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Generalized spherical and simplicial coordinates

Wolf-Dieter Richter *

University of Rostock, Mathematical Institut, Universitätsplatz 1, D-18051 Rostock, Germany

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Abstract

Elementary trigonometric quantities are defined in $l_{2,p}$ analogously to that in $l_{2,2}$, the sine and cosine functions are generalized for each $p>0$ as functions $\sin_p$ and $\cos_p$ such that they satisfy the basic equation $|\cos_p(\varphi)|^p+|\sin_p(\varphi)|^p=1$. The p-generalized radius coordinate of a point $\xi\in R^n$ is defined for each $p>0$ as $r_p=(\sum_{i=1}^n|\xi_i|^p)^{1/p}$. On combining these quantities, $l_{n,p}$-spherical coordinates are defined. It is shown that these coordinates are nearly related to $l_{n,p}$-simplicial coordinates. The Jacobians of these generalized coordinate transformations are derived. Applications and interpretations from analysis deal especially with the definition of a generalized surface content on $l_{n,p}$-spheres which is nearly related to a modified co-area formula and an extension of Cavalieri's and Torricelli's indivisibeln method, and with differential equations. Applications from probability theory deal especially with a geometric interpretation of the uniform probability distribution on the $l_{n,p}$-sphere and with the derivation of certain generalized statistical distributions.

Keywords: Generalized radius coordinate; Generalized trigonometric functions; $l_{2,p}$-Trigonometry; p-Generalized Pythagoras type equation; $l_{2,p}$-Generalized polar coordinates; $l_{n,p}$-Spherical coordinates; $l_{2,p}$-Generalized triangle coordinates; $l_{n,p}$-Simplicial coordinates; Jacobians; $l_{n,p}$-Ball volume; $l_{n,p}$-Generalized indivisibeln method; Generalized surface content; $l_{n,p}$-Generalized uniform distribution on the sphere; Modified co-area formula; Disintegration of Lebesgue measure; p-Generalized χ^2- and Student-distributions; $l_{n,p}$-Norm symmetric distributions

* Fax: +49 381 498 6553.
E-mail address: wolf-dieter.richter@uni-rostock.de.

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1. Introduction

The use of suitably defined coordinates is of great importance in numerous areas of mathematics. It is therefore not necessary to mention here many concrete situations where $l_{n,p}$-spherical and simplicial coordinates apply. Instead, it suffices to refer to Section 4 which is devoted to a number of subjects where the new coordinates presented here will be considerably successfully used. Although in principal being standard it is sometimes anything but obvious in which way coordinates should be defined. As to mention here just one such situation we refer to [1] where the author defines generalized spherical coordinates just with the aim for dealing with the uniform distribution on the $l_{n,p}$-sphere. But nevertheless he says as a certain resume of his mathematically deep study that it seems that the word "uniform" does not refer to real, geometrical uniformity of the probability mass on the surface of the unit sphere. Section 4.3 deals with this question. Our results include now a certain geometric interpretation. But note that this interpretation needs the new notion of $l_{2,p}$-generalized trigonometric functions which will be introduced in Definition 1.

Level sets of the functions $T_p(x, y) = (|x|^p + |y|^p)^{1/p}$, $(x, y) \in R^2$, $p > 0$, can be easily described by the equation $r_p = c$ if one makes use of polar or standard triangle coordinates in the cases $p = 2$ and $p = 1$, respectively. In this paper, we shall consider r_1 and r_2 as special cases of the p-generalized radius coordinate $r_p = (|x|^p + |y|^p)^{1/p}$, $p > 0$, and the functions $\sin\varphi$ and $\cos\varphi$ as special cases of certain p-generalized trigonometric functions. Such functions will be introduced in Section 2. Section 3 deals with $l_{n,p}$-spherical and simplicial coordinates. In Section 4, we shall present several analytical and probabilistic applications and interpretations of these coordinates with an emphasis on the definition of a p-generalized surface content of the $l_{n,p}$-sphere, on differential equations, on a geometric interpretation of the uniform probability distribution on the $l_{n,p}$-sphere and on deriving certain generalized exact statistical distributions. The final Section 5 contains most of the proofs.

2. The $l_{2,p}$-generalized trigonometric functions

The following definition is basic for all what follows in this paper. Geometrically, it means that we consider the right-angled triangle $Tr = ((0, 0), (x, 0), (x, y))$ from R^{+2} as a subset of $l_{2,p}$ for an arbitrary but fixed chosen p, $p > 0$, and introduce elementary $l_{2,p}$-trigonometric quantities analogously to that in R^2. The side length of Tr needed for defining the $l_{2,p}$-sine and cosine functions can be written with the help of the $l_{2,p}$-distance d_p as $d_p((0, 0), (x, y)) = (x^p + y^p)^{1/p}$, $d_p((0, 0), (x, 0)) = x$ and $d_p((x, 0), (x, y)) = y$. Hence, Tr satisfies the p-generalized Pythagoras type equation $x^p + y^p = r^p$ with $r = d_p((0, 0), (x, y))$. This concept is directed, e.g., to dealing with situations where an ϵ-enlargement of a set, e.g. for the purpose of taking the derivative of its volume w.r.t. a suitably defined parameter, will not be defined through parallel sets of thickness ϵ but through "blowing up" the set by the factor $1 + \epsilon$ and where the set has a surface with a curvature-behavior like in the "world" of a $l_{n,p}$-ball with $p \neq 2$. The following definition is therefore basic for, e.g., understanding the relation between an $l_{n,p}$-ball's volume and its $l_{n,p}$-generalized surface content in Sections 4.2 and 4.3.

Definition 1. The p-generalized sine and cosine values of an angle $\varphi \in [0, 2\pi)$ between the directions of the positive x-axes and the line through the points $(0, 0)$ and $(x, y) \in R^2$ are defined for each $p > 0$ as

$$\sin_p(\varphi) = \frac{y}{(|x|^p + |y|^p)^{1/p}} \quad \text{and} \quad \cos_p(\varphi) = \frac{x}{(|x|^p + |y|^p)^{1/p}}.$$

These functions will be alternatively called $l_{2,p}$-sine and cosine functions, respectively.

Obviously, it holds $|\cos_p(\varphi)| \leqslant 1$, $|\sin_p(\varphi)| \leqslant 1$ and

$$\left|\cos_p(\varphi)\right|^p + \left|\sin_p(\varphi)\right|^p = 1. \tag{$*$}$$

From Definition 1 it follows that for each $p > 0$, $\varphi \in [0, 2\pi)$ it holds

$$\sin_p(\varphi) = \frac{\sin\varphi}{N_p(\varphi)} \quad \text{and} \quad \cos_p(\varphi) = \frac{\cos\varphi}{N_p(\varphi)} \quad \text{where } N_p(\varphi) = \left(|\sin\varphi|^p + |\cos\varphi|^p\right)^{1/p}.$$

For $\varphi \neq k\pi/2$, $k \in \{1, 2, 3\}$, the first order derivatives of $\sin_p$ and $\cos_p$ are

$$\sin_p'(\varphi) = \cos_p(\varphi)\frac{|\cos\varphi|^{p-2}}{(N_p(\varphi))^p} \quad \text{and} \quad \cos_p'(\varphi) = -\sin_p(\varphi)\frac{|\sin\varphi|^{p-2}}{(N_p(\varphi))^p}.$$

3. The $l_{n,p}$-generalized coordinates

On combining the p-generalized radius coordinate $r_p = (\sum_{i=1}^n |x_i|^p)^{1/p}$ in n dimensions with the $l_{2,p}$-generalized sine and cosine functions, we define $l_{n,p}$-spherical coordinates by replacing the familiar trigonometric functions in the well-known definition of n-dimensional polar coordinates with their $l_{2,p}$-generalized extensions from Definition 1.

Definition 2. The $l_{n,p}$-spherical coordinate transformation $SPH_p : M_n \to R^n$, $M_n = [0, \infty) \times M_n^*$, $M_n^* = [0, \pi)^{\times(n-2)} \times [0, 2\pi)$, is defined for each $p > 0$ by $x_1 = r\cos_p(\varphi_1)$, $x_2 = r\sin_p(\varphi_1)\cos_p(\varphi_2), \ldots, x_{n-1} = r\sin_p(\varphi_1) \cdot \cdots \cdot \sin_p(\varphi_{n-2})\cos_p(\varphi_{n-1})$, $x_n = r\sin_p(\varphi_1) \cdot \cdots \cdot \sin_p(\varphi_{n-2})\sin_p(\varphi_{n-1})$.

If $n = 2$ then this transformation is called $l_{2,p}$-generalized polar coordinate transformation.

The following theorem shows that the angles $\varphi_1, \ldots, \varphi_{n-1}$ from Definition 2 can be interpreted analogously to the special case $p = 2$ and the interpretation of the variable r is that of the p-generalized radius r_p.

Let $\arccos_p$ denote the inverse function of $\cos_p$.

Theorem 1. *The map SPH_p is almost one-to-one and its inverse is given for each $p > 0$ by*

$$r = \left(\sum_{i=1}^n |x_i|^p\right)^{1/p}, \qquad \varphi_i = \arccos_p\left(\frac{x_i}{(\sum_{j=i}^n |x_j|^p)^{1/p}}\right), \quad i = 1, \ldots, n-2,$$

$$\varphi_{n-1} = \arctan\frac{x_n}{x_{n-1}}.$$

In the next definition we introduce to $l_{n,p}$-simplicial coordinates. These coordinates will be used in the proof of Theorem 2.

Definition 3. The $l_{n,p}$-simplicial coordinate transformations $\mathrm{Sim}_{p,n}^{+(-)} : N_n \to R^{n-1} \times R^{+(-)}$ with $N_n = [0, \infty) \times N_n^*$, $N_n^* = [-1, 1]^{\times(n-1)}$ are defined for each $p > 0$ by

$$x_i = \tilde{r}\left[\prod_{j=1}^{i-1}\left(1-|\mu_j|^p\right)^{\frac{1}{p}}\right]\mu_i, \quad i=1,\dots,n-1,$$

$$x_n = +(-)\tilde{r}\left(1-|\mu_1|^p\right)^{\frac{1}{p}}\cdot\dots\cdot\left(1-|\mu_{n-1}|^p\right)^{\frac{1}{p}}.$$

If $n=2$ then this transformation is called $l_{2,p}$-generalized standard triangle coordinate transformation. For finding the inverse maps of $\mathrm{Sim}_{p,n}^{+(-)}$ we refer to Theorem 1. Because of the basic equation $(*)$, one has only to put $\cos_p(\varphi_i)=\mu_i$, $i=1,\dots,n-1$, and can consider then $\cos_p(\varphi_i)$ in Definition 2 playing the role of the simplicial coordinate μ_i in Definition 3 and $\sin_p(\varphi_i)$ that of $(1-|\mu_i|^p)^{1/p}$. In other words, $l_{n,p}$-simplicial coordinates can be interpreted in terms of $l_{2,p}$-generalized trigonometric functions. This connection between the two types of coordinates arranges a corresponding connection between "round balls" and "angular simplicia" and will be exploited in the proof of the following theorem which is basic in many applications of the new coordinates.

Theorem 2. *The $l_{n,p}$-spherical coordinate transformation is almost on-to-one and its Jacobian satisfies for each $p>0$ the representation*

$$J(SPH_p)(r,\varphi)=r^{n-1}J^*(SPH_p)(\varphi), \quad (r,\varphi)\in M_n,\ \varphi_i\neq k\pi/2,\ \forall i, \forall k\in N,$$

$$J^*(SPH_p)(\varphi)=\prod_{i=1}^{n-1}(\sin\varphi_i)^{n-1-i}/\left(N_p(\varphi_i)\right)^{n+1-i}.$$

Remark 1. The Jacobians $J^*(SPH_p)$ and $J^*(SPH_2)$ are connected by the equation

$$J^*(SPH_p)(\varphi)=J^*(SPH_2)(\varphi)\Big/\prod_{i=1}^{n-1}\left(N_p(\varphi_i)\right)^{(n+1-i)/p}.$$

4. Applications and interpretations

4.1. The p-generalized elliptical coordinates

The set $E_p(\kappa)=\{(x,y)\in R^2\colon |\frac{x}{a}|^p+|\frac{y}{b}|^p\leqslant\kappa^p\}$, $p>0$, $\kappa>0$; $a>0$, $b>0$, may be considered as a p-generalized ellipse. Let the p-generalized elliptical coordinate transformation $POL_p^{a,b}:[0,\infty)\times[0,2\pi)\to R^2$ be defined by $x=ar\cos_p(\varphi)$ and $y=br\sin_p(\varphi)$. Then $r=(|\frac{x}{a}|^p+|\frac{y}{b}|^p)^{1/p}$, $\varphi=\arctan(\frac{ay}{bx})$ and the Jacobian is $J(POL_p^{a,b})(r,\varphi)=abr$. The p-generalized ellipse allows the representation $E_p(\kappa)=POL_p^{a,b}(\{(r,\varphi)\colon r\leqslant\kappa\})$.

4.2. Volume and surface content of the p-generalized ball

Let λ_n be the Lebesgue measure in R^n, $M_n(\varrho)=[0,\varrho)\times M_n^*$ and $K_{n,p}(\varrho)=SPH_p(M_n(\varrho))$ the centered $l_{n,p}$-ball of p-generalized radius ϱ. Then

$$\lambda_n\left(K_{n,p}(\varrho)\right)=\int_{M_n(\varrho)} J(SPH_p)\,d\varphi_{n-1}\,d\varphi_{n-2}\dots d\varphi_1\,dr=\frac{\varrho^n}{n}\omega_{n,p}$$

where $\omega_{n,p} = \int_{M_n^*} J^*(SPH_p)(\varphi)\,d\varphi$. It follows

$$\frac{d}{d\varrho}\lambda_n\big(K_{n,p}(\varrho)\big) = \varrho^{n-1}\omega_{n,p}.$$

It is well known that if $p = 2$ then $\frac{d}{d\varrho}\lambda_n(K_{n,p}(\varrho))$ coincides with the surface content of the $l_{n,p}$-sphere $S_{n,p}(\varrho)$ which is the same as ϱ^{n-1} times the surface content of the unit sphere $S_{n,p}(1)$. However, this is not so for arbitrary $p > 0$. To see this, it suffices to deal with the case $n = 2$. To this end, let us consider the special case of the p-generalized circle area

$$K_{2,p}(\varrho) = \bigcup_{0 \leqslant r \leqslant \varrho} \mathfrak{C}_{2,p}(r)$$

where $\mathfrak{C}_{2,p}(r) = \{(x, y) \in R^2\colon\ |x|^p + |y|^p = r^p\} = \partial K_{2,p}(r)$, $r > 0$, is the p-generalized circle line with the p-generalized radius r. The corresponding area content and the arc-length are

$$\lambda_2\big(K_{2,p}(\varrho)\big) = \int\limits_{K_{2,p}(\varrho)} dx\,dy \quad \text{and} \quad \mathfrak{L}\big(\mathfrak{C}_{2,\mathfrak{p}}(\mathfrak{r})\big) = 8\int\limits_0^{\pi/4} \big(x'^2(r,\varphi) + y'^2(r,\varphi)\big)^{1/2}\,d\varphi,$$

respectively, where prime denotes the derivative with respect to the angle φ. Changing Cartesian with $l_{2,p}$-generalized polar coordinates

$$x = x(r,\varphi) = r\cos_p(\varphi), \qquad y = y(r,\varphi) = r\sin_p(\varphi), \quad 0 \leqslant \varphi \leqslant \pi/2,\ 0 \leqslant r \leqslant \varrho,$$

we get

$$r = \big(|x|^p + |y|^p\big)^{1/p}, \quad \varphi = \arctan\left(\frac{y}{x}\right) \quad \text{and} \quad \frac{D(x,y)}{D(r,\varphi)} = \frac{r}{(N_p(\varphi))^2}.$$

Hence,

$$\lambda_2\big(K_{2,p}(\varrho)\big) = 8\int\limits_0^{\varrho} r\left[\int\limits_0^{\pi/4} \frac{d\varphi}{(N_p(\varphi))^2}\right]dr = 4\varrho^2 \int\limits_0^{\pi/4} \frac{d\varphi}{(N_p(\varphi))^2}$$

and

$$\frac{d}{d\varrho}\lambda_2\big(K_{2,p}(\varrho)\big) = 8\varrho \int\limits_0^{\pi/4} \frac{d\varphi}{(N_p(\varphi))^2}.$$

With the relations

$$x'(\varphi) = -r\sin_p(\varphi)\frac{(\sin\varphi)^{p-2}}{(N_p(\varphi))^p} \quad \text{and} \quad y'(\varphi) = r\cos_p(\varphi)\frac{(\cos\varphi)^{p-2}}{(N_p(\varphi))^p},$$

it follows that

$$\mathfrak{L}\big(\mathfrak{C}_{2,p}(\varrho)\big) = 8\varrho \int\limits_0^{\pi/4} \frac{[(\sin_p(\varphi))^2(\sin\varphi)^{2(p-2)} + (\cos_p(\varphi))^2(\cos\varphi)^{2(p-2)}]^{1/2}}{(N_p(\varphi))^p}\,d\varphi$$

$$= 8\varrho \int\limits_0^{\pi/4} \frac{(1 + (\tan\varphi)^{2p-2})^{1/2}}{(1 + (\tan\varphi)^p)^{1-1/p}}\,\frac{d\varphi}{(N_p(\varphi))^2}.$$

Note that $0 \leqslant \tan\varphi \leqslant 1$ for $0 \leqslant \varphi \leqslant \pi/4$. Further, we have $2p-2 > p$ iff $p > 2$ and $p > 2$ iff $1 - 1/p > 1/2$. In that case,

$$\left(1 + (\tan\varphi)^{2p-2}\right)^{1/2} < \left(1 + (\tan\varphi)^{p}\right)^{1-1/p}.$$

The following theorem has thus been proved.

Theorem 3. *The relations* $\mathfrak{L}(\mathfrak{C}_{2,p}(\varrho)) < (=)(>) \frac{d}{d\varrho}\lambda_2(K_{2,p}(\varrho))$ *hold if and only if* $p > (=)(<)\, 2$*, respectively.*

Remark 2. In the "flat" case $p = 1$, we have the proportionality relation

$$\mathfrak{L}\big(\mathfrak{C}_{2,1}(\varrho)\big) = \sqrt{2}\frac{d}{d\varrho}\lambda_2\big(K_{2,1}(\varrho)\big).$$

4.3. The $l_{n,p}$-generalized surface content of the $l_{n,p}$-sphere

Methods for determining volumes of multi-dimensional solids and methods for determining surface contents of the solid's intersections with suitably chosen (hyper-)planes are nearly connected with each other through Cavalieri's famous indivisibeln method. This is analogously true in Torricelli's extension of this method using $l_{n,2}$-spheres as indivisibeln. An additional weight function for the indivisibeln was introduced in [2] and [3] for deriving a geometric measure representation of the Gaussian law. For an extension of this approach to $l_{n,2}$-spherical distributions see [4] and also [5].

This subsection deals with a further extension of the indivisibeln method in the sense of two aspects. The first one is that we shall consider indivisibeln which are, in general, not parallel sets as in the case $p = 2$, but are $l_{n,p}$-spheres being nonparallel for different p-generalized radii if $p \neq 2$. From this basic effect, however, it results the following seemingly insuperable conflict. On the one hand, in accordance with the method of Cavalieri and Torricelli, the derivation of the volume of the $l_{n,2}$-ball with respect to its radius equals the surface area content. On the other hand, however, as we have seen in the preceding subsection, we cannot expect an analogous relation in the case of arbitrary $p > 0$, i.e., if we consider the volume of a $l_{n,p}$-ball and take its derivative with respect to the p-generalized radius $r = (\sum_i |x_i|^p)^{1/p}$ for $p \neq 2$. In other words, the volume is not still the "sum" or, more precisely, the integral of the surface contents of the indivisibeln if $p \neq 2$. This circumstance is in some dual sense expressed in the so-called co-area formula of analysis where the integral of suitably defined surface contents does not coincide with the corresponding volume, in general. Instead, this integral can be interpreted as a certain generalized volume. Hence, it may be a natural way out of this conflict that we shall consider the derivation of the volume of the $l_{n,p}$-ball with respect to the p-generalized radius as the p-generalized surface content of the $l_{n,p}$-sphere. This is the second aspect of the present extension of the method of Cavalieri and Torricelli. Before defining this formally, we introduce the following notions. The Borel σ-field on $S_{n,p}(1)$ will be denoted by $\mathfrak{B}_{S,p}$. The set

$$CPC_p(A) = \left\{x \in R^n\colon\ x \Big/ \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p} \in A\right\}$$

will be called the central projection cone of the set $A \in \mathfrak{B}_{S,p}$ and the set $\mathrm{sector}_p(A, \varrho) = CPC_p(A) \cap K_{n,p}(\varrho)$ will be called a sector of the ball $K_{n,p}(\varrho)$, $\varrho > 0$. Further, let $\mathfrak{B}^n \cap S_{n,p}(r) =: \mathfrak{B}_{n,p}(r)$ be the Borel σ-field on the sphere $S_{n,p}(r)$ with p-generalized radius $r > 0$.

Definition 4. The finite measure $\nu_{n,p}^{(r)} : \mathfrak{B}_{n,p}(r) \to R^+$ which is defined for each $p > 0$ as $\nu_{n,p}^{(r)}(A) = \frac{d}{d\varrho}\lambda_n(\mathrm{sector}_p(\frac{1}{r}A, \varrho))|_{\varrho=r}$, $A \in \mathfrak{B}_{n,p}(r)$, $r > 0$, will be called the $l_{n,p}$-generalized surface measure on the $l_{n,p}$-sphere $S_{n,p}(r)$.

At this stage of investigation, Definition 4 may still seem to be formally. This yet possible impression will change, however, at the latest after having read the remaining part of this subsection as well as Definition 8 and Remark 4 at the end of Section 4.5.

The following theorem can be interpreted as a modified co-area formula or disintegration formula for the n-dimensional Lebesgue measure λ_n. Differently from known such formulas, the left-hand side of our equation represents a volume and the right-hand side makes use of the just defined $l_{n,p}$-generalized surface measure which does, generally speaking, not coincide with the ordinary surface content.

Theorem 4. *For all Borel measurable sets $B \subset R^n$, it is true that*

$$\lambda_n(B) = \int_0^\infty \nu_{n,p}^{(r)}\big(B \cap S_{n,p}(r)\big)\,dr.$$

Note that this formula overcomes the seemingly insuperable conflict described at the beginning of this subsection. It follows from the consideration in the preceding subsection that

$$\nu_{n,p}^{(r)}\big(B \cap S_{n,p}(r)\big) = r^{n-1} Q_p\left(\frac{1}{r}B \cap S_{n,p}(1)\right), \quad B \in \mathfrak{B}_{n,p}(r)$$

where the finite measure $Q_p : \mathfrak{B}_{S,p} \to R^+$ is defined with $SPH_p^*(\varphi) = SPH_p(1, \varphi)$ as

$$Q_p(A) = \int\limits_{SPH_p^{*-1}(A)} J^*(SPH_p)(\varphi)\,d\varphi, \quad A \in \mathfrak{B}_{S,p}.$$

Theorem 4 can therefore be reformulated as

$$\mathrm{vol}(B) = \omega_{n,p} \int_0^\infty r^{n-1}\mathfrak{F}_{B,p}(r)\,dr, \quad B \in \mathfrak{B}^n.$$

Here, according to [2–5,10,11], we call

$$r \to \mathfrak{F}_{B,p}(r) = Q_p\left(\frac{1}{r}B \cap S_{n,p}(1)\right) \Big/ Q_p\big(S_{n,p}(1)\big)$$

the $l_{n,p}$-sphere intersection percentage function of the set B.

Thus, if we look at the triple (volume, surface, radius) then the definition of the volume remains unchanged here, as usual, but the definitions of the radius and the surface content have been changed simultaneously.

Example. For a possible technical application of the notion of the p-generalized surface content imagine the following situation. Let a workpiece being produced on a machine tool which is constructed like an $l_{2,p}$-pair of compasses, i.e. a "pair of compasses" moving a tool along a $l_{2,p}$-circle line of fixed p-generalized radius $r > 0$. For refining the surface structure, the machine moves in a second step another tool along the $l_{2,p}$-circle line of radius $r+\varepsilon$, creating thereby a thin protective coat by applying a special material to the workpiece's surface. The consumed material has then approximately, that is for small ε, the volume $\varepsilon \cdot \nu_{2,p}^{(r)}(S_{2,p}(r))$, i.e., ε times the p-generalized arc-length of the p-generalized $l_{2,p}$-circle line of p-generalized radius r. Hence, the $l_{n,p}$-generalized surface measure allows a true geometric interpretation in real life applications.

4.4. *Formal definitions of generalized sine and cosine functions*

4.4.1. *A characteristic first order differential equation system*

Recall that there are different possibilities besides the geometric one to define the trigonometric functions sine and cosine. One of them is to solve a suitable differential equation system. To this end, let f and g denote real functions of a real variable and assume that they satisfy the differential equations $f'(x)=g(x)$, $g'(x)=-f(x)$ and the initial conditions $f(0)=0$, $g(0)=1$. Then $f(x)=\sin x$, $g(x)=\cos x$.

Which differential equation system should one start from if one would like to define the p-generalized trigonometric functions in an analogous way? The following consideration deals with this question. It follows from Section 2 that for $\varphi \in (0,2\pi)$, $\varphi \neq k\pi/2$, $k\in\{1,2,3\}$, we have

$$\begin{aligned}
\sin_p'(\varphi) &= \cos_p(\varphi)\left|\cos_p(\varphi)\right|^{p-2} N_p(\varphi)^{-2},\\
\cos_p'(\varphi) &= -\sin_p(\varphi)\left|\sin_p(\varphi)\right|^{p-2} N_p(\varphi)^{-2},\\
N_p'(\varphi) &= N_p(\varphi)\left[\cos_p(\varphi)\sin_p(\varphi)\left|\sin_p(\varphi)\right|^{p-2} - \sin_p(\varphi)\cos_p(\varphi)\left|\cos_p(\varphi)\right|^{p-2}\right].
\end{aligned}$$

In other words, the functions looked for satisfy the differential equation system

$$\begin{aligned}
&f'(x) = g(x)\frac{|g(x)|^{p-2}}{(h(x))^2}, \qquad g'(x) = -f(x)\frac{|f(x)|^{p-2}}{(h(x))^2},\\
&h'(x) = h(x)f(x)g(x)\left[\left|f(x)\right|^{p-2} - \left|g(x)\right|^{p-2}\right]
\end{aligned}$$

with initial conditions $f(0)=0$, $g(0)=1$ and $h(0)=1$.

4.4.2. *Integral representation for the inverse p-generalized sine function*

Another possibility besides the geometric one is to consider the integral

$$AS(x) = \int_0^x \frac{dy}{\sqrt{1-y^2}}, \quad -1<x<1,$$

and to define the sine function as the inverse of the AS function. If one replaces the integral AS by an elliptic integral then one gets a well known generalization of the sine function. In the following theorem, we show that the inverse of the p-generalized sine function is another generalization of the AS function.

Theorem 5. *The p-generalized sine function is the inverse of the function*

$$AS_p(x) = \int_0^x \left(y^2\left(1+\left[\frac{1}{|y|^p}-1\right]^{2/p}\right)\left(1-|y|^p\right)^{1-\frac{1}{p}}\right)^{-1} dy.$$

4.5. The l_p-generalizations of classical statistical distributions

Let $X_1, \dots, X_n$ denote independent and identically distributed random variables and assume that their common density function is

$$f(x) = C_p \exp\left\{-\frac{|x|^p}{p}\right\}, \quad x \in R \text{ with } C_p = p^{1-\frac{1}{p}} \Big/ \left[2\Gamma\left(\frac{1}{p}\right)\right].$$

This means that the distribution N_p of the random vector $X_{(n)} = (X_1, \dots, X_n)$ is the so-called p-generalized n-dimensional normal distribution which was introduced in [7]. In other words, $X_{(n)}$ follows the $l_{n,p}$-spherical or, if $p \geqslant 1$, the $l_{n,p}$-norm symmetric distribution N_p. With the notation $\xi_p = (|X_1|^p + \dots + |X_n|^p)^{1/p}$, we consider the following p-generalization of the well-known χ-distribution function with n degrees of freedom

$$F_p(x) = P(\xi_p < x) = C_p^n \int\limits_{K_{n,p}(x)} \exp\left\{-\frac{1}{p}\sum_{i=1}^n |x_i|^p\right\} dx_1 \dots dx_n, \quad x \in R.$$

Changing Cartesian with $l_{n,p}$-spherical coordinates, we get

$$F_p(x) = C_p^n \omega_{n,p} \int\limits_0^x r^{n-1} \exp\left\{-\frac{r^p}{p}\right\} dr.$$

The corresponding probability density function is

$$f_p(x) = \frac{d}{dx} F_p(x) = I_{(0,\infty)}(x) C_p^n \omega_{n,p} x^{n-1} \exp\left\{-\frac{x^p}{p}\right\}.$$

Note that this distribution is a special Kotz type distribution. From our approach, however, it turns out to be natural to consider it as a p-generalization of the well-known χ-distribution. Similarly, the distribution of ξ_p^p may be considered as a p-generalization of the χ^2-distribution. Its density is

$$\frac{d}{dx} P\left(\xi_p^p < x\right) = \frac{1}{p} x^{\frac{1}{p}-1} f_p\left(x^{\frac{1}{p}}\right).$$

From the relation $1 = \lim_{x\to\infty} F_p(x)$ it follows

$$(\omega_{n,p})^{-1} = C_p^n \int\limits_0^\infty r^{n-1} \exp\left\{-\frac{r^p}{p}\right\} dr$$

and with

$$I_{n,p} = \int\limits_0^\infty r^{n-1} \exp\left\{-\frac{r^p}{p}\right\} dr = p^{\frac{n}{p}-1} \Gamma\left(\frac{n}{p}\right)$$

we get

$$\omega_{n,p} = \frac{2^n \Gamma(\frac{1}{p})^n}{p^{n-1}\Gamma(\frac{n}{p})}.$$

The following definition has thus been well motivated and the subsequent theorem has been just derived.

Definition 5. A continuous random variable Z is said to be distributed according to the p-generalized χ-distribution with n degrees of freedom (d.f.) if its density function is

$$f_p(x) = I_{(0,\infty)}(x)\frac{1}{p^{\frac{n}{p}-1}\Gamma(\frac{n}{p})}x^{n-1}\exp\left\{-\frac{x^p}{p}\right\}, \quad x \in R,$$

symbolically $Z \sim \chi(p,n)$. A continuous random variable Y is said to be distributed according to the χ^p-distribution (or p-generalized χ^2-distribution) with n d.f. if its density function is

$$f_{n,p}(x) = I_{(0,\infty)}(x)\frac{1}{p^{\frac{n}{p}}\Gamma(\frac{n}{p})}x^{\frac{n}{p}-1}\exp\left\{-\frac{x}{p}\right\}, \quad x \in R,$$

symbolically $Y \sim \chi^p(n)$. Here, n is a natural and p is a positive real number.

Theorem 6. *If $X_{(n)}$ is a N_p-distributed random sample then*

$$\xi_p \sim \chi(p,n) \quad \textit{and} \quad \xi_p^p \sim \chi^p(n).$$

Remark 3. If

$$f(x|n) = \frac{x^{\frac{n}{2}-1}e^{-x/2}}{2^{n/2}\Gamma(\frac{n}{2})}I_{(0,\infty)}(x), \quad x \in R,$$

denotes the well-known chi-square density function with n d.f. then it is possible to represent the χ^p-density formally with the help of the usual chi-square density function as

$$f_{n,p}(x) = \frac{2}{p}f\left(\frac{2}{p}x\,\middle|\,\frac{2}{p}n\right).$$

From an analytical point of view, one could like therefore to say that the definition of the ordinary χ^2-distribution is extended here to the case that the degree of freedom is not necessary a natural number. From a statistical as well as from a geometrical point of view, however, one should prefer to call the quantity $\frac{2n}{p}$ under this circumstances simply a positive parameter instead of a degree of freedom. The following definition could therefore be considered as a possible alternative to Definition 5.

Definition 6. A continuous random variable X is said to be distributed according to a two parameter chi-square distribution with parameters $n \in N$ and $p > 0$ if its density function is $f_{n,p}$.

Applications of spherical coordinates to the derivation of statistical distributions can be found in [8,9] and [2,10]. For another generalization of spherical coordinates we refer to [1]. Multidimensional simplicial or Jacobi coordinates and their application to statistical distributions were introduced in [11]. Further applications of these coordinates to the derivation of statistical distributions can be found also in [12,13].

We consider now the following p-generalized Student type statistic:

$$T(p) = \frac{X_1}{(\frac{1}{n-1}[|X_2|^p + \cdots + |X_n|^p])^{1/p}}.$$

Definition 7. A continuous random variable Z is said to be distributed according to the p-generalized Student- (or t-)distribution with n d.f. if its density is

$$f(x) = \frac{p\Gamma(\frac{n+1}{p})}{2n^{1/p}\Gamma(\frac{n}{p})\Gamma(\frac{1}{p})}\left(1+\frac{x^p}{n}\right)^{-\frac{n+1}{p}}, \quad x \in R;$$

symbolically $Z \sim t_n(p)$.

Theorem 7. *If $X_{(n)}$ is a N_p-distributed random sample then the p-generalized Student type statistic follows the p-generalized t-distribution with $n-1$ d.f., i.e., $T(p) \sim t_{n-1}(p)$.*

As announced in Section 4.3, we come finally back to Definition 4 and the $l_{n,p}$-generalized surface measure introduced there.

Definition 8. The normalized measure $U_{n,p}^{(r)}(A) = \nu_{n,p}^{(r)}(A)/\nu_{n,p}^{(r)}(S_{n,p}(r))$, $A \in \mathfrak{B}_{n,p}(r)$, will be called the $l_{n,p}$-generalized uniform probability distribution on the sphere $S_{n,p}(r)$.

Remark 4. The $l_{n,p}$-generalized uniform probability distribution $U_{n,p}^{(1)}$ on the unit sphere $S_{n,p}(1)$ as it was defined in Definition 8 coincides with a distribution having a similar name, i.e., with the uniform distribution on $S_{n,p}(1)$ which was introduced in [1] and [6] in a different way and without any geometric interpretation. Note that the $l_{n,p}$-generalized uniform distribution $U_{n,p}^{(1)}$ on the unit sphere $S_{n,p}(1)$ can be interpreted as a geometric probability measure in terms of the $l_{n,p}$-generalized surface content but not in terms of the (usual) surface content, unless in the case $p = 2$. The name uniform distribution on $S_{n,p}(1)$ could therefore be used instead for the geometric probability measure on $S_{n,p}(1)$ in terms of the usual surface content. For details, see the proof of this remark.

5. Proofs

Proof of Theorem 1. It follows from the equation $|\sin_p(\varphi_{n-1})|^p + |\cos_p(\varphi_{n-1})|^p = 1$ that

$$|x_{n-1}|^p + |x_n|^p = r^p\left|\sin_p(\varphi_1)\right|^p \cdot \dots \cdot \left|\sin_p(\varphi_{n-2})\right|^p.$$

Iteratively, we see that

$$|x_{n-2}|^p + |x_{n-1}|^p + |x_n|^p = r^p\left|\sin_p(\varphi_1)\right|^p \cdot \dots \cdot \left|\sin_p(\varphi_{n-3})\right|^p, \quad \dots,$$
$$\sum_{j=i}^{n} |x_j|^p = r^p\left|\sin_p(\varphi_1)\right|^p \cdot \dots \cdot \left|\sin_p(\varphi_{i-1})\right|^p, \quad i = 2, \dots, n-1.$$

Finally, we have $|x_1|^p + \dots + |x_n|^p = r^p$, i.e., r is uniquely defined as $r = (\sum_{i=1}^n |x_i|^p)^{1/p}$. It follows from the definition of the transformation SPH_p that $\cos_p(\varphi_1) = x_1/r$, hence $\varphi_1 \in [0, \pi)$ is uniquely defined as $\varphi_1 = \arccos_p(x_1/r)$. It follows also from the definition of SPH_p that $|x_2|^p = r^p(1 - |\cos_p(\varphi_1)|^p)|\cos_p(\varphi_2)|^p$. On combining this with

$$\sum_{j=2}^{n} |x_j|^p = r^p - |x_1|^p = r^p\left(1 - \left|\cos_p(\varphi_1)\right|^p\right),$$

we get

$$\left|\cos_p(\varphi_2)\right| = |x_2| \Big/ \left(\sum_{j=2}^{n} |x_j|^p\right)^{1/p}.$$

Furthermore, $\operatorname{sign}\{\cos_p(\varphi_2)\} = \operatorname{sign}\{x_2\}$, i.e.

$$\cos_p(\varphi_2) = x_2 \Big/ \left(\sum_{j=2}^{n} |x_j|^p\right)^{1/p}.$$

This means that $\varphi_2 \in [0, \pi)$ is uniquely defined as

$$\varphi_2 = \arccos_p\left(x_2 \Big/ \left(\sum_{j=2}^{n} |x_j|^p\right)^{1/p}\right).$$

Continuing this way, we see finally that $\varphi_{n-2} \in [0, \pi)$ is uniquely defined as

$$\varphi_{n-2} = \arccos_p\left(x_{n-2} \Big/ \left(\sum_{j=n-2}^{n} |x_j|^p\right)^{1/p}\right).$$

The last assertion of the theorem follows because $\varphi_{n-1} \in [0, 2\pi)$ satisfies $\operatorname{sign}\{\cos_p(\varphi_{n-1})\} = \operatorname{sign}\{x_{n-1}\}$, $\operatorname{sign}\{\sin_p(\varphi_{n-1})\} = \operatorname{sign}\{x_n\}$. $\square$

Proof of Theorem 2. The proof will be given in three steps. First, we recall that in the same way as the Jacobian is derived, e.g., in [14] for the usual $l_{n,2}$-spherical coordinates, one can check that the $l_{n,p}$-simplicial coordinate transformation has the Jacobian

$$J\left(\mathrm{Sim}_{p,n}^{+(-)}\right) = \left|\frac{D(x_1, \ldots, x_n)}{D(\tilde{r}, \mu_1, \ldots, \mu_{n-1})}\right| = \tilde{r}^{n-1} J^*\left(\mathrm{Sim}_{p,n}^{+(-)}\right),$$
$$J^*\left(\mathrm{Sim}_{p,n}^{+(-)}\right) = \prod_{i=1}^{n-1} \left(1 - |\mu_i|^p\right)^{(n-p-i)/p}.$$

Changing now $l_{n,p}$-simplicial coordinates with $l_{n,p}$-spherical coordinates means to consider in the second step the transformation $SPHSIM_p : M_n \to N_n$ being defined by $\tilde{r} = r$, $\mu_i = \cos_p(\varphi_i)$, $i = 1, \ldots, n-1$. Its Jacobian is

$$\begin{aligned}
J(SPHSIM_p) &= \left|\frac{D(\tilde{r}, \mu_1, \ldots, \mu_{n-1})}{D(r, \varphi_1, \ldots, \varphi_{n-1})}\right| \\
&= \left|\det \operatorname{diag}\left(\frac{d}{d\varphi_1} \cos_p(\varphi_1), \ldots, \frac{d}{d\varphi_{n-1}} \cos_p(\varphi_{n-1}), 1\right)\right| \\
&= \prod_{i=1}^{n-1} \left|\sin_p(\varphi_i)\right| \frac{|\sin\varphi_i|^{p-2}}{|\sin\varphi_i|^p + |\cos\varphi_i|^p}.
\end{aligned}$$

On combining these two maps, we get in the third step

$$J(SPH_p) = J\big(SIM^{+(-)}_{p,n}\big)J(SPHSIM_p)$$
$$= r^{n-1}\prod_{i=1}^{n-1}\big|\sin_p(\varphi_i)\big|^{n-p-i+1}\frac{|\sin\varphi_i|^{p-2}}{|\sin\varphi_i|^p+|\cos\varphi_i|^p}, \quad \text{i.e.,}$$
$$J^*(SPH_p)(\varphi) = \prod_{i=1}^{n-1}\big|\sin_p(\varphi_i)\big|^{n-1-i}/\big(N_p(\varphi_i)\big)^2.$$

Finally, $\sin\varphi_i \geqslant 0$ because $\varphi_i \in [0,\pi]$ for $i = 1,\dots,n-2$. $\square$

Proof of Theorem 4. The collection $\mathfrak{S}_p$ of all sets of the type

$$\text{sector}_p(D,\varrho_2)\setminus\text{sector}_p(D,\varrho_1) =: A_p(D;\varrho_1,\varrho_2), \quad 0<\varrho_1<\varrho_2<\infty,\ D\in\mathfrak{B}_{S,p},$$

is a semi-ring. We consider a finite, additive set function on it by

$$\lambda^*_{n,p}\big(A_p(D,\varrho_1,\varrho_2)\big) = \int_{\varrho_1}^{\varrho_2}\nu^{(r)}_{n,p}\big(A_p(D,\varrho_1,\varrho_2)\cap S_{n,p}(r)\big)\,dr.$$

Denote the smallest ring including $\mathfrak{S}_p$ by $\mathfrak{R}_p$. If (A_k) is a sequence from $\mathfrak{R}_p$ satisfying $A_{k+1}\subset A_k$, $\forall k$ and $\bigcap_{k=1}^{\infty}A_k=\emptyset$ then $\lambda^*_{n,p}(A_k)\to 0$ as $k\to\infty$, i.e., $\lambda^*_{n,p}$ is continuous at $\emptyset$ and therefore countable additive on $\mathfrak{R}_p$. Because

$$\nu^{(r)}_{n,p}\big(A_p(D,\varrho_1,\varrho_2)\cap S_{n,p}(r)\big) = r^{n-1}Q_p(D), \quad r\in(\varrho_1,\varrho_2),$$

there holds

$$\lambda^*_{n,p}\big(A_p(D,\varrho_1,\varrho_2)\big) = Q_p(D)\left(\frac{\varrho_2^n}{n}-\frac{\varrho_1^n}{n}\right).$$

Changing Cartesian with $l_{n,p}$-spherical coordinates, we get further

$$\text{vol}\big([\varrho_1,\varrho_2]\times D\big) = Q_p(D)\left(\frac{\varrho_2^n}{n}-\frac{\varrho_1^n}{n}\right).$$

Hence, $\lambda^*_{n,p}$ and the Lebesgue measure $\lambda_n = \text{vol}$ coincide on $\mathfrak{R}_p$. It follows by measure extension theorem that $\lambda^*_{n,p}$ and λ_n coincide on the σ-algebra of all Borel measurable sets $B\subset R^n$. $\square$

Proof of Theorem 5. Let $\Phi^{-1}(x)$ be the inverse of the differentiable function $y=\Phi(x)$, then

$$\frac{d}{dx}\Phi^{-1}(x) = \frac{1}{\Phi'(\Phi^{-1}(x))}.$$

With $\Phi(x)=\sin_p(x)$ and $\Phi^{-1}(x)=\arcsin_p(x)$, it follows that

$$\frac{d}{dx}\arcsin_p(x) = \big(\sin_p'(y)\big)^{-1}\big|_{y=\arcsin_p(x)} = \frac{N_p^2(\arcsin_p(x))}{[\cos_p(\arcsin_p(x))]^{p-1}}.$$

It follows from Eq. $(*)$ that $|\cos_p(\arcsin_p(x))|^p = 1-|x|^p$. It remains now to consider $N_p^2(\arcsin_p(x))$. We have

$$\sin\big(\arcsin_p(x)\big) = \sin_p\big(\arcsin_p(x)\big)\big(|\sin\varphi|^p+|\cos\varphi|^p\big)^{1/p}$$

with $\varphi = \arcsin_p(x)$. If we put $a = \sin\varphi$ and $b = \cos\varphi$ then it follows $a = x(|a|^p + |b|^p)^{1/p}$. On combining this with the equation $a^2 + b^2 = 1$, it follows

$$|b|^p = \left(1 - a^2\right)^{p/2}, \qquad |a| = \left(1 + \left(\frac{1}{|x|^p} - 1\right)^{2/p}\right)^{-\frac{1}{2}} = \left|\sin\left(\arcsin_p(x)\right)\right|,$$

$$|b| = \frac{\left(\frac{1}{|x|^p} - 1\right)^{1/p}}{\left(1 + \left(\frac{1}{|x|^p} - 1\right)^{2/p}\right)^{1/2}} = \left|\cos\left(\arcsin_p(x)\right)\right|.$$

Thus, we have

$$N_p^2\left(\arcsin_p(x)\right) = \left(|a|^p + |b|^p\right)^{2/p} = \frac{1}{x^2 + (1 - |x|^p)^{2/p}},$$

$$\frac{d}{dx}\arcsin_p(x) = \frac{1}{(x^2 + (1 - |x|^p)^{2/p})(1 - |x|^p)^{(p-1)/p}}. \qquad \square$$

Proof of Theorem 7. Let us consider distribution function $G_p(t) = P(T(p) < t)$, $t \in R$. Obviously,

$$G_p(t) = P(X_1 \leqslant 0) + C_p^n \int\limits_{C_p(t)} \exp\left\{-\frac{1}{p}\sum_{i=1}^n |x_i|^p\right\} dx_1 \ldots dx_n, \quad t > 0,$$

where the set

$$C_p(t) = \left\{x \in R^n\colon\ x_1 > 0,\ \frac{x_1}{(|x_2|^p + \cdots + |x_n|^p)^{1/p}} < \frac{t}{(n-1)^{1/p}}\right\}$$

is a cone in R^n. Using $l_{n,p}$-spherical coordinates, from Section 2 we get

$$x_i \Big/ \left(\sum_{j=i}^n |x_j|^p\right)^{1/p} = \cos_p(\varphi_i), \quad i = 1, \ldots, n-1.$$

Consequently,

$$\left|\sin_p(\varphi_i)\right| = \left(\sum_{j=i+1}^n |x_j|^p\right)^{1/p} \Big/ \left(\sum_{j=i}^n |x_j|^p\right)^{1/p}, \quad i = 1, \ldots, n-1.$$

Recall that $|\sin_p(\varphi_i)| = \sin_p(\varphi_i)$ for $i = 1, \ldots, n-2$, because $\varphi_i \in [0, \pi)$, $i = 1, \ldots, n-2$. On combining the last two equations it follows

$$\cot\varphi_i = x_i \Big/ \left(\sum_{j=i+1}^n |x_j|^p\right)^{1/p}, \quad i = 1, \ldots, n-2,$$

and therefore

$$\begin{aligned} C_p(t) &= SPH_p\left(\left\{(r, \varphi_1, \ldots, \varphi_{n-1}) \in M_n\colon\ 0 \leqslant \cot\varphi_1 < \frac{t}{(n-1)^{1/p}}\right\}\right) \\ &= SPH_p\left(\left\{(r, \varphi_1, \ldots, \varphi_{n-1})\colon\ 0 \leqslant r < \infty,\ 0 \leqslant \cot\varphi_1 < \frac{t}{(n-1)^{1/p}},\right.\right. \\ &\qquad \left.\left. 0 \leqslant \varphi_i < \pi,\ i = 2, \ldots, n-2,\ 0 \leqslant \varphi_{n-1} < 2\pi\right\}\right). \end{aligned}$$

In other words, we have

$$C_p(t) = SPH_p\left(\left\{(r, \varphi_1, \ldots, \varphi_{n-1}) \in M_n\colon \operatorname{arccot}\left(\frac{t}{(n-1)^{1/p}}\right) < \varphi_1 \leqslant \pi/2\right\}\right).$$

This gives

$$G_p(t) - \frac{1}{2} = C_p^n \int_0^\infty r^{n-1} \exp\left\{-\frac{r^p}{p}\right\} dr \int\limits_{M_n^* \cap \{\operatorname{arccot}(\frac{t}{(n-1)^{1/p}}) < \varphi_1 \leqslant \pi/2\}} J^*(SPH_p)(\varphi)\, d\varphi$$

$$= \frac{\omega_{n-1,p}}{\omega_{n,p}} \int\limits_{\operatorname{arccot}(\frac{t}{(n-1)^{1/p}})}^{\pi/2} \frac{(\sin_p(\varphi_1))^{n-2}}{(N_p(\varphi_1))^2}\, d\varphi_1.$$

The probability density function corresponding to this distribution is

$$g_p(t) = \frac{p\Gamma(\frac{n}{p})}{2\Gamma(\frac{n-1}{p})\Gamma(\frac{1}{p})} \frac{(\sin_p(\operatorname{arccot}(\frac{t}{(n-1)^{1/p}})))^{n-2}}{(N_p(\operatorname{arccot}(\frac{t}{(n-1)^{1/p}})))^2} \left(\frac{1}{1 + \frac{t^2}{(n-1)^{2/p}}}\right) \frac{1}{(n-1)^{1/p}}.$$

It follows from the relations

$$\sin(\operatorname{arccot} x) = \frac{1}{\sqrt{1+x^2}} \quad \text{and} \quad \cos(\operatorname{arccot} x) = \frac{x}{\sqrt{1+x^2}}$$

that

$$N_p(\operatorname{arccot} x) = \left(\left(\frac{1}{\sqrt{1+x^2}}\right)^p + \left(\frac{x}{\sqrt{1+x^2}}\right)^p\right)^{1/p} = \left(\frac{1+x^p}{(1+x^2)^{p/2}}\right)^{1/p}$$

and

$$\sin_p(\operatorname{arccot} x) = \frac{1}{(1+|x|^p)^{1/p}}.$$

Hence,

$$g_p(t) = \frac{p\Gamma(\frac{n}{p})}{2\Gamma(\frac{n-1}{p})\Gamma(\frac{1}{p})(n-1)^{1/p}} \left(1 + \frac{t^p}{n-1}\right)^{-\frac{n}{p}}.$$

The last formula is true with t replaced by $|t|$ for all $t \in R$ because of the symmetry of the N_p-distribution. □

Proof of Remark 4. Let us consider the random vector $\theta = X_{(n)}/\xi_p$. Obviously, it takes values on the unit sphere $S_{n,p}(1)$. The distribution $U_{n,p}^*$ of θ, i.e., $U_{n,p}^*(A) = P(\theta \in A)$, $A \in \mathfrak{B}_{S,p}$, is formally dealt with in [1] and [6] in different ways as just the uniform distribution on $S_{n,p}(1)$. It can be written as

$$U_{n,p}^*(A) = P\big(N_p \in CPC_p(A)\big) = C_p^n \int\limits_{CPC_p(A)} \exp\left\{-\frac{1}{p}\sum_{i=1}^n |x_i|^p\right\} dx_1 \ldots dx_n$$

$$= C_p^n Q_p(A) \int_0^\infty r^{n-1} \exp\left\{-\frac{r^p}{p}\right\} dr = Q_p(A)/\omega_{n,p}$$

$$= \nu_{n,p}^{(1)}(A)/\nu_{n,p}^{(1)}\big(S_{n,p}(1)\big) = U_{n,p}(A). \qquad □$$

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