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Smoothing effects for classical solutions of the relativistic Landau–Maxwell system

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ABSTRACT

The relativistic Landau–Maxwell system is one of the most fundamental and complete models for describing the dynamics of a dilute hot plasma in which particles interact through Coulomb collisions and their self-consistent electromagnetic field. In this work, we prove that the classical solutions obtained by Strain and Guo become immediately smooth with respect to all variable under the extra assumption of the electromagnetic field. As a by-product, we also prove that the classical solutions to the relativistic Landau–Poisson system and the relativistic Landau equation have the same property without any extra assumption.

1. Introduction

The relativistic Landau–Maxwell system is one of the most fundamental and complete models for describing the dynamics of a dilute hot plasma in which particles interact through Coulomb collisions and their self-consistent electromagnetic field. For notational simplicity, we consider the following normalized Landau–Maxwell system:

$$\partial_t F_+ + \frac{p}{p_0}\cdot\nabla_x F_+ + \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_+ = \mathcal{C}(F_+, F_+) + \mathcal{C}(F_+, F_-), \tag{1.1}$$

$$\partial_t F_- + \frac{p}{p_0}\cdot\nabla_x F_- - \left(E + \frac{p}{p_0}\times B\right)\cdot\nabla_p F_- = \mathcal{C}(F_-, F_-) + \mathcal{C}(F_-, F_+), \tag{1.2}$$

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with initial condition $F_\pm(0,x,p) = F_{0,\pm}(x,p)$. Here $F_\pm(t,x,p) \geqslant 0$ are the spatially periodic number density functions for ions (+) and electrons (−) at time $t \geqslant 0$, position $x \in \mathbf{T}^3 = [-\pi,\pi]^3$ and momentum $p = (p_1, p_2, p_3) \in \mathbf{R}^3$. The energy of a particle is given by $p_0 = \sqrt{1+|p|^2}$. To completely describe a dilute plasma, the electromagnetic field $E(t,x)$ and $B(t,x)$ is coupled with $F_\pm(t,x,p)$ through the celebrated Maxwell system:

$$\partial_t E - \nabla_x \times B = -\int_{\mathbf{R}^3} \frac{p}{p_0}\{F_+ - F_-\}\,dp, \qquad \nabla_x \cdot B = 0, \tag{1.3}$$

$$\partial_t B + \nabla_x \times E = 0, \qquad \nabla_x \cdot E = \int_{\mathbf{R}^3} \{F_+ - F_-\}\,dp, \tag{1.4}$$

with initial condition $E(0,x) = E_0(x)$ and $B(0,x) = B_0(x)$. The collision between particles is modeled by the relativistic Landau collision operator and its normalized form is given by

$$\mathcal{C}(g,h)(p) = \nabla_p \cdot \left\{ \int_{\mathbf{R}^3} \Phi(P,Q)\big\{\nabla_p g(p)h(q) - g(p)\nabla_q h(q)\big\}\,dq \right\}, \tag{1.5}$$

where the four-vectors are $P = (p_0, p_1, p_2, p_3)$ and $Q = (q_0, q_1, q_2, q_3)$. Moreover, the collision kernel is given the 3×3 non-negative matrix

$$\Phi(P,Q) = \frac{\Lambda(P,Q)}{p_0 q_0} S(P,Q), \tag{1.6}$$

with

$$\Lambda(P,Q) = (P\cdot Q)^2\big\{(P\cdot Q)^2 - 1\big\}^{-3/2},$$
$$S(P,Q) = \big\{(P\cdot Q)^2 - 1\big\} I_3 - (p-q)\otimes(p-q) + \big\{(P\cdot Q) - 1\big\}(p\otimes q + q\otimes p).$$

Here the Lorentz inner product is $P\cdot Q = p_0 q_0 - p\cdot q$.

From the relativistic Landau–Maxwell system, we can obtain:

$$\frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3} F_+(t) = \frac{d}{dt}\int_{\mathbf{T}^3\times\mathbf{R}^3} F_-(t) = 0,$$
$$\frac{d}{dt}\left\{ \int_{\mathbf{T}^3\times\mathbf{R}^3} \big(pF_+(t) + pF_-(t)\big) + \int_{\mathbf{T}^3} E(t)\times B(t) \right\} = 0,$$
$$\frac{d}{dt}\left\{ \frac{1}{2}\int_{\mathbf{T}^3\times\mathbf{R}^3} \big(p_0F_+(t) + p_0F_-(t)\big) + \int_{\mathbf{T}^3} \big|E(t)\big|^2 + \big|B(t)\big|^2 \right\} = 0.$$

From the Maxwell system and the periodic boundary condition of $E(t,x)$, there exists a constant $\overline{B}$ such that $\frac{1}{|\mathbf{T}^3|}\int_{\mathbf{T}^3} B(t,x)\,dx = \overline{B}$.

We introduce the normalized relativistic Maxwellian $J(p) = e^{-p_0}$, and the standard perturbation $f_\pm(t,x,p)$ to $J(p)$ as

$$F_\pm = J + \sqrt{J} f_\pm.$$

We introduce the notation $F=[F_+,F_-]$. By assuming that $[F_0,E_0,B_0]$ has the same mass, total momentum and total energy as the steady state $[J,0,\overline{B}]$, then we have

$$\int_{\mathbf{T}^3\times\mathbf{R}^3}\sqrt{J}f_+(t)=\int_{\mathbf{T}^3\times\mathbf{R}^3}\sqrt{J}f_-(t)=0, \tag{1.7}$$

$$\frac{d}{dt}\left\{\int_{\mathbf{T}^3\times\mathbf{R}^3}\big(p\sqrt{J}f_+(t)+p\sqrt{J}f_-(t)\big)+\int_{\mathbf{T}^3}E(t)\times B(t)\right\}=0, \tag{1.8}$$

$$\frac{d}{dt}\left\{\frac{1}{2}\int_{\mathbf{T}^3\times\mathbf{R}^3}\big(p_0\sqrt{J}f_+(t)+p_0\sqrt{J}f_-(t)\big)+\int_{\mathbf{T}^3}\big|E(t)\big|^2+\big|B(t)-\overline{B}\big|^2\right\}=0. \tag{1.9}$$

For notational simplicity, we omit the integrating domains $\mathbf{T}^3$ and $\mathbf{R}^3$, which correspond to variable x and variable p respectively. For example, we sometimes write $L^2_{x,p}$ instead of $L^2_x(\mathbf{T}^3;L^2_p(\mathbf{R}^3))$. For any integer $N\geqslant 0$, we define the Sobolev space

$$H^N_x=\left\{f(x):\ \sum_{|\alpha|\leqslant N}\big\|\partial^\alpha_x f\big\|_{L^2_{x,p}}<\infty\right\},\qquad H^N_{x,p}=\left\{f(x,p):\ \sum_{|\alpha|+|\beta|\leqslant N}\big\|\partial^\alpha_x\partial^\beta_p f\big\|_{L^2_{x,p}}<\infty\right\},$$

and for N, $s\geqslant 0$, we define the weighted Sobolev space

$$H^{N,s}_{x,p}=\left\{f(x,p):\ \sum_{|\alpha|+|\beta|\leqslant N}\big\|\partial^\alpha_x\partial^\beta_p f\big(1+|p|^2\big)^{s/2}\big\|_{L^2_{x,p}}<\infty\right\},$$

where the multi-index $\alpha=(\alpha_1,\alpha_2,\alpha_3)$, $|\alpha|=\alpha_1+\alpha_2+\alpha_3$ and $\partial^\alpha_x=\partial^{\alpha_1}_{x_1}\partial^{\alpha_2}_{x_2}\partial^{\alpha_3}_{x_3}$ with $x=(x_1,x_2,x_3)$. The notations for β are the same. It is obvious that $H^{N,0}_{x,p}=H^N_{x,p}$. We also define $H^\infty_{x,p}$ and $H^{\infty,s}_{x,p}$ by

$$H^\infty_{x,p}=\bigcap_{N\geqslant 0}H^N_{x,p},\qquad H^{\infty,s}_{x,p}=\bigcap_{N\geqslant 0}H^{N,s}_{x,p}.$$

The global classical solution of the relativistic Landau–Maxwell system around the above steady state in the periodic box has been obtained in [17]. The authors in [13] obtained the global solution of the relativistic Landau equation near relativistic Maxwellian in the whole space. Recently, author in [20] also proved that the relativistic Landau–Maxwell system near steady state in the whole space has a global classical solution in the Sobolev space $H^N_{x,p}(\mathbf{R}^3\times\mathbf{R}^3)$ $(N\geqslant 4)$ by the techniques in [12,13, 16,17]. By the results in [17,20], we easily obtain the following existence results of the global classical solution.

Theorem 1.1. *(See Strain and Guo [17].) Let $N\geqslant 4$ and $F_{0,\pm}(x,p)=J+\sqrt{J}f_{0,\pm}(x,p)\geqslant 0$. Assume that $[f_0,E_0,B_0]$ satisfies the conservation laws* (1.7), (1.8) *and* (1.9)*. There exists $\kappa_0>0$ such that if*

$$\|f_0\|_{H^N_{x,p}}+\big\|[E_0,B_0]\big\|_{H^N_x}\leqslant\kappa_0,$$

then there exists a unique global solution $[F(t,x,p),E(t,x),B(t,x)]$ to the relativistic Landau–Maxwell system (1.1)–(1.4)*. Moreover, there is a constant $C_0>0$ such that*

$$\|f\|_{L^\infty_t([0,\infty);H^N_{x,p})}+\big\|[E,B]\big\|_{L^\infty_t([0,\infty);H^N_x)}\leqslant C_0\kappa_0.$$

It is obvious that for any $s \geqslant 0$, there exist constant $C_1 > 0$ (depending on C_0 and s) and $C_2 > 0$ (depending on s) such that

$$\|F\|_{L_t^\infty([0,\infty);H_{x,p}^{N,s})} + \big\|[E,B]\big\|_{L_t^\infty([0,\infty);H_x^N)} \leqslant C_1\kappa_0 + C_2.$$

Our main theorem states that the classical solutions obtained thanks to Theorem 1.1 satisfy the following smoothness.

Theorem 1.2. *Assume that $[f_0, E_0, B_0]$ satisfies the conservation laws* (1.7), (1.8) *and* (1.9) *and that $F_{0,\pm}(x,p) = J + \sqrt{J} f_{0,\pm}(x,p) \geqslant 0$. There exist $C_3 > 0$ and $\epsilon_0 \in [0, \kappa_0]$ such that*

$$\|f_0\|_{H_{x,p}^4} + \big\|[E_0, B_0]\big\|_{H_x^4} \leqslant \epsilon_0,$$

the unique classical solution to the system (1.1)–(1.4) *given by Theorem* 1.1 *satisfies (for any $0 < \tau_1 < \tau < T < \infty$ and $s \geqslant 0$):*

$$F \in C_t^\infty\big([\tau,T]; H_{x,p}^{\infty,s}\big(\mathbf{T}^3 \times \mathbf{R}^3\big)\big), \tag{1.10}$$

under the assumptions that

$$\big\|E(\tau_1)\big\|_{H_x^\infty} + \big\|B(\tau_1)\big\|_{H_x^\infty} \leqslant C_3. \tag{1.11}$$

Remark 1.3. If $E(t,x) = 0$, $B(t,x) = 0$ and $F_+ = F_-$, the system (1.1) and (1.2) turn into the simple relativistic Landau equation. Theorem 1.2 says that classical solutions to the relativistic Landau equation given by Theorem 1.1 become immediately smooth with respect to all variables.

Remark 1.4. If $E(t,x) = \nabla_x\phi(t,x)$ and $B(t,x) = 0$, the system (1.1)–(1.4) turn into the relativistic Landau–Poisson system. Theorem 1.2 tells us that classical solutions to the relativistic Landau–Poisson system given by Theorem 1.1 also become immediately smooth with respect to all variables by the elliptic estimates (see Remark 3.6) without any extra assumption. This fact shows that nature that the Maxwell system is hyperbolic may lead us to impose the assumption (1.11). Whether this assumption is removed or not will be considered for the future study.

The smoothness of the solutions to the spatially homogeneous Landau equation has been investigated in [1]. Desvillettes and Villani [8] proved global existence, uniqueness and smoothness of classical solutions to the spatially homogeneous Landau equation for hard potentials and a large class of initial data. Guo [11] constructed global classical solutions near Maxwellians for a general Landau equation in a periodic box by the nice energy method. Recently, Chen, Desvillettes and He [6] proved that the classical solutions to the Landau equation obtained by Guo [11] become immediately smooth with respect to all variables by some hints of [2,7] and [8]: roughly speaking, smoothness (in the velocity variable) is produced by the elliptic property of the diffusive matrix to the Landau operator in [8] and smoothness (in the position variable) is produced by the classical averaging lemma [2,3,9].

Although the classical (non-relativistic) Landau equation has been heavily studied [1,6,8,11,14,19], the relativistic version has received scant attention [13,15,17]. Lemou studied the linearized relativistic equation in [15]. In [17], Strain and Guo presented the global existence of classical solutions to the relativistic Landau–Maxwell system (1.1)–(1.4) near steady state. Thanks to this breakthrough, we possibly improve smoothness of the solutions to the relativistic Landau–Maxwell system in this work. Although our main results are proved by using the novel idea of [6], there are several major new difficulties in this paper. The first new difficulty is due to the complexity of the relativistic Landau kernel (1.5). Unlike the classical Landau kernel, the relativistic version is not convolution-like and we use the technique of the novel integration by parts introduced in [17] to overcome this difficulty. The second one is the lower regularity of the solutions ($N \geqslant 4$) in Theorem 1.1 and the N-derivative of the vector $(b_i^{\pm})$ is bigger than N. We have to analyze detailedly the diffusive matrix $(a_{ij}^{\pm})$ and the vector $(b_i^{\pm})$ and obtain the corresponding estimates. Lastly, the matrix $(a_{ij}^{\pm})$ is different to the classical version due to the relativistic kernel and the proof of the ellipticity property of the classical version in [6] cannot be applied here. We use the important findings of Lemou [15] to obtain the key ellipticity property.

2. Basic estimates

In this section, we deduce some basic estimates which will be used repeatedly in the proof of Theorem 1.2. If we write

$$a_{ij}^{\pm}(t,x,p)=\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)F_{\pm}(t,x,q)\,dq,\qquad b_i^{\pm}(t,x,p)=\sum_{j=1}^{3}\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)\partial_{q_j}F_{\pm}(t,x,q)\,dq,$$

then the system (1.1) and (1.2) can be rewritten as

$$\begin{aligned}
&\partial_t F_+ + \frac{p}{p_0}\cdot\nabla_x F_+ + \left(E+\frac{p}{p_0}\times B\right)\cdot\nabla_p F_+\\
&\quad=\nabla_p\cdot\left(a^+\nabla_p F_+ - b^+F_+\right)+\nabla_p\cdot\left(a^-\nabla_p F_+ - b^-F_+\right),
\end{aligned}\tag{2.1}$$

$$\begin{aligned}
&\partial_t F_- + \frac{p}{p_0}\cdot\nabla_x F_- - \left(E+\frac{p}{p_0}\times B\right)\cdot\nabla_p F_-\\
&\quad=\nabla_p\cdot\left(a^-\nabla_p F_- - b^-F_-\right)+\nabla_p\cdot\left(a^+\nabla_p F_- - b^+F_-\right).
\end{aligned}\tag{2.2}$$

Here $a^{\pm}$ (respectively $b^{\pm}$) is the matrix (respectively the vector) with coefficients $(a_{ij}^{\pm})_{ij}$ (respectively $(b_i^{\pm})_i$). In the following we will prove some L^∞ estimates of $a_{ij}^{\pm}$ and $b_i^{\pm}$. We first define the relativistic differential operator:

$$\Theta_\alpha(p,q)=\left(\partial_{p_1}+\frac{q_0}{p_0}\partial_{q_1}\right)^{\alpha_1}\left(\partial_{p_2}+\frac{q_0}{p_0}\partial_{q_2}\right)^{\alpha_2}\left(\partial_{p_3}+\frac{q_0}{p_0}\partial_{q_3}\right)^{\alpha_3},\tag{2.3}$$

which is originally introduced in [17] and plays a key role in proving L^∞ estimates of $a_{ij}^{\pm}$ and $b_i^{\pm}$. We begin with the following integration by parts lemma.

Lemma 2.1. *Given $|\beta|>0$, we have*

$$\partial_p^\beta\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)f(q)\,dq=\sum_{|\beta_1|+|\beta_2|\leqslant|\beta|}\int_{\mathbf{R}^3}\Theta_{\beta_1}\Phi^{ij}(P,Q)\partial_q^{\beta_2}f(q)\phi^{\beta}_{\beta_1,\beta_2}(p,q)\,dq,\tag{2.4}$$

where $\phi^{\beta}_{\beta_1,\beta_2}(p,q)$ is a smooth function which satisfies

$$\left|\partial_q^{\nu_1}\partial_p^{\nu_2}\phi^{\beta}_{\beta_1,\beta_2}(p,q)\right|\leqslant Cq_0^{|\beta|-|\nu_1|}p_0^{|\beta_1|-|\beta|-|\nu_2|},\tag{2.5}$$

for all multi-indices ν_1 and ν_2.

Proof. This lemma is similar as Lemma 3 in [17]. We prove (2.4) by an induction over the number of derivatives $|\beta|$. Assume $\beta=e^i$ $(i=1,2,3)$. We write

$$\partial_{p_i}=-\frac{q_0}{p_0}\partial_{q_i}+\left(\partial_{p_i}+\frac{q_0}{p_0}\partial_{q_i}\right)=-\frac{q_0}{p_0}\partial_{q_i}+\Theta_{e_i}.$$

Instead of hitting $\Phi^{ij}(P,Q)$ with ∂_{p_i}, we apply the r.h.s. term above and integrate by parts over $-\frac{q_0}{p_0}\partial_{q_i}$ to obtain

$$\partial_p^\beta \int_{\mathbf{R}^3} \Phi^{ij}(P,Q) f(q)\,dq = \int_{\mathbf{R}^3} \Theta_{e^i}\Phi^{ij}(P,Q) f(q)\,dq + \int_{\mathbf{R}^3} \Phi^{ij}(P,Q)\frac{q_0}{p_0}\partial_{q_i} f(q)\,dq.$$

We can write the above in the form (2.4) with the coefficients given by

$$\phi^{e_i}_{e_i,0}(p,q) = 1, \qquad \phi^{e_i}_{0,e_i}(p,q) = \frac{q_0}{p_0}.$$

Note that these coefficients satisfy the decay (2.5). This establishes the fist step in the induction.

Assume the result holds for all $|\beta| \leqslant n$. Fix an arbitrary β' such that $|\beta'| = n+1$ and write $\partial_p^{\beta'} = \partial_{p_m}\partial_p^\beta$ for some multi-index β and

$$m = \max\{j:\ (\beta')^j > 0\}.$$

By the induction assumption, we have

$$\partial_p^{\beta'} \int_{\mathbf{R}^3} \Phi^{ij}(P,Q) f(q)\,dq = \sum_{|\bar\beta_1|+|\bar\beta_2|\leqslant|\beta|} \partial_{p_m} \int_{\mathbf{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)\partial_q^{\bar\beta_2} f(q)\phi^\beta_{\bar\beta_1,\bar\beta_2}(p,q)\,dq.$$

We apply the last derivative the same as the case $|\beta| = 1$ above. We have

$$= \sum_{|\bar\beta_1|+|\bar\beta_2|\leqslant|\beta|} \int_{\mathbf{R}^3} \Theta_{e_m}\Theta_{\bar\beta_1}\Phi^{ij}(P,Q)\partial_q^{\bar\beta_2} f(q)\phi^\beta_{\bar\beta_1,\bar\beta_2}(p,q)\,dq \tag{2.6}$$

$$+ \sum_{|\bar\beta_1|+|\bar\beta_2|\leqslant|\beta|} \int_{\mathbf{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)\frac{q_0}{p_0}\partial_{q_m}\partial_q^{\bar\beta_2} f(q)\phi^\beta_{\bar\beta_1,\bar\beta_2}(p,q)\,dq \tag{2.7}$$

$$+ \sum_{|\bar\beta_1|+|\bar\beta_2|\leqslant|\beta|} \int_{\mathbf{R}^3} \Theta_{\bar\beta_1}\Phi^{ij}(P,Q)\partial_q^{\bar\beta_2} f(q)\left(\partial_{p_m} + \frac{q_0}{p_0}\partial_{q_m}\right)\phi^\beta_{\bar\beta_1,\bar\beta_2}(p,q)\,dq. \tag{2.8}$$

We collect all the terms above and write it to the following form

$$= \sum_{\beta_1+\beta_2\leqslant\beta'} \int_{\mathbf{R}^3} \Theta_{\beta_1}\Phi^{ij}(P,Q)\partial_q^{\beta_2} f(q)\phi^{\beta'}_{\beta_1,\beta_2}(p,q)\,dq.$$

We will check (2.5) by the terms (2.6), (2.7) and (2.8). For (2.6), the order of differentiation is

$$\beta_1 = \bar\beta_1 + e_m, \qquad \beta_2 = \bar\beta_2,$$

and by the induction assumption, we have

$$\left|\partial_q^{\nu_1}\partial_p^{\nu_2}\phi^\beta_{\bar\beta_1,\bar\beta_2}(p,q)\right| \leqslant C q_0^{|\beta|-|\nu_1|} p_0^{|\bar\beta_1|-|\beta|-|\nu_2|} \leqslant C q_0^{|\beta'|-|\nu_1|} p_0^{|\beta_1|-|\beta'|-|\nu_2|}.$$

For (2.7), the order of differentiation is

$$\beta_1 = \bar\beta_1, \qquad \beta_2 = \bar\beta_2 + e_m,$$

and by the induction assumption, we have

$$\left|\partial_q^{\nu_1}\partial_p^{\nu_2}\left(\frac{q_0}{p_0}\phi^{\beta}_{\bar\beta_1,\bar\beta_2}(p,q)\right)\right| \leqslant Cq_0^{|\beta|+1-|\nu_1|}p_0^{|\bar\beta_1|-|\beta|-1-|\nu_2|} = Cq_0^{|\beta'|-|\nu_1|}p_0^{|\beta_1|-|\beta'|-|\nu_2|}.$$

For (2.8), the order of differentiation is

$$\beta_1 = \bar\beta_1, \qquad \beta_2 = \bar\beta_2,$$

and by the induction assumption and Leibniz rule, we have

$$\left|\partial_q^{\nu_1}\partial_p^{\nu_2}\left\{\left(\partial_{p_m} + \frac{q_0}{p_0}\partial_{q_m}\right)\phi^{\beta}_{\bar\beta_1,\bar\beta_2}(p,q)\right\}\right| \leqslant Cq_0^{|\beta|+1-|\nu_1|}p_0^{|\bar\beta_1|-|\beta|-1-|\nu_2|} \leqslant Cq_0^{|\beta'|-|\nu_1|}p_0^{|\beta_1|-|\beta'|-|\nu_2|}. \qquad \square$$

Lemma 2.2. *Let $F(t,x,p)$ be the non-negative solution of the system* (1.1)–(1.4) *in Theorem* 1.1. *Then there exists a constant $C > 0$ such that for any multi-indices α and β, we have*

$$\left|\partial_x^\alpha\partial_p^\beta a_{ij}^{\pm}(t,x,p)\right| \leqslant C\sum_{\beta_1\leqslant\beta}\left\|\partial_x^\alpha\partial_p^{\beta_1}F_{\pm}\left(1+|p|^2\right)^{(|\beta|+10)/2}\right\|_{L_p^2}(t,x), \tag{2.9}$$

$$\left|\partial_x^\alpha\partial_p^\beta b_{i}^{\pm}(t,x,p)\right| \leqslant C\sum_{|\beta_1|\leqslant|\beta|+1}\left\|\partial_x^\alpha\partial_p^{\beta_1}F_{\pm}\left(1+|p|^2\right)^{(|\beta|+10)/2}\right\|_{L_p^2}(t,x). \tag{2.10}$$

Proof. We know from Lemma 2.1 that when $|\beta| > 0$, we have

$$\partial_x^\alpha\partial_p^\beta a_{ij}^{\pm}(t,x,p) = \partial_p^\beta\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)\partial_x^\alpha F_{\pm}(q)\,dq = \sum_{|\beta_1|+|\beta_2|\leqslant|\beta|}\int_{\mathbf{R}^3}\Theta_{\beta_1}\Phi^{ij}(P,Q)\partial_q^{\beta_2}\partial_x^\alpha F_{\pm}(q)\phi^{\beta}_{\beta_1,\beta_2}(p,q)\,dq,$$

where $\phi^{\beta}_{\beta_1,\beta_2}(p,q)$ satisfies

$$\left|\phi^{\beta}_{\beta_1,\beta_2}(p,q)\right| \leqslant Cq_0^{|\beta|}p_0^{|\beta_1|-|\beta|}.$$

We will use the following splitting:

$$\mathcal{A} = \left\{|p-q| + |p\times q| \geqslant \left[|p|+1\right]/2\right\}, \qquad \mathcal{B} = \left\{|p-q| + |p\times q| \leqslant \left[|p|+1\right]/2\right\}. \tag{2.11}$$

From Lemma 2 in [17], on the set $\mathcal{A}$ we have the estimate

$$\left|\Theta_{\beta_1}\Phi^{ij}(P,Q)\right| \leqslant Cp_0^{-|\beta_1|}q_0^6, \tag{2.12}$$

and on the set $\mathcal{B}$ we have the estimate

$$\left|\Theta_{\beta_1}\Phi^{ij}(P,Q)\right| \leqslant Cp_0^{-|\beta_1|}q_0^7|p-q|^{-1}. \tag{2.13}$$

Thus on the set $\mathcal{A}$, the Hölder inequality implies

$$\left| \int_{\mathcal{A}} \Theta_{\beta_1} \Phi^{ij}(P,Q) \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm}(q) \phi^{\beta}_{\beta_1,\beta_2}(p,q)\, dq \right|$$
$$\leqslant C \int_{\mathbf{R}^3} p_0^{-|\beta|} q_0^{|\beta|+6} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm}(q) \right| dq$$
$$\leqslant C p_0^{-|\beta|} \left(\int_{\mathbf{R}^3} \left(1+|q|^2\right)^{-4} dq \right)^{1/2} \left(\int_{\mathbf{R}^3} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm} \right|^2 \left(1+|q|^2\right)^{(|\beta|+10)} dq \right)^{1/2}$$
$$\leqslant C p_0^{-|\beta|} \left(\int_{\mathbf{R}^3} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm} \right|^2 \left(1+|q|^2\right)^{(|\beta|+10)} dq \right)^{1/2}.$$

On the set $\mathcal{B}$, We use (2.13) to obtain

$$\left| \int_{\mathcal{B}} \Theta_{\beta_1} \Phi^{ij}(P,Q) \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm}(q) \phi^{\beta}_{\beta_1,\beta_2}(p,q)\, dq \right|$$
$$\leqslant C \int_{\mathbf{R}^3} p_0^{-|\beta|} q_0^{|\beta|+7} |p-q|^{-1} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm}(q) \right| dq$$
$$\leqslant C p_0^{-|\beta|} \left(\int_{\mathbf{R}^3} |p-q|^{-2} \left(1+|q|^2\right)^{-3} dq \right)^{1/2} \left(\int_{\mathbf{R}^3} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm} \right|^2 \left(1+|q|^2\right)^{(|\beta|+10)} dq \right)^{1/2}$$
$$\leqslant C p_0^{-|\beta|} \left(\int_{\mathbf{R}^3} \left| \partial_q^{\beta_2} \partial_x^{\alpha} F_{\pm} \right|^2 \left(1+|q|^2\right)^{(|\beta|+10)} dq \right)^{1/2},$$

where we have used the inequality

$$\int_{\mathbf{R}^3} |p-q|^{-2} \left(1+|q|^2\right)^{-3} dq \leqslant C.$$

Therefore, we can obtain that

$$\left| \partial_x^{\alpha} \partial_p^{\beta} a_{ij}^{\pm}(t,x,p) \right| \leqslant C \sum_{\beta_1 \leqslant \beta} \left\| \partial_p^{\beta_1} \partial_x^{\alpha} F_{\pm} \left(1+|p|^2\right)^{(|\beta|+10)/2} \right\|_{L_p^2}(t,x).$$

We know that

$$\partial_x^{\alpha} \partial_p^{\beta} b_i^{\pm}(t,x,p) = \sum_{j=1}^{3} \partial_p^{\beta} \int_{\mathbf{R}^3} \Phi^{ij}(P,Q) \partial_x^{\alpha} \partial_{q_j} F_{\pm}(t,x,q)\, dq.$$

This form is similar as the form of $a_{ij}^{\pm}$. By the arguments similar as the above, we can prove the second relation. □

Lemma 2.3. *Let $F(t,x,p)$ be the non-negative solution of the system* (1.1)–(1.4) *in Theorem* 1.1. *Then there exists a constant $C>0$ such that for any multi-indices α, and $|\beta|>0$, we have*

$$\left|\partial_x^\alpha \partial_p^\beta b_i^\pm(t,x,p)\right| \leqslant C \sum_{|\beta_1|\leqslant|\beta|-1} \Big(\left|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,p)\right| + p_0^{1/2-|\beta_1|} \left\|\partial_x^\alpha \partial_p^{\beta_1} F_\pm \big(1+|p|^2\big)^{(|\beta_1|+10)/2}\right\|_{L_p^2}(t,x)\Big).$$

Proof. By Lemma 4 in [17], we know

$$-\partial_{p_i} \int_{\mathbf{R}^3} \Phi^{ij}(P,Q)\partial_{q_j} f(q)\,dq = 4\int_{\mathbf{R}^3} \overline{\Phi}(P,Q) f(q)\,dq + \kappa(p) f(p), \tag{2.14}$$

where $\kappa(p) = 2^{7/2}\pi p_0 \int_0^\pi (1+|p|^2\sin^2\theta)^{-3/2}\sin\theta\,d\theta$ and

$$\overline{\Phi}(P,Q) = \frac{P\cdot Q}{p_0 q_0}\big\{(P\cdot Q)^2 - 1\big\}^{-1/2}.$$

By the similar arguments as Lemma 2.1, we have

$$\partial_p^\beta \int_{\mathbf{R}^3} \overline{\Phi}(P,Q) f(q)\,dq = \sum_{|\beta_1|+|\beta_2|\leqslant|\beta} 4\int_{\mathbf{R}^3} \Theta_{\beta_1}\overline{\Phi}(P,Q)\partial_q^{\beta_2} f(q)\bar{\phi}^\beta_{\beta_1,\beta_2}(p,q)\,dq, \tag{2.15}$$

where $\phi^\beta_{\beta_1,\beta_2}(p,q)$ is a smooth function which satisfies

$$\left|\partial_q^{\nu_1}\partial_p^{\nu_2}\bar{\phi}^\beta_{\beta_1,\beta_2}(p,q)\right| \leqslant C q_0^{|\beta|-|\nu_1|} p_0^{|\beta_1|-|\beta|-|\nu_2|}, \tag{2.16}$$

for all multi-indices ν_1 and ν_2. On the other hand, we also know

$$\Theta_{\beta_1}\overline{\Phi}(P,Q) = (P\cdot Q)\big\{(P\cdot Q)^2-1\big\}^{-1/2}\Theta_{\beta_1}\left(\frac{1}{p_0q_0}\right).$$

We will use the following inequality in [10] and [17],

$$\frac{|p-q|^2+|p\times q|^2}{2p_0q_0} \leqslant P\cdot Q - 1 \leqslant \frac{1}{2}|p-q|^2. \tag{2.17}$$

On the set $\mathcal{A}$, we have

$$2|p-q|^2 + 2|p\times q|^2 \geqslant \big(|p-q|+|p\times q|\big)^2 \geqslant \frac{1}{4}p_0^2 + \frac{|p|}{2} \geqslant \frac{1}{4}p_0^2.$$

By (2.17) and the last display we have

$$P\cdot Q + 1 \geqslant P\cdot Q - 1 \geqslant \frac{1}{16}\frac{p_0}{q_0}. \tag{2.18}$$

From the Cauchy–Schwartz inequality we also have

$$0 \leqslant P\cdot Q - 1 \leqslant P\cdot Q \leqslant p_0q_0 + |p\cdot q| \leqslant 2p_0q_0. \tag{2.19}$$

We use these last two inequalities to get

$$\begin{aligned}\left|\Theta_{\beta_1}\overline{\Phi}(P,Q)\right| &= \left|(P\cdot Q)\left\{(P\cdot Q)^2-1\right\}^{-1/2}\Theta_{\beta_1}\left(\frac{1}{p_0q_0}\right)\right| \\ &\leqslant C(p_0q_0)\{P\cdot Q-1\}^{-1}q_0^{-1}p_0^{-1-|\beta_1|} \leqslant Cp_0^{-1-|\beta_1|}q_0. \end{aligned} \tag{2.20}$$

By (2.17) we can obtain

$$\frac{|p-q|^2}{2p_0q_0} \leqslant P\cdot Q-1 \leqslant \frac{1}{2}|p-q|^2. \tag{2.21}$$

On the set $\mathcal{B}$, we have

$$\begin{aligned}\left|\Theta_{\beta_1}\overline{\Phi}(P,Q)\right| &= \left|(P\cdot Q)\left\{(P\cdot Q)^2-1\right\}^{-1/2}\Theta_{\beta_1}\left(\frac{1}{p_0q_0}\right)\right| \\ &\leqslant C(p_0q_0)\{P\cdot Q-1\}^{-1/2}\{P\cdot Q+1\}^{-1/2}q_0^{-1}p_0^{-1-|\beta_1|} \\ &\leqslant Cp_0^{1/2-|\beta_1|}q_0^{1/2}|p-q|^{-1}. \end{aligned} \tag{2.22}$$

Recalling (2.20), the integral (2.15) on the set $\mathcal{A}$ is bounded by

$$\begin{aligned} &\left|\int\limits_{\mathcal{A}} \Theta_{\beta_1}\overline{\Phi}(P,Q)\partial_q^{\beta_2}f(q)\bar{\phi}^{\beta}_{\beta_1,\beta_2}(p,q)\,dq\right| \\ &\quad\leqslant Cp_0^{-1-|\beta|}\int\limits_{\mathbf{R}^3}\left|q_0^{1+|\beta|}\partial_q^{\beta_2}f(q)\right|dq \\ &\quad\leqslant Cp_0^{-1-|\beta|}\left(\int\limits_{\mathbf{R}^3}\left(1+|q|^2\right)^{-9}dq\right)^{1/2}\left(\int\limits_{\mathbf{R}^3}\left|\partial_q^{\beta_2}f(q)\right|^2\left(1+|q|^2\right)^{(|\beta|+10)}dq\right)^{1/2} \\ &\quad\leqslant Cp_0^{-1-|\beta|}\left(\int\limits_{\mathbf{R}^3}\left|\partial_q^{\beta_2}f(q)\right|^2\left(1+|q|^2\right)^{(|\beta|+10)}dq\right)^{1/2}. \end{aligned}$$

Recalling (2.22), the integral (2.15) on the set $\mathcal{B}$ is bounded by

$$\begin{aligned} &\left|\int\limits_{\mathcal{B}} \Theta_{\beta_1}\overline{\Phi}(P,Q)\partial_q^{\beta_2}f(q)\bar{\phi}^{\beta}_{\beta_1,\beta_2}(p,q)\,dq\right| \\ &\quad\leqslant Cp_0^{1/2-|\beta|}\int\limits_{\mathbf{R}^3}\left|q_0^{\frac{1}{2}+|\beta|}|p-q|^{-1}\partial_q^{\beta_2}f(q)\right|dq \\ &\quad\leqslant Cp_0^{1/2-|\beta|}\left(\int\limits_{\mathbf{R}^3}|p-q|^{-2}\left(1+|q|^2\right)^{-9}dq\right)^{1/2}\left(\int\limits_{\mathbf{R}^3}\left|\partial_q^{\beta_2}f(q)\right|^2\left(1+|q|^2\right)^{(|\beta|+10)}dq\right)^{1/2} \\ &\quad\leqslant Cp_0^{1/2-|\beta|}\left(\int\limits_{\mathbf{R}^3}\left|\partial_q^{\beta_2}f(q)\right|^2\left(1+|q|^2\right)^{(|\beta|+10)}dq\right)^{1/2}. \end{aligned}$$

By the above inequalities, we can obtain

$$\left|\int_{\mathbf{R}^3} \Theta_{\beta_1}\overline{\Phi}(P,Q)\partial_q^{\beta_2} f(q)\overline{\phi}^{\beta}_{\beta_1,\beta_2}(p,q)\,dq\right| \leqslant Cp_0^{1/2-|\beta|}\left(\int_{\mathbf{R}^3}\left|\partial_q^{\beta_2} f(q)\right|^2\left(1+|q|^2\right)^{(|\beta|+10)}dq\right)^{1/2}.$$

By the similar arguments as the above, we also obtain

$$\left|\int_{\mathbf{R}^3}\overline{\Phi}(P,Q)f(q)\right| \leqslant Cp_0^{1/2}\left(\int_{\mathbf{R}^3}\left|f(q)\right|^2\left(1+|q|^2\right)^{10}dq\right)^{1/2}.$$

Finally, we can obtain

$$\begin{aligned}\left|\partial_x^\alpha\partial_p^\beta b_i^{\pm}(t,x,p)\right| \leqslant C\sum_{|\beta_1|\leqslant|\beta|-1}&\Big(\left|\partial_x^\alpha\partial_p^{\beta_1}F_{\pm}(t,x,p)\right| \\ &+ p_0^{1/2-|\beta_1|}\left\|\partial_x^\alpha\partial_p^{\beta_1}F_{\pm}\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_p^2}(t,x)\Big).\end{aligned}$$

Here we have used the fact that $|\partial_p^\beta\kappa(p)|\leqslant C$. □

Lemma 2.4. *Let $F(t,x,p)$ be the non-negative solution of the system* (1.1)–(1.4) *in Theorem* 1.1*. Then there exists a constant $C>0$ such that for any multi-indices α, we have*

$$\left|\partial_x^\alpha b_i^{\pm}(t,x,p)\right| \leqslant Cp_0^{9/2}\left\|\partial_x^\alpha F_{\pm}(t,x,\cdot)\left(1+|p|^2\right)^5\right\|_{L_p^2}.$$

Proof. By the integral by parts, we have

$$\begin{aligned}\partial_x^\alpha b_i^{\pm}(t,x,p) &= \int_{\mathbf{R}^3}\Phi^{ij}(P,Q)\partial_x^\alpha\partial_{q_j}F_{\pm}(t,x,q)\,dq \\ &= -\int_{\mathbf{R}^3}\partial_{q_j}\Phi^{ij}(P,Q)\partial_x^\alpha F_{\pm}(t,x,q)\,dq.\end{aligned}$$

On the other hand, we know from Lemma 3 in [17] that

$$\sum_j\partial_{q_j}\Phi^{ij}(P,Q) = 2\frac{\Lambda(P,Q)}{p_0q_0}(P\cdot Q p_i - q_i).$$

On the set $\mathcal{A}$, by (2.18) and (2.19), we get

$$\begin{aligned}\left|\frac{\Lambda(P,Q)}{p_0q_0}(P\cdot Qp_i-q_i)\right| &\leqslant C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}\left(p_0+p_0^{-1}\right) \\ &\leqslant C(p_0q_0)^2\{P\cdot Q-1\}^{-3}p_0 \leqslant C(p_0q_0)^2\left(\frac{p_0}{q_0}\right)^{-3}p_0 = Cq_0^5.\end{aligned}$$

We see that

$$\begin{aligned}&\left|\int\limits_{\mathcal{A}} \Phi^{ij}(P,Q)\partial_x^\alpha \partial_{q_j} F_\pm(t,x,q)\,dq\right|\\&\quad\leqslant C\left(\int\limits_{\mathbf{R}^3}\left(1+|q|^2\right)^{-5}dq\right)^{1/2}\left(\int\limits_{\mathbf{R}^3}\left|\partial_x^\alpha F_\pm(t,x,q)\right|^2\left(1+|q|^2\right)^{10}dq\right)^{1/2}\\&\quad\leqslant C\left(\int\limits_{\mathbf{R}^3}\left|\partial_x^\alpha F_\pm(t,x,q)\right|^2\left(1+|q|^2\right)^{10}dq\right)^{1/2}.\end{aligned}$$

On the set $\mathcal{B}$, by (2.19) and (2.21), we get

$$\begin{aligned}&\left|\frac{\Lambda(P,Q)}{p_0q_0}(P\cdot Qp_i-q_i)\right|\\&\quad\leqslant C(P\cdot Q)^2\left\{(P\cdot Q)^2-1\right\}^{-3/2}\left(p_0+p_0^{-1}\right)\\&\quad\leqslant C(p_0q_0)^2\{P\cdot Q-1\}^{-3/2}p_0\\&\quad\leqslant C(p_0q_0)^2(p_0q_0)^{3/2}|p-q|^{-3}p_0\\&\quad= Cp_0^{9/2}q_0^{7/2}|p-q|^{-3}.\end{aligned}$$

We see that

$$\begin{aligned}&\left|\int\limits_{\mathcal{B}} \Phi^{ij}(P,Q)\partial_x^\alpha \partial_{q_j} F_\pm(t,x,q)\,dq\right|\\&\quad\leqslant Cp_0^{9/2}\left(\int\limits_{\mathbf{R}^3}\left(1+|q|^2\right)^{7/2-10}|p-q|^{-6}dq\right)^{1/2}\left(\int\limits_{\mathbf{R}^3}\left|\partial_x^\alpha F_\pm(t,x,q)\right|^2\left(1+|q|^2\right)^{10}dq\right)^{1/2}\\&\quad\leqslant Cp_0^{9/2}\left(\int\limits_{\mathbf{R}^3}\left|\partial_x^\alpha F_\pm(t,x,q)\right|^2\left(1+|q|^2\right)^{10}dq\right)^{1/2}. \qquad \square\end{aligned}$$

Now we will prove the key ellipticity property for the matrix $a^\pm=(a_{ij}^\pm)$.

Lemma 2.5. *Let F be a non-negative solution of the system* (1.1)–(1.4) *given by Theorem* 1.1. *If $\|f_0\|_{H^N_{x,p}}\leqslant\epsilon_0$ for ϵ_0 small enough, then there exists a constant $K>0$ such that for any $\xi\in\mathbf{R}^3$,*

$$\sum_{i,j=1}^{3}a_{ij}^\pm(t,x,p)\xi_i\xi_j\geqslant K|\xi|^2. \tag{2.23}$$

Proof. We will use the idea in [15] to prove this lemma. We first define

$$\sigma^{ij}(p)=\int\limits_{\mathbf{R}^3}\Phi^{ij}(P,Q)J(q)\,dq. \tag{2.24}$$

For any $p\in\mathbf{R}^3$, $\sigma(p)=(\sigma^{ij}(p))$ is a symmetric positive-definite matrix satisfying $\sigma(Rp)=R\sigma(p)R^T$ for any transformation R belonging to the orthogonal groups $O_3(\mathbf{R})$. Let $(p,r,\bar{r})$ be an orthogonal basis of $\mathbf{R}^3$. Using the invariance of $\sigma(p)$ under orthogonal transformation of $\mathbf{R}^3$, we easily deduce that

$$\sum_{i,j=1}^{3}\sigma^{ij}(p)p_i r_j=\sum_{i,j=1}^{3}\sigma^{ij}(p)p_i\bar{r}_j=0.$$

This means that $(p,r,\bar{r})$ is an orthogonal basis of eigen-vectors of $\sigma(p)$. Thus, we get that $\sigma(p)$ has a simple eigenvalue $\lambda_1(p)=(1/|p|^2)\sum\sigma^{ij}(p)p_ip_j$ associated with the eigen-vector p and a double eigenvalue $\lambda_2(p)=(1/|r|^2)\sum\sigma^{ij}(p)r_ir_j$ associated with the eigen-space $p^{\perp}$, that is,

$$\lambda_1(p)=\frac{1}{|p|^2}\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}\Phi^{ij}(P,Q)p_ip_jJ(q)\,dq,$$

$$\lambda_2(p)=\frac{1}{|r|^2}\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}\Phi^{ij}(P,Q)r_ir_jJ(q)\,dq.$$

Therefore, we can obtain that

$$\sigma^{ij}(p)=\lambda_1(p)\frac{p_ip_j}{|p|^2}+\lambda_2(p)\left\{\delta_{ij}-\frac{p_ip_j}{|p|^2}\right\}. \tag{2.25}$$

On the other hand, we know from [15] that there are constants c_1, $c_2>0$ such that, as $|p|\to\infty$, $\lambda_1(p)\to c_1$ and $\lambda_2(p)\to c_2$ and that there exists a constant $\nu_0>0$ such that

$$\lambda(p)=\min\bigl(\lambda_1(p),\lambda_2(p)\bigr)\geqslant\nu_0,\quad\text{for all } p\in\mathbf{R}^3.$$

By the definition of $a_{ij}^{\pm}(t,x,p)$, we have

$$a_{ij}^{\pm}(t,x,p)=\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)J(q)\,dq+\int_{\mathbf{R}^3}\Phi^{ij}(P,Q)\sqrt{J(q)}\,f_{\pm}(t,x,q)\,dq.$$

By Theorem 1.1, we know that there is a constant $S>0$ such that

$$f_{\pm}(t,x,q)\geqslant -S\epsilon_0.$$

By the non-negativity of the matrix $\Phi(P,Q)$, we have

$$\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}\Phi^{ij}(P,Q)\xi_i\xi_j\sqrt{J(q)}f_{\pm}(t,x,q)\,dq\geqslant -S\epsilon_0\int_{\mathbf{R}^3}\sum_{i,j=1}^{3}\Phi^{ij}(P,Q)\xi_i\xi_j\sqrt{J(q)}\,dq.$$

Let $\bar{\sigma}^{ij}(p)$ be the integral of the r.h.s. of the above inequality. Then $\bar{\sigma}^{ij}(p)$ has the same properties as $\sigma^{ij}(p)$ and there exist $\bar{\lambda}_1(p)$, $\bar{\lambda}_2(p)$ such that

$$\bar{\sigma}^{ij}(p)=\bar{\lambda}_1(p)\frac{p_ip_j}{|p|^2}+\bar{\lambda}_2(p)\left\{\delta_{ij}-\frac{p_ip_j}{|p|^2}\right\}, \tag{2.26}$$

with

$$\bar{\lambda}_1(p) = \frac{1}{|p|^2} \int_{\mathbf{R}^3} \sum_{i,j=1}^{3} \Phi^{ij}(P,Q) p_i p_j \sqrt{J(q)}\, dq,$$

$$\bar{\lambda}_2(p) = \frac{1}{|r|^2} \int_{\mathbf{R}^3} \sum_{i,j=1}^{3} \Phi^{ij}(P,Q) r_i r_j \sqrt{J(q)}\, dq.$$

By (2.12), (2.13) and (2.24), we know that there is a constant $\nu_1 > 0$ such that

$$\bar{\lambda}(p) = \min\bigl(\bar{\lambda}_1(p), \bar{\lambda}_2(p)\bigr) \leqslant \nu_1, \quad \text{for all } p \in \mathbf{R}^3.$$

Therefore, we have

$$\sum_{i,j=1}^{3} a_{ij}^{\pm}(t,x,p)\xi_i\xi_j \geqslant \sum_{i,j=1}^{3} \sigma^{ij}(p)\xi_i\xi_j - S\epsilon_0 \sum_{i,j=1}^{3} \bar{\sigma}^{ij}(p)\xi_i\xi_j \geqslant (\nu_0 - S\epsilon_0\nu_1)|\xi|^2.$$

If choosing ϵ_0 small enough and $K = \nu_0 - S\epsilon_0\nu_1$, this completes the proof of Lemma 2.5. □

Remark 2.6. By the similar arguments as the above and using the spectrum theory about the classical Landau operator in [7] we can also obtain the elliptic estimate corresponding to the classical Landau operator, that is, Lemma 2.1 in [6].

3. Smoothness effects for classical solutions

In this section we shall prove Theorem 1.2. The proof will consist in an induction on the number of derivatives (in x and p) which can be controlled. We first prove regularity of p variable.

Lemma 3.1. *Let $N \geqslant 4$ be a given integer and $F = [F_+, F_-]$ be a smooth non-negative solution of the system* (1.1)–(1.4) *given by Theorem* 1.1. *We suppose that for any $T > 0$, $s \geqslant 0$,*

$$\|F\|_{L_t^\infty([0,T];H_{x,p}^{N,s})} \leqslant K_0, \quad \text{and} \quad \|F\|_{L_t^2([0,T];H_{x,p}^{N,s})} \leqslant K_0 T^{1/2},$$

where $K_0 = K_0(s,T)$ is a constant.

Then there exists a constant $\tilde{C}_1 > 0$, which depends on N, s, T, and K_0, such that

$$\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \bigl|\nabla_p\bigl(\partial_x^\alpha \partial_p^\beta F\bigr)\bigl(1+|p|^2\bigr)^{s/2}\bigr|^2\, dt\, dx\, dp \leqslant \tilde{C}_1, \tag{3.1}$$

with $|\alpha| + |\beta| \leqslant N$.

Proof. We only consider the regularity of $F_+(t,x,p)$ about p because the regularity of $F_-(t,x,p)$ can be proved by the similar arguments. By using Eq. (2.1), we know that $h = (\partial_x^\alpha \partial_p^\beta F_+)(1+|p|^2)^{s/2}$ satisfies the following equation (using Einstein's convention for the repeated indices and denoting $g = \partial_x^\alpha \partial_p^\beta F_+$):

$$\partial_t h + \frac{p}{p_0} \cdot \nabla_x h = I + II + III, \tag{3.2}$$

where

$$I=-\sum_{\alpha_1\leqslant\alpha}C_\alpha^{\alpha_1}\left(\partial^{\alpha_1}E+\frac{p}{p_0}\times\partial^{\alpha_1}B\right)\cdot\nabla_p\partial^{\alpha-\alpha_1}F_+\big(1+|p|^2\big)^{s/2},\quad \text{for } |\beta|=0,$$

$$\begin{aligned}
I=&-\sum_{|\beta_1|<|\beta|}C_\beta^{\beta_1}\partial_p^{\beta-\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial_x^\alpha\partial_p^{\beta_1}F_+\big(1+|p|^2\big)^{s/2}\\
&-\nabla_p\cdot\Bigg\{\Bigg(\sum_{\alpha_1\leqslant\alpha}C_\alpha^{\alpha_1}\partial_x^{\alpha_1}E\times\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\\
&+\sum_{\alpha_1\leqslant\alpha,\,\beta_1\leqslant\beta}C_\alpha^{\alpha_1}C_\beta^{\beta_1}\partial_p^{\beta_1}\left(\frac{p}{p_0}\right)\times\partial_x^{\alpha_1}B\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\Bigg)\big(1+|p|^2\big)^{s/2}\Bigg\}\\
&-s\big(1+|p|^2\big)^{s/2-1}p\cdot\Bigg(\sum_{\alpha_1\leqslant\alpha}C_\alpha^{\alpha_1}\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\\
&+\sum_{\alpha_1\leqslant\alpha,\,\beta_1\leqslant\beta}C_\alpha^{\alpha_1}C_\beta^{\beta_1}\partial_p^{\beta_1}\left(\frac{p}{p_0}\right)\times\partial_x^{\alpha_1}B\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\Bigg),\quad \text{for } |\beta|>0,
\end{aligned}$$

$$\begin{aligned}
II=&\,\partial_{p_i}\big(a_{ij}^+\partial_{p_j}h\big)-sa_{ij}^+(\partial_{p_j}g)\big(1+|p|^2\big)^{s/2-1}p_i-\partial_{p_i}\big[sa_{ij}^+g\big(1+|p|^2\big)^{s/2-1}p_j\big]\\
&-\partial_{p_i}\big(b_i^+h\big)+sb_i^+g\big(1+|p|^2\big)^{s/2-1}p_i-\partial_{p_i}\big(b_i^-h\big)+sb_i^-g\big(1+|p|^2\big)^{s/2-1}p_i\\
&+\partial_{p_i}\big(a_{ij}^-\partial_{p_j}h\big)-sa_{ij}^-(\partial_{p_j}g)\big(1+|p|^2\big)^{s/2-1}p_i-\partial_{p_i}\big[sa_{ij}^-g\big(1+|p|^2\big)^{s/2-1}p_j\big],
\end{aligned}$$

$$\begin{aligned}
III=&\sum_{\alpha_1+\alpha_2=\alpha,\,|\alpha_1|\geqslant1}C_\alpha^{\alpha_1}\big\{\partial_{p_i}\big[\big(\partial_x^{\alpha_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\\
&-s\big(\partial_x^{\alpha_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i\\
&-\partial_{p_i}\big[\big(\partial_x^{\alpha_1}b_i^+\big)\big(\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]+s\big(\partial_x^{\alpha_1}b_i^+\big)\big(\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i\\
&+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]-s\big(\partial_x^{\alpha_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i\\
&-\partial_{p_i}\big[\big(\partial_x^{\alpha_1}b_i^-\big)\big(\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]+s\big(\partial_x^{\alpha_1}b_i^-\big)\partial_x^{\alpha_2}F_+\big(1+|p|^2\big)^{s/2-1}p_i\big\},\quad \text{for } |\beta|=0,
\end{aligned}$$

$$\begin{aligned}
III=&\sum_{\alpha_1+\alpha_2=\alpha,\,\beta_1+\beta_2=\beta,\,|\alpha_1|+|\beta_1|\geqslant1}C_\alpha^{\alpha_1}C_\beta^{\beta_1}\big\{\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\\
&-s\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\\
&-s\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\\
&-s\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^-\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\\
&-s\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^-\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2-1}p_i\big\},\quad \text{for } |\beta|>0.
\end{aligned}$$

We only consider the case $|\beta|>0$ because the case $|\beta|=0$ is similar. Multiplying (3.2) by h and integrating on (t,x,p), we estimate the resulting equation term by term.

We see that

$$\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\left[\partial_t h+\frac{p}{p_0}\cdot\nabla_x h\right]h\,dt\,dx\,dp=\frac{1}{2}\left(\|h(T)\|^2_{L^2_{x,p}}-\|h(0)\|^2_{L^2_{x,p}}\right)\geqslant-\frac{1}{2}K_0^2.$$

For the first term I_1 of I, the Hölder's inequality entails that

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} I_1 h\,dt\,dx\,dp\right|\leqslant C\|h\|_{L^2_t([0,T];L^2_{x,p})}\|F_+\|_{L^2_t([0,T];H^{N,s}_{x,p})}\leqslant CTK_0^2. \tag{3.3}$$

For the second term I_2 of I, when $|\alpha_1|\leqslant N/2$, integral by parts and the fact that $L^\infty(\mathbf{T}^3)\subset H^2(\mathbf{T}^3)$ imply that

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} I_2 h\,dt\,dx\,dp\right|\\
&\quad\leqslant C\left\|\partial_x^{\alpha_1}[E,B]\right\|_{L^\infty_{t,x}}\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\left(\left|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\right|+\left|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\right|\right)\left|(\nabla_p g)\left(1+|p|^2\right)^s\right|dt\,dx\,dp\\
&\quad\leqslant C\sum_{|\alpha_0|\leqslant 1}\sup_t\left\|\partial_x^{\alpha_0+\alpha_1}[E,B]\right\|_{L^2_x}\|F_+\|_{L^2_t([0,T];H^{N,s}_{x,p})}\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|_{L^2_t([0,T];L^2_{x,p})}\\
&\quad\leqslant C_\epsilon\|F_+\|^2_{L^2_t([0,T];H^{N,s}_{x,p})}+\epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|^2_{L^2_t([0,T];L^2_{x,p})},
\end{aligned}$$

where we have used the fact that $\|[E,B]\|_{L^\infty_t(H^N_x)}\leqslant C_0$.

If $|\alpha_1|\geqslant N/2$, then $|\alpha-\alpha_1|+|\beta|\leqslant N/2$ and we have

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} I_2 h\,dt\,dx\,dp\right|\\
&\quad\leqslant\int_{[0,T]\times\mathbf{R}^3}\left|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\right|_{L^\infty_x}\left\|\partial_x^{\alpha_1}[E,B]\right\|_{L^2_x}\left\|\nabla_p\left(g\left(1+|p|^2\right)^s\right)\right\|_{L^2_x}dt\,dp\\
&\quad\leqslant C\sup_t\left\|\partial_x^{\alpha_1}[E,B]\right\|_{L^2_x}\int_{[0,T]\times\mathbf{R}^3}\left\|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\right\|_{H^1(\mathbf{T}^3)}\left\|\nabla_p\left(g\left(1+|p|^2\right)^s\right)\right\|_{L^2_x}dt\,dp\\
&\quad\leqslant C\sup_t\left\|\partial_x^{\alpha_1}[E,B]\right\|_{L^2_x}\|F_+\|_{L^2_t([0,T];H^{N,s}_{x,p})}\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|_{L^2_t([0,T];L^2_{x,p})}\\
&\quad\leqslant C_\epsilon\|F_+\|^2_{L^2_t([0,T];H^{N,s}_{x,p})}+\epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|^2_{L^2_t([0,T];L^2_{x,p})}.
\end{aligned}$$

In any case, we can get

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} I_2 h\,dt\,dx\,dp\right|\leqslant C_\epsilon K_0^2 T+\epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|^2_{L^2_t([0,T];L^2_{x,p})}. \tag{3.4}$$

Similarly we have that the term I_3 has the following estimate:

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} I_3 h\,dt\,dx\,dp\right| \leqslant C_\epsilon K_0^2 T + \epsilon \left\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right\|^2_{L_t^2([0,T];L^2_{x,p})}.$$

For the term containing II, we write $II = \sum_{i=1}^{10} II_i$, and then estimate term by term.

$$\begin{aligned}
\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_1 h\,dt\,dx\,dp &= -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^+(\partial_{p_j}h)(\partial_{p_i}h)\,dt\,dx\,dp \\
&= -\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^+(\partial_{p_j}g)(\partial_{p_i}g)\big(1+|p|^2\big)^s\,dt\,dx\,dp \\
&\quad - 2s\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^+(\partial_{p_j}g)g\big(1+|p|^2\big)^{s-1}p_j\,dt\,dx\,dp \\
&\quad - s^2\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^+|g|^2\big(1+|p|^2\big)^{s-2}p_ip_j\,dt\,dx\,dp.
\end{aligned}$$

Denote the r.h.s. of the above identity as $A_1 + A_2 + A_3$. Thanks to Lemma 2.5, we get

$$A_1 \leqslant -K\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right|^2 dt\,dx\,dp. \tag{3.5}$$

By Lemma 2.2 and the Sobolev embedding theorem, we see that

$$\sup_{t,x,p}\left|a_{ij}^+(t,x,p)\right| \leqslant C\sum_{|\alpha_1|\leqslant 2}\left\|\partial_x^{\alpha_1}F_+\big(1+|p|^2\big)^5\right\|_{L^\infty([0,T];L^2_{x,p})} \leqslant CK_0. \tag{3.6}$$

By (3.6) and the Hölder's inequality, we get

$$\begin{aligned}
|A_2| &\leqslant CK_0\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} |\nabla_p g||g|\big(1+|p|^2\big)^{s-1/2}\,dt\,dx\,dp \\
&\leqslant CK_0\Big(\epsilon\left\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right\|^2_{L_t^2([0,T];L^2_{x,p})} + CTK_0^2\Big),
\end{aligned} \tag{3.7}$$

and

$$|A_3| \leqslant CK_0\|F_+\|^2_{L_t^2([0,T];H^{N,s}_{x,p})} \leqslant CTK_0^3. \tag{3.8}$$

By (3.6), we obtain that

$$\left| \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_2 h\,dt\,dx\,dp \right| \leqslant C \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left| a_{ij}^{+}(\partial_{p_j} g)\big(1+|p|^2\big)^{s-1/2} g \right| dt\,dx\,dp$$
$$\leqslant C K_0\big(\epsilon \big\| (\nabla_p g)\big(1+|p|^2\big)^{s/2}\big\|^2_{L^2_t([0,T];L^2_{x,p})} + C T K_0^2\big). \tag{3.9}$$

For the term containing II_3, we write

$$\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_3 h\,dt\,dx\,dp = s \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^{+}(\partial_{p_j} h) g\big(1+|p|^2\big)^{s/2-1} p_j\,dt\,dx\,dp$$
$$= s \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^{+}(\partial_{p_j} g) g\big(1+|p|^2\big)^{s-1} p_j\,dt\,dx\,dp$$
$$+ s^2 \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} a_{ij}^{+}|g|^2\big(1+|p|^2\big)^{s-2} p_i p_j\,dt\,dx\,dp.$$

By (3.6) and the above arguments, we easily get

$$\left| \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_3 h\,dt\,dx\,dp \right| \leqslant C K_0\big(\epsilon \big\| (\nabla_p g)\big(1+|p|^2\big)^{s/2}\big\|^2_{L^2_t([0,T];L^2_{x,p})} + C T K_0^2\big). \tag{3.10}$$

Then we consider the term II_4. Integral by parts imply that

$$\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_4 h\,dt\,dx\,dp = -\frac{1}{2} \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_{p_i} b_i^{+}\big)|h|^2\,dt\,dx\,dp.$$

By Lemma 2.2 and the Sobolev embedding theorem, it is easy to know

$$\sup_{t,x,p}\left|\partial_{p_i} b_i^{+}(t,x,p)\right| \leqslant C \sum_{|\alpha_1|\leqslant 2,\,|\beta_1|\leqslant 2} \big\| \partial_x^{\alpha_1}\partial_p^{\beta_1} F_{+}(t,x,p)\big(1+|p|^2\big)^6 \big\|_{L^\infty_t(L^2_{x,p})}.$$

Therefore, we have the following estimate:

$$\left| \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} II_4 h\,dt\,dx\,dp \right| \leqslant C K_0^3 T.$$

Because a_{ij}^{+} shares the same properties as a_{ij}^{-}, the remaining terms of II has the similar as the above. Now we turn to the term III. We write $III = \sum_{i=1}^{8} III_i$. By integration by parts,

$$\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1} a_{ij}^{+}\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2} F_{+}\big)\big(1+|p|^2\big)^{s/2}\big] h\,dt\,dx\,dp$$
$$= -\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1} a_{ij}^{+}\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2} F_{+}\big)(\partial_{p_i} g)\big(1+|p|^2\big)^{s}\,dt\,dx\,dp$$

$$- s \int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right)\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right) g\left(1+|p|^2\right)^{s-1} p_i \, dt\, dx\, dp.$$

For the first term, when $1 \leqslant |\alpha_1| + |\beta_1| \leqslant N/2$ with $N \geqslant 4$, Lemma 2.2 implies that

$$\sup_{t,x,p}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right\| \leqslant C \sum_{|\alpha_0|\leqslant 2,\, \beta_0\leqslant\beta_1} \left\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}F_+\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_t^\infty(L^2_{x,p})} \leqslant CK_0,$$

so that

$$\begin{aligned}
&\left|\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right)\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right)\left(\partial_{p_i}g\right)\left(1+|p|^2\right)^{s} dt\, dx\, dp\right| \\
&\quad \leqslant CK_0\left(C_\epsilon\|F_+\|^2_{L_t^2([0,T];H^{N,s}_{x,p})} + \epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|^2_{L_t^2([0,T];L^2_{x,p})}\right).
\end{aligned}$$

By Lemma 2.2, we have

$$\left|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}(t,x,\cdot)\right|_{L_p^\infty} \leqslant C\sum_{\beta_0\leqslant\beta_1}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_p^2}. \tag{3.11}$$

When $|\alpha_1| + |\beta_1| > N/2$ and then $|\alpha_2| + |\beta_2| < N/2$, we have the following estimate

$$\begin{aligned}
&\left|\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right)\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right)\left(\partial_{p_i}g\right)\left(1+|p|^2\right)^{s} dt\, dx\, dp\right| \\
&\quad\leqslant \int\limits_{[0,T]\times\mathbf{T}^3} \left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right\|_{L_p^\infty}\left\|\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_p^2}\left\|\left(\partial_{p_i}g\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_p^2} dx\, dt \\
&\quad\leqslant C\sum_{\beta_0\leqslant\beta_1}\int\limits_{[0,T]\times\mathbf{T}^3}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_p^2} \\
&\qquad\times\left\|\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_p^2}\left\|\left(\partial_{p_i}g\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_p^2} dx\, dt \\
&\quad\leqslant C\sum_{\beta_0\leqslant\beta_1}\int\limits_{[0,T]}\sum_{\beta_0\leqslant\beta_1}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L^2_{x,p}} \\
&\qquad\times\left\|\left(\nabla_p\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_x^\infty(L_p^2)}\left\|(\nabla_p g)\left(1+|p|^2\right)^{s}\left(1+|p|^2\right)^{s/2}\right\|_{L^2_{x,p}} dx\, dt \\
&\quad\leqslant C\sum_{|\alpha_0|\leqslant 2}\left\|\left(\nabla_p\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}F_+\right)\left(1+|p|^2\right)^{s/2}\right\|_{L_t^\infty([0,T];L^2_{x,p})} \\
&\qquad\times\left(\epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s}\left(1+|p|^2\right)^{s/2}\right\|^2_{L_t^2([0,T];L^2_{x,p})} + C_\epsilon\sum_{\beta_0\leqslant\beta_1}\|F\|^2_{L_t^2([0,T];H^{N,|\beta_0|+10}_{x,p})}\right) \\
&\quad\leqslant C_\epsilon K_0^2\|F\|^2_{L_t^2([0,T];H^{N,s}_{x,p})} + \epsilon\left\|(\nabla_p g)\left(1+|p|^2\right)^{s/2}\right\|^2_{L_t^2([0,T];L^2_{x,p})},
\end{aligned} \tag{3.12}$$

where we have used the facts that $|\alpha_2| + |\beta_2| + 2 \leqslant N$ and $L^\infty(\mathbf{T}^3) \subset H^2(\mathbf{T}^3)$. By the similar arguments as the above, we can prove that the second term of III_1 shares the same bound.

As for the term III_3, integration by parts implies that

$$\begin{aligned}
&\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]h\,dt\,dx\,dp \\
&\quad= -\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}g)\big(1+|p|^2\big)^{s}\,dt\,dx\,dp \\
&\qquad- s\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)g\big(1+|p|^2\big)^{s-1}p_i\,dt\,dx\,dp.
\end{aligned}$$

We only consider the first term of III_3 and then the second term is treated similarly. When $1 \leqslant |\alpha_1| + |\beta_1| \leqslant N/2 - 1$, by the proof of Lemma 2.2, we easily get

$$\begin{aligned}
&\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big\|_{L^\infty([0,T];L^\infty_{x,p})} \\
&\quad\leqslant C\sum_{|\alpha_0|\leqslant 2}\ \sum_{|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_p^{\beta_1}\partial_x^{\alpha_0+\alpha_1}F_\pm\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_t^\infty([0,t];L^2_{x,p})}.
\end{aligned} \tag{3.13}$$

By using the above inequality, we have

$$\begin{aligned}
&\Bigg|\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}g)\big(1+|p|^2\big)^{s}\,dt\,dx\,dp\Bigg| \\
&\quad\leqslant C\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big\|_{L^\infty([0,T];L^\infty_{x,p})}\Big(C_\epsilon\big\|\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big\|^2_{L^2([0,T];L^2_{x,p})} \\
&\qquad+\epsilon\big\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\big\|^2_{L_t^2([0,T];L^2_{x,p})}\Big) \\
&\quad\leqslant C_\epsilon\|F\|^2_{L_t^2([0,T];H^{N,s}_{x,p})}+\epsilon\big\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\big\|^2_{L_t^2([0,T];L^2_{x,p})}.
\end{aligned}$$

Thanks to Lemma 2.2, we have

$$\big|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(t,x,\cdot)\big|_{L_p^\infty}\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(t,x,p)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2}. \tag{3.14}$$

By using (3.14), the similar arguments as (3.12) imply that, when $N/2 \leqslant |\alpha_1| + |\beta_1| \leqslant N-1$

$$\begin{aligned}
&\Bigg|\int\limits_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}g)\big(1+|p|^2\big)^{s}\,dt\,dx\,dp\Bigg| \\
&\quad\leqslant C_\epsilon K_0^3T_0+\epsilon\big\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\big\|^2_{L_t^2([0,T];L^2_{x,p})}.
\end{aligned}$$

The case that $|\alpha_1| + |\beta_1| = N$ and $|\alpha_2| + |\beta_2| = 0$ require more care because the term b_i^+ contains the p-derivative of F_+. In this case the total-derivatives of F_+ will be $N+1$, which is impossible to be controlled by Lemma 2.2 and our assumption. Thus we devise Lemmas 2.3 and 2.4 in order to treat it.

By Lemma 2.3 and $|\beta| > 0$, we know that

$$\left|\partial_x^\alpha \partial_p^\beta b_i^\pm(t,x,p)\right| \leqslant C \sum_{|\beta_1|\leqslant|\beta|-1} \Big(\left|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,p)\right| + p_0^{1/2-|\beta_1|} \left\|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,\cdot)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\right\|_{L_p^2}\Big). \tag{3.15}$$

By the above inequality, we have

$$\begin{aligned}
&\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \big(\partial_x^\alpha \partial_p^\beta b_i^+\big) F_+ (\partial_{p_i} g)\big(1+|p|^2\big)^s \, dt\, dx\, dp\right| \\
&\leqslant C \sum_{|\beta_1|\leqslant|\beta|-1} \bigg(\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,p)\right| \left|F_+ (\partial_{p_i} g)\big(1+|p|^2\big)^s\right| dt\, dx\, dp \\
&\quad + \int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} p_0^{1/2} \left\|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,\cdot)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\right\|_{L_p^2} \left|F_+ (\partial_{p_i} g)\big(1+|p|^2\big)^s\right| dt\, dx\, dp \bigg) \\
&\leqslant C \left\|F_+\big(1+|p|^2\big)^{s/2}\right\|_{L_t^\infty([0,T];L_{x,p}^\infty)} \Big(C_\epsilon \|F_+\|^2_{L^2([0,T];H_{x,p}^N)} + \epsilon \left\|\nabla_p g\big(1+|p|^2\big)^{s/2}\right\|^2_{L^2([0,T];H_{x,p}^N)}\Big) \\
&\quad + \sum_{|\beta_1|\leqslant|\beta|-1} C \int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left\|\partial_x^\alpha \partial_p^{\beta_1} F_\pm(t,x,\cdot)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\right\|_{L_p^2} \\
&\quad \times \left\|F_+\big(1+|p|^2\big)^{(1+s)/2}\right\|_{L_p^2} \left\|(\partial_{p_i} g)\big(1+|p|^2\big)^s\right\|_{L_p^2} dt\, dx \\
&\leqslant C \left\|F_+\big(1+|p|^2\big)^{s/2}\right\|_{L_t^\infty([0,T];H_{x,p}^4)} \Big(C_\epsilon \|F_+\|^2_{L^2([0,T];H_{x,p}^N)} + \epsilon \left\|\nabla_p g\big(1+|p|^2\big)^{s/2}\right\|^2_{L^2([0,T];L_{x,p}^2)}\Big) \\
&\quad + C \left\|F_+\big(1+|p|^2\big)^{(1+s)/2}\right\|_{L^\infty([0,T];L_x^\infty(L_p^2))} \Big(\|F_+\|^2_{L^2([0,T];H_{x,p}^{N,N+4})} + \epsilon \left\|\nabla_p g\big(1+|p|^2\big)^{s/2}\right\|^2_{L^2([0,T];L_{x,p}^2)}\Big) \\
&\leqslant C_\epsilon K_0^2 T + \epsilon \left\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right\|^2_{L_t^2([0,T];L_{x,p}^2)}.
\end{aligned}$$

Here we have used the Sobolev embedding theorem,

$$\left\|F_+\big(1+|p|^2\big)^{s/2}\right\|_{L_t^\infty([0,T];L_{x,p}^\infty)} \leqslant C \left\|F_+\big(1+|p|^2\big)^{s/2}\right\|_{L_t^\infty([0,T];H_{x,p}^4)} \leqslant C K_0. \tag{3.16}$$

By the similar arguments as the above, we can prove that the second term of III_3 shares the same bound. The other terms of III can be treated similarly and we omit it. Finally, we obtain that III has the estimate:

$$\left|\int_{[0,T]\times\mathbf{T}^3\times\mathbf{R}^3} IIIh \, dt\, dx\, dp\right| \leqslant C_\epsilon K_0^2 T + C\epsilon \left\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right\|^2_{L_t^2([0,T];L_{x,p}^2)}.$$

By using the above estimates, we see that

$$\left\|(\nabla_p g)\big(1+|p|^2\big)^{s/2}\right\|^2_{L_t^2([0,T];L_{x,p}^2)} \leqslant \frac{C K_0^2 + C_\epsilon K_0^3 T}{K - C\epsilon}.$$

If we take $\epsilon > 0$ small enough, we conclude the proof of Lemma 3.1. □

We now turn to prove the regularity for x variable. From [6] and the case near vacuum about the Boltzmann equation [2], we will resort to the averaging lemma of the relativistic version [3,9]. Now we begin with

Lemma 3.2. *Let $N \geqslant 4$ be a given integer and $F = [F_+, F_-]$ be a smooth non-negative solution of the system* (1.1)–(1.4) *given by Theorem* 1.1. *We suppose that for any $T > 0$, $s \geqslant 0$,*

$$\|F\|_{L_t^\infty([0,T];H_{x,p}^{N,s})} \leqslant K_0, \quad \textit{and} \quad \|F\|_{L_t^2([0,T];H_{x,p}^{N,s})} \leqslant K_0 T^{1/2},$$

where $K_0 = K_0(s,T)$ is a constant. Then there exists a constant $\tilde{C}_2 > 0$, which depends on N, s, T, and K_0, such that

$$\int\limits_{[0,T]\times\mathbf{R}^3} \left\| \left(\partial_x^\alpha \partial_p^\beta F\right)\left(1+|p|^2\right)^{s/2} \right\|_{\dot{H}_x^{1/20}}^2 dt\,dp \leqslant \tilde{C}_2, \tag{3.17}$$

with $|\alpha| + |\beta| \leqslant N$.

Proof. Let $\rho(t,x,p) = \partial_x^\alpha \partial_p^\beta F_+(1+|p|^2)^{(s+4)/2}$. To prove (3.17), we only need to prove

$$\int\limits_{[0,T]\times\mathbf{R}^3} \left(1+|p|^2\right)^{-4}\left(\sum_{m\in\mathbf{Z}^3} |m|^{1/10}\left|\hat{\rho}(t,m,p)\right|^2\right) dt\,dp \leqslant \tilde{C}_2, \tag{3.18}$$

where $\hat{\rho}(t,m,p)$ is the Fourier coefficient of ρ with respect to the x variable.

Let $\chi := \chi(p) \in C_c^\infty(\mathbf{R}^3)$ be a cutoff function which satisfies $\chi(p) \geqslant 0$ and $\int_{\mathbf{R}^3} \chi(p)\,dp = 1$. We set $\chi_\epsilon(p) = \epsilon^{-3}\chi(\frac{p}{\epsilon})$ and write

$$\hat{\rho}(t,m,p) = \left[\hat{\rho}(t,m,p) - \left(\hat{\rho}(t,m,\cdot) *_p \chi_\epsilon\right)(p)\right] + \left(\hat{\rho}(t,m,\cdot) *_p \chi_\epsilon\right)(p).$$

For the first term of the r.h.s. of the above equality, we get

$$\begin{aligned}
&\int\limits_{\mathbf{R}^3} \left(1+|p|\right)^{-4}\left|\hat{\rho}(t,m,p) - \left(\hat{\rho}(t,m,\cdot) *_p \chi_\epsilon\right)(p)\right|^2 dp \\
&\quad\leqslant \int\limits_{\mathbf{R}^3}\left|\int\limits_{\mathbf{R}^3}\left[\hat{\rho}(t,m,p) - \hat{\rho}(t,m,p-q)\right]\chi_\epsilon(q)\,dq\right|^2 dp \\
&\quad\leqslant \left(\int\limits_{\mathbf{R}^3}\left(\int\limits_{\mathbf{R}^3}\left|\hat{\rho}(t,m,p) - \hat{\rho}(t,m,p-q)\right|^2 dp\right)^{1/2}\chi_\epsilon(q)\,dq\right)^2 \\
&\quad\leqslant C\left(\int\limits_{\mathbf{R}^3}\chi_\epsilon(q)|q|\,dq\right)^2\int\limits_{\mathbf{R}^3}\left|\nabla_p\hat{\rho}(t,m,p)\right|^2 dp \leqslant C\epsilon^2\int\limits_{\mathbf{R}^3}\left|\nabla_p\hat{\rho}(t,m,p)\right|^2 dp
\end{aligned}$$

so that

$$\int_{[0,T]\times\mathbf{R}^3} \big(1+|p|\big)^{-4}\Big(\sum_{m\in\mathbf{Z}^3}|m|^{1/10}\big|\hat{\rho}(t,m,p)-\big(\hat{\rho}(t,m,\cdot)*_p\chi_\epsilon\big)(p)\big|^2\Big)\,dt\,dp$$

$$\leqslant C\int_{[0,T]\times\mathbf{R}^3}\sum_{m\in\mathbf{Z}^3}|m|^{1/10}\epsilon^2\big|\nabla_p\hat{\rho}(t,m,p)\big|^2\,dt\,dp. \tag{3.19}$$

Remembering that $\rho(t,x,p)=h(1+|p|)^2$, we see that ρ satisfies (3.2) with s replaced by $s+4$. Then we can write the equation satisfied by ρ under the form

$$\partial_t\rho+\frac{p}{p_0}\cdot\rho=\rho_1+\nabla_p\cdot\rho_2. \tag{3.20}$$

Here $\nabla_p\cdot\rho_2$ is the sum of the terms I_2, II_i $(i=1,3,4,6,8,10)$ and III_i $(i=1,3,4,5)$, while ρ_1 is the sum of the remaining terms.

We claim that $\rho_1,\rho_2\in L^2_t([0,T];L^2_{x,p})$. We only present the estimates for the terms I, II_1, II_5, III_1 and III_3 in the case $|\beta|>0$, the others being similar.

It is obvious that

$$\|I_1\|_{L^2_t([0,T];L^2_{x,p})}\leqslant C\|F_+\|_{L^2_t([0,T];H^{N,s+4}_{x,p})}\leqslant CK_0T^{1/2}.$$

As for the term I_2, when $|\alpha_1|\leqslant N/2$, we use the Sobolev's inequality and Theorem 1.1 to obtain

$$\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3}\bigg|\bigg(\partial_x^{\alpha_1}E\,\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_++\partial_p^{\beta_1}\bigg(\frac{p}{p_0}\bigg)\times\partial_x^{\alpha_1}B\,\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\bigg)\bigg|^2\big(1+|p|^2\big)^{(s+4)}\,dt\,dx\,dp$$

$$\leqslant C\big\|\partial_x^{\alpha_1}[E,B]\big\|^2_{L^\infty_{t,x}}\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3}\big(\big|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big|^2+\big|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\big|^2\big)\big(1+|p|^2\big)^{(s+4)}\,dt\,dx\,dp$$

$$\leqslant C\big\|\partial_x^{\alpha_1}[E,B]\big\|^2_{L^\infty_t(H^2_x)}\|F_+\|^2_{L^2_t([0,T];H^{N,s+4}_{x,p})}\leqslant CK_0^2T.$$

When $|\alpha_1|>N/2$ and then $|\alpha-\alpha_1|+|\beta|\leqslant N/2$, we use the Sobolev's inequality and Theorem 1.1 to obtain

$$\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3}\bigg|\bigg(\partial_x^{\alpha_1}E\,\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_++\partial_p^{\beta_1}\bigg(\frac{p}{p_0}\bigg)\times\partial_x^{\alpha_1}B\,\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\bigg)\bigg|^2\big(1+|p|^2\big)^{(s+4)}\,dt\,dx\,dp$$

$$\leqslant C\int_{[0,T]}\big(\big\|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big(1+|p|^2\big)^{(s+4)/2}\big\|^2_{L^\infty_x(L^2_p)}$$

$$+\big|\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\big(1+|p|^2\big)^{(s+4)/2}\big|^2_{L^\infty_x(L^2_p)}\big)\big\|\partial_x^{\alpha_1}[E,B]\big\|^2_{L^2_x}\,dt$$

$$\leqslant C\big\|\partial_x^{\alpha_1}[E,B]\big\|^2_{L^\infty_t(L^2_x)}\int_{[0,T]}\sum_{|\alpha_0|\leqslant 2}\big(\big\|\partial_x^{\alpha-\alpha_1+\alpha_0}\partial_p^{\beta}F_+\big(1+|p|^2\big)^{(s+4)/2}\big\|^2_{L^2_{x,p}}$$

$$+\big\|\partial_x^{\alpha-\alpha_1+\alpha_0}\partial_p^{\beta-\beta_1}F_+\big(1+|p|^2\big)^{(s+4)/2}\big\|^2_{L^2_{x,p}}\big)\,dt$$

$$\leqslant C\big\|\partial_x^{\alpha_1}[E,B]\big\|^2_{L^\infty_t(L^2_x)}\|F_+\|^2_{L^2_t([0,T];H^{N,s+4}_{x,p})}\leqslant CK_0^2T.$$

For the term I_3, the similar arguments imply that it shares the same bound. As for the term II_1, we know from (3.6) that

$$\|a_{ij}^{+}\partial_{p_j}\rho\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0\|\partial_{p_j}\rho\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0\sqrt{\tilde{C}_1},$$

where the last inequality holds thanks to Lemma 3.1 (applied to ρ).

As for the term II_5, by Lemma 2.2 and the Sobolev embedding theorem, it is easy to get

$$\sup_{t,x,p}\left|b_i^{+}(t,x,p)\right| \leqslant C\sum_{|\alpha_1|\leqslant 2,\,|\beta_1|\leqslant 1}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}F_{+}(t,x,p)\left(1+|p|^2\right)^6\right\|_{L_t^\infty(L_{x,p}^2)} \leqslant CK_0.$$

As a consequence, we have

$$\|II_5\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant C\left\|b_i^{+}g\left(1+|p|^2\right)^{s/2+1}p_i\right\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0^2T^{1/2}.$$

As for the term III_1, when $1 \leqslant |\alpha_1|+|\beta_1| \leqslant N/2$ with $N \geqslant 4$, we know from Lemma 2.2 that

$$\sup_{t,x,p}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right\| \leqslant C\sum_{|\alpha_0|\leqslant 2,\,\beta_0\leqslant\beta_1}\left\|\partial_x^{\alpha_0+\alpha_1}\partial^{\beta_0}\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_t^\infty(L_{x,p}^2)} \leqslant CK_0.$$

This implies that

$$\begin{aligned}
&\left\|\left(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right)\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right\|_{L_t^2([0,T];L_{x,p}^2)} \\
&\quad\leqslant C\left\|\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0^2T^{1/2}.
\end{aligned}$$

From the proof of Lemma 2.2, we see that

$$\left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right\|_{L_p^\infty} \leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{+}(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_p^2}.$$

When $|\alpha_1|+|\beta_1| > N/2$, then $|\alpha_1|+|\beta_1| < N/2$ and we obtain from (3.11) that

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{R}^3\times\mathbf{T}^3}\left|\left(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right)\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right|^2 dt\,dx\,dp \\
&\quad\leqslant \int_{[0,T]\times\mathbf{T}^3}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{+}\right\|_{L_p^\infty}^2\left\|\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right\|_{L_p^2}^2 dt\,dx \\
&\quad\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\int_{[0,T]\times\mathbf{T}^3}\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{+}(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_p^2}^2 \\
&\qquad\times\left\|\left(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right\|_{L_p^2}^2 dt\,dx \\
&\quad\leqslant C\sum_{|\alpha_0|\leqslant 2}\left\|\left(\partial_{p_j}\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}F_{+}\right)\left(1+|p|^2\right)^{(s+4)/2}\right\|_{L_t^\infty([0,T];L_{x,p}^2)}^2 \\
&\qquad\times\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{+}(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L^2([0,T];L_{x,p}^2)}^2 \\
&\quad\leqslant C\|F_{+}\|_{L_t^\infty([0,T];H_{x,p}^{N,s+4})}^2\left\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{+}(t,x,p)\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L^2([0,T];L_{x,p}^2)}^2 \leqslant CK_0^4T.
\end{aligned}$$

In any case we know that III_1 has the estimate

$$\|III_1\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0^2T^{1/2}.$$

For the term III_3, we proceed like for the corresponding term in the proof of Lemma 3.1. We use (3.13), (3.14) and (3.15) and consider the cases $|\alpha_1|+|\beta_1|<N/2$, $N/2\leqslant|\alpha_1|+|\beta_1|<N$ and $|\alpha_1|+|\beta_1|=N$ separately in order to get

$$\big\|\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{(s+4)/2}\big\|_{L_t^2([0,T];L_{x,p}^2)} \leqslant CK_0^2T^{1/2}.$$

Finally, we indeed see that $\rho_1,\rho_2\in L_t^2([0,T];L_{x,p}^2)$.

Now according to (2.23) in Theorem 2.3 (averaging lemma of the relativistic version) of [2], we can prove that

$$\begin{aligned}
m^{1/2}\int_0^T\big|\big(\hat\rho(t,m,p)*_p\chi\big)(p)\big|^2\,dt \leqslant{}& C\big(\big\|\chi_\epsilon(p-q)\big(1+|q|^2\big)\big\|_{L_q^\infty}+\big\|\nabla\chi_\epsilon(p-q)\big(1+|q|^2\big)\big\|_{L_q^\infty}\big)^2\\
&\times\big(\big\|\hat\rho(0,m,\cdot)\big\|_{L_p^2}^2+\big\|\hat\rho(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2\\
&+\big\|\hat\rho_1(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2+\big\|\hat\rho_2(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2\big).
\end{aligned}$$

Since $\|\chi_\epsilon(p-q)(1+|q|^2)\|_{L_q^\infty}\leqslant C\epsilon^{-3}(1+|p|^2)$, we see that

$$\begin{aligned}
&\int_{[0,T]\times\mathbf{R}^3}\big(1+|p|^2\big)^{-4}\sum_{m\in\mathbf{Z}^3}|m|^{1/10}\big|\hat\rho(t,m,p)*_p\chi(p)\big|^2\,dt\,dp\\
&\quad\leqslant C\sum_{m\in\mathbf{Z}^3}|m|^{\frac{1}{10}-\frac12}\big(\epsilon^{-6}+\epsilon^{-8}\big)\big(\big\|\hat\rho(0,m,\cdot)\big\|_{L_p^2}^2+\big\|\hat\rho(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2\\
&\qquad+\big\|\hat\rho_1(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2+\big\|\hat\rho_2(\cdot,m,\cdot)\big\|_{L_t^2([0,T];L_p^2)}^2\big).
\end{aligned}\tag{3.21}$$

If we choose $\epsilon=|m|^{-1/20}$, we can bound (3.19) by using Lemma 3.1. Then we can bound (3.21) by the facts that $\rho(0,\cdot,\cdot)\in L_{x,p}^2$ and $\rho,\rho_1,\rho_2\in L_t^2([0,T];L_{x,p}^2)$. This completes the proof of Lemma 3.2. □

The following lemma will improve the regularity of F about x variable.

Lemma 3.3. *Let $N\geqslant 4$ be a given integer and $F=[F_+,F_-]$ be a smooth non-negative solution of the system* (1.1)–(1.4) *given by Theorem* 1.1. *We suppose that for any $T>0$, $s\geqslant 0$,*

$$\|F\|_{L_t^\infty([0,T];H_{x,p}^{N,s})}\leqslant K_0,\quad\text{and}\quad\|F\|_{L_t^2([0,T];H_{x,p}^{N,s})}\leqslant K_0T^{1/2},$$

where $K_0=K_0(s,T)$ is a constant. Then for any $T>0$, $t_1\in[0,T]$ and $s\geqslant 0$, there exists a constant $\tilde C_3>0$, which depends on N, s, T, t_1 and K_0, such that

$$\int_{[t_1,T]\times\mathbf{R}^3}\big\|\big(\partial_x^\alpha\partial_p^\beta F\big)\big(1+|p|^2\big)^{s/2}\big\|_{\dot H_x^{1/10}}^2\,dt\,dp\leqslant\tilde C_3,$$

with $|\alpha|+|\beta|\leqslant N$.

Proof. We denote $\delta = 1/20$ for notational simplicity. Let $g_{\delta,k}(t,x,p) = \Delta_k h(t,x,p)|k|^{-\delta-\frac{3}{2}}$ for $k \in \mathbf{T}^3$, where $\Delta_k h(t,x,p) = h(t,x+k,p) - h(t,x,p)$ and $h(t,x,p) = (\partial_x^\alpha \partial_p^\beta F)(1+|p|^2)^{s/2}$. Lemma 3.2 tells us that $\|g_{\delta,k}\|_{L^2_{x,p,k}} \in L^2([0,T])$ so that for some $t_0 \in [0,t_1]$, $\|g_{\delta,k}(t_0)\|^2_{L^2_{x,p,k}} \leqslant \tilde{C}_2/t_1$. Noticing that

$$\int_{\mathbf{T}^3} |\hat{g}_{\delta,k}|^2\,dk = C|m|^{2\delta}\big|\hat{h}(m)\big|^2.$$

In order to prove Lemma 3.3, we only need to prove

$$\int_{[t_1,T]\times\mathbf{R}^3} \sum_{m\in\mathbf{Z}^3} |m|^{4\delta}\big|\hat{h}(m)\big|^2\,dt\,dp \leqslant \tilde{C}_3,$$

$$\int_{[t_1,T]\times\mathbf{R}^3\times\mathbf{T}^3} \sum_{m\in\mathbf{Z}^3} |m|^{4\delta}\big|\hat{h}(m)\big|^2\,dt\,dp\,dk \leqslant \tilde{C}_3,$$

or

$$\int_{[t_1,T]\times\mathbf{R}^3} \|g_{\delta,k}\|^2_{H^\delta_x}\,dt\,dp \leqslant \tilde{C}_3. \tag{3.22}$$

In order to prove (3.22), we first prove that

$$\int_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \sum_{m\in\mathbf{Z}^3} |\nabla_p g_{\delta,k}|^2\,dt\,dx\,dp\,dk \leqslant \tilde{C}_4. \tag{3.23}$$

This step is similar to the proof of (3.1), the main difference being that we now have an additional integration with respect to k.

We now write down the equation satisfied by $g_{\delta,k}$:

$$\partial_t g_{\delta,k} + \frac{p}{p_0}\nabla_x g_{\delta,k} = (IV + V + VI)|k|^{-\delta-\frac{3}{2}}, \tag{3.24}$$

where

$$IV = -\nabla_p \cdot \left\{\sum_{\alpha_1\leqslant\alpha} C_\alpha^{\alpha_1}\left(\Delta_k\big(\partial^{\alpha_1}E\partial^{\alpha-\alpha_1}F_+\big) + \frac{p}{p_0}\times\Delta_k\big(\partial^{\alpha_1}B\partial^{\alpha-\alpha_1}F_+\big)\right)\right\}\big(1+|p|^2\big)^{s/2}, \quad |\beta| = 0,$$

$$\begin{aligned}
IV = {}& -\sum_{|\beta_1|<|\beta|} C_\beta^{\beta_1}\partial_p^{\beta-\beta_1}\left(\frac{p}{p_0}\right)\cdot\nabla_x\partial_x^\alpha\partial_p^{\beta_1}\Delta_k F_+\big(1+|p|^2\big)^{s/2} - \nabla_p\cdot\left\{\left(\sum_{\alpha_1\leqslant\alpha} C_\alpha^{\alpha_1}\Delta_k\big(\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\big)\right.\right.\\
&\left.\left. + \sum_{\alpha_1\leqslant\alpha,\,\beta_1\leqslant\beta} C_\alpha^{\alpha_1}C_\beta^{\beta_1}\partial_p^{\beta_1}\left(\frac{p}{p_0}\right)\times\Delta_k\big(\partial_x^{\alpha_1}B\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\big)\right)\big(1+|p|^2\big)^{s/2}\right\}\\
&- s\big(1+|p|^2\big)^{s/2-1}p\cdot\left(\sum_{\alpha_1\leqslant\alpha} C_\alpha^{\alpha_1}\Delta_k\big(\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\big)\right.\\
&\left. + \sum_{\alpha_1\leqslant\alpha,\,\beta_1\leqslant\beta} C_\alpha^{\alpha_1}C_\beta^{\beta_1}\partial_p^{\beta_1}\left(\frac{p}{p_0}\right)\times\Delta_k\big(\partial_x^{\alpha_1}B\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\big)\right), \quad \text{for } |\beta| > 0,
\end{aligned}$$

$$\begin{aligned}V = {} & \partial_{p_i}\big(\Delta_k\big(a_{ij}^+\partial_{p_j}h\big)\big) - s\Delta_k\big(a_{ij}^+(\partial_{p_j}g)\big)\big(1+|p|^2\big)^{s/2-1}p_i - \partial_{p_i}\big[s\Delta_k\big(a_{ij}^+g\big)\big(1+|p|^2\big)^{s/2-1}p_j\big] \\ & - \partial_{p_i}\big(\Delta_k\big(b_i^+h\big)\big) + s\Delta_k\big(b_i^+g\big)\big(1+|p|^2\big)^{s/2-1}p_i - \partial_{p_i}\big(\Delta_k\big(b_i^-h\big)\big) + s\Delta_k\big(b_i^-g\big)\big(1+|p|^2\big)^{s/2-1}p_i \\ & + \partial_{p_i}\big(\Delta_k\big(a_{ij}^-\partial_{p_j}h\big)\big) - s\Delta_k\big(a_{ij}^-(\partial_{p_j}g)\big)\big(1+|p|^2\big)^{s/2-1}p_i - \partial_{p_i}\big[s\Delta_k\big(a_{ij}^-g\big)\big(1+|p|^2\big)^{s/2-1}p_j\big],\end{aligned}$$

$$\begin{aligned}VI = {} & \sum_{\alpha_1+\alpha_2=\alpha,\,|\alpha_1|\geqslant 1} C_\alpha^{\alpha_1}\big\{\partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i - \Delta_k\big(\partial_{p_i}\big[\big(\partial_x^{\alpha_1}b_i^+\big)\big(\partial_x^{\alpha_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]\big) \\ & + s\Delta_k\big(\big(\partial_x^{\alpha_1}b_i^+\big)\big(\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i + \partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i - \partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}b_i^-\big)\big(\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & + s\Delta_k\big(\big(\partial_x^{\alpha_1}b_i^-\big)\big(\partial_x^{\alpha_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i\big\}, \quad \text{for } |\beta| = 0,\end{aligned}$$

$$\begin{aligned}VI = {} & \sum_{\alpha_1+\alpha_2=\alpha,\,\beta_1+\beta_2=\beta,\,|\alpha_1|+|\beta_1|\geqslant 1} C_\alpha^{\alpha_1}C_\beta^{\beta_1}\big\{\partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i \\ & + \partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i \\ & + \partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^-\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i \\ & + \partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^-\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big] \\ & - s\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^-\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2-1}p_i\big\}, \quad \text{for } |\beta| > 0.\end{aligned}$$

We still consider the case $|\beta| \geqslant 1$. We multiply Eq. (3.24) by $g_{\delta,k}$ and then on (t,x,p,k) in the domain $[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3$. We only estimate the terms containing the terms of IV, the first and fourth term of V and the first term of VI, denoted by IV_1, IV_2, IV_3, V_1, V_4 and VI_1 respectively, since the estimates for the other terms are similar. For the first term IV_1, we get

$$\begin{aligned}& \bigg|\int_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} IV_1 g_{\delta,k}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\bigg| \\ & \quad \leqslant C\bigg(\sum_{|\beta_1|<|\beta|}\big\|\nabla_x\partial_x^\alpha\partial_p^{\beta_1}\Delta_kF_+\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2([t_0,T];L^2_{x,p,k})} + \|g_{\delta,k}\|^2_{L^2([t_0,T];L^2_{x,p,k})}\bigg).\end{aligned} \tag{3.25}$$

From (3.17), we easily know for any $s \geqslant 0$ and $|\alpha|+|\beta| \leqslant N$

$$\int_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \big|\Delta_k\partial_x^\alpha\partial_p^\beta F\big(1+|p|^2\big)^{s/2}\big|^2|k|^{-\frac{1}{10}-3}\,dt\,dx\,dp\,dk \leqslant \tilde{C}_2. \tag{3.26}$$

By using (3.26) we can prove that the r.h.s. of (3.25) is bounded by some constant $C > 0$. For any function $f(x)$ and $g(x)$,

$$\Delta_k\big(f(x)g(x)\big) = f(x+k)\Delta_k g(x) + g(x)\Delta_k f(x).$$

For the term IV_2, we use integral by parts to obtain

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} IV_2 g_{\delta,k}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk \\
&\quad= \int\bigg\{\bigg(\sum_{\alpha_1\leqslant\alpha}\Delta_k\big(\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big) + \sum_{\alpha_1\leqslant\alpha,\,\beta_1\leqslant\beta}\partial_p^{\beta_1}\bigg(\frac{p}{p_0}\bigg)\times\Delta_k\big(\partial_x^{\alpha_1}B\partial_x^{\alpha-\alpha_1}\partial_p^{\beta-\beta_1}F_+\big)\bigg) \\
&\qquad\times\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\bigg\}\nabla_p g_{\delta,k}\,dt\,dx\,dp\,dk.
\end{aligned}$$

We only estimate the first part of the r.h.s. of the above equation and then the second part is treated similarly. We know that

$$\Delta_k\big(\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big) = \partial_x^{\alpha_1}E(x+k)\Delta_k\big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big) + \big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big)\Delta_k\big(\partial_x^{\alpha_1}E\big).$$

We only consider the first part of the r.h.s. of the above equation also because the second part can be estimated in the same way. Then when $|\alpha_1| \leqslant N/2$, we know that

$$\big\|\partial_x^{\alpha_1}E(x+k)\big\|_{L_t^\infty([t_0,T];L_{x,k}^\infty)} \leqslant C\big\|\partial_x^{\alpha_1}E(x+k)\big\|_{L_t^\infty([t_0,T];L_k^\infty(H_x^2))} \leqslant C.$$

By using the above inequality we can obtain

$$\begin{aligned}
&\bigg|\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\partial_x^{\alpha_1}E(x+k)\Delta_k\big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\nabla_p g_{\delta,k}\,dt\,dx\,dp\,dk\bigg| \\
&\quad\leqslant C\big\|\Delta_k\big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L_t^2([t_0,T];L_{x,p,k}^2)} + \epsilon\|\nabla_p g_{\delta,k}\|^2_{L_t^2([t_0,T];L_{x,p,k}^2)} \\
&\quad\leqslant C_\epsilon\tilde{C}_2 + \epsilon\|\nabla_p g_{\delta,k}\|^2_{L^2([t_0,T];L_{x,p,k}^2)}.
\end{aligned}\tag{3.27}$$

When $|\alpha_1| > N/2$, we can obtain

$$\begin{aligned}
&\bigg|\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\partial_x^{\alpha_1}E(x+k)\Delta_k\big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\nabla_p g_{\delta,k}\,dt\,dx\,dp\,dk\bigg| \\
&\quad\leqslant \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3}\big\|\partial_x^{\alpha_1}E(x+k)\big\|_{L_x^2}\big\|\Delta_k\big(\partial_x^{\alpha-\alpha_1}\partial_p^{\beta}F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_x^\infty}\|\nabla_p g_{\delta,k}\|_{L_x^2}\,dt\,dx\,dp\,dk \\
&\quad\leqslant \big\|\partial_x^{\alpha_1}E(x+k)\big\|_{L_t^\infty([t_0,T];L_x^2)}\bigg(C_\epsilon\sum_{|\alpha_0|\leqslant 2}\big\|\Delta_k\big(\partial_x^{\alpha-\alpha_1+\alpha_0}\partial_p^{\beta}F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\qquad\times|k|^{-\delta-\frac{3}{2}}\big\|^2_{L_t^2([t_0,T];L_{p,k,x}^2)} + \epsilon\|\nabla_p g_{\delta,k}\|^2_{L_t^2([t_0,T];L_{p,k,x}^2)}\bigg) \\
&\quad\leqslant C_\epsilon\tilde{C}_2 + \epsilon\|\nabla_p g_{\delta,k}\|^2_{L^2([t_0,T];L_{x,p,k}^2)}.
\end{aligned}\tag{3.28}$$

We will consider the first term V_1 of V. We can write

$$V_1|k|^{-\delta-\frac{3}{2}} = \partial_{p_i}\big(a_{ij}^{+}(x+k)(\partial_{p_j} g_{\delta,k})\big) + \partial_{p_i}\big(\Delta_k\big(a_{ij}^{+}\big)(\partial_{p_j} h)|k|^{-\delta-\frac{3}{2}}\big).$$

We denote the r.h.s. of the above equality by $B_1 + B_2$. After integration by parts,

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} B_1 g_{\delta,k}\,dt\,dx\,dp\,dk \\
&= -\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} a_{ij}^{+}(x+k)(\partial_{p_j} g_{\delta,k})(\partial_{p_i} g_{\delta,k})\,dt\,dx\,dp\,dk \\
&= -\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} a_{ij}^{+}(x+k)\big[\partial_{p_i}(\Delta_k g)\big]\big[\partial_{p_j}(\Delta_k g)\big]\big(1+|p|^2\big)^s|k|^{-2\delta-3}\,dt\,dx\,dp\,dk \\
&\quad - 2s\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} a_{ij}^{+}(x+k)\big[\partial_{p_i}(\Delta_k g)\big](\Delta_k g)\big(1+|p|^2\big)^{s-1}p_j|k|^{-2\delta-3}\,dt\,dx\,dp\,dk \\
&\quad - s^2\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} a_{ij}^{+}(x+k)\big|(\Delta_k g)\big|^2\big(1+|p|^2\big)^{s-2}p_i p_j|k|^{-2\delta-3}\,dt\,dx\,dp\,dk,
\end{aligned}$$

where $g = \partial_x^\alpha \partial_p^\beta F_+$. We write the r.h.s. of the above equality as $B_{1,1} + B_{1,2} + B_{1,3}$. Since F_+ is a spatially periodic function, we obtain from Lemma 2.5 that

$$B_{1,1} \leqslant -K\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \big|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big|^2\,dt\,dx\,dp\,dk. \tag{3.29}$$

Thanks to Lemma 2.2, we know that

$$\big\|a_{ij}^{+}(x+k)\big\|_{L_t^\infty([0,T];L_{x,k,p}^\infty)} \leqslant \sum_{|\alpha_0|\leqslant 2}\big\|\partial_x^{\alpha_0}F_+\big(1+|p|^2\big)^5\big\|_{L_t^\infty([0,T];L_{x,p}^2)} \leqslant CK_0.$$

Using the Hölder's inequality and Lemma 3.2, we get

$$\begin{aligned}
|B_{1,2}| &\leqslant CK_0\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\big|\nabla_p(\Delta_k g)\big|\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big) \\
&\quad\times\big(\big|(\Delta_k g)\big|\big(1+|p|^2\big)^{(s+1)/2}|k|^{-\delta-\frac{3}{2}}\big)\,dt\,dx\,dp\,dk \\
&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2([t_0,T];L^2_{x,p,k})} + C_\epsilon\tilde{C}_2\big).
\end{aligned}$$

Similarly we can obtain

$$|B_{1,3}| \leqslant C\tilde{C}_2K_0.$$

As for the term containing B_2, we use (3.26), integration by parts and the following inequality

$$\big\|\Delta_k a_{ij}^{+}\big\|_{L_{x,p}^\infty} \leqslant C\sum_{|\alpha_0|\leqslant 2}\big\|\big(\Delta_k\partial_x^{\alpha_0}F_+\big)\big(1+|p|^2\big)^5\big\|_{L_{x,p}^2} \leqslant CK_0,$$

so that we can obtain

$$\left|\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} B_2 g_{\delta,k}\,dt\,dx\,dp\,dk\right| \leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).$$

As for the term containing V_4, we use (3.26), integration by parts and the following inequality

$$\|\Delta_k b_i^+\|_{L^\infty_{x,p}} \leqslant C\sum_{|\alpha_0|\leqslant 2}\big\|\big(\Delta_k\partial_x^{\alpha_0}\partial_{p_j}F_+\big)\big(1+|p|^2\big)^5\big\|_{L^2_{x,p}} \leqslant CK_0,$$

so that we can obtain

$$\left|\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} V_4 g_{\delta,k}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\right| \leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).$$

As for the terms containing VI_1, we write

$$\begin{aligned}
&\partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}}\\
&\quad=\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+(x+k)\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}}\\
&\qquad+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}},
\end{aligned}$$

and we denote it by D_1+D_2. By integration by parts,

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} D_1 g_{\delta,k}\,dt\,dx\,dp\,dk\\
&\quad=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+(x+k)\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)(\partial_{p_i}g_{\delta,k})\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\\
&\quad=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+(x+k)\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(\partial_{p_i}(\Delta_k g)\big)\big(1+|p|^2\big)^{s}|k|^{-2\delta-3}\,dt\,dx\,dp\,dk\\
&\qquad-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+(x+k)\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)(\Delta_k g)\big(1+|p|^2\big)^{s-1}p_i|k|^{-2\delta-3}\,dt\,dx\,dp\,dk.
\end{aligned}$$

We write the r.h.s. of the above equality as $D_{1,1}+D_{1,2}$. When $|\alpha_1|+|\beta_1|\leqslant N/2$, Lemma 2.2 ensures that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^+(x+k)\big\|_{L^\infty_t([0,T];L^\infty_{x,k})} \leqslant \sum_{|\alpha_0|\leqslant 2,\,\beta_0\leqslant\beta_1}\big\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}F_+\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^\infty_t([0,T];L^2_{x,p})}.$$

Thanks to (3.26), we get

$$\begin{aligned}|D_{1,1}| &\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}\\&\quad + C_\epsilon\big\|\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}\big)\\&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})} + C_\epsilon\tilde{C}_2\big).\end{aligned}$$

When $|\alpha_1|+|\beta_1|>N/2$, we obtain from Lemma 2.2 that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a^+_{ij}(t,x,p)\big\|_{L^\infty_p}\leqslant C\sum_{\beta_0\leqslant\beta_1}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^2_p},$$

so that

$$\begin{aligned}|D_{1,1}| &\leqslant C\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{T}^3}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}a^+_{ij}(x+k)\big\|_{L^\infty_p}\big\|\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_p}\\&\quad\times\big\|\big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-2\delta-3}\big\|_{L^2_p}\,dt\,dx\,dk\\&\leqslant C\sum_{\beta_0\leqslant\beta_1}\int\limits_{[t_0,T]\times\mathbf{T}^3}\big\|\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^\infty_x(L^2_p)}\\&\quad\times\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^2_{x,p}}\big\|\big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-2\delta-3}\big\|_{L^2_{x,p}}\,dt\,dk\\&\leqslant C\sum_{|\alpha_0|\leqslant 2,\,\beta_0\leqslant\beta_1}\int\limits_{[t_0,T]}\big\|\big(\partial_{p_j}\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_{x,p,k}}\\&\quad\times\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^2_{x,p}}\big\|\big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-2\delta-3}\big\|_{L^2_{x,p,k}}\,dt\\&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})} + C_\epsilon\tilde{C}_2\big).\end{aligned}$$

Thus in any case we get

$$|D_{1,1}|\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})} + C_\epsilon\tilde{C}_2\big). \tag{3.30}$$

Similarly, we can obtain

$$|D_{1,2}|\leqslant CK_0\big(K_0^2T+\tilde{C}_2\big).$$

As for D_2, by integration by parts, we get

$$\begin{aligned}&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} D_2 g_{\delta,k}\,dt\,dx\,dp\,dk\\&\quad= -\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k a^+_{ij}\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)(\partial_{p_i}g_{\delta,k})\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\\&\quad= -\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k a^+_{ij}\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}\Delta_k g)\big(1+|p|^2\big)^{s}|k|^{-2\delta-3}\,dt\,dx\,dp\,dk\end{aligned}$$

$$- s \int_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k a_{ij}^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\Delta_k g)\big(1+|p|^2\big)^{s-1}p_i|k|^{-2\delta-3}\,dt\,dx\,dp\,dk.$$

We denote the r.h.s. of the above equality by $D_{2,1}+D_{2,2}$. When $|\alpha_1|+|\beta_1|\leqslant N/2$, Lemma 2.2 ensures that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}\big(\Delta_k a_{ij}^+\big)\big\|_{L^\infty_{x,p}} \leqslant \sum_{|\alpha_0|\leqslant 2,\ \beta_0\leqslant\beta_1} \big\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}(\Delta_k F_+)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^2_{x,p}}.$$

Thanks to the above inequality we can obtain

$$\begin{aligned}
|D_{2,1}| &\leqslant C \int_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}\big(\Delta_k a_{ij}^+\big)\big\|_{L^\infty_x} \\
&\quad\times \big\|\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big\|_{L^2_x}\big\|(\partial_{p_i}\Delta_k g)\big\|_{L^2_x}\big(1+|p|^2\big)^s|k|^{-2\delta-3}\,dt\,dp\,dk \\
&\leqslant C \sum_{|\alpha_0|\leqslant 2,\ \beta_0\leqslant\beta_1} \int_{[t_0,T]\times\mathbf{T}^3} \big\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}(\Delta_k F_+)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_{x,p}} \\
&\quad\times \big\|\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big(1+|p|^2\big)^{s/2}\big\|_{L^2_{x,p}}\big\|(\partial_{p_i}\Delta_k g)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_{x,p}}\,dt\,dk \\
&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).
\end{aligned}$$

When $|\alpha_1|+|\beta_1|>N/2$, we obtain from Lemma 2.2 that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}\big(\Delta_k a_{ij}^+\big)\big\|_{L^\infty_p} \leqslant C\sum_{\beta_0\leqslant\beta_1} \big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}(\Delta_k F_+)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L^2_p}.$$

Thanks to the above inequality we can obtain

$$\begin{aligned}
|D_{2,1}| &\leqslant C \int_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}\big(\Delta_k a_{ij}^+\big)\big\|_{L^\infty_p} \\
&\quad\times \big\|\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big(1+|p|^2\big)^{s/2}\big\|_{L^2_p}\big\|(\partial_{p_i}\Delta_k g)\big(1+|p|^2\big)^{s/2}\big\|_{L^2_p}|k|^{-2\delta-3}\,dt\,dx\,dk \\
&\leqslant C \sum_{|\alpha_0|\leqslant 2,\ \beta_0\leqslant\beta_1} \int_{[t_0,T]\times\mathbf{T}^3} \big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}(\Delta_k F_+)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_{x,p}} \\
&\quad\times \big\|\partial_{p_j}\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}F_+\big(1+|p|^2\big)^{s/2}\big\|_{L^2_{x,p}}\big\|(\partial_{p_i}\Delta_k g)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L^2_{x,p}}\,dt\,dk \\
&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).
\end{aligned}$$

Similarly we can get

$$|D_{2,2}| \leqslant CK_0\big(K_0^2T+\tilde{C}_2\big).$$

Finally the term VI_1 has the following estimate

$$\left| \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} VI_1 g_{\delta,k}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\right| \leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\big(K_0^2T+\tilde{C}_2\big)\big).$$

As for the terms containing VI_3, we write

$$\begin{aligned}
&\partial_{p_i}\big[\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}}\\
&\quad=\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_kF_+\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}}\\
&\qquad+\partial_{p_i}\big[\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_kb_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{s/2}\big]|k|^{-\delta-\frac{3}{2}},
\end{aligned}$$

and we denote it by E_1+E_2. By integration by parts,

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} E_1 g_{\delta,k}\,dt\,dx\,dp\,dk\\
&=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_kF_+\big)(\partial_{p_i}g_{\delta,k})\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\\
&=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_kF_+\big)\big(\partial_{p_i}(\Delta_kg)\big)\big(1+|p|^2\big)^{s}|k|^{-2\delta-3}\,dt\,dx\,dp\,dk\\
&\quad-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_kF_+\big)(\Delta_kg)\big(1+|p|^2\big)^{s-1}p_i|k|^{-2\delta-3}\,dt\,dx\,dp\,dk.
\end{aligned}$$

We write the r.h.s. of the above equality as $E_{1,1}+E_{1,2}$. When $|\alpha_1|+|\beta_1|\leqslant N/2$, Lemma 2.2 ensures that

$$\begin{aligned}
&\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big\|_{L_t^\infty([0,T];L^\infty_{x,k,p})}\\
&\quad\leqslant\sum_{|\alpha_0|\leqslant 2,\,|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}F_+\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_t^\infty([0,T];L^2_{x,p})}\leqslant CK_0.
\end{aligned}$$

Thanks to (3.26) and the Hölder inequality, we get

$$|E_{1,1}|\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_kg)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).$$

When $N/2<|\alpha_1|+|\beta_1|\leqslant N-1$, we obtain from Lemma 2.2 that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big\|_{L_p^\infty}\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2},$$

so that

$$
\begin{aligned}
|E_{1,1}| &\leqslant C \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{T}^3} \big\| \partial_x^{\alpha_1}\partial_p^{\beta_1} b_i^+(x+k)\big\|_{L_p^\infty} \big\| \big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_p^2} \\
&\quad \times \big\| \big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_p^2}\, dt\, dx\, dk \\
&\leqslant C \sum_{|\beta_0|\leqslant|\beta_1|+1} \int\limits_{[t_0,T]\times\mathbf{T}^3} \big\| \big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_x^\infty(L_p^2)} \\
&\quad \times \big\| \partial_x^{\alpha_1}\partial_p^{\beta_0} F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_{x,p}^2} \big\| \big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_{x,p}^2}\, dt\, dk \\
&\leqslant C \sum_{|\alpha_0|\leqslant 2,\ |\beta_0|\leqslant|\beta_1|+1} \int\limits_{[t_0,T]} \big\| \big(\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_{x,p,k}^2} \\
&\quad \times \big\| \partial_x^{\alpha_1}\partial_p^{\beta_0} F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_{x,p}^2} \big\| \big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s}|k|^{-\delta-\frac{3}{2}}\big\|_{L_{x,p,k}^2}\, dt \\
&\leqslant CK_0\big(\epsilon \big\| \big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L_t^2([t_0,T];L_{x,p,k}^2)} + C_\epsilon \tilde{C}_2\big).
\end{aligned}
$$

When $|\alpha_1|+|\beta_1|=N$ with $|\beta_1|>0$, we obtain from Lemma 2.3 that

$$
\begin{aligned}
\big|\partial_x^{\alpha_1}\partial_p^{\beta_1} b_i^+(x+k)\big| &\leqslant C \sum_{|\beta_0|\leqslant|\beta_1|-1} \Big(\big|\partial_x^{\alpha_1}\partial_p^{\beta_0} F_+(x+k)\big| \\
&\quad + p_0^{1/2-|\beta_0|} \big\| \partial_x^{\alpha}\partial_p^{\beta_0} F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2}\Big).
\end{aligned}
$$

By using the above inequality, we have

$$
\begin{aligned}
&\bigg| \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \big(\partial_x^{\alpha_1}\partial_p^{\beta_1} b_i^+(x+k)\big)(\Delta_k F_+)\big(\partial_{p_i}(\Delta_k g)\big)\big(1+|p|^2\big)^{s}|k|^{-\delta-\frac{3}{2}}\, dt\, dx\, dp\, dk\bigg| \\
&\quad \leqslant C \sum_{|\beta_0|\leqslant|\beta_1|-1} \Bigg(\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \big|\partial_x^{\alpha_1}\partial_p^{\beta_0} F_+(x+k)\big| \big|(\Delta_k F_+)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-3/2}\big| \\
&\qquad \times \big|\big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big|\, dt\, dx\, dp\, dk \\
&\qquad + C \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} p_0^{1/2} \big\| \partial_x^{\alpha}\partial_p^{\beta_0} F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2} \\
&\qquad \times \big|(\Delta_k F_+)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-3/2}\big| \big|\big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big|\, dt\, dx\, dp\, dk \\
&\quad \leqslant C \sum_{|\bar\alpha|+|\bar\beta|\leqslant 4,\ |\beta_0|\leqslant|\beta_1|-1} \big\| \partial_x^{\alpha_1}\partial_p^{\beta_0} F_+\big\|_{L_t^\infty([t_0,T];L_{x,p}^2)} \big\| \partial_x^{\bar\alpha}\partial_p^{\bar\beta}(\Delta_k F_+)\big(1+|p|^2\big)^{s/2} \\
&\qquad \times |k|^{-\delta-\frac{3}{2}}\big\|_{L_t^2([t_0,T];L_{x,p,k}^2)} \big\| \big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_t^2([t_0,T];L_{x,p,k}^2)} \\
&\qquad + C \sum_{|\beta_1|\leqslant|\beta|-1} \int\limits_{[t_0,T]\times\mathbf{T}^3} \big\| \partial_x^{\alpha}\partial_p^{\beta_1} F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_{x,p}^2} \\
&\qquad \times \big\| (\Delta_k F_+)\big(1+|p|^2\big)^{(1+s)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_x^\infty(L_p^2)} \big\| \big(\nabla_p(\Delta_k g)\big)\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_{x,p}^2}\, dt\, dk\Bigg)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big)\\
&\quad+CK_0\bigg(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}\\
&\quad+C_\epsilon\sum_{|\bar{\alpha}|\leqslant 2}\big\|\partial^{\bar{\alpha}}(\Delta_k F_+)\big(1+|p|^2\big)^{(1+s)/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2([0,T];L^2_{x,p,k})}\bigg)\\
&\leqslant \tilde{C}_2C_\epsilon K_0+\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}.
\end{aligned}$$

Here we have used (3.26) and the Sobolev embedding inequality:

$$\big\|F_+\big(1+|p|^2\big)^{s/2}\big\|_{L^\infty_t([0,T];L^\infty_{x,p})}\leqslant C\big\|F_+\big(1+|p|^2\big)^{s/2}\big\|_{L^\infty_t([0,T];H^4_{x,p})}\leqslant CK_0.$$

Thus in any case we get

$$|E_{1,1}|\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big). \tag{3.31}$$

Similarly, we can obtain

$$|E_{1,2}|\leqslant CK_0\big(K_0^2T+\tilde{C}_2\big). \tag{3.32}$$

As for E_2, by integration by parts, we get

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} E_2 g_{\delta,k}\,dt\,dx\,dp\,dk\\
&=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k b_i^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}g_{\delta,k})\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\,dt\,dx\,dp\,dk\\
&=-\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k b_i^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\partial_{p_i}\Delta_k g)\big(1+|p|^2\big)^{s}|k|^{-2\delta-3}\,dt\,dx\,dp\,dk\\
&\quad-s\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k b_i^+\big)\big(\partial_{p_j}\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)(\Delta_k g)\big(1+|p|^2\big)^{s-1}p_i|k|^{-2\delta-3}\,dt\,dx\,dp\,dk.
\end{aligned}$$

We denote the r.h.s. of the above equality by $E_{2,1}+E_{2,2}$. These two terms can be estimated in the same way. We do not detail the computation, but we should notice that the cases $|\alpha_1|+|\beta_1|\leqslant N/2$, $N/2<|\alpha_1|+|\beta_1|\leqslant N-1$ and $|\alpha_1|+|\beta_1|=N$ are considered separately. As a consequence,

$$|E_2|\leqslant CK_0\big(\epsilon\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}+C_\epsilon\tilde{C}_2\big).$$

Finally, we can obtain that

$$\big\|\big[\nabla_p(\Delta_k g)\big]\big(1+|p|^2\big)^{s/2}|k|^{-\delta-\frac{3}{2}}\big\|^2_{L^2_t([t_0,T];L^2_{x,p,k})}\leqslant\frac{C_\epsilon\tilde{C}_2K_0+CK_0^3T}{K-C\epsilon}. \tag{3.33}$$

If we take $\epsilon>0$ small enough, we can prove (3.23).

In order to prove (3.22), we will use the method of the proof of Lemma 3.2. We introduce $\rho_{\delta,k} = g_{\delta,k}(1+|p|^2)^2$ and write

$$\hat{\rho}_{\delta,k}(t,m,p) = \left[\hat{\rho}_{\delta,k}(t,m,p) - \left(\hat{\rho}_{\delta,k}(t,m,\cdot) *_p \chi_\epsilon\right)(p)\right] + \left(\hat{\rho}_{\delta,k}(t,m,\cdot) *_p \chi_\epsilon\right)(p),$$

where $\hat{\rho}(t,m,p)$ is the Fourier coefficient of ρ with respect to the x variable and the parameter ϵ will be chosen.

Following the proof of (3.19), we can obtain

$$\int\limits_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \left(1+|p|\right)^{-4}\left(\sum_{m\in\mathbf{Z}^3} |m|^{1/10}\left|\hat{\rho}_{\delta,k}(t,m,p) - \left(\hat{\rho}(t,m,\cdot) *_p \chi_\epsilon\right)(p)\right|^2\right) dt\,dp\,dk$$
$$\leqslant C \int\limits_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \sum_{m\in\mathbf{Z}^3} |m|^{1/10}\epsilon^2\left|\nabla_p\hat{\rho}(t,m,p)\right|^2 dt\,dp\,dk. \tag{3.34}$$

Notice that $\rho_{\delta,k}$ is the solution of Eq. (3.20) where s is replaced by $s+4$. Thus we have

$$\partial_t\rho_{\delta,k} + \frac{p}{p_0}\cdot\rho_{\delta,k} = \left(\rho^{(1)}_{\delta,k} + \nabla_p\cdot\rho^{(2)}_{\delta,k}\right)|k|^{-\delta-\frac{3}{2}}. \tag{3.35}$$

Here $\nabla_p\cdot\rho^{(2)}_{\delta,k}$ is the sum of the terms IV_2, V_i $(i=1,3,4,6,8,10)$ and VI_i $(i=1,3,4,6)$, while $\rho^{(1)}_{\delta,k}$ is the sum of the remaining terms.

We claim that $\rho^{(1)}_{\delta,k}|k|^{-\delta-\frac{3}{2}}$, $\rho^{(2)}_{\delta,k}|k|^{-\delta-\frac{3}{2}} \in L^2_t([t_0,T]; L^2_{x,p,k})$. We only present the estimates of IV_2 and VI_3 in the case $|\beta|\geqslant 1$ here, the others being similar. We write

$$\Delta_k\left(\partial_x^{\alpha_1}E\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right) = \partial_x^{\alpha_1}E(x+k)\Delta_k\left(\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right) + \left(\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right)\Delta_k\left(\partial_x^{\alpha_1}E\right).$$

We only consider the first part of the r.h.s. of the above equation also because the second part can be estimated in the same way. Then when $|\alpha_1|\leqslant N/2$, we know that

$$\left\|\partial_x^{\alpha_1}E(x+k)\right\|_{L^\infty([t_0,T];L^\infty_{x,k})} \leqslant C\left\|\partial_x^{\alpha_1}E(x+k)\right\|_{L^\infty([t_0,T];L^\infty_k(H^2_x))} \leqslant C\epsilon.$$

By using (3.26) and the above inequality we can obtain

$$\left\|\partial_x^{\alpha_1}E(x+k)\Delta_k\left(\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right)\left(1+|p|^2\right)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\right\|_{L^2_t([t_0,T];L^2_{x,p,k})} \leqslant C\sqrt{\tilde{C}_2}.$$

When $|\alpha_1| > N/2$, we can obtain

$$\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \left|\partial_x^{\alpha_1}E(x+k)\Delta_k\left(\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right)\left(1+|p|^2\right)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\right|^2 dt\,dx\,dp\,dk$$
$$\leqslant \int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3} \left\|\partial_x^{\alpha_1}E(x+k)\right\|_{L^2_x}\left\|\Delta_k\left(\partial_x^{\alpha-\alpha_1}\partial_p^\beta F_+\right)\left(1+|p|^2\right)^{s/2}|k|^{-\delta-\frac{3}{2}}\right\|_{L^\infty_x(L^2_p)} dt\,d\,dk$$
$$\leqslant \left\|\partial_x^{\alpha_1}E(x+k)\right\|^2_{L^\infty([t_0,T];L^2_x)}\left(\sum_{|\alpha_0|\leqslant 2}\left\|\Delta_k\left(\partial_x^{\alpha-\alpha_1+\alpha_0}\partial_p^\beta F_+\right)\left(1+|p|^2\right)^{s/2}|k|^{-\delta-\frac{3}{2}}\right\|^2_{L^2_t([t_0,T];L^2_{p,k,x})}\right).$$

Now we consider the term VI_3. We write

$$\begin{aligned}
&\Delta_k\big(\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\\
&\quad=\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\\
&\qquad+\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}\Delta_k b_i^+\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}},
\end{aligned}$$

and we only estimate the first term of r.h.s. of the above equality, the second being similar. When $|\alpha_1|+|\beta_1|\leqslant N/2$, Lemma 2.2 ensures that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big\|_{L_t^\infty([0,T];L_{x,k}^\infty)}\leqslant\sum_{|\alpha_0|\leqslant 2,\,|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_0+\alpha_1}\partial_p^{\beta_0}F_+\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_t^\infty([0,T];L_{x,p}^2)}.$$

Thanks to (3.26), we get

$$\big\|\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_t^2([t_0,T];L_{x,p,k}^2)}\leqslant CK_0\sqrt{\tilde{C}_2}.$$

When $N/2<|\alpha_1|+|\beta_1|\leqslant N-1$, we obtain from Lemma 2.2 that

$$\big\|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big\|_{L_p^\infty}\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2},$$

so that

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big|\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\big|^2\,dt\,dx\,dp\,dk\\
&\quad\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\int\limits_{[t_0,T]\times\mathbf{T}^3}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_{x,p}^2}^2\\
&\qquad\times\big\|\big(\partial_x^{\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_x^\infty(L_p^2)}^2\,dt\,dx\,dk\\
&\quad\leqslant C\sum_{|\alpha_0|\leqslant 2,\,|\beta_0|\leqslant|\beta_1|+1}\big\|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_t^\infty([t_0,T];L_{x,p}^2)}^2\\
&\qquad\times\big\|\big(\partial_x^{\alpha_0+\alpha_2}\partial_p^{\beta_2}\Delta_k F_+\big)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\big\|_{L_t^2([t_0,T];L_{x,p,k}^2)}^2\leqslant CK_0\tilde{C}_2.
\end{aligned}$$

When $|\alpha_1|+|\beta_1|=N$ with $|\beta_1|>0$, we obtain from Lemma 2.3 that

$$\begin{aligned}
\big|\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big|\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|-1}&\Big(\big|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big|\\
&+p_0^{1/2-|\beta_0|}\big\|\partial_x^{\alpha}\partial_p^{\beta_0}F_+(x+k)\big(1+|p|^2\big)^{(|\beta_1|+10)/2}\big\|_{L_p^2}\Big).
\end{aligned}$$

By the above inequality, we have

$$\begin{aligned}
&\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big|\big(\partial_x^{\alpha_1}\partial_p^{\beta_1}b_i^+(x+k)\big)(\Delta_k F_+)\big(1+|p|^2\big)^{(s+4)/2}|k|^{-\delta-\frac{3}{2}}\big|^2\,dt\,dx\,dp\,dk\\
&\quad\leqslant C\sum_{|\beta_0|\leqslant|\beta_1|-1}\Bigg(\int\limits_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3}\big|\partial_x^{\alpha_1}\partial_p^{\beta_0}F_+(x+k)\big|^2\big|(\Delta_k F_+)\big(1+|p|^2\big)^{(s+4)}\big||k|^{-2\delta-3}\,dt\,dx\,dp\,dk
\end{aligned}$$

$$+ C \int_{[t_0,T]\times\mathbf{T}^3\times\mathbf{R}^3\times\mathbf{T}^3} \left\| \partial_x^\alpha \partial_p^{\beta_0} F_+(x+k)\left(1+|p|^2\right)^{(|\beta_1|+10)/2} \right\|_{L_p^2}$$

$$\times \left|(\Delta_k F_+)\left(1+|p|^2\right)^{(s+5)}\right| |k|^{-2\delta-3}\, dt\, dx\, dp\, dk \Big)$$

$$\leqslant C \sum_{|\bar\alpha|+|\bar\beta|\leqslant 4,\ |\beta_0|\leqslant|\beta_1|-1} \left\| \partial_x^{\alpha_1} \partial_p^{\beta_0} F_+ \right\|^2_{L^\infty([t_0,T];L^2_{x,p})}$$

$$\times \left\| \partial_x^{\bar\alpha} \partial_p^{\bar\beta} (\Delta_k F_+)\left(1+|p|^2\right)^{(s+5)/2} |k|^{-\delta-\frac{3}{2}} \right\|^2_{L^2([0,T];L^2_{x,p,k})}$$

$$+ C \sum_{|\beta_1|\leqslant|\beta|-1} \int_{[0,T]\times\mathbf{T}^3} \left\| \partial_x^\alpha \partial_p^{\beta_1} F_+(x+k)\left(1+|p|^2\right)^{(|\beta_1|+10)/2} \right\|^2_{L^2_{x,p}}$$

$$\times \left\| (\Delta_k F_+)\left(1+|p|^2\right)^{(s+5)/2} |k|^{-\delta-\frac{3}{2}} \right\|^2_{L_x^\infty(L_p^2)} dt\, dk$$

$$\leqslant C K_0^2 \tilde C_2 + C K_0^2 \sum_{|\bar\alpha|\leqslant 2} \left\| \partial_x^{\bar\alpha} (\Delta_k F_+)\left(1+|p|^2\right)^{(s+5)/2} |k|^{-\delta-\frac{3}{2}} \right\|^2_{L^2([0,T];L^2_{x,p,k})}.$$

Here we have used (3.26) and the Sobolev embedding inequality:

$$\left\| F_+\left(1+|p|^2\right)^{s/2} \right\|_{L_t^\infty([0,T];L^\infty_{x,p})} \leqslant C \left\| F_+\left(1+|p|^2\right)^{s/2} \right\|_{L_t^\infty([0,T];H^4_{x,p})} \leqslant C K_0.$$

Proceeding like in the estimate for (3.21), we can get

$$|m|^{1/2} \int_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \left(1+|p|^2\right)^{-4} \sum_{m\in\mathbf{Z}^3} |m|^{2\delta} \left| \hat\rho_{\delta,k}(t,m,p) *_p \chi(p) \right|^2 dt\, dp\, dk$$

$$\leqslant C \sum_{m\in\mathbf{Z}^3} |m|^{2\delta} \left(\epsilon^{-6}+\epsilon^{-8}\right)\Big(\left\| \hat\rho_{\delta,k}(0,m,\cdot) \right\|^2_{L_p^2} + \left\| \hat\rho_{\delta,k}(\cdot,m,\cdot) \right\|^2_{L_t^2([0,T];L^2_{p,k})}$$

$$+ \left\| \hat\rho^{(1)}_{\delta,k}(\cdot,m,\cdot) \right\|^2_{L_t^2([0,T];L^2_{p,k})} + \left\| \hat\rho^{(1)}_{\delta,k}(\cdot,m,\cdot) \right\|^2_{L_t^2([0,T];L^2_{p,k})}\Big). \tag{3.36}$$

Choosing $\epsilon = |m|^{-\delta}$ and using (3.34) and (3.23), we get

$$\int_{[t_0,T]\times\mathbf{R}^3\times\mathbf{T}^3} \left(1+|p|^2\right)^{-4} \sum_{m\in\mathbf{Z}^3} |m|^{2\delta} \left| \hat\rho_{\delta,k}(t,m,p) \right|^2 dt\, dp\, dk \leqslant \tilde C_3,$$

which implies the estimate (3.22). This ends the proof of Lemma 3.3. □

Now if iterating Lemma 3.3 twenty times, we end up with a time $\tau_1 > 0$ small enough such that $\|F\|_{L^2([\tau_1,T];H^{N+1,s}_{x,p})} < \infty$. We need to transform L^2 into L^∞ estimate. For this, we consider $\bar\rho = (\partial_x^\alpha \partial_p^\beta F)(1+|p|^2)^{s/2}$ for $|\alpha|+|\beta| \leqslant N+1$.

From the Maxwell system (1.3) and (1.4), we have

$$\int_{\mathbf{R}^3\times\mathbf{T}^3} \partial_x^\alpha E \cdot \frac{p}{p_0} \left\{ \partial_x^\alpha F_+ - \partial_x^\alpha F_- \right\} dx\, dp = \frac{d}{dt} \left\{ \left\| \partial_x^\alpha E \right\|^2_{L_x^2} + \left\| \partial_x^\alpha B \right\|^2_{L_x^2} \right\}.$$

Then for any $|\alpha| \leqslant N+1$ and $t \in [\tau_1, T]$, we can obtain from (1.11) that

$$\begin{aligned}&\|\partial_x^\alpha E(t)\|^2_{L_t^\infty([\tau_1,T];L^2_{x,p})}+\|\partial_x^\alpha B(t)\|^2_{L_t^\infty([\tau_1,T];L^2_{x,p})}\\&\quad\leqslant C\|\partial_x^\alpha E(\tau_1)\|^2_{L^2_x}+\|\partial_x^\alpha B(\tau_1)\|^2_{L^2_x}+CT\|\{\partial_x^\alpha F_+-\partial_x^\alpha F_-\}(1+|p|^2)^4\|^2_{L^\infty([\tau_1,T];L^2_{x,p})}\leqslant\tilde{C}_5.\end{aligned}\tag{3.37}$$

Remark 3.4. If $E(t,x)=\nabla_x\phi(t,x)$ and $B(t,x)=0$, we use the spatially periodic condition to obtain

$$\sum_{|\alpha_1|\leqslant|\alpha|}\|\partial_x^{\alpha_1}\nabla_x\phi(t)\|_{L_t^\infty([\tau_1,T];L^2_x)}\leqslant C\sum_{|\alpha_1|\leqslant|\alpha|-1}\|\partial_x^{\alpha_1}F(t)\|_{L_t^\infty([\tau_1,T];L^2_{x,p})},\tag{3.38}$$

$$\|\nabla_x\phi(t)\|_{L_t^\infty([\tau_1,T];L^2_x)}\leqslant C\|F\|_{L_t^\infty([\tau_1,T];L^2_{x,p})}.\tag{3.39}$$

By the similar arguments as Lemma 3.1, we easily know

$$\nabla_p\bar{\rho}\in L_t^2([\tau_1,T];L^2_{x,p}).$$

We claim that $\bar{\rho}$ satisfies an equation of the type

$$\partial_t\bar{\rho}+\frac{p}{p_0}\cdot\nabla_x\bar{\rho}=\bar{\rho}_1+\nabla_p\cdot\bar{\rho}_2,$$

with $\bar{\rho}_1,\bar{\rho}_2\in L_t^2([\tau_1,T];L^2_{x,p})$. Indeed, since we know that $[E,B]\in L_t^\infty([\tau_1,T];H_x^{N+1})$, $F\in L_t^\infty([0,T];H^{N,s}_{x,p})$ and $L_t^2([\tau_1,T];H^{N+1,s}_{x,p})$ and $\nabla_p\bar{\rho}\in L_t^2([\tau_1,T];L^2_{x,p})$, we can follow the proof of the fact that ρ_1 and ρ_2 (in (3.20)) belong to $L_t^2([\tau_1,T];L^2_{x,p})$.

Multiplying now the above equation by $\bar{\rho}$, and integrating on $[t_*,T]\times\mathbf{T}^3\times\mathbf{R}^3$, where $t_*>\tau_1$ small enough such that $\bar{\rho}(t_*,\cdot,\cdot)\in L^2_{x,p}$, we get

$$\begin{aligned}\frac{1}{2}\left(\|\bar{\rho}(t)\|^2_{L^2_{x,p}}-\|\bar{\rho}(t_*)\|^2_{L^2_{x,p}}\right)&\leqslant\|\bar{\rho}_1\|_{L_t^2([t_*,t];L^2_{x,p})}\|\bar{\rho}\|_{L_t^2([t_*,t];L^2_{x,p})}\\&\quad+\|\bar{\rho}_2\|_{L_t^2([t_*,t];L^2_{x,p})}\|\nabla_p\bar{\rho}\|_{L_t^2([t_*,t];L^2_{x,p})}.\end{aligned}$$

We thus obtain $\bar{\rho}\in L_t^\infty([t_*,t];L^2_{x,p})$.

By Lemmas 3.1–3.3 and the above arguments, we can have the following proposition:

Proposition 3.5. *Let $N\geqslant 4$ be a given integer and $F=[F_+,F_-]$ be a smooth non-negative solution of the system* (1.1)–(1.4) *given by Theorem* 1.1. *We suppose that for any $T>0$, $s\geqslant 0$, $\|F\|_{L_t^\infty([0,T];H^{N,s}_{x,p})}\leqslant K_0$, where $K_0=K_0(s,T)$ is a constant. Then for any $T>0$, $t_*\in[0,T]$ and $s\geqslant 0$, there exists a constant $\tilde{C}_6>0$, which depends on N, s, T, t_* and K_0, such that*

$$\|F\|_{L_t^\infty([t_*,T];H^{N+1,s}_{x,p})}\leqslant\tilde{C}_6.\tag{3.40}$$

Proof of Theorem 1.2. By using Proposition 3.5 and the assumption (1.11) repeatedly, we get by induction of N that for any $0<\tau<T<\infty$ and $s\geqslant 0$,

$$f\in L_t^\infty([\tau,T];H^{\infty,s}_{x,p}).$$

We now prove by induction on n that $\partial_t^nF\in L_t^\infty([\tau,T];H^{\infty,s}_{x,p})$. By the above, this is true for $n=0$. Let us assume that the induction hypothesis holds for any integer $k\leqslant n$. Then for all multi-indices α and β, any $s\geqslant 0$,

$$\begin{aligned}
&\big[\partial_x^\alpha \partial_p^\beta \big(\partial_t^{n+1} F_+\big)\big]\big(1+|p|^2\big)^{s/2} + \left(\partial_x^\alpha \partial_p^\beta \left[\frac{p}{p_0}\cdot \nabla_x\big(\partial_t^n F_+\big)\right]\right)\big(1+|p|^2\big)^{s/2} \\
&\quad + \sum_{\alpha_1+\alpha_2=\alpha,\, \beta_1+\beta_2=\beta} \sum_{l=0}^{n} C_\alpha^{\alpha_1} C_\beta^{\beta_1} C_n^l \nabla_p \cdot \bigg(\partial_t^l \partial_x^{\alpha_1} E \partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^\beta F_+ \\
&\quad + \partial_p^{\beta_1}\left(\frac{p}{p_0}\right) \times \partial_t^l \partial_x^{\alpha_1} B \partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} F_+\bigg)\big(1+|p|^2\big)^{s/2} \\
&= \sum_{\alpha_1+\alpha_2=\alpha,\, \beta_1+\beta_2=\beta} \sum_{l=0}^{n} C_\alpha^{\alpha_1} C_\beta^{\beta_1} C_n^l \Big[\big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} a_{ij}^+\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_i} \partial_{p_j} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad + \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} \partial_{p_i} a_{ij}^+\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_j} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad - \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} b_i^+\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_i} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad - \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} \partial_{p_i} b_i^+\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad + \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} a_{ij}^-\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_i} \partial_{p_j} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad + \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} \partial_{p_i} a_{ij}^-\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_j} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad - \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} b_i^-\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} \partial_{p_i} F_+\big)\big(1+|p|^2\big)^{s/2} \\
&\quad - \big(\partial_t^l \partial_x^{\alpha_1} \partial_p^{\beta_1} \partial_{p_i} b_i^-\big)\big(\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^{\beta_2} F_+\big)\big(1+|p|^2\big)^{s/2}\Big].
\end{aligned} \tag{3.41}$$

By (1.11), (3.41), $\partial_t^n F \in L_t^\infty([\tau,T]; H_{x,p}^{\infty,s})$ and the Maxwell system (1.3) and (1.4), we get, for any multi-index α_0 and $\tau_1 \leqslant \tau \leqslant t \leqslant T$, that

$$\big\|\partial_t \partial_x^{\alpha_0} B\big\|_{L_t^\infty([\tau,T];L_x^2)} \leqslant \big\|\nabla_x \partial_x^{\alpha_0} E\big\|_{L_t^\infty([\tau,T];L_x^2)} \leqslant \tilde{C}_7,$$

$$\begin{aligned}
&\big\|\partial_t \partial_x^{\alpha_0} E\big\|_{L_t^\infty([\tau,T];L_x^2)} \\
&\quad \leqslant \big\|\nabla_x \partial_x^{\alpha_0} B\big\|_{L_t^\infty([\tau,T];L_x^2)} + C\big\|\big\{\partial_x^{\alpha_0} F_+ - \partial_x^{\alpha_0} F_-\big\}\big(1+|p|^2\big)^4\big\|_{L_t^\infty([\tau,T];L_{x,p}^2)} \leqslant \tilde{C}_7.
\end{aligned}$$

By the above inequalities, we can obtain, for any integer l and any α_0, that

$$\big\|\partial_t^l \partial_x^{\alpha_0}[E,B]\big\|_{L_t^\infty([\tau,T];L_x^2)} \leqslant \tilde{C}_8. \tag{3.42}$$

We first estimate the second term of the l.h.s. of (3.41). We see that

$$\left\|\left(\partial_x^\alpha \partial_p^\beta \left[\frac{p}{p_0}\cdot \nabla_x\big(\partial_t^n F_+\big)\right]\right)\big(1+|p|^2\big)^{s/2}\right\|_{L_t^\infty(L_{x,p}^2)} \leqslant C \sum_{\beta_1 \leqslant \beta} \big\|\partial_t^n \partial_x^\alpha \partial_p^{\beta_1} F_+\big(1+|p|^2\big)^{s/2}\big\|_{L_t^\infty(L_{x,p}^2)}.$$

By the induction hypothesis the above term is bounded.

The third term of the l.h.s. of (3.41) can be bounded by

$$\begin{aligned}
&C\big\|\partial_t^l \partial_x^{\alpha_1}[E,B]\big\|_{L_t^\infty(L_x^\infty)} \big\|\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^\beta F_+\big\|_{L_t^\infty(L_{x,p}^2)} \\
&\quad \leqslant C\big\|\partial_t^l \partial_x^{\alpha_1}[E,B]\big\|_{L_t^\infty(H_x^2)} \big\|\partial_t^{n-l} \partial_x^{\alpha_2} \partial_p^\beta F_+\big\|_{L_t^\infty(L_{x,p}^2)}.
\end{aligned}$$

By the induction hypothesis and (3.42), we know the above term is bounded.

By Lemma 2.2 and the Sobolev embedding theorem, we know

$$\left\|\partial_t^l\partial_x^{\alpha_1}\partial_p^{\beta_1}a_{ij}^{\pm}\right\|_{L_t^\infty(L_{x,p}^\infty)} \leqslant C\sum_{\beta_0\leqslant\beta_1}\left\|\partial_t^l\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{\pm}\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_t^\infty(H_{x,p}^2)},$$

$$\left\|\partial_t^l\partial_x^{\alpha_1}\partial_p^{\beta_1}b_{i}^{\pm}\right\|_{L_t^\infty(L_{x,p}^\infty)} \leqslant C\sum_{|\beta_0|\leqslant|\beta_1|+1}\left\|\partial_t^l\partial_x^{\alpha_1}\partial_p^{\beta_0}F_{\pm}\left(1+|p|^2\right)^{(|\beta_1|+10)/2}\right\|_{L_t^\infty(H_{x,p}^2)}.$$

By the induction hypothesis, we know the above term is bounded. By using these two bounded terms, we can prove that the other terms of r.h.s. of (3.41) are in $L_t^\infty([\tau,T];L_{x,p}^2)$. Therefore, we know that for some constant $\tilde{C}_9>0$,

$$\left\|\left[\partial_x^\alpha\partial_p^\beta\left(\partial_t^{n+1}F_+\right)\right]\left(1+|p|^2\right)^{s/2}\right\|_{L_t^\infty(L_{x,p}^2)}\leqslant\tilde{C}_9.$$

This ends the proof of Theorem 1.2. □

Remark 3.6. By using (3.38) and (3.39), the similar arguments as the above imply that classical solutions to the relativistic Landau–Poisson system satisfy (1.10) without any extra assumption.

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