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Algebras associated to acyclic directed graphs

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ABSTRACT

We construct and study a class of algebras associated to generalized layered graphs, i.e. directed graphs with a ranking function on their vertices and edges. Each finite acyclic directed graph admits countably many structures of a generalized layered graph. We construct linear bases in such algebras and compute their Hilbert series. Our interest to generalized layered graphs and algebras associated to those graphs is motivated by their relations to factorizations of polynomials over noncommutative rings.

0. Introduction

By a generalized layered graph we mean a pair $\Gamma = (G, |.|)$ where $G = (V, E)$ is a directed graph and $|.| : V \to \mathbf{Z}_{\geqslant 0}$ satisfies $|v| > |w|$ whenever $v, w \in V$ and there is an edge $e \in E$ from v to w. We call $|.|$ the rank function of Γ. We write $l(e) = |v| - |w|$ and call this the length of the edge e. We will see that if G is any acyclic directed graph then there are countably many rank functions $|.|$ such that $(G, |.|)$ is a generalized layered graph.

In this paper we construct and study a class of algebras $A(\Gamma)$ associated to generalized layered graphs Γ. Generators of our algebras are elements $a_1(e), a_2(e), \dots, a_{l(e)}(e)$ associated to edges e of Γ.

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The relations are defined as follows. Let sequences of edges $e_1, e_2, \ldots, e_p$ and $f_1, f_2, \ldots, f_q$ define paths with the same end and the same origin. Then they define a relation given by the identity

$$U_{e_1}(\tau)U_{e_2}(\tau)\ldots U_{e_p}(\tau) = U_{f_1}(\tau)U_{f_2}(\tau)\ldots U_{f_q}(\tau),$$

where τ is a formal central variable and

$$U_e(\tau) = \tau^{l(e)} - a_1(e)\tau^{l(e)-1} + a_2(e)\tau^{l(e)-2} - \cdots \pm a_{l(e)}(e)$$

for any edge e of Γ. We will show that if $\Gamma = (G, |.|)$ the structure of $A(\Gamma)$ depends on the rank function $|.|$ as well as directed graph G.

Our interest to generalized layered graphs and algebras associated to those graphs is motivated by their relations to factorizations of polynomials over noncommutative rings. Let R be a unital algebra, $P(\tau)$ a monic polynomial over R, and $\mathcal{P}$ be a set of monic right divisors of $P(\tau)$, i.e. $\mathcal{P}$ consists of monic polynomials $Q(\tau) \in R[\tau]$ such that $P(\tau) = U(\tau)Q(\tau)$ for some $U(\tau) \in R[\tau]$. In papers [4–6,10] we studied subalgebras $R_{\mathcal{P}}$ of R generated by the coefficients of polynomials $U(\tau)$ for certain sets $\mathcal{P}$. These studies led us to our notion of universal algebras of type $R_{\mathcal{P}}$ defined via generalized layered graphs.

To any set $\mathcal{P}$ we associate a generalized layered graph $\Gamma(\mathcal{P})$ constructed in the following way. The vertices of rank m in Γ are polynomials $Q(\tau) \in \mathcal{P}$ of degree m. An edge e of length k goes from a vertex $Q_1(\tau)$ to a vertex $Q_2(\tau)$ if and only if $Q_1(\tau) = U(\tau)Q_2(\tau)$ where $U(\tau) \in R[\tau]$ is a polynomial of degree k. A set of polynomials $Q_j(\tau) \in \mathcal{P}$, $j = 1, 2, \ldots, p$, such that $Q_j(\tau) = U_j(\tau)Q_{j+1}(\tau)$ for $j = 1, 2, \ldots, p-1$ defines a path in $\Gamma(\mathcal{P})$ from vertex $Q_1(\tau)$ to a vertex $Q_p(\tau)$. If a set of polynomials $S_k(\tau)$, $k = 1, 2, \ldots, q$, defines another path (say $S_k(\tau) = W_k(\tau)S_{k+1}(\tau)$) in this graph with the same origin (i.e. $Q_1(\tau) = S_1(\tau)$) and the same end (i.e. $Q_p(\tau) = S_q(\tau)$) then $U_1(\tau)U_2(\tau)\ldots U_{p-1}(\tau) = W_1(\tau)W_2(\tau)\ldots W_{q-1}(\tau)$.

If $\mathcal{P}$ contains polynomials 1 and $P(\tau)$ then the graph $\Gamma(\mathcal{P})$ contains exactly one minimal vertex $1 \in R$ of rank 0 and exactly one maximal vertex $P(\tau)$ of rank $\deg P(\tau)$.

Note that there is a canonical homomorphism $\phi : A(\Gamma(\mathcal{P})) \to R$ defined in the following way. For any edge e in $\Gamma(\mathcal{P})$ of length k there are corresponding generators $a_1(e), a_2(e), \ldots, a_k(e)$ in $A(\Gamma(\mathcal{P}))$. Since the edge goes from a vertex associated to some polynomial $Q(\tau)$ to a vertex associated to some polynomial $W(\tau)$ there is a polynomial

$$S_e(\tau) = \tau^k - b_1\tau^{k-1} + b_2\tau^{k-2} - \cdots + (-1)^k b_k \in R[\tau]$$

such that $Q(\tau) = S_e(\tau)W(\tau)$. Then there is a unique ϕ satisfying $\phi(a_i(e)) = b_i$, $i = 1, 2, \ldots, k$. The image of ϕ is a subalgebra $R_{\mathcal{P}}$ of R which records information about factorizations of the initial polynomial $P(\tau)$.

This paper continues our investigations started in [4,5,8–10] where we defined and studied properties of the algebras $A(\Gamma)$ for layered graphs Γ, i.e. graphs with edges of length one. These algebras correspond to factorizations of polynomials into products of linear factors. The algebras $A(\Gamma)$ for layered graphs have a deep interesting structure. In this paper we show that many results about the algebras $A(\Gamma)$ for a layered graph Γ admit natural generalization to the much wider class of generalized layered graphs.

For reasons of clarity we have required that the polynomials in the foregoing discussion be monic. However, in the body of the paper, in order to be able to write inverses easily in terms of geometric series, we will work instead with polynomials with constant term 1.

The paper is organized as follows. In Section 1 we prove that any acyclic directed graph may be given the structure of a generalized layered graph in countably many ways and define the algebras $A(\Gamma)$ for generalized layered graphs. In Section 2 we construct a spanning set of monomials in the algebra $A(\Gamma)$. In Section 3 we prove that this spanning set is, in fact, a linear basis in $A(\Gamma)$. Our proof is based on the main result from [5] and an analysis of the behavior of algebras $A(\Gamma)$ under a natural operation on Γ (adding a vertex). We also study the behavior of these algebras under other operations (adding edges, etc.) and show that our construction gives a functor from the category of generalized layered graphs to the category of associative algebras. In Section 4 we compute Hilbert series for the algebras $A(\Gamma)$. This generalize the main result for layered graphs from [9]. In Section 5 we study the behavior of the Hilbert series under operations on graphs. In particular, we use these results to construct a natural family of noncommutative complete intersections (in the sense of [1,7]).

1. Basic definitions

Let F be a field and for any set S let $T(S)$ denote the free associative algebra on S over F.

Let $G=(V,E)$ be a directed graph. Thus V is a set whose elements are called the *vertices* of G, E is a set whose elements are called the *edges* of G, and there are two functions $h,t: E\to V$. For $e\in E, t(e)$ is called the *tail* of e and $h(e)$ is called the *head* of e. We say that a pair $\Gamma=(G,|.|)$ is a *generalized layered graph* if $G=(V,E)$ is a directed graph and $|.|: V\to \mathbf{Z}_{\geqslant 0}$ satisfies $|v|>|w|$ whenever there is an edge e from v to w. If Γ is a generalized layered graph and V_i is the set of $v\in V$ such that $|v|=i$, then V is the disjoint union $\bigcup_{i=0}^{\infty} V_i$. If $e\in E$ we define $|e|=|t(e)|$ and $l(e)$, the *length* of e, to be $|t(e)|-|h(e)|$. Recall that if $l(e)=1$ for all $e\in E$, then Γ is a *layered graph*.

While the requirement that a directed graph is layered is quite restrictive, the requirement that a graph be generalized layered is not.

Proposition 1.1. *Let $G=(V,E)$ be a finite acyclic directed graph. Then there are countably many ranking functions $|.|$ such that $(G,|.|)$ is a generalized layered graph.*

Proof. The result clearly holds if $|V|=1$. We will proceed by induction on $|V|$ Let $v'\in V$ satisfy $\{e\in E \mid h(e)=v'\}=\emptyset$. Set $V'=V\setminus\{v'\}$ and $E'=\{e\in E\mid t(e)\neq v'\}$. Then (V',E') is a finite acyclic directed graph and so, by the induction assumption, has countably many ranking functions. Let $|.|'$ be one such and let $|.|: V\to\mathbf{Z}_{\geqslant 0}$ extend $|.|'$. Then $|.|$ is a ranking function on G if and only if $|v'|>|w|$ for all $w\in V'$ such that there is an edge from v' to w. Thus $|.|'$ has countably many extensions to ranking functions of G. $\square$

Remark 1.2. The above argument shows that if $|v|_{can}$ is defined to be 0 when $\{e\in E\mid t(e)=v\}=\emptyset$ and otherwise defined to be the largest s such that there exist edges $e_1,\dots,e_s\in E$ with $t(e_1)=v$ and $h(e_i)=t(e_{i+1})$ for $1\leqslant i<s$, then $(G,|.|_{can})$ is a generalized layered graph and if $(G,|.|)$ is any generalized layered graph, then $|v|\geqslant |v|_{can}$ for all $v\in V$.

Let $G=(V,E)$ be a acyclic directed graph. Let $\mathcal{G}(G)$ denote the (countable) set of all generalized layered graphs $(G,|.|)$. For $(G,|.|_1)$, $(G,|.|_2)\in\mathcal{G}(G)$ we write $(G,|.|_1)\geqslant (G,|.|_2)$ if $|e|_1\geqslant |e|_2$ for all $e\in E$. With this definition, $\mathcal{G}(G)$ becomes a partially ordered set.

We will now define the algebras which are the primary objects of study in this paper. If $G=(V,E)$ and $\Gamma=(G,|.|)$ is a generalized layered graph, let $E^\sharp$ denote the subset

$$\bigcup_{e\in E}\{e\}\times\bigl[1,l(e)\bigr]$$

of $E\times\mathbf{Z}$. Denote the ordered pair $(e,i)\in E^\sharp$ by $a_i(e)$. For $a_i(e)\in E^\sharp$ define $\deg(a_i(e))=i$. Let t be a central variable and define $P_e(t)$ to be the polynomial

$$1+\sum_{j=1}^{l(e)}(-1)^j a_j(e)t^j$$

in $T(E^\sharp)[t]$. Set $a_0(e)=1$. Then *deg* gives $T(E^\sharp)$ the structure of a graded algebra

$$T\bigl(E^\sharp\bigr)=\bigoplus T\bigl(E^\sharp\bigr)_{[i]}.$$

Note that if we set $\deg(t) = -1$ then $P_e(t)$ is homogeneous of degree 0 in the graded algebra $T(E^\sharp)[t]$.

Furthermore, we may also give $T(E^\sharp)$ the structure of a filtered algebra. Set

$$f(a, j) = ja - j(j-1)/2$$

and for $(e, i) \in E^\sharp$ set $|a_i(e)| = \sum_{j=1}^{i}(|e| - j + 1) = f(|e|, i)$. Then setting

$$T\big(E^\sharp\big)_i = span\big\{a_{j_1}(e_1)\dots a_{j_r}(e_r) \bigm| r \geqslant 0, \big|a_{j_1}(e_1)\big| + \cdots + \big|a_{j_r}(e_r)\big| \leqslant i\big\}$$

gives $T(E^\sharp)$ the structure of a filtered algebra.

As usual, we say that a sequence $\pi = (e_1, \dots, e_r)$ of edges of Γ is a *(directed) path* from v to w if $t(e_1) = v, h(e_r) = w$ and $h(e_j) = t(e_{j+1})$ for all j, $1 \leqslant j < r$. If π is such a path, we write $v > w$, define $t(\pi) = t(e_1)$ and call this the tail of π, and define $h(\pi) = h(e_r)$ and call this the head of π. We define $|\pi| = |t(\pi)|$ and $l(\pi) = |t(e_1)| - |h(e_r)| = \sum_{i=1}^{r} l(e_i)$. For any path $\pi = (e_1, \dots, e_r)$ in Γ define

$$P_\pi(t) = P_{e_1}(t)P_{e_2}(t)\dots P_{e_r}(t) \in T\big(E^\sharp\big)[t]$$

and write

$$P_\pi(t) = 1 + \sum_{j=1}^{l(\pi)}(-1)^j e(\pi, j)t^j.$$

Let R denote the ideal in $T(E^\sharp)$ generated by

$$\big\{e(\pi_1, j) - e(\pi_2, j) \bigm| t(\pi_1) = t(\pi_2),\ h(\pi_1) = h(\pi_2),\ 1 \leqslant j \leqslant l(\pi_1)\big\}.$$

Definition 1.3. For a generalized layered graph $\Gamma = (G, |.|)$ where $G = (V, E)$ we define

$$A(\Gamma) = T\big(E^\sharp\big)/R.$$

Thus the images of $P_{\pi_1}(t)$ and $P_{\pi_2}(t)$ in $A(\Gamma)[t]$ are equal whenever $t(\pi_1) = t(\pi_2)$, $h(\pi_1) = h(\pi_2)$.

Since each $P_\pi(t)$ is homogeneous of degree 0, the ideal R is generated by homogeneous elements and so is a graded ideal. Thus

$$A(\Gamma) = \bigoplus A(\Gamma)_{[i]}$$

has the structure of a graded algebra where

$$A(\Gamma)_{[i]} = \big(T\big(E^\sharp\big)_{[i]} + R\big)/R.$$

$A(\Gamma)$ also has the structure of a filtered algebra where

$$A(\Gamma)_i = \big(T\big(E^\sharp\big)_i + R\big)/R.$$

When the meaning is clear from context we will denote elements of $A(\Gamma)$ or $A(\Gamma)[t]$ by their representatives in $T(E^\sharp)$ or $T(E^\sharp)[t]$.

Recall that $f(a, j) = ja - j(j-1)/2$.

Lemma 1.4. *$e(\pi, j) \in A(\Gamma)_{f(|\pi|, j)}$ for $1 \leqslant j \leqslant l(\pi)$.*

Proof. Write $\pi = (e_1, \dots, e_r)$. Let U denote the set of all sequences of integers $(i_1, \dots, i_r)$ with $i_1 + \dots + i_r = j$ and $0 \leqslant i_k \leqslant l(e_k)$ for $1 \leqslant k \leqslant r$. Then

$$e(\pi, j) = \sum_{(i_1,\dots,i_r)\in U} a_{i_1}(e_1)\dots a_{i_r}(e_r).$$

For $(i_1, \dots, i_r) \in U$, write

$$S(i_1, \dots, i_r) = \sum_{k=1}^{r} \bigl(|e_k| + \bigl(|e_k| - 1\bigr) + \dots + \bigl(|e_k| - i_k + 1\bigr)\bigr).$$

Then

$$a_{i_1}(e_1)\dots a_{i_r}(e_r) \in A(\Gamma)_{S(i_1,\dots,i_r)}.$$

Now if $(i_1, \dots, i_r), (i_1, \dots, i_{k-1}, i_k + 1, i_{k+1} - 1, i_{k+2}, \dots, i_r) \in U$ we have $S(i_1, \dots, i_r) \leqslant S(i_1, \dots, i_{k-1}, i_k + 1, i_{k+1} - 1, i_{k+2}, \dots, i_r)$. Then writing $j = l(e_1) + l(e_2) + \dots + l(e_s) + j'$ where $0 \leqslant s < r$ and $0 \leqslant j' \leqslant l(e_{s+1})$ we see that for any $(i_1, \dots, i_r) \in U$ we have

$$S(i_1, \dots, i_r) \leqslant S\bigl(l(e_1), \dots, l(e_s), j', 0, \dots, 0\bigr).$$

Since $S(l(e_1), \dots, l(e_s), j', 0, \dots, 0) = j|\pi| - (j(j-1)/2) = f(|\pi|, j)$ the lemma is proved. $\square$

We may also define a sequence of related algebras, $A(k, \Gamma), k \geqslant 1$, by requiring that the images of $P_{\pi_1}(t)$ and $P_{\pi_2}(t)$ in $A(\Gamma)[t]/(t^k)$ are equal whenever $t(\pi_1) = t(\pi_2), h(\pi_1) = h(\pi_2)$. Thus we define R_k to be the ideal in $T(E^\sharp)$ generated by

$$\bigl\{e(\pi_1, j) - e(\pi_2, j) \bigm| 1 \leqslant j < k,\ t(\pi_1) = t(\pi_2),\ h(\pi_1) = h(\pi_2)\bigr\}$$

and define

$$A(k, \Gamma) = T\bigl(E^\sharp\bigr)/R_k.$$

Then

$$A(1, \Gamma) \to A(2, \Gamma) \to \cdots$$

and if $V = \coprod_{j=0}^{n} V_j$ we have $A(\Gamma) = A(m, \Gamma)$ whenever $m > n$. Note that the image of $P_\pi(t)$ in $T(E^\sharp)[t]/(t^j)$ is invertible for any path π and for any $j > 0$ and that its inverse is given by the image of the geometric series:

$$P_\pi(t)^{-1} = \sum_{m=0}^{j-1} \bigl(1 - P_\pi(t)\bigr)^m.$$

Remark 1.5. One might require $P_e(t)$ to be a monic polynomial (instead of requiring that the constant term be 1). This gives an equivalent definition. We have chosen to require that the constant term be 1 in order to have the above easy expressions for inverses.

Example 1.6. Let $\Gamma = (G, |.|)$ and assume $G = (V, E)$ is acyclic as a (nondirected) graph, i.e., a tree. Let $E = \{e_1, \dots, e_n\}$ and recall that $l(e)$ denotes the length of an edge e. Write $K = \sum_{i=1}^{n} l(e_i)$ and assume that the edges have been ordered so that $l(e_i) \leqslant l(e_{i+1})$ for $1 \leqslant i < n$ so that $(l_1, \dots, l_n)$ is a partition of K. Let $(m_1, \dots, m_r)$ be the partition conjugate to $(l_1, \dots, l_n)$. Then $R = (0)$ and so $A(\Gamma) = T(E^\sharp)$ is the free algebra on K generators, m_j of which have degree j.

Algebras associated with generalized layered graphs occur naturally in the study of certain modules for the symmetric group. The following example is taken from Duffy [2,3].

Example 1.7. Let $[n]$ denote the set $\{1, \dots, n\}$. For $\sigma \in Sym_n$ define $[n, \sigma]$ to be the set of σ-orbits in $[n]$. Let $\mathcal{P}([n, \sigma])$ denote the power set of $[n, \sigma]$ and $G([n, \sigma])$ denote the Hasse graph of the partially ordered set $\mathcal{P}([n, \sigma])$. For $v = \{v_1, \dots, v_k\} \in \mathcal{P}([n, \sigma])$ let $|v|$ denote the cardinality of $\bigcup_{i=1}^{k} v_i$. Then $\Gamma([n, \sigma]) = (G([n, \sigma]), |.|)$ is a generalized layered graph. In particular, $\Gamma([3, (12)])$ has four vertices: $a = \{\{1, 2\}, \{3\}\}, b = \{\{1, 2\}\}, c = \{\{3\}\}, * = \emptyset$ and edges e_1 from a to b, e_2 from a to c, e_3 from b to $*$, and e_4 from c to $*$. Here $|a| = 3$, $|b| = 2$, $|c| = 1$, $|*| = 0$, $l(e_1) = l(e_4) = 1$, $l(e_2) = l(e_3) = 2$.

We define now the category of generalized layered graphs. Let $\Gamma = (G, |.|)$ and $\Gamma' = (G', |.|')$ be generalized layered graphs where $G = (V, E)$ and $G' = (V', E')$ are directed graphs. A morphism $\phi : \Gamma \to \Gamma'$ is a pair $\phi = (\phi_V, \phi_E)$ where

$$\phi_V : V \to V',$$
$$\phi_E : E \to E',$$

such that for all $e \in E$

$$t\big(\phi_E(e)\big) = \phi_V\big(t(e)\big),$$
$$h\big(\phi_E(e)\big) = \phi_V\big(h(e)\big),$$

and

$$l\big(\phi_E(e)\big) \leqslant l(e).$$

We denote by $\mathcal{GLG}$ the category with generalized layered graphs as objects and with morphisms as defined above.

Let $\phi \in \hom(\Gamma, \Gamma')$ where $\Gamma = (G, |.|), \Gamma' = (G', |.|')$. For a ground field F define

$$\tilde{\phi}_E : E^\sharp \to F E'^\sharp$$

by

$$\tilde{\phi}_E\big(a_i(e)\big) = a_i\big(\phi_E(e)\big)$$

if $0 \leqslant i \leqslant l(\phi_E(e))$, and

$$\tilde{\phi}_E\big(a_i(e)\big) = 0$$

if $i > l(\phi_E(e))$.

Then $\tilde{\phi}_E$ induces a homomorphism, again denoted $\tilde{\phi}_E$ from $T(E^\sharp) \to T(E'^\sharp)$ and hence also induces a homomorphism, still denoted $\tilde{\phi}_E$, from $T(E^\sharp)[t] \to T(E'^\sharp)[t]$. Clearly $\tilde{\phi}_E(P_e)(t) = P_{\phi_E(e)}(t)$ and so, if π is a path, $\tilde{\phi}_E(P_\pi(t)) = P_{\phi_E(\pi)}(t)$. Therefore $\tilde{\phi}(R_\Gamma) \subseteq R_{\Gamma'}$ and hence $\tilde{\phi}$ induces a homomorphism of associative algebras

$$A(\phi): A(\Gamma) \to A(\Gamma').$$

Proposition 1.8. *The mappings*

$$\Gamma \mapsto A(\Gamma)$$

and

$$\phi \mapsto A(\phi)$$

define a functor from $\mathcal{GLG}$ to the category of associative algebras.

Addition of a single edge gives an interesting example of a morphism of generalized layered graphs. Thus let $G = (V, E)$ be a directed graph and $v, w \in V$, $|v| > |w|$. Define $G^e = (V, E^e)$ by $E^e = E \cup \{e\}$, $t(e) = v$, $h(e) = w$. Let $|.|$ be a ranking function for G and $\Gamma = (G, |.|)$, $\Gamma^e = (G', |.|)$. Let i_V denote the identity map on V and i_E^e denote the injection of E into E^e. Then $i^e = (i_V, i_E^e)$ is a morphism of Γ to Γ^e. We will consider properties of the corresponding homomorphism $A(i^e)$ in Section 3.

2. Spanning set for $A(\Gamma)$

Let $\Gamma = (G, |.|)$ where $G = (V, E)$ be a generalized layered graph and assume that V_0 is the set of minimal vertices of Γ. Write $V_+ = V \setminus V_0$.

For each vertex $v \in V_+$ fix (arbitrarily) an edge $e_v \in E$ with $t(e_v) = v$. Then there is a path π_v defined by $\pi_v = (e_1, \ldots, e_r)$ with $e_1 = e_v$, $e_{i+1} = e_{h(e_i)}$ for $1 \leqslant i < r$, $h(e_r) \in V_0$. Set $P_v(t) = P_{\pi_v}(t)$. Let $V^\sharp$ denote the subset

$$\bigcup_{v \in V_+} \{v\} \times \bigl[1, |v|\bigr]$$

of $V_+ \times \mathbf{Z}$ and, for $(v, j) \in V^\sharp$ set $e(v, j) = e(\pi_v, j)$. Define a partial order on $V \times \mathbf{Z}$ by $(v, k) > (w, l)$ if $v > w$ and $k = |v| - |w|$.

Lemma 2.1. *Let π be a path with $t(\pi) = v$ and $l(\pi) = m$.*

$$e(\pi, m) \equiv e(v, m) \quad \bigl(\operatorname{mod} A(\Gamma)_{f(|v|, m) - 1}\bigr).$$

Proof. Set $w = h(\pi)$. Then $P_\pi(t) = P_v(t) P_w(t)^{-1}$ in $A(\Gamma)[[t]]$ and so (since the constant term of $P_w(t)$ is 1)

$$e(\pi, m) = \sum (-1)^r e(v, i_0) e(w, i_1) \ldots e(w, i_r)$$

where the sum is taken over all sequences of integers $(i_0, \ldots, i_r)$ with $r \geqslant 0, i_0 \geqslant 0$, $i_1, \ldots, i_r \geqslant 1$ and $i_0 + \cdots + i_r = m$. Let

$$M(i_0, \ldots, i_r) = |v| + \bigl(|v| - 1\bigr) + \cdots + \bigl(|v| - i_0 + 1\bigr) + \sum_{j=1}^{r} \bigl(|w| + \bigl(|w| - 1\bigr) + \cdots + \bigl(|w| - i_j + 1\bigr)\bigr).$$

Clearly $M(i_0,\dots,i_r) \leqslant M(i_0) + (m - i_0)|w|$ and as $|v| - m = |w|$ $M(i_0) + (m - i_0)|w| \leqslant M(m)$ with equality if and only if $i_0 = m$ (or, equivalently, $r = 0$). Since $e(v,i_0)e(w,i_1)\dots e(w,i_r) \in A(\Gamma)_{M(i_0,\dots,i_r)}$ and $M(m) = f(|v|,m)$, the lemma is proved. □

Proposition 2.2. *If $(v,k) > (w,l)$ then*

$$e(v,k)e(w,l) \equiv e(v,k+l) \quad \bigl(\mathrm{mod}\ A(\Gamma)_{f(|v|,k+l)-1}\bigr).$$

Proof. Let π be a path from v to w. Then $P_v(t) = P_\pi(t)P_w(t)$. Since $l(\pi) = |v| - |w| = k$, $P_\pi(t)$ is a polynomial of degree k and so $e(v,k+l)$, the coefficient of $(-t)^{k+l}$ in $P_v(t)$, is equal to

$$\sum_{j=0}^{k} e(\pi,j)e(w,k+l-j).$$

By Lemma 1.4,

$$e(\pi,j)e(w,k+l-j) \in A(\Gamma)_{f(|\pi|,j)+f(|w|,k+l-j)}.$$

Now, as $0 \leqslant j \leqslant k$,

$$f\bigl(|\pi|,j\bigr) + f\bigl(|w|,k+l-j\bigr) \leqslant f\bigl(|v|,k+l\bigr)$$

with equality if and only if $j = k+l$. Thus

$$e(v,k+l) \equiv e(\pi,k)e(w,l) \quad \bigl(\mathrm{mod}\ A(\Gamma)_{f(|v|,k+l)-1}\bigr)$$

and so the proposition follows from Lemma 2.1. □

Define

$$\mathbf{B}_1(\Gamma) = \bigl\{\bigl((v_1,k_k),\dots,(v_r,k_r)\bigr) \bigm| r \geqslant 0,\ v_1,\dots,v_r \in V_+,\ 1 \leqslant k_i \leqslant |v_i|\bigr\}$$

and

$$\mathbf{B}(\Gamma) = \bigl\{\bigl((v_1,k_1),\dots,(v_r,k_r)\bigr) \in \mathbf{B}_1(\Gamma) \bigm| (v_j,i_j) \not> (v_{j+1},i_{j+1}),\ -1 \leqslant j < r\bigr\}.$$

Define

$$\epsilon : \mathbf{B}_1(\Gamma) \to A(\Gamma)$$

by

$$\epsilon : \bigl((v_1,k_1),\dots,(v_r,k_r)\bigr) \mapsto e(v_1,k_1)\dots e(v_r,k_r).$$

Proposition 2.3. *Let Γ be a generalized layered graph with a unique minimal vertex. Then $\epsilon(\mathbf{B}(\Gamma))$ spans $A(\Gamma)$.*

Proof. Let $*$ denote the unique minimal vertex of Γ and let $e \in E$. If $h(e) \neq *$ then $1 + \sum_{j=1}^{l(e)}(-1)^j a_j(e)t^j = P_e(t) = P_{t(e)}(t)P_{h(e)}(t)^{-1}$ and so each $a_i(e)$ is in the subalgebra of $A(\Gamma)$ generated by the coefficients of $P_{t(e)}(t)$ and $P_{h(e)}(t)$, and hence in the subalgebra generated by all $e(v,j)$, $v \in V_+$, $1 \leqslant j \leqslant |v|$. If $h(e) = *$ then $P_e(t) = P_{t(e)}(t)$ and so $a_i(e) = e(t(e),i)$ for $1 \leqslant i \leqslant l(e)$. Thus $A(\Gamma)$ is generated by $\{e(v,i) \mid v \in V_+, 1 \leqslant i \leqslant |v|\}$. Therefore $A(\Gamma)$ is spanned by the set of all products of these elements. Proposition 2.2 then gives the result. □

3. Operations on graphs

We define several operations on graphs (adding a vertex, adding an edge, inversion, formation of bouquets) and discuss the relations between the corresponding algebras. As a consequence of our results on adding a vertex we see (Theorem 3.2) that the spanning set of Proposition 2.3 is a linear basis for $A(\Gamma)$.

Let $\Gamma = (G, |.|)$, where $G = (V, E)$, be a generalized layered graph, $e \in E$ be an edge of length greater than 1, and i an integer, $0 < i < l(e)$. We define a new graph $G^w = (V^w, E^w)$ by placing a new vertex, w, on the edge e. Thus

$$V^w = V \cup \{w\}$$

and

$$E^w = \big(E \setminus \{e\}\big) \cup \{e_1, e_2\}$$

where

$$t(e_1) = t(e), \qquad h(e_1) = t(e_2) = w, \qquad h(e_2) = h(e).$$

Now extend the rank function $|.|$ to V^w by $|w| = |h(e_2)| + i$. Then $\Gamma^w = (G^w, |.|)$ is a generalized layered graph and

$$l(e_1) = l(e) - i, \qquad l(e_2) = i.$$

Define a map

$$\tilde{\imath} : E^\sharp \to T\big(E^{w\sharp}\big)$$

by

$$\tilde{\imath} : a_j(f) \mapsto a_j(f)$$

for all $f \in E$, $f \neq e$, $1 \leqslant j \leqslant l(f)$, and

$$\tilde{\imath} : a_j(e) \mapsto \sum_{\max(0, i+j-l(e_1)) \leqslant k \leqslant \min(i,j)} a_{j-k}(e_1) a_k(e_2)$$

for $1 \leqslant j \leqslant l(e)$. The map $\tilde{\imath}$ extends to homomorphisms of graded algebras

$$\tilde{\imath} : T\big(E^\sharp\big) \to T\big(E^{w\sharp}\big)$$

and

$$\tilde{\imath} : T\big(E^\sharp\big)[t] \to T\big(E^{w\sharp}\big)[t]$$

and we have

$$\tilde{\imath}\big(P_e(t)\big) = P_{e_1}(t) P_{e_2}(t).$$

Let $\pi = (f_1, \dots, f_r)$ be a path in Γ. If $f_1, \dots, f_r \neq e$ define $\tilde{\iota}(\pi)$ to be the path $(f_1, \dots, f_r)$ in Γ^w. If $f_j = e$ define $\tilde{\iota}(\pi)$ to be the path $(f_1, \dots, f_{j-1}, e_1, e_2, f_{j+1}, \dots, f_r)$ in Γ^w. Then we have

$$\tilde{\iota}\big(P_\pi(t)\big) = P_{\tilde{\iota}(\pi)}(t)$$

for any path π in E. If π_1, π_2 are paths in Γ with $t(e_1) = t(e_2)$, $h(e_1) = h(e_2)$, then $\tilde{\iota}\pi_1$, $\tilde{\iota}\pi_2$ are paths in Γ^w with $t(\tilde{\iota}e_1) = t(\tilde{\iota}e_2)$, $h(\tilde{\iota}e_1) = h(\tilde{\iota}e_2)$. Thus the generators of the ideal R in $T(E^\sharp)$ are mapped by $\tilde{\iota}$ to generators of the ideal R^w in $T(E^{w\sharp})$ and so $\tilde{\iota}$ induces a homomorphism

$$\iota : A(\Gamma) \to A\big(\Gamma^w\big).$$

For $v \in V^w$, $1 \leqslant j \leqslant |v|$ define $e^w(v, j)$ by

$$P_v(t) = \sum_{j=0}^{|v|} (-1)^j e^w(v, j) t^j.$$

Lemma 3.1.

(a) *If $v \in V_+$, then $\iota(e(v, j)) = e^w(v, j)$.*
(b) *$\iota(\epsilon(\mathbf{B}(\Gamma)) \subseteq \epsilon(\mathbf{B}(\Gamma^w))$.*

Proof. Part (a) follows from the definition of $e^w(v, j)$. Part (b) then follows since $e(v_1, j_1) \dots e(v_r, j_r) \in \epsilon(\mathbf{B}(\Gamma))$ if and only if $e^w(v_1, j_1) \dots e^w(v_r, j_r) \in \epsilon(\mathbf{B}(\Gamma^w))$. $\square$

Theorem 3.2. *Let Γ have a unique minimal vertex. Then*

(a) *$\epsilon(\mathbf{B}(\Gamma))$ is a basis for $A(\Gamma)$;*
(b) *$\iota : A(\Gamma) \to A(\Gamma^w)$ is an injection.*

Proof. Set $s(\Gamma) = \sum_{e \in E} (l(e) - 1)$. Then $s(\Gamma) = 0$ if and only if every edge of Γ has length one, i.e., if and only if Γ is a layered graph (in the sense of [5]). Then, by Theorem 4.3 of [5], $\epsilon(\mathbf{B}(\Gamma))$ is a basis for $A(\Gamma)$ if $s(\Gamma) = 0$.

We proceed by induction on $s(\Gamma)$, assuming that $\epsilon(\mathbf{B}(\Gamma'))$ is a basis for $A(\Gamma')$ whenever $s(\Gamma') < s(\Gamma)$. Clearly $s(\Gamma^w) = s(\Gamma) - 1$ and so we have that $\epsilon(\mathbf{B}(\Gamma^w))$ is linearly independent. Then, by Lemma 3.1, ι maps $\epsilon(\mathbf{B}(\Gamma))$ into the linearly independent set $\epsilon(\mathbf{B}(\Gamma^w))$ in $A(\Gamma^w)$ and hence $\epsilon(\mathbf{B}(\Gamma))$ is linearly independent. In view of Proposition 2.3, $\epsilon(\mathbf{B}(\Gamma))$ is a basis for $A(\Gamma)$ and ι is injective. $\square$

Let $\Gamma = (G, |.|)$ where $G = (V, E)$ be a generalized layered graph and $v, w \in V$ be vertices with $|v| > |w|$. Recall (from Section 1) the definition of the graph obtained by adding a new edge from v to w: $G^e = (V, E^e)$, $E^e = E \cup \{e\}$, $t(e) = v$, $h(e) = w$, $\Gamma^e = (G^e, |.|)$. Then $i^e = (i_V, i^e_E)$, where i_V is the identity map on V and i^e_E is the injection of E into E^e, is a morphism of Γ to Γ^e and so there is a corresponding homomorphism $A(i^e) : A(\Gamma) \to A(\Gamma^e)$.

Proposition 3.3. *Let Γ have a unique minimal vertex. Then $A(i^e)$ is surjective.*

Proof. Since Γ has a unique minimal vertex, $h(\pi_v) = h(\pi_w)$. Therefore $P_v(t) = P_e(t) P_w(t)$ in $A(\Gamma^e)[t]$ and so

$$P_e(t) = 1 + \sum_{j=1}^{l(e)} (-1)^j a_j(e) t^j = P_v(t) P_w(t)^{-1}.$$

Since the coefficients of $P_v(t)$ and $P_w(t)$ are in the image of $A(i^e)$ we have that $E^{e\sharp}$ is contained in the image of $A(i^e)$, giving the result. $\square$

We will later (Corollary 5.3) see that if $v > w$ in G then $A(i^e)$ is an isomorphism.

Let $\Gamma = (V, E)$ be a generalized layered graph with $V = \coprod_{i=0}^n V_i$. We define the inverted graph $\check{\Gamma} = (\check{V}, \check{E})$ by reversing all edges. Thus

$$\check{V} = \coprod_{i=0}^{n} \check{V}^i$$

where

$$\check{V}^i = V_{n-i}$$

and

$$\check{E} = \{\check{e} \mid e \in E\}$$

where

$$t(\check{e}) = h(e), \qquad h(\check{e}) = t(e).$$

Note that $l(\check{e}) = l(e)$. Define

$$\tilde{\eta} : E^\sharp \to \check{E}^\sharp$$

by

$$\tilde{\eta} : a_i(e) \to a_i(\check{e}).$$

Then $\tilde{\eta}$ extends to anti-isomorphisms of graded algebras

$$\tilde{\eta} : T\big(E^\sharp\big) \to T\big(\check{E}^\sharp\big)$$

and

$$\tilde{\eta} : T\big(E^\sharp\big)[t] \to T\big(\check{E}^\sharp\big)[t].$$

If $\pi = (e_1, \ldots, e_r)$ is a path in Γ we set $\tilde{\eta}(\pi) = (\check{e_r}, \ldots, \check{e_1})$, a path in $\check{\Gamma}$. Then

$$\tilde{\eta}\big(P_\pi[t]\big) = P_{\tilde{\eta}(\pi)}[t]$$

and so

$$\tilde{\eta}(R) = \check{R}.$$

Thus $\tilde{\eta}$ induces a anti-isomorphism

$$\eta : A(\Gamma) \to A(\check{\Gamma}).$$

Finally, let Γ_1 and Γ_2 be generalized layered graphs with unique minimal vertices $*_1$ and $*_2$. We define $\Gamma_1 \vee \Gamma_2$ to be the "bouquet" obtained by identifying the minimal vertices. Clearly, $A(\Gamma_1 \vee \Gamma_2)$ is the free product of $A(\Gamma_1)$ and $A(\Gamma_2)$.

4. Hilbert series

We will derive an expression for $H(A(\Gamma), z)$, the Hilbert series of $A(\Gamma)$, when Γ is a generalized layered graph with a unique minimal element. While this expression is identical to that of [9] for layered graphs, we will present it in a form which is better suited to applications.

For convenience we write $h(z) = H(A(\Gamma), z)$, where Γ is a generalized layered graph with unique minimal element $*$ of level 0. For $\mathbf{b} = ((v_1, k_1), \ldots, (v_r, k_r)) \in \mathbf{B}_1(\Gamma)$ define

$$|\mathbf{b}| = \sum_{s=1}^{r} k_s$$

and set

$$\mathbf{B}_1(\Gamma)_{[j]} = \bigl\{\mathbf{b} \in \mathbf{B}_1(\Gamma) \bigm| |\mathbf{b}| = j\bigr\}.$$

For any subset $X \subseteq \mathbf{B}_1(\Gamma)$ set

$$X_{[j]} = \mathbf{B}_1(\Gamma)_{[j]} \cap X$$

and

$$\|X\| = \sum_{j \geqslant 0} |X_{[j]}| z^j.$$

Now $\epsilon(\mathbf{B}_1(\Gamma)_{[j]}) \subseteq A(\Gamma)_{[j]}$ and $\epsilon(\mathbf{B}(\Gamma)_{[j]})$ is a basis for $A(\Gamma)_{[j]}$. If $\mathbf{b} = ((v_1, k_1), \ldots, (v_r, k_r))$, $\mathbf{c} = ((w_1, l_1), \ldots, (w_s, l_s)) \in \mathbf{B}_1(\Gamma)$ define $\mathbf{b} \circ \mathbf{c} = ((v_1, k_1), \ldots, (v_r, k_r), (w_1, l_1), \ldots, (w_s, l_s))$. For $v \in V_+$, define

$$\mathbf{C}_v(\Gamma) = \bigcup_{k=1}^{|v|} (v, k) \circ \mathbf{B}(\Gamma),$$

$$\mathbf{B}_v(\Gamma) = \mathbf{C}_v(\Gamma) \cap \mathbf{B}(\Gamma),$$

and

$$\mathbf{D}_v(\Gamma) = \mathbf{C}_v(\Gamma) \setminus \mathbf{B}_v(\Gamma).$$

Then $\mathbf{B}(\Gamma) = \{\emptyset\} \cup \bigcup_{v \in V_+} \mathbf{B}_v(\Gamma)$. Let $h_v(z) = \|\mathbf{B}_v(\Gamma)\|$. Then

$$h(z) = \bigl\|\mathbf{B}(\Gamma)\bigr\| = 1 + \sum_{v \in V_+} h_v(z).$$

Now

$$\|\mathbf{C}_v\| = \bigl(z + \cdots + z^{|v|}\bigr) h(z) = z\left(\frac{z^{|v|} - 1}{z - 1}\right) h(z).$$

Since

$$\mathbf{D}_v = \bigl\{\bigl((v, k), (v_1, k_1) \ldots (v_r, k_r)\bigr) \bigm| 1 \leqslant k \leqslant |v|,\ (v, k) > (v_1, k_1),$$
$$\bigl((v_1, k_1) \ldots (v_l, k_l)\bigr) \in \mathbf{B}(\Gamma)\bigr\}$$

we have

$$\mathbf{D}_v = \bigcup_{v>v_1>*} \big(v, |v| - |v_1|\big) \circ \mathbf{B}_{v_1}(\Gamma).$$

Then $\|\mathbf{D}_v\| = \sum_{v>v_1>*} z^{|v|-|v_1|} h_{v_1}(z)$ and so

$$h_v(z) = z\left(\frac{z^{|v|}-1}{z-1}\right)h(z) - \sum_{v>w>*} z^{|v|-|w|} h_w(z).$$

This equation may be written in matrix form. Arrange the elements of V_+ in increasing order and index the elements of vectors and matrices by this ordered set. Let $\mathbf{h}(z)$ denote the column vector with entry $h_v(z)$ in the v-position, let $\mathbf{s}$ denote the column vector with entry $\frac{z(z^{|v|}-1)}{z-1}$ in the v-position, let $\mathbf{1}$ denote the column vector all of whose entries are 1, and let $\zeta(z)$ denote the matrix with entries $\zeta_{v,w}(z)$ for $v, w \in V_+$ where $\zeta_{v,w}(z) = z^{|v|-|w|}$ if $v \geqslant w$ and 0 otherwise. Then we have

$$\zeta(z)\mathbf{h}(z) = \mathbf{s}h(z).$$

Now $N(z) = \zeta(z) - I$ is a strictly lower triangular matrix and so $\zeta(z)$ is invertible. In fact, $\zeta(z)^{-1} = I - N(z) + N(z)^2 - \cdots$. For $v, w \in V_+$ set

$$\mu(v, w) = \sum_{v=v_1>\cdots>v_l=w} (-1)^{l+1},$$

the well-known Möbius function of the partially ordered set V_+ (see Ch. 3 in [11]). Then the (v, w)-entry of $\zeta(z)^{-1}$ is

$$\mu(v, w) z^{|v|-|w|}.$$

Now

$$1 - h(z) = -\mathbf{1}^T \mathbf{h}(z) = -\mathbf{1}^T \zeta(z)^{-1} \mathbf{s} h(z).$$

Solving for $h(z)$ gives

$$h(z) = \frac{1}{1 - \mathbf{1}^T \zeta(z)^{-1} \mathbf{s}}.$$

Using the expression given above for $\zeta(z)^{-1}$, we obtain the following result.

Theorem 4.1. *Let Γ be a generalized layered graph with unique minimal element $*$ of level* 0 *and $h(z)$ denote the Hilbert series of $A(\Gamma)$. Then*

$$h(z) = \frac{1-z}{1 - z + \sum_{v_1>v_2>\cdots>v_l>*} (-1)^l (z^{|v_1|-|v_l|+1} - z^{|v_1|+1})}.$$

This result can be restated using the important definition

$$\mathcal{M}(\Gamma)(z) = \sum_{v>w\geqslant *} \mu(v, w) z^{|v|-|w|}.$$

Corollary 4.2. *Let Γ be a generalized layered graph with unique minimal element $*$ of level 0 and $h(z)$ denote the Hilbert series of $A(\Gamma)$. Then*

$$h(z) = \frac{1-z}{1-z\mathcal{M}(\Gamma)}.$$

Proof. We show

$$\frac{1-z}{h(z)} = 1 - z + \sum_{v_1>v_2>\cdots>v_l>*} (-1)^l \big(z^{|v_1|-|v_l|+1} - z^{|v_1|+1}\big)$$
$$= 1 + \sum_{v_1>v_2>\cdots>v_l\geqslant*} (-1)^l z^{|v_1|-|v_l|+1} = 1 - z\mathcal{M}(\Gamma)(z).$$

The first equality is immediate from the theorem and the second follows by writing

$$\sum_{v_1>v_2>\cdots>v_l\geqslant*} (-1)^l z^{|v_1|-|v_l|+1}$$

as

$$\sum_{v_1>v_2>\cdots>v_l>*} (-1)^l z^{|v_1|-|v_l|+1} + \sum_{v_1>v_2>\cdots>v_l=*} (-1)^l z^{|v_1|-|v_l|+1}. \qquad \square$$

As noted in Example 1.6, if $\Gamma = (G, |.|)$ where G is a rooted tree, then $A(\Gamma)$ is the free algebra $T(E^\sharp)$. By applying Corollary 4.2 in this situation we recover the well-known expression for the Hilbert series of a free algebra.

Corollary 4.3. *Let $\Gamma = (G, |.|)$ be a generalized layered graph where $G = (V, E)$ is a rooted tree. Let m_j, $1 \leqslant j \leqslant r$, denote the number of edges of length $\geqslant j$. Then*

$$\frac{1}{H(\Gamma, z)} = 1 - \sum_{j=1}^{r} m_j z^j.$$

Proof. Let $E = \{e_1, \dots, e_n\}$ and recall that $l(e)$ denotes the length of an edge e. Now let $v, w \in V$, $v > w$ and let $S(v, w) = \{u \in V \mid v \geqslant u \geqslant w\}$. Then

$$\mu(v, w) = \sum_{\emptyset\neq T\subseteq S(v,w)} (-1)^{|T|}.$$

Thus $\mu(v, w) = 1$ if $v = w$, $\mu(v, w) = -1$ if there is an edge e from v to w, and $\mu(v, w) = 0$ otherwise. Therefore $\mathcal{M}(\Gamma)(z) = |V| - \sum_{i=1}^{n} z^{l(e_i)}$. Hence

$$1 - z\mathcal{M}(\Gamma)(z) = (1-z) - z\left(|E| - \sum_{i=1}^{n} z^{l(e_i)}\right) = (1-z) - z\left(\sum_{i=1}^{n}\big(1 - z^{l(e_i)}\big)\right)$$
$$= (1-z)\left(1\sum_{i=1}^{n}\big(z + \cdots + z^{l(e_i)}\big)\right) = (1-z)\left(1 - \sum_{j=1}^{r} m_j z^j\right). \qquad \square$$

5. Hilbert series and operations on graphs

If Γ has a unique minimal vertex $v_{\min}$, we define

$$\mathcal{M}_\circ(\Gamma)(z) = \sum_{v \in V} \mu(v, v_{\min}) z^{|v|-|v_{\min}|}.$$

Similarly, if Γ has a unique maximal vertex $v_{\max}$ we define

$$\mathcal{M}^\circ(\Gamma)(z) = \sum_{v \in V} \mu(v_{\max}, v) z^{|v_{\max}|-|v|}.$$

Let $e \in E$ be an edge of length greater than 1 with $t(e) = u$ and $h(e) = v$. Recall that for any = i, $1 \leqslant i < l(e)$, we have defined a new graph $\Gamma^w = (G^w, |.|)$ where $G^w = (V^w, E^w)$ by adding a new vertex w on the edge e with $|w| - |v| = i$. We define two related graphs, $G_-^w = (V_-^w, E_-^w)$ and $G_+^w = (V_+^w, E_+^w)$ by $V_-^w = \{v \in V^w \mid w \geqslant v\}$, $E_-^w = \{e \in E^w \mid t(e), h(e) \in V_-^w\}$, $V_+^w = \{v \in V^w \mid v \geqslant w\}$, $E_+^w = \{e \in E^w \mid t(e), h(e) \in V_+^w\}$. Then the restrictions of the rank function $|.|$ to V_-^w and to V_+^w (which we continue to denote by $|.|$) are rank functions. Thus we have generalized layered graphs $\Gamma_-^w = (G_-^w, |.|)$ and $\Gamma_+^w = (G_+^w, |.|)$.

Proposition 5.1.

$$\mathcal{M}\big(\Gamma^w\big)(z) = \mathcal{M}(\Gamma)(z) + \mathcal{M}_\circ\big(\Gamma_+^w\big)(z) \cdot \mathcal{M}^\circ\big(\Gamma_-^w\big)(z).$$

Proof. For $u', v' \in V^w, u' \geqslant w \geqslant v'$ define

$$\mu^w(u', v') = \sum_{\substack{u' = x_1 > \cdots > x_l = v' \\ w \in \{x_1, \ldots, x_l\}}} (-1)^{l+1}.$$

Then

$$\mathcal{M}\big(\Gamma^w\big)(z) - \mathcal{M}(\Gamma)(z) = \sum_{u' > w > v'} \mu^w(u', v') z^{|u'|-|v'|} + \sum_{u' > w} \mu^w(u', w) z^{|u'|-|w|} + \sum_{w > v'} \mu^w(w, v') z^{|w|-|v'|} + \mu^w(w, w).$$

Now

$$\mathcal{M}_\circ\big(\Gamma_+^w\big)(z) - 1 = \sum_{u' > w} \mu^w(u', w) z^{|u'|-|w|} = -\sum_{u' \geqslant u'' \geqslant u} \mu(u', u'') z^{|u'|-|w|},$$

and

$$\mathcal{M}^\circ\big(\Gamma_-^w\big)(z) - 1 = \sum_{w > v'} \mu^w(w, v') z^{|w|-|v'|} = -\sum_{v \geqslant v'' \geqslant v'} \mu(v'', v') z^{|w|-|v'|}.$$

Also, if $u' > w > v'$, we have

$$\begin{aligned}\mu^w(u',v')z^{|u'|-|v'|} &= \sum_{u'\geqslant u''\geqslant u>v\geqslant v''\geqslant v'} \mu(u',u'')\mu(v'',v')z^{|u'|-|v'|}\\ &= \Big(\sum_{u'\geqslant u''\geqslant u}\mu(u',u'')z^{|u'|-|w|}\Big)\Big(\sum_{v\geqslant v''\geqslant v'}\mu(v'',v')z^{|w|-|v'|}\Big)\\ &= \big(\mathcal{M}_\circ\big(\Gamma_+^w\big)(z)-1\big)\big(\mathcal{M}^\circ\big(\Gamma_-^w\big)(z)-1\big),\end{aligned}$$

giving the result. □

Now let $\Gamma=(V,E)$, $v,w\in V$, $|v|>|w|$. Recall that we have defined a graph $\Gamma^e=(V^e,E^e)$ by adjoining an edge e to E with $t(e)=v$, $h(e)=w$. If $a,b\in V$ write $a>b$ if there is a path in E from a to b. Then the following proposition is immediate from the definition of $\mathcal{M}$.

Proposition 5.2.

$$\mathcal{M}\big(\Gamma^e\big)(z)-\mathcal{M}(\Gamma)(z)=\sum(-1)^{l+m}z^{|v_1|-|w_m|}$$

where the sum is taken over all sequences $v_1>\cdots>v_l\geqslant v$, $w\geqslant w_1>\cdots>w_m$, such that there is no path in E from v_l to w_1.

Corollary 5.3. *If Γ^e is obtained from Γ by adjoining an edge from v to w where there is a path in Γ from v to w then $\mathcal{M}(\Gamma^e)(z)=\mathcal{M}(\Gamma)(z)$. Hence $H(A(\Gamma^e),z)=H(A(\Gamma),z)$ so $A(i^e):A(\Gamma)\to A(\Gamma^e)$ is an isomorphism.*

Proof. If there is a path from v to w in Γ then the sum occurring in the proposition is vacuous. □

Next let $\Gamma=(G,|.|)$ where $G=(V,E)$ have unique maximal vertex and a unique minimal vertex. We have defined the inverted graph $\check{\Gamma}$. Since $v_1>\cdots>v_l$ in V if and only if $v_l>\cdots>v_1$ in $\check{V}$, the following proposition is immediate from the definition of $\mathcal{M}$.

Proposition 5.4. *Let Γ have a unique maximal vertex and a unique minimal vertex. Then $\mathcal{M}(\check{\Gamma})(z)=\mathcal{M}(\Gamma)(z)$ and so $H(A(\check{\Gamma}),z)=H(A(\Gamma),z)$.*

If Γ_1 and Γ_2 are generalized layered graphs with unique minimal vertices $*_1$ and $*_2$, we have defined $\Gamma_1\vee\Gamma_2$ to be the “bouquet” obtained by identifying the minimal vertices. The following proposition is clear.

Proposition 5.5. $\mathcal{M}(\Gamma_1\vee\Gamma_2)(z)=\mathcal{M}(\Gamma_1)(z)+\mathcal{M}(\Gamma_2)(z)-1$.

We remark that this implies

$$\frac{1}{H(A(\Gamma_1\vee\Gamma_2),z)}=\frac{1}{H(A(\Gamma_1),z)}+\frac{1}{H(A(\Gamma_2),z)}-1.$$

Of course, this already followed from our previous observation that $A(\Gamma_1\vee\Gamma_2)$ is the free product of $A(\Gamma_1)$ and $A(\Gamma_2)$.

Finally, let Γ_1 and Γ_2 be generalized layered graphs with unique minimal vertices $v_{\min,1}$ and $v_{\min,2}$ of level 0 and unique maximal vertices $v_{\max,1}$ and $v_{\max,2}$ of level d. We define $\Gamma_1\diamondsuit\Gamma_2$ to be the “double bouquet” obtained by identifying $v_{\min,1}$ with $v_{\min,2}$ and identifying $v_{\max,1}$ with $v_{\max,2}$.

Proposition 5.6. $\mathcal{M}(\Gamma_1\diamondsuit\Gamma_2)(z)=\mathcal{M}(\Gamma_1)(z)+\mathcal{M}(\Gamma_2)(z)+2-z^d$.

Proof. Let S_i denote the set of sequences $u_1 > \cdots > u_l$ in V_i for $i = 1, 2$ and let S_{12} denote the set of sequences $u_1 > \cdots > u_l$ in $V_1 \diamondsuit V_2$. Write $v_{\max} = v_{\max,1} = v_{\max,2}$ and $v_{\min} = v_{\min,1} = v_{\min,2}$ in $V_1 \diamondsuit V_2$. Clearly $S_{12} = S_1 \cup S_2$ and $S_1 \cap S_2 = \{(v_{\max}), (v_{\min}), (v_{\max}, v_{\min})\}$, giving the result. □

Let $\Delta(d)$ denote the generalized layered graph with two vertices max and min of levels d and 0 respectively and with one edge e with $t(e) = \max$, $h(e) = \min$.

Corollary 5.7.

$$\frac{1}{H(A(\Gamma_1 \diamondsuit \Gamma_2), z)} = \frac{1}{H(A(\Gamma_1), z)} + \frac{1}{H(A(\Gamma_2), z)} - \frac{1}{H(A(\Delta(d), z)}.$$

Recall that according to [7] if V is a graded vector space with graded dimension $H(V, z)$ and R is a graded subspace of the tensor algebra $T(V)$ with graded dimension $H(R, z)$, the quotient $A = T(V)/\langle R\rangle$ is said to be a noncommutative complete intersection if

$$H(A, z) = \frac{1}{1 - H(V, z) + H(R, z)}.$$

Corollary 5.8. *Let Γ_1 and Γ_2 be generalized layered graphs with unique minimal vertices $v_{\min,1}$ and $v_{\min,2}$ of level* 0 *and unique maximal vertices $v_{\max,1}$ and $v_{\max,2}$ of level d. Assume that $A(\Gamma_1)$ and $A(\Gamma_2)$ are noncommutative complete intersections. Then $A(\Gamma_1 \diamondsuit \Gamma_2)$ is a noncommutative complete intersection.*

Proof. Let G_i denote space of generators for $A(\Gamma_i)$ and R_i denote the space of relations. Since $A(\Delta(d))$ is a free algebra on generators of degrees $1, 2, \ldots, d$ we have $H(A(\Delta(d), z)^{-1} = 1 - z - z^2 - \cdots - z^d$. Thus

$$\begin{aligned}\frac{1}{H(A(\Gamma_1 \diamondsuit \Gamma_2), z)} &= 1 - H(G_1, z) + H(R_1, z) + 1 - H(G_2, z) + H(R_2, z) - 1 + z + \cdots + z^d \\ &= 1 - H(G_1 + G_2, z) + H(R_1 + R_2, z) + z + \cdots + z^d.\end{aligned}$$

Now the generators for $\Gamma_1 \diamondsuit \Gamma_2$ are just the generators for Γ_1 and for Γ_2 and the relations for $\Gamma_1 \diamondsuit \Gamma_2$ are just the relations for Γ_1 and for Γ_2 together with the relations stating that $P_{\pi_1}(t) = P_{\pi_2}(t)$ where π_i is a path from $v_{\max,i}$ to $v_{\min,i}$. Since $P_{\pi_1}(t)$ has degree d, there is one such relation of degree j for $j = 1, \ldots, d$. Thus the space of relations for $\Gamma_1 \diamondsuit \Gamma_2$ has graded dimension $H(R_1 + R_2, z) + z + \cdots + z^d$ and the corollary is proved. □

Example 5.9. Using Corollaries 4.3 and 5.7 we see that the Hilbert series for the algebra $A(\Gamma([3, (12)])$ (defined in Example 1.7) is $\frac{1}{1-3z-z^2+z^3}$.

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