).

Theorem 3.2. *Let* $0 < p, q, r \leqslant \infty$, $\Theta = (0, 1)$, *and let* $M(\theta)$ *be tempered on the interval* $(0, 1)$. *Then*

$$\delta^{(p)}_{\theta\in\Theta}\left(M(\theta)\left\langle\vec{A}_{\theta,r}\right\rangle\right) = \delta^{(p)}_{\theta\in\Theta}\left(M(\theta)\left\langle\vec{A}_{\theta,q}\right\rangle\right).$$

Proof. As in the proof of Theorem 2.9 it is enough to consider the embeddings at the end points. More precisely, it is sufficient to prove the following embeddings:

$$\vec{A}_{\sigma/2,\infty} \cap \vec{A}_{2\sigma,\infty} \subset \sigma^{1/r}\vec{A}_{\sigma,r} \quad \text{if } \sigma \sim 0 \tag{64}$$

and

$$\vec{A}_{2\sigma-1,\infty} \cap \vec{A}_{(1+\sigma)/2,\infty} \subset (1-\sigma)^{1/r}\vec{A}_{\sigma,r} \quad \text{if } \sigma \sim 1. \tag{65}$$

In turn (64) and (65) will follow from

$$\vec{A}_{\sigma/2,\infty} \cap \vec{A}_{2\sigma,\infty} \subset (\vec{A}_{\sigma/2,\infty}, \vec{A}_{2\sigma,\infty})_{1/3,r}$$

and

$$\vec{A}_{2\sigma-1,\infty} \cap \vec{A}_{(1+\sigma)/2,\infty} \subset (\vec{A}_{2\sigma-1,\infty}, \vec{A}_{(1+\sigma)/2,\infty})_{2/3,r}.$$

Indeed, let $B_0 = \vec{A}_{\sigma/2,\infty}$, $B_1 = \vec{A}_{2\sigma,\infty}$, then by Holmsted's formula we get

$$K(t, f; \vec{B}) \geqslant ct^{-1/3}K(t^{\frac{2}{3\sigma}}, f; \vec{A}),$$

whence

$$\int_0^\infty t^{-r/3}K^r(t, f; \vec{B})dt/t \geqslant c\int_0^\infty t^{-2r/3}K^r(t^{\frac{2}{3\sigma}}, f; \vec{A})\,dt/t$$
$$= c\sigma\int_0^\infty t^{-\sigma r}K^r(t, f; \vec{A})\,dt/t.$$

Therefore,

$$(\vec{A}_{\sigma/2,\infty}, \vec{A}_{2\sigma,\infty})_{1/3,r} \subset \sigma^{1/r}\vec{A}_{\sigma,r}.$$

Analogously, if $C_0 = \vec{A}_{2\sigma-1,\infty}$, $C_1 = \vec{A}_{(1+\sigma)/2,\infty}$, then

$$K(t, f; \vec{C}) \geqslant ct^{-\frac{2}{3}\frac{2\sigma-1}{1-\sigma}}K(t^{\frac{2}{3}\frac{1}{1-\sigma}}, f; \vec{A}).$$

It follows that

$$(\vec{A}_{2\sigma-1,\infty}, \vec{A}_{(1+\sigma)/2,\infty})_{2/3,r} \subset (1-\sigma)^{1/r} \vec{A}_{\sigma,r}.$$

The same proof gives □

Theorem 3.3. *Let* $M(\theta)$ *be tempered on the interval* $(0,1)$, *and let* $0<\theta_0<\theta_1<1$. *Then*

$$\Delta^{(p)}_{(0,\theta_1)}\left(M(\theta)\left\langle \vec{A}_{\theta,r}\right\rangle\right) = \Delta^{(p)}_{(0,\theta_1)}\left(M(\theta)\left\langle \vec{A}_{\theta,q}\right\rangle\right);$$

$$\Delta^{(p)}_{(\theta_0,1)}\left(M(\theta)\left\langle \vec{A}_{\theta,r}\right\rangle\right) = \Delta^{(p)}_{(\theta_0,1)}\left(M(\theta)\left\langle \vec{A}_{\theta,q}\right\rangle\right).$$

In the case $0<\theta_0<\theta_1<1$ we have

Theorem 3.4. *Let* $\vec{A}=(A_0,A_1)$ *be a pair of quasi-Banach spaces, and let* M *be tempered on the interval* (θ_0,θ_1), $0<\theta_0<\theta_1<1$, $|1/p-1/r(\theta)|\leqslant c/|\ln(\theta-\theta_0)(\theta_1-\theta)|$. *Then*

$$\Delta^{(p)}_{\Theta}(M(\theta)\vec{A}_{\theta,r(\theta)}) = \Delta^{(p)}_{\Theta}(M(\theta)\vec{A}_{\theta,p}).$$

Analogous results are of course valid for one-sided spaces or the discrete method.

The analog of Corollary 2.3 is

Corollary 3.1. *Let* $M(\theta)$ *be tempered on the interval* $0<\theta_0<\theta_1<1$, *and let* $r>p$. *Then*

$$\Delta^{(p)}_{\Theta}(M(\theta)[(\theta-\theta_0)(\theta_1-\theta)]^{1/r-1/p}\vec{A}_{\theta,r}) \subset \Delta^{(p)}_{\Theta}(M(\theta)\vec{A}_{\theta,p}).$$

The following example shows that in general Theorem 3.4 is not true if $r\neq p$ is fixed.

Example 3.1. *Let* $L^p=L^p(0,1)$, $0<p<r\leqslant\infty$, $a>0$, $0<\varepsilon<\theta_1<1$, $p_0=(1-\theta_1)p$. *Then*

$$\Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,p}) \neq \Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,r}).$$

Proof. Let $\alpha>0$, $1/r<a-\alpha<1/p$. Define $f(t)=t^{-1/p}(1-\ln t)^\alpha$, $0<t<1$. According to Theorem 4.7 below

$$\Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,p}) = L^p(\log L)_{-a}$$

and

$$\|f\|^p_{L^p(\log L)_{-a}} = \int_0^1 (1-\ln t)^{(-a+\alpha)p}\frac{dt}{t} = \infty.$$

On the other hand, $(L^{p_0},L^\infty)_{\theta_1-\sigma,r} = L^{q,r}$, $1/q=1/p+\sigma/p_0$ (uniformly w.r.t. to $\sigma\in(0,\varepsilon)$). Therefore

$$\|f\|^r_{L^{q,r}} \approx \int_0^1 e^{r\sigma\ln t/p_0}(1-\ln t)^{\alpha r}\frac{dt}{t} = \int_1^\infty e^{r\sigma(1-s)/p_0}s^{\alpha r}\,ds \approx \sigma^{-\alpha r-1}.$$

Consequently

$$\|f\|^p_{\Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,r})} \approx \int_0^\varepsilon \sigma^{(a-\alpha-1/r)p}\frac{d\sigma}{\sigma} < \infty.$$

We also have (cf. Corollary 3.1) □

Remark 3.5. Let $L^p = L^p(0,1)$, $0 < p < r \leqslant \infty, a > 0, 0 < \varepsilon < \theta_1 < 1, p_0 = (1-\theta_1)p$. Then

$$\Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}\sigma^{1/r-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,p}) \neq \Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,r}).$$

Proof. Let $1/r < \alpha < 1/p, 1/p-\alpha < a \leqslant 1/p - 1/r$. Define $f(t) = t^{-1/p}(1-\ln t)^{-\alpha}$, $0<t<1$. According to Theorem 4.7

$$\Delta^{(p)}_{(0,\varepsilon)}(\sigma^{a-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,p}) = L^p(\log L)_{-a}$$

and

$$\|f\|^p_{L^p(\log L)_{-a}} = \int_0^1 (1-\ln t)^{(-a+\alpha)p}\frac{dt}{t} < \infty.$$

On the other hand, $(L^{p_0},L^\infty)_{\theta_1-\sigma,r} = L^{q,r}, 1/q = 1/p+\sigma/p_0$ (uniformly w.r.t. to $\sigma\in(0,\varepsilon)$) and

$$\|f\|^r_{L^{q,r}} \approx \int_0^1 e^{r\sigma\ln t/p_0}(1-\ln t)^{-\alpha r}\frac{dt}{t} = \int_1^\infty e^{r\sigma(1-s)/p_0}s^{-\alpha r}ds \approx 1.$$

Hence

$$\|f\|^p_{\Delta^{(p)}_{0,\varepsilon}(\sigma^{a-1/p}\sigma^{1/r-1/p}(L^{p_0},L^\infty)_{\theta_1-\sigma,r})} \approx \int_0^\varepsilon \sigma^{(a+1/r-1/p)p}d\sigma/\sigma = \infty. \qquad \square$$

Now we characterize the $\Delta^{(p)}$ spaces in the case $M=1$.

Remark 3.6. Let $0\leqslant\theta_0<\theta_1\leqslant 1, W_j(t) = t^{-\theta_j}(1+|\ln t|)^{-1/p}$, $j=0,1$. Then

$$\Delta^{(p)}_\Theta(\vec{A}_{\theta,p}) = \vec{A}_{W_0,p}\cap\vec{A}_{W_1,p}.$$

Proof. Use (60). □

Here is another variant:

Remark 3.7. Let $0<\theta_0<\theta_1<1$. Then for $0<q\leqslant\infty$,

$$\Delta^{(\infty)}_\Theta(\vec{A}_{\theta,q}) = \vec{A}_{\theta_0,q}\cap\vec{A}_{\theta_1,q}.$$

Proof. Since $\vec{A}_{\theta,q} = \left\langle (B_0,B_1)_{\sigma,q}\right\rangle$, $\theta = (1-\sigma)\theta_0+\sigma\theta_1$, $B_j = \vec{A}_{\theta_j,q}$, we have

$$\Delta^{(\infty)}_\Theta(\vec{A}_{\theta,q}) \supset \vec{A}_{\theta_0,q}\cap\vec{A}_{\theta_1,q}.$$

Conversely, note that for all $\theta \in (\theta_0, \theta_1)$,

$$\|f\|_{\vec{A}_{\theta,q}} \leqslant \|f\|_{\Delta_{\Theta}^{(\infty)}(\vec{A}_{\theta,q})}.$$

We complete the proof by means of letting $\theta \to \theta_j$, $j = 0$ or $j = 1$ (if $q < \infty$ we use Fatou's lemma). □

An analogous result, with similar proof, is valid for the discrete method.

Remark 3.8. Let $0 < \theta_0 < \theta_1 < 1$. Then for $0 < q \leqslant \infty$,

$$\delta_{\Theta}^{(\infty)}(\vec{A}_{\theta,q}) = \vec{A}_{\theta_0,q} \cap \vec{A}_{\theta_1,q}.$$

As we have seen, the $\Delta^{(p)}$ space is an intersection of two "end'' spaces,

$$\Delta^{(p)}(M(\theta)\vec{A}_{\theta,p}) = \vec{A}_{W_0,p} \cap \vec{A}_{W_1,p}, \tag{66}$$

where the weights W_0, and W_1 are given by

$$W_0(t) = \left\{\int_{\theta_0}^{\alpha} [t^{-\theta}M(\theta)]^p d\theta\right\}^{1/p}, \ W_1(t) = \left\{\int_{\beta}^{\theta_1} [t^{-\theta}M(\theta)]^p d\theta\right\}^{1/p}.$$

These weights satisfy the following properties

$$W_1(t) \leqslant cW_0(t) \quad \text{if } t > 1,$$

$$W_0(t) \leqslant cW_1(t) \quad \text{if } 0 < t < 1.$$

It follows that

$$\|f\|_{\vec{A}_{W_0,p}\cap\vec{A}_{W_1,p}} \approx \left\{\int_0^1 [W_1(t)K(t,f;\vec{A})]^p \frac{dt}{t} + \int_1^\infty [W_0(t)K(t,f;\vec{A})]^p \frac{dt}{t}\right\}^{1/p}.$$

Therefore in the computation of $\|f\|_{\vec{A}_{W_0,p}\cap\vec{A}_{W_1,p}}$ the weight $W_1(t)$ plays a role only for t in the range $0 < t < 1$ and the weight $W_0(t)$ plays a role only for $t > 1$.

3.2. Extrapolation theorems

We now write down a prototype extrapolation theorem for the $\Delta^{(p)}$ method (analogous results are valid for the discrete methods).

Theorem 3.5 (Δ *extrapolation theorem*). *Suppose that* $\{A_\theta\}$ *and* $\{B_\theta\}$ *are scales of spaces such that* $\Delta_A \subset A_\theta \subset \Sigma_A$, $\Delta_B \subset B_\theta \subset \Sigma_B$ *uniformly with respect to* $\theta \in \Theta$. *Let* T *be an operator* (*not necessarily linear*) *such that* $T : A_\theta \to B_\theta$ *is bounded with quasi-norm* 1 *for all* $\theta \in \Theta$. *Then, for all functions* $M(\theta)$ *such that* $\{\int_\Theta [M(\theta)]^p d\theta\}^{1/p} < \infty$, *we have*

$$T : \Delta^{(p)}(M(\theta)A_\theta) \to \Delta^{(p)}(M(\theta)B_\theta)$$

with quasi-norm 1.

In the applications we need some variants of the previous results.

Corollary 3.2. *Let* $0 < \theta_0 < 1$, $0 < \sigma < \alpha < 1$, $0 < p \leqslant \gamma_B$, *and let* $T : \vec{A}_{\theta_0+\sigma,p} \to \vec{B}_{\sigma,\infty}$, *be a bounded linear operator with* $\|T\|_{\vec{A}_{\theta_0+\sigma,p}\to\vec{B}_{\sigma,\infty}} \leqslant M(\sigma)$ *for all* σ. *Let* w_η^* *be given by*

$$\frac{1}{w_\eta^*(t)} = \left\{ \sum_{0<\sigma\leqslant\eta} \left[\frac{t^{\theta_0+\sigma}}{M(\sigma)} \right]^{p^*} \right\}^{1/p^*}, \quad t > 0. \tag{67}$$

Then T *can be extended as a bounded linear operator*

$$T : \varDelta_{(0,\alpha)}^{(\infty)}(N(\eta)\vec{A}_{w_\eta^*,p}) \to \varDelta_{(0,\alpha)}^{(\infty)}\left(N(\eta)\left(B_0 + \left\langle \vec{B}_{\eta,p} \right\rangle\right)\right). \tag{68}$$

Proof. Use Corollary 2.4. □

In our applications to the theory of log Sobolev inequalities we shall need still another type of extrapolation theorem involving the $\varDelta$ and Σ methods.

Theorem 3.6 ($\varDelta - \Sigma$ *extrapolation*). *Let* $\{A_\theta\}$ *and* $\{B_\theta\}$ *be scales of Banach spaces such that* $\varDelta_A \subset A_\theta \subset \Sigma_A$, *and* $B_\theta \subset \Sigma_B$ *uniformly with respect to* $\theta \in (0,\varepsilon)$. *Suppose that* $\{T(\theta)\}_{\theta\in(0,\varepsilon)}$ *is a family of bounded linear operators,* $T(\theta) : A_\theta \to B_\theta$ *and let* $\|T\|_{A_\theta\to B_\theta} = M(\theta)$, $\theta \in (0,\varepsilon)$. *Suppose that* $M(\theta)$ *satisfies* $\int_0^\varepsilon M(\theta)d\theta/\theta < \infty$, *and let* $\bar{T}$ *be the operator defined on* $\varDelta_A$ by $\bar{T} = \int_0^\varepsilon T(\theta)d\theta/\theta$. *Then,* (i) $\bar{T} : \varDelta_A \to \Sigma_B$ *is a bounded operator.* (ii) *Suppose that* $N(\theta)$ *is a positive function satisfying* $\{\int_0^\varepsilon [N(\theta)]^{-p^*} d\theta/\theta\}^{1/p^*} < \infty$, $\{\int_0^\varepsilon [N(\theta)M(\theta)]^p \frac{d\theta}{\theta}\}^{1/p} < \infty$. *Then, if* $\varDelta_A$ *is dense in* $\varDelta_{(0,\varepsilon)}^{(p)}(M(\theta)N(\theta)\theta^{-1/p}A_\theta)$, *it follows that* $\bar{T}$ *has a norm-one bounded extension*

$$\bar{T} : \varDelta_{(0,\varepsilon)}^{(p)}(M(\theta)N(\theta)\theta^{-1/p}A_\theta) \to \int_{p,0,\varepsilon} (N(\theta)B_\theta).$$

Proof. Follows immediately from the definitions. □

3.3. Duality

Now we give a duality result

Proposition 3.1. *Let* $\vec{A}$ *be a Banach pair and let* $1 \leqslant p < \infty$. *If* $A_0 \cap A_1$ *is dense in* A_j, $j = 0, 1$ *and* $\vec{A}^* = (A_0^*, A_1^*)$ *is the dual pair, then*

$$\left\{ \int_{p,0,\varepsilon} (M(\sigma)\vec{A}_{\sigma+\theta_0,p;J}) \right\}^* = \Delta_{(0,\varepsilon)}^{(p^*)} \left(\frac{\sigma^{-1/p^*}}{M(\sigma)} \vec{A}^*_{\sigma+\theta_0,p^*} \right).$$

Proof. According to Remark 2.7, the left-hand side is the space $\{\vec{A}_{v,p;J}\}^* = \vec{A}^*_{h,p^*}$, where (cf. [6]) $h(t) = \frac{1}{v(1/t)}$. It remains to apply formula (59).

4. Computations

4.1. $\Sigma^{(p)}$ spaces

Let $M(\theta) = (\theta - \theta_0)^{-a_0}(\theta_1 - \theta)^{-a_1}$, $a_j > 0$, $j = 0, 1$ with $0 \leqslant \theta_0 < \theta < \theta_1 \leqslant 1$ (and $a_j \geqslant 0$ if $p \leqslant \gamma_A$). Our first goal will be to characterize the $\Sigma^{(p)}$ extrapolation spaces for the scale $\{M(\theta)(A_0, A_1)_{\theta,p}\}_{\theta \in \Theta}$. We will then apply the general results to explicitly compute the corresponding $\Sigma^{(p)}$ extrapolation spaces for pairs of L^p spaces.

To apply the results of previous sections (e.g. Theorems 2.2, 2.3, etc.) we need to compute the weights $\{w_0, w_1\}$ defined by,

$$[w_0(t)]^{-1} = \left\{ \sum_{\theta_0 < \theta < \alpha} \left[\frac{t^\theta}{M(\theta)} \right]^q \right\}^{1/q}, \ 0 < t < 1,$$

$$[w_1(t)]^{-1} = \left\{ \sum_{\beta < \theta < \theta_1} \left[\frac{t^\theta}{M(\theta)} \right]^q \right\}^{1/q}, \ t > 1,$$

where [14] $q = p^+$ or $q = \infty$ and $\alpha = \theta_0 + 2^{-n_1}$, $\beta = \theta_1 - 2^{n_0}$, where $n_0 < 0$, and $n_1 > 0$.

We shall treat in detail only the case $q < \infty$. The necessary modifications for the case $q = \infty$ are left to the reader.

It is readily seen (by monotonicity) that

$$\begin{aligned} c_1 \int_{\theta_0}^{\tilde{\alpha}} t^{\theta q} (\theta - \theta_0)^{a_0 q - 1} d\theta &\leqslant [w_0(t)]^{-q} \\ &\leqslant c_2 \int_{\theta_0}^{\alpha} t^{\theta q} (\theta - \theta_0)^{a_0 q - 1} d\theta, \quad 0 < t < 1, \quad \tilde{\alpha} = \alpha - 2^{-n_1 - 1} \end{aligned}$$

and

$$\begin{aligned} c_1 \int_{\tilde{\beta}}^{\theta_1} t^{\theta q} (\theta_1 - \theta)^{a_1 q - 1} d\theta &\leqslant [w_1(t)]^{-q} \\ &\leqslant c_2 \int_{\beta}^{\theta_1} t^{\theta q} (\theta_1 - \theta)^{a_1 q - 1} d\theta, \quad t > 1, \quad \tilde{\beta} = \beta + 2^{n_0 - 1}. \end{aligned}$$

By a change of variables we have to calculate the integrals:

$$\int_0^{\sigma_0} t^{\theta_0 q} t^{\sigma q} \sigma^{a_0 q - 1} d\sigma, \quad 0 < t < 1$$

and

$$\int_0^{\sigma_1} t^{\theta_1 q} t^{-\sigma q} \sigma^{a_1 q - 1} d\sigma, \quad t > 1.$$

[14] See (10).

Set $\lambda = q|\ln t|$, then $\lambda \to \infty$ as $t \to 0$ or $t \to \infty$. We have to find the asymptotic properties of the integral

$$I(\lambda, a) = \int_0^c e^{-\sigma\lambda}\sigma^{aq-1}d\sigma.$$

We have

$$I(\lambda, a) \approx \lambda^{-aq}.$$

Therefore, for t near 0,

$$[w_0(t)]^{-q} \approx t^{\theta_0 q}|\ln t|^{-a_0 q},$$

it follows that

$$w_0(t) \approx (1 - \ln t)^{a_0} t^{-\theta_0}, \quad 0 < t < 1. \tag{69}$$

Similarly,

$$w_1(t) \approx (1 + \ln t)^{a_1} t^{-\theta_1}, \quad 1 < t < \infty. \tag{70}$$

The previous discussion leads to

Theorem 4.1 ($\Sigma^{(p)}$ *space*). *Let* $0 < \theta_0 < \theta < \theta_1 < 1, 0 < p \leqslant \infty, a_i > 0, i = 0, 1$. *Let* $w_j(t) = (1 + |\ln t|)^{a_j} t^{-\theta_j},\ j = 0, 1$. *Then*

$$\Sigma^{(p)}((\theta - \theta_0)^{-a_0}(\theta_1 - \theta)^{-a_1}(A_0, A_1)_{\theta,p}) = \vec{A}_{w_0,p} + \vec{A}_{w_1,p}.$$

Remark 4.1 ($\Sigma^{(p)}$ *space*, $0 < p \leqslant \gamma_{\vec{A}}$). If $0 < p \leqslant \gamma_{\vec{A}}$ then $p^+ = p^* = \infty$ and the same proof shows that Theorem 4.1 remains valid for $a_j \geqslant 0,\ j = 0, 1$.

Analogous results are valid for the one-sided extrapolation spaces $\Sigma^{(p)-}$ and $\Sigma^{(p)+}$. For example,

Theorem 4.2 ($\Sigma^{(p)-}$ *space*). *Let* $0 < \theta_0 < \theta \leqslant \alpha, 0 < p \leqslant \infty, a > 0,\ w_0(t) = (1 + |\ln t|)^a t^{-\theta_0}$. *Then*

$$\Sigma^{(p)-}_{(\theta_0,\alpha)}((\theta - \theta_0)^{-a}(A_0, A_1)_{\theta,p}) = \vec{A}_{w_0,p} + \vec{A}_{\alpha,p}.$$

Proof. We only need to compute the weight

$$[w(t)]^{-1} = \left\{ \sum_{\theta_0 < \theta \leqslant \alpha} [t^\theta(\theta - \theta_0)^a]^q \right\}^{1/q}, 1 < t < \infty.$$

Let $q < \infty$. Since $\alpha = \theta_0 + 2^{-n_1}$ and $\theta_n - \theta_0 = 2^{-n}$ we need to estimate:

$$I(t) := \sum_{n \geqslant n_1} t^{q\theta_n}(\theta_n - \theta_0)^{qa}, \quad t > 1.$$

We have

$$\begin{aligned} I(t) &= t^{q\theta_0} \sum_{n \geqslant n_1} t^{q2^{-n}} 2^{-qan} \\ &= t^{q\theta_0} 2^{-qan_1} \sum_{k \geqslant 0} t^{q2^{-n_1}2^{-k}} 2^{-qak} \\ &= t^{q\theta_0} 2^{-qan_1} t^{q2^{-n_1}} \sum_{k \geqslant 0} t^{q2^{-n_1}(2^{-k}-1)} 2^{-qak} \\ &= t^{q\alpha} 2^{-qan_1} g(t), \end{aligned}$$

where

$$g(t) := \sum_{k \geqslant 0} t^{q2^{-n_1}(2^{-k}-1)} 2^{-qak}.$$

Since $g(t) \approx 1$ for $t > 1$ we get $w(t) \approx t^{-\alpha}$ if $t > 1$. The result follows. □

Theorem 4.3 ($\Sigma^{(p)}$ *space*). *Let* $0 < \theta_0 < \theta < \theta_1 < 1, 0 < p_0 \leqslant p \leqslant \infty, a > 0, b > 0$, $1/r_i = (1 - \theta_i)/p_0, i = 0, 1$. *Let* Ω *be a* σ*-finite measure space and let* $(L^{p_0}, L^\infty) = (L^{p_0}(\Omega), L^\infty(\Omega))$. *Then*

$$\Sigma^{(p)}((\theta - \theta_0)^{-a}(\theta_1 - \theta)^{-b}(L^{p_0}, L^\infty)_{\theta,p}) = L^{r_0,p}(\log L)_a + L^{r_1,p}(\log L)_b,$$

where the logarithmic Lorentz–Zygmund spaces $L^{r,p}(\log L)_a, 0 < p, r \leqslant \infty, a \in R$, *are defined by the quasi-norm* (cf. [4])

$$\|f\|_{L^{r,p}(\log L)_a} = \left\{ \int_0^\infty (1 + |\ln t|)^{ap} t^{p/r} [f^*(t)]^p \frac{dt}{t} \right\}^{1/p}.$$

Analogously

Theorem 4.4 ($\Sigma^{(p)-}$ *space*). *Let* $0 < \theta_0 < \theta < \alpha, 0 < p_0 \leqslant p \leqslant \infty, a > 0$, $1/r_0 = (1 - \theta_0)/p_0$, $1/r_1 = (1 - \alpha)/p_0$. *Then*

$$\Sigma^{(p)-}_{(\theta_0,\alpha)}((\theta - \theta_0)^{-a}(L^{p_0}, L^\infty)_{\theta,p}) = L^{r_0,p}(\log L)_a + L^{r_1,p}.$$

Proof of Theorems 4.3 and 4.4. It is sufficient to prove that for $0 < \theta \leqslant 1$, $w(t) = (1 + |\ln t|)^a t^{-\theta}$, $1/r = (1 - \theta)/p_0$, $p \geqslant p_0$,

$$(L^{p_0}, L^\infty)_{w,p} = L^{r,p}(\log L)_a. \tag{71}$$

Moreover, note that the embedding

$$(L^{p_0}, L^\infty)_{w,p} \subset L^{r,p}(\log L)_a \tag{72}$$

is valid if $\theta = 0$. Using the well-known formula (cf. [5])

$$K(t, f; L^{p_0}, L^\infty) \approx \left\{ \int_0^{t^{p_0}} [f^*(s)]^{p_0} ds \right\}^{1/p_0}, \quad p_0 > 0 \tag{73}$$

and the fact that f^* decreases, we find

$$c\|f\|_{(L^{p_0},L^\infty)_{w,p}} \geqslant \left\{\int_0^\infty t^{-\theta p}(1+|\ln t|)^{ap}[f^*(t^{p_0})t]^p \frac{dt}{t}\right\}^{1/p}.$$

Consequently

$$(L^{p_0}, L^\infty)_{w,p} \subset L^{r,p}(\log L)_a,\ 1/r = (1-\theta)/p_0. \tag{74}$$

Let $K(t, f) = K(t, f; L^{p_0}, L^\infty)$. Then by (73),

$$\int_0^\infty t^{-\theta p}(1+|\ln t|)^{ap}[K(t, f)]^p \frac{dt}{t} \leqslant \int_0^\infty t^{(1-\theta)p}(1+|\ln t|)^{ap}\left(\int_0^1 [f^*(st^{p_0})]^{p_0} ds\right)^{p/p_0} \frac{dt}{t}.$$

Applying Minkowski's inequality (recall that $p \geqslant p_0$) and making the change of variables $st^{p_0} = \sigma$ we get

$$\int_0^\infty t^{-\theta p}(1+|\ln t|)^{ap}[K(t, f)]^p \frac{dt}{t}$$

$$\leqslant c\left\{\int_0^1 \left(\int_0^\infty \sigma^{p/r}(1+|\ln(\sigma/s)|)^{ap}[f^*(\sigma)]^p \frac{d\sigma}{\sigma}\right)^{p_0/p} s^{\theta-1} ds\right\}^{p/p_0}.$$

Now we use the elementary inequalities

$$1+|\ln\sigma - \ln s| \leqslant (1+|\ln\sigma|)(1-\ln s)$$

$$1+|\ln\sigma - \ln s| \geqslant (1+|\ln\sigma|)(1-\ln s)^{-1},\ \ 0 < s < 1,$$

to conclude that

$$c\|f\|^p_{(L^{p_0},L^\infty)_{w,p}} \leqslant \int_0^\infty [f^*(\sigma)]^p \sigma^{p/r}(1+|\ln\sigma|)^{ap}\frac{d\sigma}{\sigma}\left[\int_0^1 s^{\theta-1}(1-\ln s)^{|a|p} ds\right]^{p/p_0} \leqslant c\int_0^\infty [f^*(\sigma)]^p \sigma^{p/r}(1+|\ln\sigma|)^{ap}\frac{d\sigma}{\sigma}.$$

Therefore

$$(L^{p_0}, L^\infty)_{w,p} \supset L^{r,p}(\log L)_a,\ 1/r = (1-\theta)/p_0,$$

which, combined with (74), proves the desired result. □

Remark 4.2 ($\Sigma^{(p)}$ *space*, $0 < p = p_0 \leqslant 1$). If $0 < p = p_0 \leqslant 1$ then we can choose $p^+ = p^* = \infty$ and the same proof shows that Theorem 4.3 remains valid for $a \geqslant 0$, $b \geqslant 0$.

4.2. $\Delta^{(p)}$ spaces

Let $M(\theta) = (\theta - \theta_0)^{a_0 - 1/p}(\theta_1 - \theta)^{a_1 - 1/p}$, $a_j > 0$, $j = 0, 1$, $0 < p \leqslant \infty$, with $0 \leqslant \theta_0 < \theta < \theta_1 \leqslant 1$. In this section we compute the extrapolation spaces $\Delta^{(p)}\{M(\theta)(A_0, A_1)_{\theta,p}\}_{\theta\in\Theta}$. We then apply the general results to explicitly compute the corresponding $\Delta^{(p)}$ extrapolation spaces for pairs of L^p spaces.

According to (66) if $p < \infty$ (the case $p = \infty$ can be treated analogously)

$$\Delta^{(p)}((\theta - \theta_0)^{a_0 - 1/p}(\theta_1 - \theta)^{a_1 - 1/p}\vec{A}_{\theta,p})) = \vec{A}_{W_0,p} \cap \vec{A}_{W_1,p},$$

where

$$[W_0(t)]^p = \int_{\theta_0}^{\alpha} (\theta - \theta_0)^{a_0 p - 1}(\theta_1 - \theta)^{a_1 p - 1} t^{-\theta p} d\theta,$$

$$[W_1(t)]^p = \int_{\beta}^{\theta_1} (\theta - \theta_0)^{a_0 p - 1}(\theta_1 - \theta)^{a_1 p - 1} t^{-\theta p} d\theta.$$

Moreover, by the discussion that follows Remark 3.8 we only need to compute the weight $W_0(t)$ for $t > 1$ and the weight $W_1(t)$ for $0 < t < 1$.

The arguments we gave during the course of the proof of Theorem 4.1 show that

$$W_0(t) \approx t^{-\theta_0}(1 + \ln t)^{-a_0} \quad \text{if } 1 < t < \infty,$$

$$W_1(t) \approx t^{-\theta_1}(1 - \ln t)^{-a_1} \quad \text{if } 0 < t < 1.$$

Thus we have

Theorem 4.5 ($\Delta^{(p)}$ *space*). *Let* $0 \leqslant \theta_0 < \theta < \theta_1 \leqslant 1$, $0 < p \leqslant \infty$, $a_i > 0$, $i = 0, 1$, $W_j(t) = (1 + |\ln t|)^{-a_j} t^{-\theta_j}$, $j = 0, 1$. *Then*

$$\Delta^{(p)}_{\theta\in\Theta}((\theta - \theta_0)^{a_0 - 1/p}(\theta_1 - \theta)^{a_1 - 1/p}\vec{A}_{\theta,p}) = \vec{A}_{W_0,p} \cap \vec{A}_{W_1,p}.$$

Remark 4.3 ($\Delta^{(\infty)}$ *space*). If $p = \infty$ the same proof shows that Theorem 4.5 remains valid for $a_j \geqslant 0$, $j = 0, 1$.

Analogous results are valid for one-sided extrapolation spaces. For example,

Theorem 4.6 ($\delta^{(p)-}$ *space*). *Let* $0 \leqslant \theta_0 < \theta \leqslant \theta_1 \leqslant 1$, $0 < p \leqslant \infty$, $a > 0$, $W_0(t) = (1 + |\ln t|)^{-a} t^{-\theta_0}$. *Then*

$$\delta^{(p)-}_{\Theta}((\theta - \theta_0)^{a}\vec{A}_{\theta,p}) = \vec{A}_{W_0,p} \cap \vec{A}_{\theta_1,p}.$$

As a corollary we get the following results for logarithmic Lorentz spaces.

Theorem 4.7 ($\Delta^{(p)}$ *space*). *Let* $0 < \theta_0 < \theta < \theta_1 \leqslant 1$, $0 < p_0 \leqslant p \leqslant \infty$, $a_i > 0$, $1/r_i = (1-\theta_i)/p_0$, $i = 0, 1$. *Then*

$$\Delta^{(p)}_{\theta \in \Theta}((\theta-\theta_0)^{a_0-1/p}(\theta_1-\theta)^{a_1-1/p}(L^{p_0}, L^{\infty})_{\theta,p}) \\ = L^{r_0,p}(\log L)_{-a_0} \cap L^{r_1,p}(\log L)_{-a_1}.$$

Also,

$$\delta^{(p)-}_{\Theta}((\theta-\theta_0)^{a}(L^{p_0}, L^{\infty})_{\theta,p}) = L^{r_0,p}(\log L)_{-a} \cap L^{r_1,p}.$$

Remark 4.4 ($\Delta^{(\infty)}$ *space*). If $p = \infty$ the same proof shows that Theorem 4.7 remains valid for $a_j \geqslant 0$, $j = 0, 1$.

5. Applications

5.1. Extrapolation theorems of Yano type

Theorem 5.1. *Let* $\vec{A}, \vec{B}$ *be pairs of quasi-Banach spaces, let* $0 < a_j, b_j, s_j < \infty$ $(j = 0, 1)$, $0 < \theta_0 < \theta < \theta_1 < 1$, *and let* T *be a linear operator satisfying*

$$\|Tf\|_{\vec{B}_{\theta,s_j}} \leqslant c|\theta-\theta_j|^{-a_j+b_j}\,\|f\|_{\vec{A}_{\theta,s_j}}.$$

Then

$$T : \vec{A}_{w_0,s_0} + \vec{A}_{w_1,s_1} \to \vec{B}_{v_0,s_0} + \vec{B}_{v_1,s_1},$$

where $w_j(t) = t^{-\theta_j}(1+|\ln t|)^{a_j}$, $v_j(t) = t^{-\theta_j}(1+|\ln t|)^{b_j}$ $(j = 0, 1)$.

Proof. The assumptions, combined with Theorem 2.15, imply

$$T : \vec{A}_{f_0,s_0} \to \Sigma^{(s_0)-}_{\theta_0,\theta_1}((\theta-\theta_0)^{-b_0}\vec{B}_{\theta,s_0}),$$

and

$$T : \vec{A}_{f_1,s_1} \to \Sigma^{(s_1)+}_{\theta_0,\theta_1}((\theta_1-\theta)^{-b_1}\vec{B}_{\theta,s_1}),$$

where

$$f_0(t) = \begin{cases} t^{-\theta_0}(1-\ln t)^{a_0} & 0 < t < 1, \\ t^{-\theta_1} & t > 1, \end{cases}$$

$$g_0(t) = \begin{cases} t^{-\theta_0}(1-\ln t)^{b_0} & 0 < t < 1, \\ t^{-\theta_1} & t > 1, \end{cases}$$

$$f_1(t) = \begin{cases} t^{-\theta_0} & 0 < t < 1, \\ t^{-\theta_1}(1+\ln t)^{a_1} & t > 1, \end{cases}$$

$$g_1(t) = \begin{cases} t^{-\theta_0} & 0 < t < 1, \\ t^{-\theta_1}(1+\ln t)^{b_1} & t > 1. \end{cases}$$

Thus

$$T:\vec{A}_{f_0,s_0}+\vec{A}_{f_1,s_1}\to\vec{B}_{g_0,s_0}+\vec{B}_{g_1,s_1}.$$

By Theorem 2.8,

$$\vec{A}_{f_0,s_0}+\vec{A}_{f_1,s_1}=\vec{A}_{w_0,s_0}+\vec{A}_{w_1,s_1}$$

and

$$\vec{B}_{g_0,s_0}+\vec{B}_{g_1,s_1}=\vec{B}_{v_0,s_0}+\vec{B}_{v_1,s_1}.$$

The desired result follows. □

Corollary 5.1. *Let $r_0<q<r_1<\infty$, $0<s_j,a_j,b_j<\infty$, $(j=0,1)$, and let T be a sublinear operator satisfying*

$$\|Tf\|_{L^{q,s_j}}\leqslant c|q-r_j|^{-a_j+b_j}\|f\|_{L^{q,s_j}}.$$

Then

$$T:L^{r_0,s_0}(\log L)_{a_0}+L^{r_1,s_1}(\log L)_{a_1}\to L^{r_0,s_0}(\log L)_{b_0}+L^{r_1,s_1}(\log L)_{b_1}.$$

If $s_j=s\leqslant 1$, $j=0,1$, we can prove the following result which can be considered a generalization of Yano's classical extrapolation theorem [29]. The abstract version of the result is a consequence of Theorems 2.4 and 4.2.

Theorem 5.2. *Let $a>0, 0<\sigma<\eta<1$, $0<\theta_0<1$, $0<s\leqslant\gamma_{\vec{B}}$, and let $\vec{A},\vec{B}$ be quasi-Banach pairs. Suppose that T is a linear operator satisfying*

$$\|Tf\|_{\vec{B}_{\sigma,\infty}}\leqslant c\sigma^{-a}\|f\|_{\vec{A}_{\sigma(1-\theta_0)+\theta_0,s}}.$$

Then

$$T:\vec{A}_{w_a,s}+\vec{A}_{\theta,s}\to B_0+\vec{B}_{\theta,s},$$

where $w_a(t)=t^{-\theta_0}(1+|\ln t|)^a$, $\theta=\eta(1-\theta_0)+\theta_0$.

Corollary 5.2. *Let T be a sublinear operator satisfying*

$$\|Tf\|_{L^{q,\infty}}\leqslant c(q-s)^{-a}\|f\|_{L^{q,s}},\ 0<s<q<p<\infty,\ s\leqslant 1, a>0.$$

Then

$$T:L^s(\log L)_a+L^{p,s}\to L^s+L^{p,s},$$

where $L^{q,\infty}:=(L^s,L^\infty)_{\sigma,\infty}$, $1/q=(1-\sigma)/s$.

Proof. Apply Theorem 5.2 to $\vec{A}=(L^{p_0},L^\infty)$, where $p_0=(1-\theta_0)s, 0<\theta_0<1$ and $\vec{B}=(L^s,L^\infty)$. We use

$$L^{q,s}=(L^{p_0},L^\infty)_{\sigma(1-\theta_0)+\theta_0,s},\ 1/q=(1-\sigma)/s=(1-\sigma(1-\theta_0)-\theta_0)/p_0.$$

In particular,

$$L^{p,s} = (L^{p_0}, L^\infty)_{\eta(1-\theta_0)+\theta_0,s}, \ \ 1/p = (1-\eta)/s = (1-\eta(1-\theta_0)-\theta_0)/p_0.$$

To conclude we apply Theorem 5.2 and use formula (71). □

The following result is a generalization of Theorem 4.1 [15]. To simplify the statement we shall consider L^p spaces on finite measure spaces.

Corollary 5.3. *Let T be a sublinear operator satisfying*

$$\|Tf\|_{L^{rq}} \leqslant c(q-1)^{-b} \|f\|_{L^{q,1}}, \ \ 1 < q < p < \infty, \ \ r \geqslant 1, a \geqslant b > 0.$$

Then

$$T : L(\log L)_a \to L^r(\log L)_{a-b}.$$

Proof. Let

$$L^{q,1} = (L^{p_0}, L^\infty)_{\sigma+\theta_0,1}, 1/q = 1 - \sigma/p_0, 1-\theta_0 = p_0.$$

Then $L^{rq} = (L^{rp_0}, L^\infty)_{\sigma+\theta_0,rq}$. Applying Theorems 2.15, 4.4 we get

$$T : L(\log L)_a \to \Sigma^{(1)-}(\sigma^{b-a}(L^{rp_0}, L^\infty)_{\sigma+\theta_0,rq}).$$

It remains to identify the space on the right-hand side. If $a > b$, we have

$$\Sigma^{(1)-}(\sigma^{b-a}(L^{rp_0}, L^\infty)_{\sigma+\theta_0,rq}) \subset \Sigma^{(r)-}(\sigma^{b-a}(L^{rp_0}, L^\infty)_{\sigma+\theta_0,rq})$$

and by Theorems 2.13, 4.4 this is the same as

$$\Sigma^{(r)-}(\sigma^{b-a}(L^{rp_0}, L^\infty)_{\sigma+\theta_0,r}) = L^r(\log L)_{a-b}.$$

If $a = b$ we have

$$\Sigma^{(1)-}(L^{rp_0}, L^\infty)_{\sigma+\theta_0,rq} = \Sigma^{(1)-}L^{rq} \subset \Sigma^{(1)-}L^r \subset L^r.$$

The result follows. □

Another variant of these results can be proved using Corollary 3.2 (see [9,22] for similar results for Banach pairs).

Theorem 5.3. *Let $\vec{A}, \vec{B}$ be quasi-Banach pairs, let $a > 0$, $0 < \sigma < \alpha < 1$, $0 < \theta_0 < 1$, $0 < s \leqslant \gamma_{\vec{B}}$, and let T be a linear operator satisfying*

$$\|Tf\|_{\vec{B}_{\sigma,\infty}} \leqslant c\sigma^{-a} \|f\|_{\vec{A}_{\sigma(1-\theta_0)+\theta_0,s}}.$$

Then

$$\sup_{t>0}(1+\ln^+ t)^{-a} K(t, Tf; \vec{B}) \leqslant c\|f\|_{\vec{A}_{v,s}},$$

where $v(t) = t^{-\theta_0}(1+\ln^+ 1/t)^a$, $t > 0$.

Proof. Using Corollary 3.2 we get

$$T : \Delta^{(\infty)}_{0,\alpha}(\eta^a \vec{A}_{w^*_\eta,s}) \to \Delta^{(\infty)}_{0,\alpha}\left(\eta^a \left(B_0 + \left\langle \vec{B}_{\eta,s} \right\rangle\right)\right).$$

The weight w^*_η can be estimated by

$$w^*_\eta(t) \approx \begin{cases} t^{-\theta_0}(-\ln t)^a & t < e^{-a/\eta}, \\ t^{-\theta_0-\eta(1-\theta_0)}\eta^{-a} & t > e^{-a/\eta}. \end{cases}$$

Therefore

$$w^*_\eta(t) \leqslant c v(t) \eta^{-a}.$$

Consequently

$$\vec{A}_{v,s} \subset \Delta^{(\infty)}_{0,\alpha}(\eta^a \vec{A}_{w^*_\eta,s}).$$

Moreover, we have

$$B_0 + \left\langle \vec{B}_{\eta,s} \right\rangle = \vec{B}_{h_\eta,s},$$

where

$$h_\eta(t) = \begin{cases} 0 & \text{if } 0 < t < 1, \\ t^{-\eta} & \text{if } t > 1. \end{cases}$$

As in the proof of Theorem 3.1 , we see that

$$\Delta^{(\infty)}_{0,\alpha}(\eta^a \vec{B}_{h_\eta,s}) = \Delta^{(\infty)}_{0,\alpha}(\eta^a \vec{B}_{h_\eta,\infty}). \qquad \square$$

The result follows since the quasi-norm in the space $\Delta^{(\infty)}_{0,\alpha}(\eta^a \vec{B}_{h_\eta,\infty})$ is given by $\sup_{t>0}(1+\ln^+ t)^{-a} K(t, f; \vec{B})$.

Corollary 5.4. *Let* $0 < p < q < p_1 < \infty$, $a > 0$, *and* $s = p$ *if* $p \leqslant 1$, *or* $s = 1$ *if* $p > 1$. *Let* $L^{q,\infty} := (L^p, L^\infty)_{\sigma,\infty}$, $1/q = (1-\sigma)/p$, *and let* T *be a sublinear operator such that*

$$\|Tf\|_{L^{q,\infty}} \leqslant c(q-p)^{-a} \|f\|_{L^{q,s}}.$$

Then

$$T : L^{p,s}(\log^+ 1/L)_a \to (L^p, L^\infty)_{g,\infty}, \ \text{where } g(t) = (1 + \ln^+ t)^{-a},$$

and where $L^{p,s}(\log^+ 1/L)_a$ *has the quasi-norm*

$$\left\{ \int_0^\infty [u^{1/p}(1 + \ln^+ 1/u)^a f^*(u)]^s \frac{du}{u} \right\}^{1/s}.$$

Another application of Theorem 5.3 gives

Theorem 5.4. *Let $\vec{A}, \vec{B}$ be quasi-Banach pairs, let $a > 0$, $0 < \sigma < \alpha < \theta_1 < 1$, $0 < s \leqslant \gamma_{\vec{B}}$, and let T be a linear operator satisfying*

$$\|Tf\|_{\vec{B}_{1-\sigma,\infty}} \leqslant c\sigma^{-a} \|f\|_{\vec{A}_{\theta_1-\sigma,s}} .$$

Then

$$\sup_{t>0}(1+\ln^+ 1/t)^{-a} t^{-1} K(t, Tf; \vec{B}) \leqslant c\|f\|_{\vec{A}_{h,s}},$$

where $h(t) = t^{-\theta_1}(1+\ln^+ t)^a$.

Proof. Indeed, use the relations:

$$(B_0, B_1)_{1-\sigma,\infty} = (B_1, B_0)_{\sigma,\infty}, \ (A_0, A_1)_{\theta_1-\sigma,s} = (A_1, A_0)_{1-\theta_1+\sigma,s}$$

and the formula $K(t, f; B_0, B_1) = tK(\frac{1}{t}, f; B_1, B_0)$. □

Corollary 5.5. *Let $0 < s < p_0 < q < p < \infty$, $0 < s \leqslant 1$, $a > 0$, $L^{q,\infty} := (L^s, L^p)_{1-\sigma,\infty}$, $1/q = (1-\sigma)/p + \sigma/s$. Let T be a sublinear operator such that*

$$\|Tf\|_{L^{q,\infty}} \leqslant c(q-p)^{-a} \|f\|_{L^{q,s}} .$$

Then

$$T : L^{p,s}(\log^+ L)_a \to (L^p, L^\infty)_{g,\infty}, \ \textit{where } g(t) = t^{-1}(1+\ln^+ 1/t)^{-a},$$

and where $L^{p,s}(\log^+ L)_a$ has the quasi-norm

$$\left\{ \int_0^\infty [u^{1/p}(1+\ln^+ u)^a f^*(u)]^s \frac{du}{u} \right\}^{1/s} .$$

Proof. Follows from the previous theorem, writing $L^{q,s} = (L^{p_0}, L^\infty)_{\theta_1(1-\sigma),s}$ where $1/p = \theta_1/p_0$. □

Analogous results with similar proofs are valid for the $\Delta^{(p)}$ spaces.

Theorem 5.5. *Let $\vec{A}, \vec{B}$ be quasi-Banach pairs, let $0 < \theta_0 < \theta < \theta_1 < 1$, $a_j, b_j, s_j > 0$, (or $a_j, b_j \geqslant 0$ if $s_j = \infty$), $(j = 0, 1)$. Suppose that T is an operator satisfying*

$$\|Tf\|_{\vec{B}_{\theta,s_j}} \leqslant c|\theta - \theta_j|^{-a_j+b_j} \|f\|_{\vec{A}_{\theta,s_j}}, \ (j = 0, 1).$$

Then

$$T : \vec{A}_{v_0,s_0} \cap \vec{A}_{v_1,s_1} \to \vec{B}_{w_0,s_0} \cap \vec{B}_{w_1,s_1},$$

where $w_j(t) = t^{-\theta_j}(1+|\ln t|)^{-a_j}$, $v_j(t) = t^{-\theta_j}(1+|\ln t|)^{-b_j}$ $(j = 0, 1)$.

Proof. Let $M_0(\theta) = (\theta - \theta_0)^{b_0}$, $M_1(\theta) = (\theta - \theta_0)^{a_0}$, $N_0(\theta) = (\theta_1 - \theta)^{b_1}$, $N_1(\theta) = (\theta_1 - \theta)^{a_1}$. From the assumptions it follows that

$$T : \delta_{\theta_0,\theta_1}^{(s_0)-}(M_0(\theta)\vec{A}_{\theta,s_0}) \to \delta_{\theta_0,\theta_1}^{(s_0)-}(M_1(\theta)\vec{B}_{\theta,s_0}),$$

and

$$T : \delta_{\theta_0,\theta_1}^{(s_1)+}(N_0(\theta)\vec{A}_{\theta,s_1}) \to \delta_{\theta_0,\theta_1}^{(s_1)+}(N_1(\theta)\vec{B}_{\theta,s_1}).$$

Applying Theorem 4.6, we readily obtain

$$T : \vec{A}_{g_j,s_j} \to \vec{B}_{f_j,s_j}, \ (j = 0, 1),$$

whence

$$T : \vec{A}_{g_0,s_0} \cap \vec{A}_{g_1,s_1} \to \vec{B}_{f_0,s_0} \cap \vec{B}_{f_1,s_1},$$

where

$$f_0(t) = \begin{cases} t^{-\theta_0}(1 + \ln t)^{-a_0} & t > 1, \\ t^{-\theta_1} & t < 1, \end{cases}$$

$$g_0(t) = \begin{cases} t^{-\theta_0}(1 + \ln t)^{-b_0} & t > 1, \\ t^{-\theta_1} & t < 1, \end{cases}$$

$$f_1(t) = \begin{cases} t^{-\theta_0} & t > 1, \\ t^{-\theta_1}(1 - \ln t)^{-a_1} & t < 1, \end{cases}$$

$$g_1(t) = \begin{cases} t^{-\theta_0} & t > 1, \\ t^{-\theta_1}(1 - \ln t)^{-b_1} & t < 1. \end{cases}$$

Consider the weights $u_j(t) = t^{-\theta_j}(1 + |\ln t|)^{c_j}$, $0 < \theta_0 < \theta_1 < 1$, $c_j \in R$ $(j = 0, 1)$. We need the formula

$$\|f\|_{\vec{A}_{u_0,s_0} \cap \vec{A}_{u_1,s_1}} \approx I_0 + I_1, \tag{75}$$

where

$$I_0^{s_0} = \int_1^\infty [u_0(t)K(t, f; \vec{A})]^{s_0} \frac{dt}{t}, \quad I_1^{s_1} = \int_0^1 [u_1(t)K(t, f; \vec{A})]^{s_1} \frac{dt}{t}.$$

Indeed from

$$K(t, f; \vec{A}) \leqslant c[u_1(t)]^{-1} I_0, \ t < 1,$$

$$K(t, f; \vec{A}) \leqslant c[u_0(t)]^{-1} I_1, \ t > 1,$$

it follows that

$$\int_1^\infty [u_1(t)K(t, f; \vec{A})]^{s_1} \frac{dt}{t} \leqslant c I_1^{s_1}, \quad \int_0^1 [u_0(t)K(t, f; \vec{B})]^{s_0} \frac{dt}{t} \leqslant c I_0^{s_0},$$

and (75) follows.

We therefore see that

$$\vec{B}_{f_0,s_0} \cap \vec{B}_{f_1,s_1} = \vec{B}_{w_0,s_0} \cap \vec{B}_{w_1,s_1}$$

and

$$\vec{A}_{g_0,s_0} \cap \vec{A}_{g_1,s_1} = \vec{A}_{v_0,s_0} \cap \vec{A}_{v_1,s_1}.$$

The theorem is proved. □

Corollary 5.6. *Let T be an operator satisfying*

$$\|Tf\|_{L^{q,s_j}} \leqslant c|q - r_j|^{-a_j+b_j} \|f\|_{L^{q,s_j}}, \ (j = 0, 1),$$

where $r_0 < q < r_1 \leqslant \infty$, $a_j, b_j > 0$, (or $a_j, b_j \geqslant 0$ if $s_j = \infty$), $(j = 0, 1)$. Then

$$T : L^{r_0,s_0}(\log L)_{-b_0} \cap L^{r_1,s_1}(\log L)_{-b_1} \to L^{r_0,s_0}(\log L)_{-a_0} \cap L^{r_1,s_1}(\log L)_{-a_1}.$$

In the case $s_j = \infty$, $j = 0, 1$, we can prove a variant, which generalizes Yano's theorem. Again we start with an abstract version.

Theorem 5.6. *Let $\vec{A}$, $\vec{B}$ be quasi-Banach pairs, and let T be an operator satisfying*

$$\|Tf\|_{\vec{B}_{1-\sigma,\infty}} \leqslant c\sigma^{-a} \|f\|_{\langle \vec{A}_{1-\sigma,s} \rangle}, \quad 0 < \sigma < \eta \leqslant 1, \ \ s > 0.$$

Then

$$T : \vec{A}_{1-\eta,\infty} \cap A_{1,\infty} \to \vec{B}_{1-\eta,\infty} \cap \vec{B}_{w_a,\infty},$$

where $w_a(t) = t^{-1}(1 + |\ln t|)^{-a}$.

Proof. We have to apply Theorem 3.1:

$$\|Tf\|_{\Delta^{(\infty)}(\sigma^a \vec{B}_{1-\sigma,\infty})} \leqslant c \|f\|_{\Delta^{(\infty)}(\vec{A}_{1-\sigma,\infty})}$$

and Remark 4.4.

Corollary 5.7. *Let $0 < p < q < \infty$, $s > 0$, $a > 0$ and let $L^{q,\infty} := (L^r, L^\infty)_{\theta,\infty}$, $1/q = (1-\theta)/r$, $r = \min(p, s)$. Let T be an operator satisfying*

$$\|Tf\|_{L^{q,\infty}} \leqslant cq^a \|f\|_{L^{q,s}}.$$

Then

$$T : L^p \cap L^\infty \to L^p \cap L^\infty(\log L)_{-a} \ if \ p \leqslant s$$

and

$$T : L^{p,\infty} \cap L^\infty \to L^{p,\infty} \cap L^\infty(\log L)_{-a} \ if \ p > s.$$

Proof. We apply Theorem 5.6 to $\vec{A} = \vec{B} = (L^r, L^\infty)$ and use Example 2.2:

$$\langle (L^r, L^\infty)_{1-\sigma,s} \rangle \subset L^{q,s},\ 1/q = \sigma/r,\ r \leqslant s.$$

If $p \leqslant s$ then $r = p$ and $0 < \sigma < 1$. In this case we use the relations

$$L^r = (L^r, L^\infty)_{0,\infty}, \quad L^\infty = (L^r, L^\infty)_{1,\infty}.$$

On the other hand if $p > s$ then $r = s$ and $0 < \sigma < \eta := s/p < 1$. Then we use the relation

$$L^{p,\infty} = (L^s, L^\infty)_{1-\eta,\infty}.$$

Finally, we use also formula (71). □

5.2. Logarithmic Sobolev inequalities

We consider operators on $L^2(R^n)$ provided with a Gaussian measure. Let A be a self-adjoint positive operator such that $P_t = e^{-tA}, t \geqslant 0$ is a hypercontractive semigroup on $L^p, 1 < p < \infty$. More precisely, there exists $c > 0$ such that

$$P_t : L^p \to L^p \text{ is bounded for all } t \geqslant 0,\ 1 < p < \infty \text{ and } \|P_t\|_p \leqslant ce^{-ct}, \tag{76}$$

and

$$P_t : L^p \to L^{q(t)},\ q(t) - 1 = e^t(p-1), \text{ is bounded uniformly for all } t \geqslant 0. \tag{77}$$

For example, if $B = -\Delta + |x|^2$ is the Hermite operator in $\mathbf{R}^n$ then we consider the operator $A := UBU^{-1}$, where $Uf := e^{x^2/2} f$ is the unitary mapping: $L^2(\mathbf{R}^n) \to L^2(\mathbf{R}^n, e^{-x^2} dx)$, (cf. [1,26]), and e^{-tA} is hypercontractive by Nelson's theorem (cf. [2,26]).

In [1,2], the following theorem is proved for the Hermite operator in $\mathbf{R}$.

Theorem 5.7. *Let* $1 < p < \infty, a \in R$, *then*

$$A^{-z} : L^p(\log L)_a \to L^p(\log L)_{a+\alpha}, \quad \Re z = \alpha,$$

is a bounded operator.

We give an extrapolation proof of this theorem in an abstract setting. Our proof in fact is valid for any hypercontractive semigroup. First we prove

Lemma 5.1. *Let* $1 < p < \infty, a < 0, \alpha + a > 0$, *then*

$$A^{-z} : L^p(\log L)_a \to L^p(\log L)_{a+\alpha},\ \Re z = \alpha,$$

is a bounded operator.

Proof. We start with the formula

$$A^{-z} f = \frac{1}{\Gamma(z)} \int_0^\infty t^{z-1} P_t f\, dt.$$

For any $\varepsilon > 0$ consider the operator

$$Qf = \int_{\varepsilon}^{\infty} t^{z-1} P_t f \, dt.$$

Since

$$Qf = P_{\varepsilon} \int_{0}^{\infty} (t+\varepsilon)^{z-1} P_t f \, dt,$$

it follows that for $\delta > 0$

$$Q : L^p \to L^{p+\delta}.$$

Thus it is sufficient to consider the operator

$$Q_z f = \int_0^{\varepsilon} \theta^{z-1} P_{\theta} f \, d\theta. \tag{78}$$

Let $1 < p < \infty$ be fixed, and choose $\theta_0 > 0$ small enough so that $\frac{1-\theta_0}{1-2\theta_0} < p$. Let $p_0 = (1-\theta_0)p$ and let $1/q = 1/p + \theta/p_0$, where $0 < \theta < \varepsilon$. We have

$$P_{\theta} : L^q \to L^{q(\theta)}, \quad q(\theta) = 1 + e^{\theta}(q-1).$$

If we write $1/q(\theta) = 1/p - \theta\delta(\theta)/p_0$ then, $q(\theta) - q = \theta(2p - 1 + O(\theta))$ as $\theta \to 0$, and the choice of θ_0, p_0, implies that $\delta(0) = p_0(2 - 1/p)/p - 1 > 0$. Moreover, $\delta(\theta) \approx 1$ for $0 < \theta < \varepsilon$ provided $\varepsilon < \theta_0$ is small enough. We can also assure the property $\theta_0 + \theta\delta(\theta) < \theta_1 < 1$ for all $0 < \theta < \varepsilon$. Write

$$L^q = (L^{p_0}, L^{\infty})_{\theta_0 - \theta, q}, \quad L^{q(\theta)} = (L^{p_0}, L^{\infty})_{\theta_0 + \theta\delta(\theta), q(\theta)}.$$

Applying Theorem 3.6 with $M(\theta) = \theta^{\alpha}$, $\alpha > 0$ and $N(\theta) = \theta^{-a-\alpha}$, $a + \alpha > 0$, we get

$$Q_z : \Delta^{(p)}_{(0,\varepsilon)}(\theta^{-a-1/p}(L^{p_0}, L^{\infty})_{\theta_0-\theta,q}) \to \Sigma^{(p)-}_{(0,\varepsilon)}(\theta^{-a-\alpha}(L^{p_0}, L^{\infty})_{\theta_0+\theta\delta(\theta),q(\theta)}).$$

Applying Theorems 2.13 and 3.4 it follows that

$$Q_z : \Delta^{(p)}_{(0,\varepsilon)}(\theta^{-a-1/p}(L^{p_0}, L^{\infty})_{\theta_0-\theta,p}) \to \Sigma^{(p)-}_{(0,\varepsilon)}(\theta^{-a-\alpha}(L^{p_0}, L^{\infty})_{\theta_0+\theta,p}).$$

It remains to apply Theorems 4.4 and 4.7 to conclude the proof. □

Proof of Theorem 5.7. It remains to remove the restrictions $a < 0$, $a + \alpha > 0$ imposed in Lemma 5.1. To this end we follow [1,2], where it is proved that

$$A^{-z} : L^p(\log L)_a \to L^p(\log L)_a, \Re z = 0, 1 < p < \infty, a \in \mathbf{R}. \tag{79}$$

Now we can interpolate the analytic family A^{-z} between (79) and Lemma 5.1 as in [20]. Theorem 5.7 is proved. □

5.3. Mappings of finite distortion

In this section we consider the continuity of mappings of finite distortion (cf. [16]).

Let $f = (f_1, \ldots, f_n)$ be a mapping in the Sobolev space $W^{1,1}_{\text{loc}}(\Omega, \mathbf{R}^n)$, where Ω is a domain in $\mathbf{R}^n$, $n \geqslant 2$. By definition [16], f is a map of finite distortion if there exists a measurable function $K(x) \geqslant 1$ such that

$$|Df(x)|^n \leqslant K(x) J(x, f), \quad \text{a.e.} \tag{80}$$

Here $|Df(x)|$ is the Euclidean norm of the differential of f and $J(x, f) = det\, Df(x) \geqslant 0$. Let $B = B(0, R)$ be a ball of radius R and centered at the origin. We consider the functions on B with a norm

$$\|u\|_{WL^n} = \sup_{0<s<1/2} s^{1/n} \|u\|_{L^n(\log L)_{-(s+1)/n}}. \tag{81}$$

By definition, WL^n is the closure of L^n with respect to this norm. Then, $u \in WL^n$ if and only if

$$\omega(u, s) := s^{1/n} \|u\|_{L^n(\log L)_{-(s+1)/n}} \to 0 \quad \text{as } s \to 0. \tag{82}$$

Theorem 5.8. *Let f be a mapping of finite distortion on the ball B, and let $|Df(x)| \in WL^n$. Then f is continuous, and moreover,*

$$|f(x) - f(y)| \leqslant C(n, R)\omega\left(|Df|, \frac{1}{\ln|\ln|x-y||}\right) \tag{83}$$

if $|x - y|$ is small and $x, y \in B(0, R/2)$.

Remark 5.1. If f has finite distortion on the ball B and

$$\int_0^\infty [|Df|^*(t)]^n (1 + |\ln t|)^{-1} (1 + \ln(1 + |\ln t|))^{-1} dt < \infty$$

then $|Df(x)| \in WL^n$ and hence f is continuous. In particular, Theorem 1.4 of [16] follows. Our method also gives a result similar to Theorem 1.6 of [16].

We are now ready for the proof of Theorem 5.8.

Proof. Following [16], we first show that the coordinate functions, say f_1, are weakly monotone. To see this we note that $WL^n \subset L^{n-\varepsilon}$ for all $0 < \varepsilon < 1$. Then, by definition (cf. [16]), we have to show that if $v := (f_1 - M)^+ - (m - f_1)^+$ is a limit of C_0^∞ functions in the open ball B in the norm of $W^{1,n-\varepsilon}$, for some constants $m < M$, then $v = 0$. Now since $v = 0$ on the set $\{x \in B : m \leqslant f_1(x) \leqslant M\}$ and $dv = df_1$ on its complement E, it suffices to prove that $Dg(x) = 0$, $x \in E$, where $g = (v, f_2, \ldots, f_n)$. Applying Lemma 5.1 of [16], we can write

$$\int_E |Dg(x)|^{n-\sigma} d\mu \leqslant C(n, R)\sigma \int_B |Df(x)|^{n-\sigma} dx,$$

where $d\mu := dx/K(x)$. In order to extrapolate this inequality we write

$$L^{n-\sigma} = (L^1, L^\infty)_{\theta_{1-\frac{\sigma}{n(n-\sigma)}}, n-\sigma}, \quad 0 < \sigma < 1/2.$$

Then for $\alpha > 0$ we have

$$\|Dg\|_{\Delta^{(n)}_{(0,1/2)}(\sigma^{\alpha/n-1/n}(L^1,L^\infty)_{\theta_{1-\frac{\sigma}{n(n-\sigma)}},n-\sigma})} \leqslant c\|Df\|_{\Delta^{(n)}_{(0,1/2)}(\sigma^{\alpha/n}(L^1,L^\infty)_{\theta_{1-\frac{\sigma}{n(n-\sigma)}},n-\sigma})}.$$

Let L^n_μ be the Lebesgue space $L^n(E)$ with respect to the measure μ. Theorems 3.4 and 4.7 yield

$$\|Dg\|_{L^n_\mu(\log L)_{-\alpha/n}} \leqslant c\alpha^{1/n}\|Df\|_{L^n(\log L)_{-(\alpha+1)/n}}. \tag{84}$$

The condition $|Df| \in WL^n$ means that the right-hand side in (84) goes to zero as $\alpha \to 0$. Applying Fatou's lemma to the left-hand side in (84) we conclude that $Dg = 0$ on E. Thus we have proved that $u := f_1$ is weakly monotone. Now we can use the oscillation Lemma 7.2 of [16]. Let x_0, y_0 be fixed Lebesgue points of u in the ball $B(0, R/2)$ and let $a := (x_0 + y_0)/2$, $r := |x_0 - y_0|/2$. Then for almost all t, $r < t < R/2$,

$$\left(\frac{|u(x_0) - u(y_0)|}{t}\right)^{n-\sigma} \leqslant Ct^{-n+1}\int_{S(a,t)} |\nabla u|^{n-\sigma} dx,$$

uniformly for $0 < \sigma < 1/2$, where $S(a, t)$ is the boundary of $B(a, t)$, the ball of radius t and centered at a. Consequently,

$$|u(x_0) - u(y_0)|^{n-\sigma} g(\sigma, r, R) \leqslant C\|\nabla u\|^{n-\sigma}_{L^{n-\sigma}(B)},$$

where $g(\sigma, r, R) := \int_r^{R/2} t^{\sigma-1} dt$. Since $g(\sigma, r, R) \geqslant C(R)$, uniformly with respect to $r < R/4$, $0 < \sigma < 1/2$, we conclude that

$$|u(x_0) - u(y_0)| \left(\int_r^{R/2} t^{\sigma-1} dt\right)^{1/n} \leqslant C(n, R)\|\nabla u\|_{L^{n-\sigma}(B)}.$$

Therefore, using the Δ method of extrapolation as above, we get for $\alpha > 0$,

$$|u(x_0) - u(y_0)| \left(\int_0^{1/2} [1 - (2r/R)^\sigma]\sigma^{\alpha-1} d\sigma\right)^{1/n} \leqslant C(n, R)\|\nabla u\|_{L^n(\log L)_{-(\alpha+1)/n}}.$$

For r and α small, the integral above behaves like $\frac{1-|\ln r|^{-\alpha}}{\alpha}$ and the best choice for α is $\alpha = \frac{1}{\ln|\ln r|}$ as $r \to 0$. Thus (83) follows and the theorem is proved. □

5.4. *Logarithmic Sobolev spaces*

In this section we show that the Sobolev spaces used by Donaldson and Sullivan in [12] can be replaced by logarithmic Sobolev spaces. As we know, logarithmic Sobolev spaces are built up over Σ spaces.

For concreteness sake, we shall consider only the situation relevant to [12], but the results are valid in much more generality. Let Ω be a convex bounded domain in $\mathbf{R}^n$ and let E be a Banach space of functions, locally integrable on Ω. Then the homogeneous Sobolev space, $\mathcal{W}^1 E$, is defined as a completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla f\|_E$. Analogously to Lemma 3.8 [12], we have

Theorem 5.9.

$$\mathcal{W}^1 \Sigma_{(0,1)}^{(1)-}(\sigma^{-1/n^*} L^{n+\sigma}) \subset L^\infty(\Omega), \quad 1/n^* + 1/n = 1.$$

Proof. Indeed, as in the proof of Lemma 3.8 [12], we derive

$$|f(x)| \leqslant c \int_\Omega |\nabla f(y)||x-y|^{1-n} dy, \quad x \in \Omega. \tag{85}$$

Let $f \in \mathcal{W}^1 E$, with $E = \Sigma_{(0,1)}^{(1)-}(\sigma^{-1/n^*} L^{n+\sigma})$, then we can write

$$\nabla f = \sum g_\sigma, \quad g_\sigma \in L^{n+\sigma}, \tag{86}$$

with

$$\|\nabla f\|_E \approx \sum \sigma^{-1/n^*} \|g_\sigma\|_{L^{n+\sigma}}. \tag{87}$$

Inserting (86) back in (85) and using Hölder's inequality we get

$$|f(x)| \leqslant c \sum \int_\Omega |g_\sigma(y)||x-y|^{1-n} dy,$$

$$|f(x)| \leqslant c \sum \|g_\sigma\|_{L^{n+\sigma}} \||x-y|^{1-n}\|_{L^{(n+\sigma)^*}}.$$

Finally, since Ω is bounded, we have

$$\||x-y|^{1-n}\|_{L^{(n+\sigma)^*}} \leqslant c\sigma^{-1/n^*}, \quad x \in \Omega,$$

and therefore

$$|f(x)| \leqslant c \sum \sigma^{-1/n^*} \|g_\sigma\|_{L^{n+\sigma}}.$$

The desired result now follows from (87).

Donaldson and Sullivan [12] consider the spaces $\hat{L}^p$, defined by

$$\hat{L}^p = \left\{ f = \sum_{i=1}^\infty f_i : \sum_{i=1}^\infty \rho^{-i} \|f_i\|^2_{L^{p+\varepsilon^i}(\Omega)} < \infty \right\},$$

where ε, ρ are fixed numbers in $(0, 1)$. Equipped with the norm

$$\|f\|_{\hat{L}^p} = \inf \left\{ \left(\sum_{i=1}^\infty \rho^{-i} \|f_i\|^2_{L^{p+\varepsilon^i}(\Omega)} \right)^{1/2} : f = \sum_{i=1}^\infty f_i \right\},$$

$\hat{L}^p$ becomes a Banach space. In [12] this construction plays a crucial role: it allows the authors to construct the Sobolev spaces $\mathcal{W}^1\hat{L}^4$, based on $\hat{L}^4$, where Ω is a domain in $\mathbf{R}^4$, with the crucial property that (cf. [12, Lemma 3.8]):

$$\mathcal{W}^1\hat{L}^4 \subset L^\infty(\Omega), \quad \rho < \varepsilon^{3/4}. \tag{88}$$

To show that this result is a consequence of Theorem 5.9, we first write $\rho = \varepsilon^d$ and prove the relation

$$\hat{L}^4 = \Sigma^{(2)-}_{(0,1)}(\sigma^{-d/2}L^{4+\sigma}). \tag{89}$$

Indeed, using monotonicity of the scale $L^{4+\sigma}$, we can replace the discrete definition given above by a continuous one:

$$\|f\|_{\hat{L}^4} = \inf\left\{\left(\int_0^1 \sigma^{-d}\,\|f_\sigma\|^2_{L^{4+\sigma}}\frac{d\sigma}{\sigma}\right)^{1/2} : f = \int_0^1 f_\sigma\frac{d\sigma}{\sigma}\right\}.$$

Hence (89) follows.

On the other hand, Hölder's inequality implies the embedding

$$\Sigma^{(q)-}_{(0,1)}(\sigma^{-b}L^{p+\sigma}) \subset \Sigma^{(1)-}_{(0,1)}(\sigma^{-a}L^{p+\sigma}), \quad b > a > 0, q > 1.$$

Thus

$$\hat{L}^4 \subset \Sigma^{(1)-}_{(0,1)}(\sigma^{-a}L^{4+\sigma}), \quad d > 2a.$$

Finally, applying Theorem 5.9 we get

$$\mathcal{W}^1\hat{L}^4 \subset L^\infty(\Omega)$$

if $d > 3/2$, hence if $\rho < \varepsilon^{3/2} < \varepsilon^{3/4}$. □

In [12] the precise identification of the $\mathcal{W}^1\hat{L}^4$ spaces was not important, the authors just needed suitable spaces, where (besides quasiconformal invariance) the crucial property (88) was valid, to develop their theory. It follows that instead of $\mathcal{W}^1\hat{L}^4$ we can use any of the spaces $\mathcal{W}^1\Sigma^{(q)-}_{(0,1)}(\sigma^{-b}L^{4+\sigma})$, $b > 3/4$, $q > 1$. In particular we can use the space

$$\mathcal{W}^1\Sigma^{(4)-}_{(0,1)}(\sigma^{-b}L^{4+\sigma}) = \mathcal{W}^1L^4(\log L)_b, b > 3/4,$$

that is, a logarithmic Sobolev space. Such an explicit characterization simplifies some of the analysis in [12]. For example the elliptic theory (cf. [12, Lemma 2.16]) follows from Sneiberg's extrapolation Lemma for the real method (cf. [7,30]).

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