# Morita equivalence of dual operator algebras

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## A R T I C L E   I N F O

## A B S T R A C T

We consider notions of Morita equivalence appropriate to weak* closed algebras of Hilbert space operators. We obtain new variants, appropriate to the dual algebra setting, of the basic theory of strong Morita equivalence, and new nonselfadjoint analogues of aspects of Rieffel's $W^*$-algebraic Morita equivalence.

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## 1. Introduction and notation

By definition, an *operator algebra* is a subalgebra of $B(H)$, the bounded linear operators on a Hilbert space $H$, which is closed in the norm topology. It is a *dual algebra* if it is closed in the weak* topology (also known as the $\sigma$-weak topology). In [11], the first author, Muhly, and Paulsen generalized Rieffel's strong Morita equivalence of $C^*$-algebras [24], to general operator algebras. At that time however, we were not clear about how to generalize Rieffel's variant for $W^*$-algebras [23], to dual operator algebras. Recently, two approaches have been suggested for this, in [9,17,18], each of which reflects (different) important aspects of Rieffel's $W^*$-algebraic Morita equivalence. For example, the notion introduced in [17,18] is equivalent to the very important notion of (weak*) 'stable isomorphism' [20]. The fact remains, however, that neither approach seems able to treat certain important examples, such as the second dual of a strong Morita equivalence. In the present paper we examine a framework, part of which was suggested at the end of [9], which does include all examples hitherto considered, and which represents a natural setting for the Morita equivalence of dual algebras. It is also one to which all the relevant parts of the earlier theory of strong Morita equivalence (from e.g. [11,10]) transfers in a very clean manner, indeed which may in some sense be summarized as 'just changing the tensor product involved' to one appropriate to the weak* topology.

Since many of the ideas and proofs are extremely analogous to those from our papers on related topics, principally [11, 1,10] and to a lesser extent [3–5,9], we will be quite brief in many of the proofs. That is, we assume that the reader is a little familiar with these earlier ideas and proof techniques. We will often merely indicate the modifications to weak* topologies. For a more detailed exposition see the second authors Ph. D. thesis [21], along with many other related results.

In Section 2, we develop some basic tensor product properties which we shall need. In Section 3, we define our variant of Morita equivalence, and present some of its consequences. Section 4 is centered on the 'weak linking algebra', the key tool for dealing with most aspects of Morita equivalence, and in Section 5 we prove that if $M$ and $N$ are weak* Morita equivalent dual operator algebras, then the von Neumann algebras generated by $M$ and $N$ are Morita equivalent in Rieffel's $W^*$-algebraic sense.

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Turning to notation, if $E$, $F$ are sets, then $EF$ will mean the norm closure of the span of products $zw$ for $z \in E$, $w \in F$. We reserve the letters $H$ and $K$ for Hilbert spaces. We will assume that the reader is familiar with basic notions from *operator space theory*, as may be found in any of the current texts on that subject, e.g. [15], and the application of this theory to operator algebras, as may be found in e.g. [8]. We study operator algebras from an operator space point of view. Thus an *abstract operator algebra A* is an operator space and a Banach algebra, for which there exists a Hilbert space $H$ and a completely isometric homomorphism $\pi : A \to B(H)$.

We will often abbreviate 'weak*' to '$w^*$'. A *dual operator algebra* is an operator algebra $M$ which is also a dual operator space. By well known duality principles, any $w^*$-closed subalgebra of $B(H)$, is a dual operator algebra. Conversely, it is known (see e.g. [8]), that for any dual operator algebra $M$, there exists a Hilbert space $H$ and a $w^*$-continuous completely isometric homomorphism $\pi : M \to B(H)$. In this case, the range $\pi(M)$ is a $w^*$-closed subalgebra of $B(H)$, which we may identify with $M$ in every way. In this paper $M$ and $N$ are dual operator algebras which are *unital*, that is we assume they each possess an identity of norm 1. Nondual operator algebras in this paper, in contrast, will usually be approximately unital, that is, they possess a contractive approximate identity (cai). A *normal representation* of $M$ is a $w^*$-continuous unital completely contractive homomorphism $\pi : M \to B(H)$.

For cardinals or sets $I$, $J$, we use the symbol $M_{I,J}(X)$ for the operator space of $I \times J$ matrices over $X$, whose 'finite submatrices' have uniformly bounded norm. We write $\mathbb{K}_{I,J}(X)$ for the norm closure of these 'finite submatrices'. Then $C_J^w(X) = M_{J,1}(X)$, $R_J^w(X) = M_{1,J}(X)$, and $C_J(X) = \mathbb{K}_{J,1}(X)$ and $R_J(X) = \mathbb{K}_{1,J}(X)$. We sometimes write $\mathbb{M}_I(X)$ for $M_{I,I}(X)$.

A *concrete left operator module* over an operator algebra $A$, is a subspace $X \subset B(K, H)$ such that $\pi(A)X \subset X$ for a completely contractive representation $\pi : A \to B(H)$. An *abstract operator A-module* is an operator space $X$ which is also an $A$-module, such that $X$ is completely isometrically isomorphic, via an $A$-module map, to a concrete operator $A$-module. Similarly for right modules, or bimodules. Most of the interesting modules over operator algebras are operator modules, such as *Hilbert C\*-modules*, and *Hilbert modules* (see next paragraph).

A *concrete dual operator M–N-bimodule* is a $w^*$-closed subspace $X$ of $B(K, H)$ such that $\theta(M)X\pi(N) \subset X$, where $\theta$ and $\pi$ are normal representations of $M$ and $N$ on $H$ and $K$ respectively. An *abstract dual operator M–N-bimodule* is defined to be a nondegenerate operator $M$–$N$-bimodule $X$, which is also a dual operator space, such that the module actions are separately weak* continuous. Such spaces can be represented completely isometrically as concrete dual operator bimodules, and in fact this can be done under even weaker hypotheses (see e.g. [8,9,14]). Similarly for one-sided modules (the case $M$ or $N$ equals $\mathbb{C}$). We use standard notation for module mapping spaces, e.g. $CB(X, N)_N$ (resp. $w^*CB(X, N)_N$) are the completely bounded (resp. and $w^*$-continuous) right $N$-module maps $X \to N$. Important examples of left dual operator modules over $M$, are the *(completely contractive) normal Hilbert M-modules*. By this, we mean a pair $(H, \pi)$, where $H$ is a (column) Hilbert space (see e.g. 1.2.23 in [8]), and $\pi : M \to B(H)$ is a normal representation. The module action is expressed through the equation $m \cdot \zeta = \pi(m)\zeta$. The morphisms between Hilbert $M$-modules are the bounded $M$-module maps ('intertwiners').

If $M$ is a dual operator algebra, then the *maximal W\*-cover* $W_{\max}^*(M)$ is a $W^*$-algebra containing $M$ as a $w^*$-closed subalgebra, and which is generated by $M$ as a $W^*$-algebra, and which has the universal property: any normal representation $\pi : M \to B(H)$ extends uniquely to a (unital) normal $*$-representation $\tilde{\pi} : W_{\max}^*(M) \to B(H)$ (see [13]). A normal representation $\pi : M \to B(H)$ of a dual operator algebra $M$, or the associated space $H$ viewed as an $M$-module, will be called *normal universal*, if any other normal representation is unitarily equivalent to the restriction of a 'multiple' of $\pi$ to a reducing subspace (see [13]).

**Lemma 1.1.** *A normal representation $\pi : M \to B(H)$ of a dual operator algebra $M$ is normal universal iff its extension $\tilde{\pi}$ to $W_{\max}^*(M)$ is one-to-one.*

**Proof.** The ($\Leftarrow$) direction is stated in [13]. Thus any faithful normal representation of $W_{\max}^*(M)$ restricts to a normal universal representation $\pi$ whose extension $\tilde{\pi}$ to $W_{\max}^*(M)$ is one-to-one. It is observed in [13] that any other normal universal representation $\theta$ is quasiequivalent to $\pi$. It follows that the extension $\tilde{\theta}$ to $W_{\max}^*(M)$ is quasiequivalent to $\tilde{\pi}$, and it is easy to see from this that $\tilde{\theta}$ is one-to-one. $\square$

## 2. Some tensor products

We begin by recalling the definition of the Haagerup tensor product. Suppose $X$ and $Y$ are two operator spaces. Define $\|u\|_n$ for $u \in M_n(X \otimes Y)$ as:

$$\|u\|_n = \inf \ \{\|a\|\|b\| \ : \ u = a \odot b, a \in M_{np}(X), b \in M_{pn}(Y), p \in \mathbb{N}\}.$$

Here $a \odot b$ stands for the $n \times n$ matrix whose $i, j$-entry is $\sum_{k=1}^{p} a_{ik} \otimes b_{kj}$. The algebraic tensor product $X \otimes Y$ with this sequence of matrix norms is an operator space. The completion of this operator space in the above norm is called *Haagerup tensor product*, and is denoted by $X \otimes_h Y$. The completion of an operator space is an operator space, hence $X \otimes_h Y$ is an operator space.

If $X$ and $Y$ are respectively right and left operator $A$-modules, then the *module Haagerup tensor product* $X \otimes_{hA} Y$ is defined to be the quotient of $X \otimes_h Y$ by the closure of the subspace spanned by terms of the form $xa \odot y - x \odot ay$, for $x \in X$, $y \in Y$, $a \in A$. Let $X$ be a right and $Y$ be a left operator $A$-module where $A$ is an operator algebra. We say that a bilinear map $\psi : X \times Y \to W$ is *balanced* if $\psi(xa, y) = \psi(x, ay)$ for all $x \in X$, $y \in Y$ and $a \in A$. It is well known that the module Haagerup tensor product

linearizes balanced bilinear maps which are completely contractive (or completely bounded) in the sense of Christensen and Sinclair (see e.g. 1.5.4 in [8]).

If $X$ and $Y$ are two operator spaces, then the *extended Haagerup tensor product* $X \otimes_{eh} Y$ may be defined to be the subspace of $(X^* \otimes_h Y^*)^*$ corresponding to the completely bounded bilinear maps from $X^* \times Y^* \to \mathbb{C}$ which are separately weak*-continuous. If $X$ and $Y$ are dual operator spaces, with preduals $X_*$ and $Y_*$, then this coincides with the *weak\* Haagerup tensor product* defined earlier in [12], and indeed $X \otimes_{eh} Y = (X_* \otimes_h Y_*)^*$. The *normal Haagerup tensor product* $X \otimes^{\sigma h} Y$ is the operator space dual of $X_* \otimes_{eh} Y_*$. The canonical maps are complete isometries

$$X \otimes_h Y \to X \otimes_{eh} Y \to X \otimes^{\sigma h} Y.$$

See [16] for more details.

**Lemma 2.1.** *For any dual operator spaces X and Y,* Ball$(X \otimes_h Y)$ *is $w^*$-dense in* Ball$(X \otimes^{\sigma h} Y)$.

**Proof.** Let $x \in$ Ball$(X \otimes^{\sigma h} Y) \setminus \overline{\text{Ball}(X \otimes_h Y)}^{w^*}$. By the geometric Hahn-Banach theorem, there exists a $\phi \in (X \otimes^{\sigma h} Y)_*$, and $t \in \mathbb{R}$, such that Re $\phi(x) > t >$ Re $\phi(y)$ for all $y \in$ Ball$(X \otimes_h Y)$. We view $\phi$ as a map $X \otimes_h Y \to \mathbb{C}$. It follows that Re $\phi(x) > t > |\phi(y)|$ for all $y \in$ Ball$(X \otimes_h Y)$, which implies that $\|\phi\| \leq t$. Thus $|\text{Re } \phi(x)| \leq \|\phi\| \|x\| \leq t$, which is a contradiction. $\square$

**Lemma 2.2.** *The normal Haagerup tensor product is associative. That is, if X, Y, Z are dual operator spaces then $(X \otimes^{\sigma h} Y) \otimes^{\sigma h} Z = X \otimes^{\sigma h} (Y \otimes^{\sigma h} Z)$ as dual operator spaces.*

**Proof.** This follows by the definition of the normal Haagerup tensor product and using associativity of the extended Haagerup tensor product (e.g. see [16]). $\square$

We now turn to the module version of the normal Haagerup tensor product, and review some definitions and facts from [20]. Let $X$ be a right dual operator $M$-module and $Y$ be a left dual operator $M$-module. Let $(X \otimes_{hM} Y)^*_\sigma$ denote the subspace of $(X \otimes_h Y)^*$ corresponding to the completely bounded bilinear maps from $\psi : X \times Y \to \mathbb{C}$ which are separately weak*-continuous and $M$-balanced (that is, $\psi(xm, y) = \psi(x, my)$). Define the *module normal Haagerup tensor product* $X \otimes^{\sigma h}_M Y$ to be the operator space dual of $(X \otimes_{hM} Y)^*_\sigma$. Equivalently, $X \otimes^{\sigma h}_M Y$ is the quotient of $X \otimes^{\sigma h} Y$ by the weak*-closure of the subspace spanned by terms of the form $xm \otimes y - x \otimes my$, for $x \in X, y \in Y, m \in M$. The module normal Haagerup tensor product linearizes completely contractive, separately weak*-continuous, balanced bilinear maps. That is, the universal property of $\otimes^{\sigma h}_M$ is: every completely contractive separately weak* continuous map $u : X \times Y \to Z$ such that $u(xm, y) = u(x, my)$ for $m \in M, x \in X, y \in Y$, yields a weak* continuous complete contraction $\tilde{u} : X \otimes^{\sigma h}_M Y \to Z$, whose composition with the canonical map $\otimes : X \times Y \to X \otimes^{\sigma h}_M Y$, equals $u$ (see [20, Proposition 2.2]). The map $\otimes$ here is a completely contractive, separately weak*-continuous, balanced bilinear map.

**Lemma 2.3.** *Let $X_1, X_2, Y_1, Y_2$ be dual operator spaces. If $u : X_1 \to Y_1$ and $v : X_2 \to Y_2$ are $w^*$-continuous, completely bounded, linear maps, then the map $u \otimes v$ extends to a well defined $w^*$-continuous, linear, completely bounded map from $X_1 \otimes^{\sigma h} X_2 \to Y_1 \otimes^{\sigma h} Y_2$, with $\|u \otimes v\|_{cb} \leq \|u\|_{cb} \|v\|_{cb}$.*

**Proof.** This follows by considering preduals of the maps, and using the functoriality of the extended Haagerup tensor product [16]. $\square$

**Corollary 2.4.** *Let $N$ be a dual algebra, let $Y_1$ and $Y_2$ be dual operator spaces which are left $N$-modules, and let $X_1, X_2$ be dual operator spaces which are right $N$-modules. If $u : X_1 \to X_2$ and $v : Y_1 \to Y_2$ are completely bounded, $w^*$-continuous, $N$-module maps, then the map $u \otimes v$ extends to a well defined linear, $w^*$-continuous, completely bounded map from $X_1 \otimes^{\sigma h}_N Y_1 \to X_2 \otimes^{\sigma h}_N Y_2$, with $\|u \otimes v\|_{cb} \leq \|u\|_{cb} \|v\|_{cb}$.*

**Proof.** Lemma 2.3 gives a $w^*$-continuous, completely bounded, linear map $X_1 \otimes^{\sigma h} Y_1 \to X_2 \otimes^{\sigma h} Y_2$ taking $x \otimes y$ to $u(x) \otimes v(y)$. Composing this map with the $w^*$-continuous, quotient map $X_2 \otimes^{\sigma h} Y_2 \to X_2 \otimes^{\sigma h}_N Y_2$, we obtain a $w^*$-continuous, completely bounded map $X_1 \otimes^{\sigma h} Y_1 \to X_2 \otimes^{\sigma h}_N Y_2$. It is easy to see that the kernel of the last map contains all terms of form $xn \otimes_N y - x \otimes_N ny$, with $n \in N, x \in X_1, y \in Y_1$. This gives a map $X_1 \otimes^{\sigma h}_N Y_1 \to X_2 \otimes^{\sigma h}_N Y_2$ with the required properties. $\square$

**Lemma 2.5.** *If $X$ is a dual operator $M$–$N$-bimodule and if $Y$ is a dual operator $N$–$L$-bimodule, then $X \otimes^{\sigma h}_N Y$ is a dual operator $M$–$L$-bimodule.*

**Proof.** To show e.g. it is a left dual operator $M$-module, use the canonical maps

$$M \otimes_h (X \otimes^{\sigma h} Y) \to M \otimes^{\sigma h} (X \otimes^{\sigma h} Y) \to (M \otimes^{\sigma h} X) \otimes^{\sigma h} Y \to X \otimes^{\sigma h} Y.$$

Composing the map $M \otimes^{\sigma h} (X \otimes^{\sigma h} Y) \to X \otimes^{\sigma h} Y$ above with the canonical map $M \times (X \otimes^{\sigma h} Y) \to M \otimes^{\sigma h} (X \otimes^{\sigma h} Y)$, one sees the action of $M$ on $X \otimes^{\sigma h} Y$ is separately weak* continuous (see also [20]). That $(a_1 a_2)z = a_1(a_2 z)$ for $a_i \in M, z \in X \otimes^{\sigma h} Y$, follows from the weak* density of $X \otimes Y$, and since this relation is true if $z$ is finite rank. It follows from 3.3.1 in [8], that $X \otimes^{\sigma h} Y$ is an operator $M$-module. By 3.8.8 in [8], $X \otimes^{\sigma h}_N Y$ is a dual operator $M$-module. $\square$

There is clearly a canonical map $X \otimes_{hM} Y \to X \otimes^{\sigma h}_M Y$, with respect to which:

**Corollary 2.6.** *For any dual operator M-modules X and Y, the image of* $\mathrm{Ball}(X \otimes_{hM} Y)$ *is* $w^*$*-dense in* $\mathrm{Ball}(X \otimes_M^{\sigma h} Y)$.

**Proof.** Consider the canonical $w^*$-continuous quotient map $q : X \otimes^{\sigma h} Y \to X \otimes_M^{\sigma h} Y$. If $z \in X \otimes_M^{\sigma h} Y$ with $\|z\| < 1$, then there exists $z' \in X \otimes^{\sigma h} Y$ with $\|z'\| < 1$ such that $q(z') = z$. By Lemma 2.1, there exists a net $(z_t)$ in $\mathrm{Ball}(X \otimes_h Y)$ such that $z_t \xrightarrow{w^*} z'$. Then $q(z_t) \xrightarrow{w^*} q(z') = z$. $\quad\square$

**Lemma 2.7.** *For any dual operator M-modules X and Y, and $m, n \in \mathbb{N}$, we have $M_{mn}(X \otimes_M^{\sigma h} Y) \cong C_m(X) \otimes_M^{\sigma h} R_n(Y)$ completely isometrically and weak\* homeomorphically. This is also true with $m, n$ replaced by arbitrary cardinals or sets: $M_{IJ}(X \otimes_M^{\sigma h} Y) \cong C_I^w(X) \otimes_M^{\sigma h} R_J^w(Y)$.*

**Proof.** We just prove the case that $m, n \in \mathbb{N}$, the other being similar (or can be deduced easily from Proposition 2.9). First we claim that $M_{mn}(X \otimes^{\sigma h} Y) \cong C_m(X) \otimes^{\sigma h} R_n(Y)$. Using facts from [16] and basic operator space duality, the predual of the latter space is

$$
\begin{aligned}
C_m(X)_* \otimes_{eh} R_n(Y)_* &\cong (R_m \otimes_h X_*) \otimes_{eh} (Y_* \otimes_h C_n) \\
&\cong (R_m \otimes_{eh} X_*) \otimes_{eh} (Y_* \otimes_{eh} C_n) \\
&\cong R_m \otimes_{eh} (X_* \otimes_{eh} Y_*) \otimes_{eh} C_n \\
&\cong R_m \otimes_h (X_* \otimes_{eh} Y_*) \otimes_h C_n \\
&\cong (X_* \otimes_{eh} Y_*) \bar\otimes (M_{mn})_*.
\end{aligned}
$$

We have used for example 1.5.14 in [8], 5.15 in [16], and associativity of the extended Haagerup tensor product [16]. Now $(X_* \otimes_{eh} Y_*) \bar\otimes (M_{mn})_*$ is the predual of $M_{mn}(X \otimes^{\sigma h} Y)$, by e.g. 1.6.2 in [8]. This gives the claim. If $\theta$ is the ensuing completely isometric isomorphism $C_m(X) \otimes^{\sigma h} R_n(Y) \to M_{mn}(X \otimes^{\sigma h} Y)$, it is easy to check that $\theta$ takes $[x_1\, x_2 \ldots x_m]^{\mathrm{T}} \otimes [y_1\, y_2 \ldots y_n]$ to the matrix $[x_i \otimes y_j]$. Now $C_m(X) \otimes_M^{\sigma h} R_n(Y) = C_m(X) \otimes^{\sigma h} R_n(Y)/N$ where $N = [xt \otimes y - x \otimes ty]^{-w^*}$ with $x \in C_m(X), y \in R_n(Y), t \in M$. Let $N' = [xt \otimes y - x \otimes ty]^{-w^*}$ where $x \in X, y \in Y, t \in M$, then clearly $\theta(N) = M_{mn}(N')$. Hence

$$
C_m(X) \otimes^{\sigma h} R_n(Y)/N \cong M_{mn}(X \otimes^{\sigma h} Y)/\theta(N) = M_{mn}(X \otimes^{\sigma h} Y)/M_{nm}(N'),
$$

which in turn equals $M_{mn}(X \otimes^{\sigma h} Y/N') = M_{mn}(X \otimes_M^{\sigma h} Y)$. $\quad\square$

**Corollary 2.8.** *For any dual operator M-modules X and Y, and $m, n \in \mathbb{N}$, we have that* $\mathrm{Ball}(M_{mn}(X \otimes_{hM} Y))$ *is* $w^*$*-dense in* $\mathrm{Ball}(M_{mn}(X \otimes_M^{\sigma h} Y))$.

**Proof.** If $\eta \in \mathrm{Ball}(M_{mn}(X \otimes_M^{\sigma h} Y))$, then by Lemma 2.7, $\eta$ corresponds to an element $\eta' \in C_m(X) \otimes_M^{\sigma h} R_n(Y)$. By Corollary 2.6, there exists a net $(u_t)$ in $C_m(X) \otimes_{hM} R_n(Y)$ such that $u_t \xrightarrow{w^*} \eta'$. By 3.4.11 in [8], $u_t$ corresponds to $u_t' \in \mathrm{Ball}(M_{mn}(X \otimes_{hM} Y))$ such that $u_t' \xrightarrow{w^*} \eta$. $\quad\square$

**Proposition 2.9.** *The normal module Haagerup tensor product is associative. That is, if M and N are dual operator algebras, if X is a right dual operator M-module, if Y is a M–N-dual operator bimodule, and Z is a left dual operator N-module, then $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ is completely isometrically isomorphic to $X \otimes_M^{\sigma h}(Y \otimes_N^{\sigma h} Z)$.*

**Proof.** We define $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$ to be the quotient of $X \otimes^{\sigma h} Y \otimes^{\sigma h} Z$ by the $w^*$-closure of the linear span of terms of the form $xm \otimes y \otimes z - x \otimes my \otimes z$ and $x \otimes yn \otimes z - x \otimes y \otimes nz$, with $x \in X, y \in Y, z \in Z, m \in M, n \in N$. By extending the arguments of Proposition 2.2 in [20] to the threefold normal module Haagerup tensor product, one sees that $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$ has the following universal property: if $W$ is a dual operator space and $u : X \times Y \times Z \to W$ is a separately $w^*$-continuous, completely contractive, balanced, trilinear map, then there exists a $w^*$-continuous and completely contractive, linear map $\tilde{u} : X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z \to W$ such that $\tilde{u}(x \otimes_M y \otimes_N z) = u(x, y, z)$. We will prove that $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ has the above universal property defining $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Let $u : X \times Y \times Z \to W$ be a separately $w^*$-continuous, completely contractive, balanced, trilinear map. For each fixed $z \in Z$, define $u_z : X \times Y \to W$ by $u_z(x, y) = u(x, y, z)$. This is a separately $w^*$-continuous, balanced, bilinear map, which is completely bounded. Hence we obtain a $w^*$-continuous completely bounded linear map $u_z' : X \otimes_M^{\sigma h} Y \to W$ such that $u_z'(x \otimes_M y) = u_z(x, y)$. Define $u' : (X \otimes_M^{\sigma h} Y) \times Z \to W$ by $u'(a, z) = u_z'(a)$, for $a \in X \otimes_M^{\sigma h} Y$. Then $u'(x \otimes_M y, z) = u(x, y, z)$, and it is routine to check that $u'$ is bilinear and balanced over $N$. We will show that $u'$ is completely contractive on $(X \otimes_{hM} Y) \times Z$, and then the complete contractivity of $u'$ follows from Corollary 2.8. Let $a \in M_{mn}(X \otimes_{hM} Y)$ with $\|a\| < 1$ and $z \in M_{mn}(Z)$ with $\|z\| < 1$. We want to show $\|u_n'(a, z)\| < 1$. It is well known that we can write $a = x \odot_M y$ where $x \in M_{nk}(X)$ and $y \in M_{km}(Y)$ for some $k \in \mathbb{N}$, with $\|x\| < 1$ and $\|y\| < 1$. Hence $\|u_n'(a, z)\| = \|u_n(x, y, z)\| \leq \|x\| \|y\| \|z\| < 1$, proving $u'$ is completely contractive. By Proposition 2.2 in [20], we obtain a $w^*$-continuous, completely contractive, linear map $\tilde{u} : (X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z \to W$ such that $\tilde{u}((x \otimes_M y) \otimes_N z) = u'(x \otimes_M y, z) = u(x, y, z)$. This shows that $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ has the defining universal property of $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Therefore $(X \otimes_M^{\sigma h} Y) \otimes_N^{\sigma h} Z$ is completely isometrically isomorphic and $w^*$-homeomorphic to $X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. Similarly $X \otimes_M^{\sigma h}(Y \otimes_N^{\sigma h} Z) = X \otimes_M^{\sigma h} Y \otimes_N^{\sigma h} Z$. $\quad\square$

**Lemma 2.10.** *If X is a left dual operator M-module then $M \otimes_M^{\sigma h} X$ is completely isometrically isomorphic to X.*

**Proof.** As in Lemma 3.4.6 in [8], or follows from the universal property. $\quad\square$

## 3. Morita contexts

We now define two variants of Morita equivalence for unital dual operator algebras, the first being seemingly more general than the second. There are many equivalent variants of these definitions, some of which we shall see later.

Throughout this section, we fix a pair of unital dual operator algebras, $M$ and $N$, and a pair of dual operator bimodules $X$ and $Y$; $X$ will always be a $M$–$N$-bimodule and $Y$ will always be an $N$–$M$-bimodule.

**Definition 3.1.** We say that $M$ is *weak$^*$ Morita equivalent* to $N$, with equivalence bimodules $X$ and $Y$, if $M \cong X \otimes_N^{\sigma h} Y$ as dual operator $M$-bimodules (that is, via a completely isometric $w^*$-homeomorphism which is also a $M$-bimodule map), and similarly if $N \cong Y \otimes_M^{\sigma h} X$ as dual operator $N$-bimodules. We call $(M, N, X, Y)$ a *weak$^*$ Morita context* in this case.

For the next definition, we suppose that we have separately weak$^*$-continuous completely contractive bilinear maps $(\cdot, \cdot) : X \times Y \to M$, and $[\cdot, \cdot] : Y \times X \to N$, and we will work with the 6-tuple, or *context*, $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$.

**Definition 3.2.** We say that $M$ is *weakly Morita equivalent* to $N$, if there exists a 6-tuple as above, and exist $w^*$-dense operator algebras $A$ and $B$ in $M$ and $N$ respectively, and there exists a $w^*$-dense operator $A$–$B$-submodule $X'$ in $X$, and a $w^*$-dense $B$–$A$-submodule $Y'$ in $Y$, such that the 'subcontext' $(A, B, X', Y')$, together with restrictions of the pairings $(\cdot, \cdot)$ and $[\cdot, \cdot]$, is a (strong) Morita context in the sense of [11, Definition 3.1]. In this case, we call $(M, N, X, Y)$ (or more properly the 6-tuple above the definition), a *weak Morita context*.

**Remark** Some authors use the term 'weak Morita equivalence' for a quite different notion, namely to mean that the algebras have equivalent categories of Hilbert space representations.

Weak Morita equivalence, as we have just defined it, is really nothing more than the 'weak$^*$-closure of' a strong Morita equivalence in the sense of [11]. This definition includes all examples that have hitherto been considered in the literature:

*Examples*:

(1) We shall see in Corollary 3.4 that every weak Morita equivalence is an example of weak$^*$ Morita equivalence.

(2) We shall see in Section 4 that every weak Morita equivalence arises as follows: Let $A$, $B$ be subalgebras of $B(H)$ and $B(K)$ respectively, for Hilbert spaces $H$, $K$, and let $X \subset B(K, H)$, $Y \subset B(H, K)$, such that the associated subset $\mathscr{L}$ of $B(H \oplus K)$ with $A, B, X, Y$ as 'corners', is a subalgebra of $B(H \oplus K)$, for Hilbert spaces $H$, $K$. This is the same as specifying a list of obvious algebraic conditions, such as $XY \subset A$. Assume in addition that $A$ possesses a cai $(e_t)$ with terms of the form $xy$, for $x \in \text{Ball}(R_n(X))$ and $y \in \text{Ball}(C_n(Y))$, and $B$ possessing a cai with terms of a similar form $yx$ (dictated by symmetry). Taking the weak$^*$ (that is, $\sigma$-weak) closure of all these spaces clearly yields a weak Morita equivalence of $\overline{A}^{w*}$ and $\overline{B}^{w*}$.

(3) Every weak$^*$ Morita equivalence arises similarly to the setting in (2). The main difference is that $A$, $B$ are unital, and $(e_t)$ is not a cai, but $e_t \to 1_A$ weak$^*$, and similarly for the net in $B$.

(4) Von Neumann algebras which are Morita equivalent in Rieffel's $W^*$-algebraic sense from [23], are clearly weakly Morita equivalent. We state this in the language of TROs. We recall that a TRO is a subspace $Z \subset B(K, H)$ with $ZZ^*Z \subset Z$. Rieffel's $W^*$-algebraic Morita equivalence of $W^*$-algebras $M$ and $N$ is essentially the same (see e.g. [8, Section 8.5] for more details) as having a weak$^*$ closed TRO (that is, a *WTRO*) $Z$, with $ZZ^*$ weak$^*$ dense in $M$ and $Z^*Z$ weak$^*$ dense in $N$. Recall that $Z^*Z$ denotes the norm closure of the span of products $z^*w$ for $z, w \in Z$. Here $(ZZ^*, Z^*Z, Z, Z^*)$ is the weak$^*$ dense subcontext.

(5) More generally, the 'tight Morita $w^*$-equivalence' of [9, Section 5], is easily seen to be a special case of weak Morita equivalence. In this case, the equivalence bimodules $X$ and $Y$ are 'selfdual'. Indeed, this selfduality is the reason for the approach taken in [9, Section 5].

(6) The second duals of strongly Morita equivalent operator algebras are weakly Morita equivalent. Recall that if $A$ and $B$ are approximately unital operator algebras, then $A^{**}$ and $B^{**}$ are unital dual operator algebras, by 2.5.6 in [8]. If $X$ is a non-degenerate operator $A$–$B$-bimodule, then $X^{**}$ is a dual operator $A^{**}$–$B^{**}$-bimodule in a canonical way. Let $(\cdot, \cdot)$ be a bilinear map from $X \times Y$ to $A$ that is balanced over $B$ and is an $A$-bimodule map. Then notice that by 1.6.7 in [8], there is a unique separately $w^*$-continuous extension from $X^{**} \times Y^{**}$ to $A^{**}$, which we still call $(\cdot, \cdot)$. Now the weak Morita equivalence follows easily from the Goldstine lemma.

(7) Any unital dual operator algebra $M$ is weakly Morita equivalent to $\mathbb{M}_I(M)$, for any cardinal $I$. The weak$^*$ dense strong Morita subcontext in this case is $(M, \mathbb{K}_I(M), R_I(M), C_I(M))$, whereas the equivalence bimodules $X$ and $Y$ above are $R_I^w(M)$ and $C_I^w(M)$ respectively.

(8) *TRO equivalent* dual operator algebras $M$ and $N$, or more generally $\Delta$-*equivalent algebras*, in the sense of [17,18], are weakly Morita equivalent. If $M \subset B(H)$ and $N \subset B(K)$, then TRO equivalence means that there exists a TRO $Z \subset B(H, K)$ such that $M = [Z^*NZ]^{\overline{w}^*}$ and $N = [ZMZ^*]^{\overline{w}^*}$. Eleftherakis shows that one may assume that $Z$ is a WTRO and $1_N z = z 1_M = z$ for $z \in Z$. Define $X$ and $Y$ to be the weak$^*$ closures of $MZ^*N$ and $NZM$ respectively. Define $A$ and $B$ to be, respectively, $Z^*NZ$ and $ZMZ^*$. Define $X'$ and $Y'$ to be, respectively, the norm closures of $Z^*YZ^*$ and $ZXZ$. Since $Z$ is a TRO, $Z^*Z$ is a $C^*$-algebra, and so it has a contractive approximate identity $(e_t)$ where $e_t = \sum_{k=1}^{n(t)} x_k^t y_k^t$ for some $y_k^t \in Z$, and $x_k^t = (y_k^t)^*$. It is easy to check that $(e_t)$ is a cai for $A$, and a similar statement holds for $B$. Indeed it is clear that $(A, B, X', Y')$ is a weak$^*$-dense strong Morita subcontext of $(M, N, X, Y)$. Hence $M$ and $N$ are weakly Morita equivalent. We remark that it is proved in [20] that, in our language, $M$ and $N$ are weak$^*$ Morita equivalent.

(9) Examples of weak and weak* Morita equivalence may also be easily built as at the end of [6, Section 6], from a weak* closed subalgebra $A$ of a von Neumann algebra $M$, and a strictly positive $f \in M_+$ satisfying a certain 'approximation in modulus' condition. Then the weak linking algebra of such an example is Morita equivalent in the same sense to $A$ (see Section 4), but they are probably not always weak* stably isomorphic.

(10) A beautiful example from [19] (formerly part of [17]): two 'similar' separably acting nest algebras are clearly weakly Morita equivalent by the facts presented around [19, Theorem 3.5] (Davidson's similarity theorem), indeed in this case the 'subcontext' (see Definition 3.2) equals the 'context', and the algebras are even strongly Morita equivalent. However, Eleftherakis shows they need not be '$\Delta$-equivalent' (that is, weak* stably isomorphic [20]).

In the theory of strong Morita equivalence, and also in our paper, it is very important that $N$ has some kind of 'approximate identity' $(f_s)$ of the form

$$f_s = \sum_{i=1}^{n_s} [y_i^s, x_i^s], \qquad \|[y_1^s, \ldots, y_{n_s}^s]\|\|[x_1^s, \ldots, x_{n_s}^s]^T\| < 1, \tag{3.1}$$

and similarly that $M$ has some kind of 'approximate identity' $(e_t)$ of form

$$e_t = \sum_{i=1}^{m_t} (x_i^t, y_i^t), \qquad \|[x_1^t, \ldots, x_{m_t}^t]\|\|[y_1^t, \ldots, y_{m_t}^t]^T\| < 1. \tag{3.2}$$

Here $x_i^s, x_i^t \in X$, $y_i^s, y_i^t \in Y$. Indeed, by [11], $x_i^s, x_i^t, y_i^s, y_i^t$ may be chosen in $X'$ and $Y'$ in the case of weak Morita equivalence.

In what follows, we say, for example, that $(\cdot, \cdot)$ is a *bimodule map* if $m(x, y) = (mx, y)$ and $(x, y)m = (x, ym)$ for all $x \in X, y \in Y, m \in M$.

**Theorem 3.3.** *$(M, N, X, Y)$ is a weak\* Morita context iff the following conditions hold: there exists a separately* weak\*-*continuous completely contractive $M$-bimodule map $(\cdot, \cdot) : X \times Y \to M$ which is balanced over $N$, and a separately* weak\*-*continuous completely contractive $N$-bimodule map $[\cdot, \cdot] : Y \times X \to N$ which is balanced over $M$, such that $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$ for $x, x' \in X, y, y' \in Y$; and also there exist nets $(f_s)$ in $N$ and $(e_t)$ in $M$ of the form in* (3.1) *and* (3.2) *above, with $f_s \to 1_N$ and $e_t \to 1_M$ weak\*.*

**Proof.** ($\Leftarrow$) Under these conditions, we first claim that if $\pi : X \otimes_N^{\sigma h} Y \to M$ is the canonical ($w^*$-continuous) $M$–$M$-bimodule map induced by $(\cdot, \cdot)$, then $\pi(u)x \otimes_N y = u(x, y)$ for all $x \in X, y \in Y$, and $u \in X \otimes_N^{\sigma h} Y$. To see this, fix $x \otimes_N y \in X \otimes_N^{\sigma h} Y$. Define $f, g : X \otimes_N^{\sigma h} Y \to X \otimes_N^{\sigma h} Y$: $f(u) = u(x, y)$ and $g(u) = \pi(u)x \otimes_N y$ where $u \in X \otimes_N^{\sigma h} Y$. We need to show that $f = g$. Since $X \otimes_{hN} Y$ is $w^*$-dense in $X \otimes_N^{\sigma h} Y$, and $f, g$ are $w^*$-continuous, it is enough to check that $f = g$ on $X \otimes_{hN} Y$. For $u = x' \otimes_N y'$, we have

$$u(x, y) = x' \otimes_N y'(x, y) = x' \otimes_N [y', x]y = x'[y', x] \otimes_N y = (x', y')x \otimes_N y = \pi(u)x \otimes_N y,$$

as desired in the claim.

To see that $M \cong X \otimes_N^{\sigma h} Y$, we shall show that $\pi$ above is a complete isometry. Since $M$ is the weak* closure of the span of the range of $(\cdot, \cdot)$, it will follow from the Krein-Smulian theorem that $\pi$ maps onto $M$. Choose an approximate identity $(e_t)$ for $M$ of the form in (3.2). Define $\rho_t : M \to X \otimes_N^{\sigma h} Y$: $\rho_t(m) = \sum_{i=1}^{n_t} mx_i^t \otimes_N y_i^t$. For $[u_{jk}] \in M_n(X \otimes_N^{\sigma h} Y)$, we have by the last paragraph that

$$\rho_t \circ \pi([u_{jk}]) = \left[ \sum_{i=1}^{n_t} \pi(u_{jk}) x_i^t \otimes_N y_i^t \right] = \left[ \sum_{i=1}^{n_t} u_{jk}(x_i^t, y_i^t) \right] = [u_{jk}e_t] \xrightarrow{w^*} [u_{jk}],$$

the convergence by [20, Lemma 2.3]. Since $\rho_t$ is completely contractive, we have

$$\|[u_{jk}e_t]\| = \|(\rho_t \circ \pi)([u_{jk}])\| \leq \|\pi([u_{jk}])\|.$$

As $[u_{jk}]$ is the $w^*$-limit of the net $([u_{jk}e_t])_t$, by Alaoglu's theorem we deduce that $\|[u_{jk}]\| \leq \|\pi([u_{jk}])\|$. Similarly, $N \cong Y \otimes_M^{\sigma h} X$.

($\Rightarrow$) The existence of the nets $(f_s)$ and $(e_t)$ follows from Corollary 2.6. Define $(\cdot, \cdot)$ to be the composition of the canonical map $X \times Y \to X \otimes_N^{\sigma h} Y$ with the isomorphism of the latter space with $M$. Similarly one obtains $[\cdot, \cdot]$, and these maps have all the desired properties except the relations $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$. To obtain these we have to adjust $(\cdot, \cdot)$ by multiplying it by a certain unitary in $M$, as in the proof of [5, Proposition 1.3]. Indeed that proof transfers easily to our present setting, and in fact becomes slightly simpler, since in the latter proof the map called $T$ is weak* continuous in our case, and $w^*CB_M(M) \cong M$. $\square$

**Corollary 3.4.** *Every weak Morita context is a weak\* Morita context.*

**Proof.** Let $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ be a weak Morita context with strong Morita subcontext $(A, B, X', Y')$. If $(f_s)$ is a cai for $B$ it is clear that $f_s \to 1_N$ weak*. Indeed if a subnet $f_{s_\alpha} \to f$ in the weak*-topology in $N$, then $bf = b$ for all $b \in B$. By weak*-density it follows that $bf = b$ for all $b \in N$. Similarly $fb = b$. Thus $f = 1_N$. By Lemma 2.9 in [11] we may choose $(f_s)$ of the form (3.1), and similarly $A$ has a cai $(e_t)$ of form in (3.2). That $(x, y)x' = x[y, x']$ and $y'(x, y) = [y', x]y$ for $x, x' \in X, y, y' \in Y$, follows by weak* density, and from the fact that the analogous relations hold in $X'$ and $Y'$. Similarly one sees that $(\cdot, \cdot)$ and $[\cdot, \cdot]$ are balanced bimodule maps. $\square$

A key point for us, is that the condition involving (3.1) in Theorem 3.3 becomes a powerful tool when expressed in terms of an 'asymptotic factorization' of $I_Y$ (resp. $I_{Y'}$) through spaces of the form $C_n(M)$ (resp. $C_n(A)$ in the case of a weak Morita equivalence). Indeed, define $\varphi_s(y)$ to be the column $[(x_j^s, y)]_j$ in $C_{n_s}(M)$, for $y$ in $Y$, and define $\psi_s([a_j]) = \sum_j y_j^s a_j$ for $[a_j]$ in $C_{n_s}(M)$. Then $\psi_s(\varphi_s(y)) = f_s y \to y$ weak* if $y \in Y$ (or in norm if $y \in Y'$, in the case of a weak Morita equivalence, in which case we can replace $C_{n_s}(M)$ by $C_{n_s}(A)$). Similarly, (3.1) may be expressed in terms of an 'asymptotic factorization' of $I_X$ through spaces of the form $R_n(M)$ (or $I_{X'}$ through $R_n(A)$ in the 'weak Morita' case), Similarly, the condition involving (3.2) may be expressed in terms of an 'asymptotic factorization' of $I_Y$ through spaces of the form $R_n(N)$ (or $I_{Y'}$ through $R_n(B)$ in the 'weak Morita' case), or of $I_X$ through $C_n(N)$.

Henceforth in this section, let $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ be as in Theorem 3.3. We will also refer to this 6-tuple as the weak* Morita context.

**Theorem 3.5.** *Weak\* Morita equivalent dual operator algebras have equivalent categories of dual operator modules.*

**Proof.** Write $_M\mathcal{R}$ for the category of left dual operator modules over $M$. The morphisms here are the $w^*$-continuous completely bounded $M$-module maps. If $Z \in {}_N\mathcal{R}$ and if $\mathcal{F}(Z) = X \otimes_N^{\sigma h} Z$, then $\mathcal{F}(Z)$ is a left dual operator $M$-module by Lemma 2.5. That is, $\mathcal{F}(Z) \in {}_M\mathcal{R}$. Further, if $T \in w^*CB_N(Z, W)$, for $Z, W \in {}_N\mathcal{R}$, and if $\mathcal{F}(T)$ is defined to be $I \otimes_N T : \mathcal{F}(Z) \to \mathcal{F}(W)$, then by the functoriality of the normal module Haagerup tensor product we have $\mathcal{F}(T) \in w^*CB_M(\mathcal{F}(Z), \mathcal{F}(W))$, and $\|\mathcal{F}(T)\|_{cb} \leq \|T\|_{cb}$. Thus $\mathcal{F}$ is a contractive functor from $_N\mathcal{R}$ to $_M\mathcal{R}$. Similarly, we obtain a contractive functor $\mathcal{G}$ from $_M\mathcal{R}$ to $_N\mathcal{R}$. Namely, $\mathcal{G}(W) = Y \otimes_M^{\sigma h} W$, for $W \in {}_M\mathcal{R}$, and $\mathcal{G}(T) = I \otimes_M T$ for $T \in w^*CB_M(W, Z)$ with $W, Z \in {}_M\mathcal{R}$. Similarly, it is easy to check that these functors are completely contractive; for example, $T \mapsto \mathcal{F}(T)$ is a completely contractive map on each space $w^*CB_N(Z, W)$ of morphisms. It is more complicated to show that the map $T \to \mathcal{F}(T)$ is weak* continuous: a proof of this emerged during a conversation with Jon Kraus; we will present this elsewhere. If we compose $\mathcal{F}$ and $\mathcal{G}$, we find that for $Z \in {}_N\mathcal{R}$ we have $\mathcal{G}(\mathcal{F}(Z)) \in {}_N\mathcal{R}$. By Proposition 2.9 and Lemma 2.10, we have

$$\mathcal{G}(\mathcal{F}(Z)) \cong Y \otimes_M^{\sigma h}(X \otimes_N^{\sigma h} Z) \cong (Y \otimes_M^{\sigma h} X) \otimes_N^{\sigma h} Z \cong N \otimes_N^{\sigma h} Z \cong Z.$$

where the isomorphisms are completely isometric. The rest of the proof follows as in Theorem 3.9 in [11]. $\quad\square$

**Remark** We imagine that the ideas of [5] show that the converse of the last theorem is true, and hope to pursue this in the future.

We shall adopt the convention from algebra of writing maps on the side opposite the one on which ring acts on the module. For example a left $A$-module map on $X$ will be written on the right and a right $A$-module map will be written on the left. The pairings and actions arising in the weak* Morita context give rise to eight maps:

$$R_N : N \to CB_M(X, X), \quad xR_N(b) = x \cdot b$$
$$L_N : N \to CB(Y, Y)_M, \quad L_N(b)y = b \cdot y$$
$$R_M : M \to CB_N(Y, Y), \quad yR_M(a) = y \cdot a$$
$$L_M : M \to CB(X, X)_N, \quad L_M(a)x = a \cdot x$$
$$R^M : Y \to CB_M(X, M), \quad xR^M(y) = (x, y)$$
$$L^N : Y \to CB(X, N)_N, \quad L^N(y)x = [y, x]$$
$$R^N : X \to CB_N(Y, N), \quad yR^M(x) = [y, x]$$
$$L^M : X \to CB(Y, M)_M, \quad L^M(x)y = (x, y).$$

The first four maps are completely contractive since module actions are completely contractive. Also the maps $L_N$ and $L_M$ are homomorphisms and $R_N$ and $R_M$ are anti-homomorphisms. Similar proofs to the analogous results in [11] show that $R^M, L^N, R^N$, and $L^M$ are completely contractive.

**Theorem 3.6.** *If $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ is a weak\* Morita context, then each of the maps $R^M, R^N, L^M$ and $L^N$ is a weak\* continuous complete isometry. The range of $R^M$ is $w^*CB_M(X, M)$, with similar assertions holding for $R^N, L^M$ and $L^N$. The map $L_N$ (resp. $R_N$) is a $w^*$-continuous completely isometric isomorphism (resp. anti-isomorphism) onto the $w^*$-closed left (resp. right) ideal $w^*CB(Y)_M$ (resp. $w^*CB_M(X)$). Similar results hold for $L_M$ and $R_M$.*

**Proof.** Most of this can be proved directly, as in [11, Theorem 4.1]. Instead we will deduce it from the functoriality (Theorem 3.5). For example, because of the equivalence of categories via the (completely contractive) functor $\mathcal{F} = Y \otimes_M^{\sigma h} -$, we have completely isometrically:

$$M \cong w^*CB_M(M) \cong w^*CB_N(\mathcal{F}(M)) \cong w^*CB_N(Y),$$

and the composition of these maps is easily seen to be $R_M$. Thus $R_M$ is a complete isometry. Similar proofs work for the other seven maps. To see that $L_N$ is $w^*$-continuous, for example, let $(b_t)$ be a bounded net in $N$ converging in the $w^*$-topology of $N$ to $b \in N$. Then $L_N(b_t)$ is a bounded net in $CB(Y)_M$. As the module action is separately $w^*$-continuous, it is easy to see that $L_N(b_t)$ converges to $L_N(b)$ in the $w^*$-topology. Thus $L_N$ is a $w^*$-continuous isometry with $w^*$-closed range, by the Krein-Smulian theorem. To see that its range is a left ideal simply use the weak*-density of the span of terms $[y, x]$ in $N$, and the equation $TL_N([y, x])(y') = L_N[Ty, x](y')$ for $T \in CB(Y)_M, y' \in Y$. We leave the variants for the other maps to the reader. $\quad\square$

**Remarks** (1) Note that in the case of weak Morita equivalence, $CB_A(X')$ is an operator algebra ([11], Theorem 4.9). It is not true in general that $CB_M(X)$ is an operator algebra, as we show in [21]. Nonetheless, the above shows that $w^*CB_M(X)$ is a dual operator algebra ($\cong N$).

(2) In fact $w^*CB(Y)_M$ (resp. $w^*CB_M(X)$) equals the one-sided operator space multiplier algebra (see [8, Chapter 4]) $\mathcal{M}_\ell(Y)$ (resp. $\mathcal{M}_r(X)$). See [7].

**Theorem 3.7.** *If $M$ and $N$ are weak$^*$ Morita equivalent dual operator algebras, then their centers are completely isometrically isomorphic via a $w^*$-homeomorphism.*

**Proof.** By Theorem 3.6 there is a $w^*$-continuous complete isometry $R_M : M \to w^*CB_N(Y)$. The restriction of $R_M$ to $Z(M)$ maps into $w^*CB(Y)_M \cong N$, and so we have defined a $w^*$-continuous completely isometric homomorphism $\theta : Z(M) \to N$. One easily sees that $\theta(a)(y) = ya$, for $a \in Z(M)$. It is also easy to see that this implies that $\theta$ maps into $Z(N)$, and to argue, by symmetry, that $\theta$ must be an isomorphism.   □

**Lemma 3.8.** *In the case of weak Morita equivalence, if $Z$ is a left dual operator $M$-module, then the canonical map from $Y' \otimes_{hA} Z$ into $Y \otimes_M^{\sigma h} Z$ is completely isometric, and it maps the ball onto a $w^*$-dense set in Ball $(Y \otimes_M^{\sigma h} Z)$.*

**Proof.** The canonical map here is completely contractive, let us call it $\theta$. On the other hand, let $(f_s)$ be as in (3.1), and let $\varphi_s, \psi_s$ be as defined just below Corollary 3.4, with $\psi_s(\varphi_s(y)) = f_s y \to y$. Then for $u \in M_n(Y' \otimes_A Z)$, we have

$$\|\theta_n(u)\| \geq \|(\varphi_s \otimes I)_n(\theta_n(u))\| = \|(\varphi_s \otimes I)_n(u)\| \geq \|f_s u\|.$$

Taking a limit over $s$, gives $\|\theta_n(u)\| \geq \|u\|$.

Let $u \in \text{Ball}(Y \otimes_M^{\sigma h} Z)$. By Corollary 2.6, there exists a net $(u_t)$ in the image of $\text{Ball}(Y \otimes_{hM} Z)$ such that $u_t \overset{w^*}{\to} u$. We may assume that each $u_t$ is of the form $w \odot z$, for $w \in \text{Ball}(R_n(Y))$, $z \in \text{Ball}(C_n(Z))$. We rewrite (3.1) and the lines below it, namely write each $f_s$ in the form $[y, x]$ (in suggestive notation), for $y \in \text{Ball}(R_m(Y'))$ and $x \in \text{Ball}(C_m(X'))$. Note $w \odot z$ is the weak$^*$ limit of terms $f_s w \odot z$, and $f_s w \odot z = y \odot v$, where $v$ is a column with $k$th entry $\sum_j [x_k, w_j] z_j$. It is easy to check that $\|v\| \leq 1$.   □

**Proposition 3.9.** *Weak$^*$ Morita equivalence is an equivalence relation.*

**Proof.** This follows the usual lines, for example the transitivity follows from associativity of the tensor products and Lemma 2.10.   □

**Remark** Concerning transitivity of weak Morita equivalence, it is convenient (and probably necessary) to consider Definition 3.2 as defining an equivalence between pairs $(M, A)$ and $(N, B)$, as opposed to just between $M$ and $N$. That is we also consider the weak$^*$-dense operator subalgebras. Then it is fairly routine to see that weak Morita equivalence is an equivalence relation [21].

**Theorem 3.10.** *Weak$^*$ Morita equivalent dual operator algebras have equivalent categories of normal Hilbert modules. Moreover, the equivalence preserves the subcategory of modules corresponding to completely isometric normal representations.*

**Proof.** If $H$ is a normal Hilbert $M$-module, let $K = Y \otimes_M^{\sigma h} H^c$. By the discussion just below Corollary 3.4, combined with Corollary 2.4, there are nets of maps $\varphi_s : K \to C_{n_s}(M) \otimes_M^{\sigma h} H^c \cong C_{n_s}(H^c)$, and maps $\psi_s : C_{n_s}(H^c) \to K$, with $\psi_s(\varphi_s(z)) = f_s z \to z$ weak$^*$ for all $z \in K$. Here $(f_s)$ is as in (3.1). Let $\Lambda$ be the directed set indexing $s$, and let $\mathcal{U}$ be an ultrafilter with the property that $\lim_{\mathcal{U}} z_s = \lim_{\Lambda} z_s$ for scalars $z_s$, whenever the latter limit exists. Let $H_{\mathcal{U}}$ be the ultraproduct of the spaces $C_{n_s}(H^c)$, which is a column Hilbert space, as is well known and easy to see. Define $T : K \to H_{\mathcal{U}}$ by $T(x) = (\varphi_s(x))_s$, for $x \in K$. This is a complete contraction. To see that it is an isometry, note that for any $x \in K$, $\rho \in \text{Ball}(K_*)$, we have

$$|\rho(x)| = \lim_{\mathcal{U}} |\rho(\psi_s(\varphi_s(x)))| \leq \lim_{\mathcal{U}} \|\varphi_s(x)\| = \|T(x)\|.$$

Similarly, $T$ is a complete isometry, as we leave to the reader to check. Thus $K$ is a (column) Hilbert space.

That $K = Y \otimes_M^{\sigma h} H^c$ is a normal Hilbert $N$-module now follows from Lemma 2.5. Finally, suppose that $M$ is a weak$^*$ closed subalgebra of $B(H)$, we will show that the induced representation $\rho$ of $N$ on $K$ is completely isometric. Certainly this map is completely contractive. Let $[b_{pq}] \in M_d(N)$, $[y_{kl}] \in \text{Ball}(M_m(Y))$, $[\zeta_{rs}] \in \text{Ball}(M_g(H^c))$, $[x_{ij}] \in \text{Ball}(M_n(X))$. We have $\|[y_{kl} \otimes \zeta_{rs}]\| \leq 1$ with respect to the operator space projective tensor product matrix norm, and hence also with respect to $\otimes_M^{\sigma h}$. Thus,

$$\|[\rho(b_{pq})]\| \geq \|[b_{pq} y_{kl} \otimes \zeta_{rs}]\| \geq \|[(x_{ij}, b_{pq} y_{kl}) \zeta_{rs}]\|.$$

Taking the supremum over all such $[\zeta_{rs}]$, gives

$$\|[\rho(b_{pq})]\| \geq \sup\{\|[(x_{ij}, b_{pq} y_{kl})]\| : [x_{ij}] \in \text{Ball}(M_n(X))\} = \|[b_{pq} y_{kl}]\|,$$

by Theorem 3.6. Taking the supremum over all such $[y_{kl}] \in \text{Ball}(M_m(Y))$ gives $\|[\rho(b_{pq})]\| \geq \|[b_{pq}]\|$, by Theorem 3.6 again.   □

The last result shows that weak* Morita equivalent operator algebras have equivalent categories of normal representations. It would be very interesting to characterize when two operator algebras have equivalent categories of normal representations; it seems quite possible that this happens iff they are weak* Morita equivalent.

**Corollary 3.11.** *If $H$ is a normal Hilbert module over $M$, then $Y \otimes_M^{\sigma h} H^c$ is a column Hilbert space. Also, the image of $\mathrm{Ball}(Y \otimes_{hM} H^c)$ is norm dense in $\mathrm{Ball}(Y \otimes_M^{\sigma h} H^c)$. In the case of weak Morita equivalence, $Y \otimes_M^{\sigma h} H^c = Y' \otimes_{hA} H^c = Y' \widehat{\otimes}_A H^c$ completely isometrically. Here $\widehat{\otimes}_M$ is as in 3.4.2 of* [8].

**Proof.** For the second assertion, by Corollary 2.6 the image of $\mathrm{Ball}(Y \otimes_{hM} H^c)$ is weakly dense, and hence norm dense by Mazur's theorem. In the case of weak Morita equivalence, if $\eta \in H \ominus [AH]$ and $(e_t)$ is a cai for $A$, then

$$\langle \eta, \eta \rangle = \lim_t \langle e_t \eta, \eta \rangle = 0.$$

Thus $A$ acts nondegenerately on $H$. The assertions involving $Y'$ now follow from Lemma 3.8 and Mazur's theorem again. Since $- \otimes_h H^c = - \widehat{\otimes} H^c$ (see e.g. [15, Proposition 9.3.2]), we may replace $\otimes_{hM}$ here by $\widehat{\otimes}_M$ (see 3.4.2 of [8] for this notation). □

## 4. The weak linking algebra

In this section again, $(M, N, X, Y, (\cdot, \cdot), [\cdot, \cdot])$ is a weak* Morita context. Suppose that $M$ is represented as a weak*-closed nondegenerate subalgebra of $B(H)$, for a Hilbert space $H$. Then by Corollary 3.11, $K = Y \otimes_M^{\sigma h} H^c$ is a column Hilbert space. Define a right $M$-module map $\Phi : Y \to B(H, K)$ by $\Phi(y)(\zeta) = y \otimes_M \zeta$ where $y \in Y$ and $\zeta \in H$. It is easy to see that $\Phi$ is a completely contractive $N$–$M$-bimodule map. It is weak*-continuous, since if we have a bounded net $y_t \to y$ weak* in $Y$, and if $\zeta \in H$, then $y_t \otimes_M \zeta \to y \otimes_M \zeta$ weakly. That is, $\Phi(y_t) \to \Phi(y)$ in the WOT, and it follows that $\Phi$ is weak*-continuous. If $\|\Phi(y)\| \leq 1$, and if $\zeta \in \mathrm{Ball}(H^{(n)})$, and $[x_{ij}] \in \mathrm{Ball}(M_n(X))$, then

$$\|[(x_{ij}, y)]\zeta\| = \|[x_{ij} \otimes \Phi(y)]\zeta\| \leq \|\Phi(y)\|.$$

Taking the supremum over such $\zeta$, and then over such $[x_{ij}]$, we obtain from Theorem 3.6 that $\|y\| \leq 1$. Thus $\Phi$ is an isometry, and a similar but more tedious argument shows that $\Phi$ is a complete isometry. By the Krein-Smulian theorem we deduce that the range of $\Phi$ is weak*-closed. A similar argument, which we leave to the reader, shows that the map $\Psi : X \to B(K, H)$, defined by $\Psi(x)(y \otimes \zeta) = (x, y)\zeta$, is a $w^*$-continuous completely isometric $M$–$N$-bimodule map. See also [7, Section 3.1] for more on the above points. As we said in Theorem 3.10, the induced normal representation $N \to B(K)$ is completely isometric.

We use the above to define the direct sum $M \oplus^c Y$ as follows. For specificity, the reader might want to take $H$ to be a universal normal representation of $M$, that is the restriction to $M$ of a one-to-one normal representation of $W_{\max}^*(M)$. Define a map $\theta : M \oplus Y \to B(H, K \oplus H)$ by $\theta((m, y))(\zeta) = (m\zeta, y \otimes_M \zeta)$, for $y \in Y, m \in M, \zeta \in H$. One can quickly check that $\theta$ is a one-to-one, $M$-module map, and that $\theta$ is a weak*-continuous complete isometry when restricted to each of $Y$ and $M$. Also, $W = \mathrm{Ran}(\theta)$ is easily seen to be weak*-closed. We norm $M \oplus^c Y$ by pulling back the operator space structure from $W$ via $\theta$. Thus $M \oplus^c Y$ may be identified with the weak*-closed right $M$-submodule $W$ of $B(H, H \oplus K)$; and hence it is a dual operator $M$-module. In a similar way, we define $M \oplus^r X$ to be the canonical weak*-closed left $M$-submodule of $B(H \oplus K, H)$.

We next define the 'weak linking algebra' of the context, namely

$$\mathcal{L}^w = \left\{ \begin{bmatrix} a & x \\ y & b \end{bmatrix} : a \in M, b \in N, x \in X, y \in Y \right\},$$

with the obvious multiplication. As in [11, Lemma 5.6], one easily sees that there is at most one possible sensible dual operator space structure on this linking algebra. Indeed if $\Lambda$ is the set indexing $t$ in the net in (3.2), and if $\beta, t \in \Lambda$, then define $\theta^{\beta,t}$ on the linear space $\mathcal{L}^w$ to be the map $\theta^\beta$ in [11, p. 45], but with all the $y_i^\beta$ replaced by $y_i^t$. Then a simple modification of the argument in [11, p. 50-51], and using semicontinuity of the norm in the weak* topology, yields that any 'sensible' norm assigned to $\mathcal{L}^w$ must agree with $\sup_{\beta,t} \|\theta^{\beta,t}(\cdot)\|$.

To see that such a dual operator space structure does exist, one only needs to view $\mathcal{L}^w$ as a subalgebra $\mathcal{R}$ of $B(H \oplus K)$, using the obvious pairings $X \times K \to H$ (induced by $(\cdot, \cdot)$), $Y \times H \to K$, and $N \times K \to K$ (this is the induced representation of $N$ on $K$ from Theorem 3.10). It is easy to check that $(M, \mathcal{R}, M \oplus^r X, M \oplus^c Y)$ is also a weak* Morita context (this follows from norm equalities of the kind in e.g. the centered equations in [11, Theorem 5.12]). This all may be most easily visualized by picturing both contexts as $3 \times 3$-matrices, namely as subalgebras of $B(H \oplus H \oplus K)$. Theorem 3.6 gives $\mathcal{R} \cong w^* CB(M \oplus^c Y)_M$ completely isometrically and $w^*$-homeomorphically.

Note that in a weak Morita situation, the linking operator algebra of the strong Morita context $(A, B, X', Y')$ can be identified completely isometrically as the obvious weak* dense subalgebra $\mathcal{L}$ of $\mathcal{R}$ (see e.g. [1, Proposition 6.10]). Incidentally, at this point we have already proved the assertion made at the start of Example (2) in Section 3, and indeed that every weak Morita equivalence arises as the weak* closure of a strong Morita equivalence, or can be viewed as the weak*-closure, in some representation, of the linking operator algebra of a strong Morita equivalence. We have a strong Morita context $(A, \mathcal{L}, A \oplus^r X', A \oplus^c Y')$ (see [11,10]), which can be viewed as a subcontext of $(M, \mathcal{R}, M \oplus^r X, M \oplus^c Y)$. Thus the latter is a weak Morita context.

Extracting from the last paragraphs, we have:

**Corollary 4.1.** *M is weak\* Morita equivalent to the weak linking algebra $\mathcal{L}^w$. Indeed this is a weak Morita equivalence if $(M, N, X, Y)$ is a weak Morita context.*

It is often useful here to know that:

**Proposition 4.2.** *With notation as in Theorem 3.10, we have $(M \oplus^c Y) \otimes_M^{\sigma h} H^c \cong (H \oplus K)^c$ as Hilbert spaces.*

**Proof.** We will just sketch this, since it is not used here. By Corollary 4.1, and Theorem 3.10, we have that $L = (M \oplus^c Y) \otimes_M^{\sigma h} H^c$ is a column Hilbert space. Moreover, the projections from $M \oplus^c Y$ onto $M$ and $Y$ respectively, induce by Corollary 2.4, projections $P$ and $Q$ from $L$ onto $M \otimes_M^{\sigma h} H^c \cong H^c$, and $K$, respectively, such that $P + Q = I$.  $\square$

Mimicking the proof of [11, Theorem 5.1] we have:

**Theorem 4.3.** *Let $(M, N, X, Y)$ be a weak\* Morita context. Then there is a lattice isomorphism between the $w^*$-closed M-submodules of X and the lattice of $w^*$-closed left ideals in N. The $w^*$-closed M–N-submodules of X corresponds to the $w^*$-closed two-sided ideals in N. Similar statements for Y follows by symmetry. In particular, M and N have isomorphic lattices of $w^*$-closed two-sided ideals.*

We next show, analogously to [11, Section 6], that if $M$ and $N$ are $W^*$-algebras, then they are Morita equivalent in Rieffel's sense iff they are weakly (or equivalently, weak\*) Morita equivalent in our sense. Indeed we already have remarked (Example (4) in Section 3) that Rieffel's Morita equivalence is an example of our weak Morita equivalence. The following gives the converse, and more:

**Theorem 4.4.** *Let $(M, N, X, Y)$ be a weak\* Morita context where N is a $W^*$-algebra. Then M is a $W^*$-algebra, and there is a completely isometric isomorphism $i : \bar{X} \to Y$ such that X becomes a $W^*$-equivalence M–N-bimodule (see e.g. 8.5.12 in [8]) with inner products defined by the formulas $_M\langle x_1, x_2 \rangle = (x_1, i(\bar{x}_2))$ and $\langle x_1, x_2 \rangle_N = [i(\bar{x}_1), x_2]$.*

**Proof.** First we represent the linking algebra on a Hilbert space $H \oplus K$ as above. We rechoose the net $(e_t)$ such that $e_t \to I_H$ strongly, so that $e_t^* e_t \to I_H$ weak\*, and similarly for the net $(f_s)$. To accomplish this, note that the WOT-closure of the convex hull of the $(e_t)$ equals the SOT-closure, by elementary operator theory. However it is easy to see that the form in (3.2) is preserved if we replace $e_s$ by convex combinations of the $e_t$. Now one can follow the proof of [11, Theorem 6.2] to deduce that the adjoint of any $y \in Y$ is a limit of terms in $X$. That is $Y \subset X^*$. Similarly, $X \subset Y^*$. So $X = Y^*$, and so it follows that $M$ is a $W^*$-algebra, and $X$ is a WTRO (this term was defined in Example (4) in Section 3) setting up a $W^*$-algebra Morita equivalence. We leave the rest as an exercise.  $\square$

The following is the nonselfadjoint analogue of a theorem of Rieffel. A special case of it is mentioned, with a proof sketch, at the end of [9].

**Theorem 4.5.** *Let H be a universal normal representation for M, and let K be the induced representation of N studied above. Then $M' \cong N'$; that is there is a completely isometric $w^*$-continuous isomorphism $\theta : B_M(H) \cong B_N(K)$. Writing $\mathcal{R}$ for either of these commutants, we have $X \cong B_{\mathcal{R}}(K, H)$ and $Y \cong B_{\mathcal{R}}(H, K)$ completely isometrically and as dual operator bimodules.*

**Proof.** One uses the equivalence of categories to see that $B_M(H) \cong B_N(\mathcal{F}(H)) = B_N(K)$ completely isometrically, in the notation of Theorem 3.5. That is, $M' \cong N'$ as asserted. The isomorphism $\theta$ here is essentially defined by the relation $\theta(T)\Phi(y) = \Phi(y)T$ for all $y \in Y, T \in M'$. Here $\Phi$ is as in the discussion at the start of Section 4. It follows easily from this that if $T_t \to T$ weak\* in $M'$, then $\theta(T_t)(y \otimes \zeta) \to \theta(T)(y \otimes \zeta)$ weakly. By the second assertion of Corollary 3.11, $\theta$ is weak\* continuous, even a weak\* homeomorphism, by the Krein-Smulian theorem. Now mimic the proof of 8.5.32 and 8.5.37 in [8]. The main point to bear in mind is that since $M$ is weak\* Morita equivalent to the weak linking algebra $\mathcal{L}^w$, the induced representation of $\mathcal{L}^w$ is also a universal normal representation, by easy category theoretic arguments. Thus by [13] it satisfies the double commutant theorem. Carefully computing the first, and then the second, commutants of $\mathcal{L}^w$ (as in 8.5.32 in [8] or [22]), and using the double commutant theorem, gives the result.  $\square$

**Example 4.6.** If $M$ and $N$ are *finite dimensional* then weak\* Morita equivalence equals strong Morita equivalence, and coincides also with the equivalence of [17,18,20], that is, weak\* stable isomorphism. Indeed if $(M, N, X, Y)$ is a weak\* Morita context, then it is clearly a strong Morita context in this case. By [11, Lemma 2.8] we can factor the identity map $I_Y$ through $C_n(M)$ for some $n \in \mathbb{N}$, so that $Y$ is finite dimensional. Similarly, $X$ is finite dimensional. To see that this implies that $M$ and $N$ are weak\* stably isomorphic, note that in this situation, since $M \cong X \otimes_N^{\sigma h} Y$, there is a norm 1 element in $X \otimes_h Y$ mapping to $1_M$. Similarly for $1_N$, and it is evident that one has what is called a 'quasi-unit of norm 1' in [11, Section 7]. By [11, Corollary 7.9], $M$ and $N$ are stably isomorphic, and taking second duals and using e.g. (1.62) in [8], we see that they are weak\* stably isomorphic. In the infinite dimensional case however, all these notions are distinct.

## 5. Morita equivalence of generated $W^*$-algebras

From [10] or [1], we know that a strong Morita equivalence of operator algebras in the sense of [11] 'dilates' to, or is a subcontext of, a strong Morita equivalence in the sense of Rieffel, of containing $C^*$-algebras. This happens in a very tidy way. More particularly, suppose that $(A, B, X, Y)$ is a strong Morita context of operator algebras $A$ and $B$. Then any $C^*$-algebra $C$ generated by $A$ induces a $C^*$-algebra $D$ generated by $B$, and $C$ and $D$ are strongly Morita equivalent in the sense of Rieffel [23], with equivalence bimodule the '$C^*$-dilation' (see [3]) $C \otimes_{hA} X$. Moreover the linking algebra for $A$ and $B$ is (completely isometrically) a subalgebra of the linking $C^*$-algebra for $C$ and $D$. We see next that all of this, and the accompanying theory, will extend to our present setting. Although one may use any '$W^*$-cover' in the arguments below, for specificity, the maximal $W^*$-algebra $W^*_{\max}(M)$ from [13] will take the place of $C$ above, and the 'maximal $W^*$-dilation' $W^*_{\max}(M) \otimes_M^{\sigma h} X$ will play the role of the $C^*$-dilation. One can develop a theory for this '$W^*$-dilation' in a general setting analogously to [3,10], but we shall not take the time to do this here (see [21]). We will however state that just as in [3], any (left, say) dual operator $M$-module is completely isometrically embedded in its 'maximal $W^*$-dilation', via the $M$-module map $x \mapsto 1 \otimes x$, which is weak* continuous.

Throughout this section, $(M, N, X, Y)$ is a weak* Morita context. We shall show that the 'left' and 'right' $W^*$-dilations coincide, and constitute a bimodule implementing the $W^*$-algebraic Morita equivalence between $W^*_{\max}(M)$ and $W^*_{\max}(N)$.

**Theorem 5.1.** *The $W^*$-dilation $Y \otimes_M^{\sigma h} W^*_{\max}(M)$ is a right $C^*$-module over $W^*_{\max}(M)$.*

**Proof.** With $H$ a normal universal Hilbert $M$-module as usual, we may view $W^*_{\max}(M)$ as the von Neumann algebra $\mathcal{R}$ generated by $M$ in $B(H)$. Let $K = Y \otimes_M^{\sigma h} H^c$ as usual, and let $Z = Y \otimes_M^{\sigma h} W^*_{\max}(M)$. Note that

$$Z \otimes_{W^*_{\max}(M)}^{\sigma h} H^c \cong Y \otimes_M^{\sigma h} W^*_{\max}(M) \otimes_{W^*_{\max}(M)}^{\sigma h} H^c \cong Y \otimes_M^{\sigma h} H^c = K.$$

This allows us to define a completely contractive weak*-continuous $\phi : Z \to B(H, K)$ given by $\phi(y \otimes a)(\zeta) = y \otimes a\zeta$, for $y \in Y, a \in \mathcal{R}, \zeta \in H$. Note that $\phi$ restricted to the copy of $Y$ is just the map $\Phi$ at the start of Section 4. We are following the ideas of [2, p. 286–288]. It is clear that $\phi$ is a $\mathcal{R}$-module map. By the discussion just below Corollary 3.4, combined with Corollary 2.4, there are nets of maps $\varphi_s \otimes I : Z \to C_{n_s}(M) \otimes_M^{\sigma h} W^*_{\max}(M) \cong C_{n_s}(W^*_{\max}(M))$, and maps $\psi_s \otimes I$, with $(\psi_s \otimes I)(\varphi_s \otimes I)(z) = f_s z \to z$ weak* for all $z \in Z$. Here $(f_s)$ is as in (3.1), and the last convergence follows from e.g. [20, Lemma 2.3]. We have $\|[f_s z_{ij}]\| \leq \|[(\varphi_s \otimes I)(z_{ij})]\| \leq \|[\phi(z_{ij})]\|$. This follows, as in [2, p. 287], from the fact that there is a sequence of weak* continuous complete contractions

$$B(H, K) \to B(H, C_{n_t}(M) \otimes_M^{\sigma h} W^*_{\max}(M) \otimes_{W^*_{\max}(M)}^{\sigma h} H^c) \cong B(H, C_{n_t}(H^c))$$

that maps $\phi(y \otimes a)$ to $\varphi_s(y)a$, for $y \in Y, a \in \mathcal{R}$, and hence maps $\phi(z)$ for $z \in Z$, to $(\varphi_s \otimes I)(z)$. As in [2, p. 287], it follows that $\phi$ is a complete isometry.

Define $\langle z, w \rangle = \phi(z)^*\phi(w)$ for $z, w \in Z$. To see that this is an $\mathcal{R}$-valued inner product on $Z$, we will use von Neumann's double commutant theorem (this is a well known idea (see e.g. [22])). Note that if $\Delta(A) = A \cap A^*$ is the 'diagonal' of a subalgebra $A$ of $B(H)$, then $\mathcal{R}' = \Delta(M')$, the 'prime' denoting commutants. The proof of Theorem 4.5 shows that there is a completely isometric isomorphism $\theta : M' \to N'$, such that $\Phi(y)T = \theta(T)\Phi(y)$ for $y \in Y, T \in M'$, where $\Phi(y)(\zeta) = y \otimes \zeta \in K$, for $\zeta \in H$. By 2.1.2 in [8], $\theta$ restricts to a $*$-isomorphism from $\Delta(M') = \mathcal{R}'$ onto $\Delta(N')$. It follows that, in the notation of Theorem 5.1, if $y \in Y, a \in \mathcal{R}, \zeta \in H, T \in \Delta(M')$ then

$$\phi(y \otimes a)(T\zeta) = y \otimes aT\zeta = y \otimes Ta\zeta = \Phi(y)T(a\zeta) = \theta(T)\Phi(y)(a\zeta) = \theta(T)\phi(y \otimes a)(\zeta).$$

Hence if $w, z \in Z$ and $T \in \mathcal{R}'$ then

$$\phi(z)^*\phi(w)T = \phi(z)^*\theta(T)\phi(w) = (\theta(T^*)\phi(z))^*\phi(w) = (\phi(z)T^*)^*\phi(w) = T\phi(z)^*\phi(w),$$

so that $\phi(z)^*\phi(w) \in \mathcal{R}'' = \mathcal{R}$.

Thus $Z$ is a right $C^*$-module over $W^*_{\max}(M)$, completely isometrically isomorphic to the WTRO Ran$(\phi)$. $\square$

**Theorem 5.2.** *Suppose that $(M, N, X, Y)$ is a weak* Morita context. Then $W^*_{\max}(M)$ and $W^*_{\max}(N)$ are Morita equivalent $W^*$-algebras in the sense of Rieffel, and the associated equivalence bimodule is $Y \otimes_M^{\sigma h} W^*_{\max}(M)$. Moreover, $Y \otimes_M^{\sigma h} W^*_{\max}(M) \cong W^*_{\max}(N) \otimes_N^{\sigma h} Y$ completely isometrically. Analogous assertions hold with $Y$ replaced by $X$. Finally, the $W^*$-algebra linking algebra for this Morita equivalence contains (as a weak* closed subalgebra) the linking algebra $\mathcal{L}^w$ defined earlier for the context $(M, N, X, Y)$.*

**Proof.** We use the idea in [1, p. 406–407] and [10, p. 585–586]. Let $H, K$ be as in the proof of Theorem 5.1. We consider the following subalgebras of $B(H \oplus K)$:

$$\begin{bmatrix} W^*_{\max}(M) & W^*_{\max}(M)X \\ YW^*_{\max}(M) & YW^*_{\max}(M)X \end{bmatrix}, \qquad \begin{bmatrix} XW^*_{\max}(N)Y & XW^*_{\max}(N) \\ W^*_{\max}(N)Y & W^*_{\max}(N) \end{bmatrix}.$$

Let $\mathcal{L}_1$ and $\mathcal{L}_2$ denote the weak* closures of these two subalgebras. These are dual operator algebras which are the linking algebras for a Morita equivalence of the type in the present paper. Thus by Theorem 4.4, they are actually selfadjoint. Moreover both of these can now be seen to equal the von Neumann algebra generated by $\mathcal{L}^w$, and so they are equal to

each other. Hence the weak* closures of $YW_{\max}^*(M)$ and $W_{\max}^*(N)Y$ coincide, and this constitutes an equivalence bimodule (or WTRO) setting up a $W^*$-algebraic Morita equivalence between $W_{\max}^*(M)$ and $W_{\max}^*(N)$. As we said, the $W^*$-algebraic linking algebra here is just $\mathcal{L}_1 = \mathcal{L}_2$, and this clearly contains the algebra we called $\mathcal{R}$ in the discussion in the beginning of Section 4, that is $\mathcal{L}^w$, as a subalgebra.

Finally, notice that the map $\phi$ in the proof of the last theorem is a completely isometric $W_{\max}^*(M)$-module map from $Z = Y \otimes_M^{\sigma h} W_{\max}^*(M)$ onto the weak* closure $W$ of $YW_{\max}^*(M)$ in $B(H, K)$. Similar considerations, or symmetry, shows that $V = W_{\max}^*(N) \otimes_N^{\sigma h} Y$ agrees with the weak* closure of $W_{\max}^*(N)Y$, which by the above equals $W$, and thus agrees with $Z$. Similarly for the modules involving $X$.    □

**Remark** Theorems 4 and 5 of [10] have obvious variants valid in our setting, with arbitrary $W^*$-dilations in place of $W_{\max}^*(M)$. Similarly, one can show as in [10] that $W_{\max}^*(\mathcal{L}^w) = \mathcal{L}_1$. See [21] for details.

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We thank Vern Paulsen for suggesting that we look again at the last example in [19] (which was previously part of [17]). Indeed this is clearly a weak (indeed even a strong) Morita equivalence, by Davidson's similarity theorem (see Example (10) in Section 3). However it is shown in [19] not to be a $\Delta$-equivalence. Hence it is not a weak* stable isomorphism [20]. Thus Eleftherakis' 'Delta-equivalence' and the relations considered in our paper are distinct; each seem to have their own distinct advantages (see for example the discussion on the first page of our paper). Eleftherakis' Morita contexts *contain* a $W^*$-algebraic Morita context (thus, his bimodules contain a bimodule implementing a $W^*$-algebraic Morita equivalence). Our contexts are *contained in* a $W^*$-algebraic Morita context (see Section 5).

More recently, in [7], we have defined a generalization to nonselfadjoint algebras, of the important $C^*$-algebraic notion of '$W^*$-module' (sometimes called 'selfdual $C^*$-module' over a von Neumann algebra). We have also developed their theory. This is the 'weak*' version of our earlier theory of 'rigged modules' [1].

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